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# $U(3)$ gauge theory on fuzzy extra dimensions

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In this article, we explore the low energy structure of a $U(3)$ gauge theory over spaces with fuzzy sphere(s) as extra dimensions. In particular, we determine the equivariant parametrization of the gauge fields, which transform either invariantly or as vectors under the combined action of $SU(2)$ rotations of the fuzzy spheres and those $U(3)$ gauge transformations generated by $SU(2) \subset U(3)$ carrying the spin 1 irreducible representation of $SU(2)$. The cases of a single fuzzy sphere $S_F^2$ and a particular direct sum of concentric fuzzy spheres, $S_F^{2\,\mathrm{Int}}$, covering the monopole bundle sectors with windings $\pm 1$ are treated in full and the low energy degrees of freedom for the gauge fields are obtained. Employing the parametrizations of the fields in the former case, we determine a low energy action by tracing over the fuzzy sphere and show that the emerging model is Abelian Higgs type with $U(1) \times U(1)$ gauge symmetry and possesses vortex solutions on $\mathbb{R}^2$, which we discuss in some detail. Generalization of our formulation to the equivariant parametrization of gauge fields in $U(n)$ theories is also briefly addressed.

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## I. INTRODUCTION

It is by now very well known that $N = 4$ supersymmetric $SU(\mathcal{N})$ Yang-Mills theories (SYM), deformed by the addition of cubic [soft supersymmetry breaking (SSB)] and mass terms in the scalar matter fields and relatedly $SU(\mathcal{N})$ gauge theories coupled to a triplet of scalars carrying the adjoint representation of $SU(\mathcal{N})$ as well as pure Yang-Mills (YM) matrix models with cubic and quadratic deformation terms develop fuzzy vacua; these are generically described by direct sums of products of fuzzy spheres $\mathcal{S}_F^2 \times \mathcal{S}_F^2$ $(:=\bigoplus S_F^2 \times S_F^2)$ or that of fuzzy spheres $\mathcal{S}_F^2 (:=\bigoplus S_F^2)$ [1–12]. Such fuzzy sphere vacua also appear in Berenstein-Maldacena-Nastase (BMN) matrix models, which were proposed some time ago to give a nonperturbative description of the M-theory on maximally supersymmetric pp-wave backgrounds [13,14].

For the $SU(\mathcal{N})$ YM theory on Minkowski space $\mathbb{M}^4$ coupled to a triplet of adjoint scalar fields, the fuzzy sphere $S_F^2$ vacuum was investigated in [5]. In this model, three matrices describing the $S_F^2$ are the vacuum expectation values (VEVs) of the scalar fields and the $SU(2)$ symmetry of $S_F^2$ is inherited from a global $SU(2)$ gauge symmetry of the YM model. Nonzero VEVs of the scalar fields also imply that the $SU(\mathcal{N})$ gauge symmetry is spontaneously broken down to a $U(n)$, where $\mathcal{N}$, $n$, and the level $\ell$ of the fuzzy sphere are related as $\mathcal{N} = (2\ell + 1)n$. Fluctuations around this vacuum configuration are found to have the structure of $U(n)$ gauge fields over $S_F^2$, which preliminarily indicates that the emerging model after symmetry breaking may be conjectured to be an effective gauge theory over $\mathbb{M}^4 \times S_F^2$. A Kaluza-Klein (KK)-type mode expansion of the gauge fields given in [5] places this interpretation on firm grounds.

Equivariant parametrization (EP) of gauge fields in the framework of these models endows us with a complementary viewpoint in developing the effective gauge theory interpretation and understanding the low energy limit in this and a range of other models, which we have been recently investigating in [8–12]. This method, being akin to coset space dimensional reduction techniques [15,16] (see also [17] in this context), involves imposing proper symmetry conditions on the fields of the model so that they transform covariantly under the action of the symmetry group of the extra dimensions up to the gauge transformations of the emergent model. These conditions may be solved using the representation theory of Lie groups and explicit EPs of all the fields in the model can be obtained providing strong evidence for the interpretation of such models as effective gauge theories, since, subsequently, an effective low energy action (LEA) may be obtained by integrating out (i.e., tracing over) the fuzzy extra dimensions. Models with minimal non-Abelian gauge symmetry, a $U(2)$ model for the case of $\mathcal{M} \times \mathcal{S}_F^2$, and a $U(4)$ model for $\mathcal{M} \times \mathcal{S}_F^2 \times \mathcal{S}_F^2$, where $\mathcal{M}$ denotes a Riemannian or a Lorentzian manifold, have been investigated in [8,10] and LEAs are obtained when the extra dimensions do not have the direct sum structure but are given by a single fuzzy sphere $S_F^2$ or $S_F^2 \times S_F^2$, respectively. LEAs' obtained in this manner leads to Abelian Higgs-type models with vortex solutions for $\mathcal{M} \equiv \mathbb{R}^2$. Applications of equivariant dimensional reduction method on higher dimensional YM theories are reported in [18–24]. Other recent interesting articles within this general setting that we do not want to pass without mentioning include [25–31].

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The outlined developments call for further investigations on the low energy structure around such fuzzy vacua in a diverse class of models with larger gauge groups in order to better assess the potential value of these models from a phenomenological point of view. In this article, we take a step forward and determine in full detail the equivariant field modes of a $U(3)$ gauge theory over $\mathcal{M}\times S_F^2$ and obtain the corresponding LEA by tracing over $S_F^2$. First, we find that equivariant scalars may be constructed by taking advantage of the dipole and quadrupole terms, which appear in the branching of the adjoint representation of $SU(3)$ as $\underline{8}\to\underline{5}\oplus\underline{3}$ when the $SU(2)$ subgroup is maximally embedded in $SU(3)$. More concretely, we use these considerations and other group theoretical inputs coming from the equivariance conditions to construct the invariants as "idempotents" involving intertwiners combining the spin $\ell$ irreducible representation (IRR) of $SU(2)$ generating the rotations of $S_F^2$ and those $U(3)$ gauge transformations generated by $SU(2)\subset U(3)$ carrying the spin 1 IRR of $SU(2)$. There is also another invariant proportional to the $\mathcal{N}$-dimensional identity matrix, which essentially appears due to a $U(1)$ subgroup of $U(3)\approx SU(3)\times U(1)$. Equivariant vectors are built using these invariants and the generators of $S_F^2$. These developments are presented in Sec. III, where we also show that the equivariance conditions break the $U(3)$ symmetry down to the Abelian product group $U(1)\times U(1)\times U(1)$. In Sec. IV, we obtain the LEA, which, in addition to the three Abelian gauge fields that naturally appear, contains two complex scalars each coupling to only one of the gauge fields and three real scalars interacting with the complex fields and with each other through a quartic potential. In the $\ell\to\infty$ limit, we determine the vacuum configuration of this quadric potential and use it in Sec. V to determine vortex solutions to the LEA on $\mathcal{M}\equiv\mathbb{R}^2$ in two different limits governed together by $\ell$ and the coupling constant of the constraint term in the potential, both of which are characterized by two winding numbers. Scattered throughout Secs. III–V, we indicate how the commutative limit of our results relates to the instanton solutions in self-dual $SU(3)$ Yang-Mills theory for cylindrically symmetric gauge fields of Bais and Weldon [32]. In particular, we point out the connection between the Bogomolnyi-Prasad-Sommerfeld (BPS) vortices that we obtain in a certain commutative limit in Sec. V and the instanton solution in [32]. In Sec. VI, we briefly outline the generalization of the EP of gauge fields to $U(n)$ theories over $\mathcal{M}\times S_F^2$, and show that equivariant scalars are obtained by employing the $n-1$ multipole terms, which appear in the branching of the adjoint representation of $SU(n)$ under $SU(2)$, when the latter is maximally embedded in $SU(n)$.

Section VII is devoted to the study of $U(3)$-equivariant fields over $\mathcal{M}\times S_F^{2\,\mathrm{Int}}$, where $S_F^{2\,\mathrm{Int}}\coloneqq S_F^2(\ell)\oplus S_F^2(\ell)\oplus S_F^2(\ell+\frac{1}{2})\oplus S_F^2(\ell-\frac{1}{2})$ was revealed in [11] via a certain field redefinition of the triplet of scalars as a potentially interesting vacuum configuration for the $SU(\mathcal{N})$ YM theory. The reason of interest on this vacuum is twofold. First, through its certain projections it gives us access to fuzzy monopole bundles with winding numbers $\pm1$, in a setting that is readily amenable to explicitly expressing the equivariant field modes; and secondly it naturally identifies with the bosonic part of the $N=2$ fuzzy supersphere with $OSP(2,2)$ supersymmetry as discussed in [11]. Here, we are able to express all the equivariant field modes characterizing the low energy behavior of the effective $U(3)$ theory on $\mathcal{M}\times S_F^{2\,\mathrm{Int}}$ in terms of suitable idempotents and projection operators. From a geometrical point of view the $S_F^{2\,\mathrm{Int}}$ vacuum is akin to stacks of concentric fuzzy D-branes carrying magnetic monopole fluxes, despite the fact that not all the string theoretic aspects [33] may be reproduced within the current framework [1]. Nevertheless, this viewpoint allows us to think of the equivaraint gauge field modes of the effective gauge theory as those living on the world volume of these D-branes, which may prove to be useful in an attempt to relate the effective gauge theory and the string theoretic perspectives.

## II. $U(n)$ GAUGE THEORY OVER $\mathcal{M}\times S_F^2$

In order to orient the developments, we start with briefly explaining how an $SU(\mathcal{N})$ gauge theory on a $d$-dimensional manifold $\mathcal{M}$ coupled to a triplet of adjoint scalar fields spontaneously develops extra dimensions in the form of a fuzzy sphere $S_F^2$ and how a $U(n)$, $(n<\mathcal{N})$, gauge theory on $\mathcal{M}\times S_F^2$ emerges as a consequence [5]. We are interested in the model whose action may be given as

$$S=\int_{\mathcal{M}}d^dy\mathrm{Tr}_{\mathcal{N}}\left(-\frac{1}{4g^2}F_{\mu\nu}^\dagger F^{\mu\nu}-(D_\mu\Phi_a)^\dagger(D^\mu\Phi_a)\right)-\frac{1}{\tilde{g}^2}V_1(\Phi)-\mathfrak{g}^2V_2(\Phi),\tag{2.1}$$

$$V_1(\Phi)=\mathrm{Tr}_{\mathcal{N}}(F_{ab}^\dagger F_{ab}),$$
$$V_2(\Phi)=\mathrm{Tr}_{\mathcal{N}}((\Phi_a\Phi_a+\tilde{b}\mathbf{1}_{\mathcal{N}})^2),\tag{2.2}$$

where $y_\mu(\mu=1,\ldots,d)$ are the coordinates on $\mathcal{M}$. $F_{\mu\nu}$ denotes the gauge connection associated to the $su(\mathcal{N})$ valued anti-Hermitian gauge fields $A_\mu=A_\mu(y)$ on $\mathcal{M}$. $\Phi_a=\Phi_a(y)(a=1,2,3)$ are three anti-Hermitian $\mathcal{N}\times\mathcal{N}$ matrices, whose entries are valued in $\mathcal{M}$. Thus, they are scalar fields transforming in the adjoint representation of $SU(\mathcal{N})$ as

$$\Phi_a\to U^\dagger\Phi_aU,\qquad U\in SU(\mathcal{N}),\tag{2.3}$$

and the covariant derivative in the action (2.1) is given as $D_\mu\Phi_a=\partial_\mu\Phi_a+[A_\mu,\Phi_a]$. Let us also note that in (2.1) $g,\tilde{g},\mathfrak{g},\tilde{b}$ are constants, $\mathbf{1}_{\mathcal{N}}$ stands for the $\mathcal{N}\times\mathcal{N}$ unit matrix, and $\mathrm{Tr}_{\mathcal{N}} = \mathcal{N}^{-1}\mathrm{Tr}$ indicates a normalized trace. In the potential term $V_1(\Phi)$, $F_{ab}$ is defined as

$$F_{ab} \coloneqq [\Phi_a, \Phi_b] - \epsilon_{abc}\Phi_c. \qquad (2.4)$$

In addition to the $SU(\mathcal{N})$ gauge symmetry (2.1) is also invariant under a global $SU(2)$ symmetry with respect to which the scalar fields $\Phi_a$ form a triplet. Thus, at this stage, the indices $(a = 1, 2, 3)$ pertain to this global symmetry. In what follow, we will observe how their meaning shifts after the emergence of the $U(n)$ gauge theory over $\mathcal{M} \times S_F^2$.

$V_2(\Phi)$ is a constraint term, whose purpose, as we will see below, is essentially to force the model to select the single fuzzy sphere $S_F^2$ vacuum configuration, as opposed to a vacuum given in terms of the direct sums of fuzzy spheres, say, $\mathcal{S}_F^2 \coloneqq \oplus S_F^2$.

We may also note that $\mathcal{M}$ may be selected as a manifold on which (2.1) is renormalizable. In particular, it may be taken as the four-dimensional Minkowski space or $\mathbb{R}^2$ as we do in Sec. V.

It is obvious that the potential terms $V_1(\Phi)$ and $V_2(\Phi)$ are positive definite and the minimum of potentials can be obtained by solving the equations

$$F_{ab} = [\Phi_a, \Phi_b] - \epsilon_{abc}\Phi_c = 0, \quad -\Phi_a\Phi_a = \tilde{b}\mathbf{1}_{\mathcal{N}}. \qquad (2.5)$$

A well-known solution [5] to these equations is given by taking $\tilde{b}$ as the eigenvalue of the quadratic Casimir of an IRR $\ell$ of $SU(2)$, and assuming that $\mathcal{N}$ factorizes as $\mathcal{N} = (2\ell + 1)n$. Then, up to the gauge transformations (2.3) the matrices

$$\Phi_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_n, \qquad (2.6)$$

where $X_a^{(2\ell+1)}$ are the anti-Hermitian generators of $SU(2)$ in the irreducible representation $\ell$ with the commutation relation

$$[X_a^{(2\ell+1)}, X_b^{(2\ell+1)}] = \epsilon_{abc}X_c^{(2\ell+1)}, \qquad (2.7)$$

satisfy (2.5).

Evidently, this vacuum configuration spontaneously breaks the $SU(\mathcal{N})$ symmetry down to the gauge group $U(n)$. In addition, we see that it may be interpreted as the fuzzy sphere at level $\ell$ since the latter, at level $\ell$, is the algebra of $(2\ell + 1) \times (2\ell + 1)$ matrices generated by the three Hermitian coordinate functions,

$$\hat{x}_a \coloneqq \frac{i}{\sqrt{\ell(\ell + 1)}} X_a^{(2\ell+1)}, \qquad (2.8)$$

satisfying

$$[\hat{x}_a, \hat{x}_b] = \frac{i}{\sqrt{\ell(\ell + 1)}}\epsilon_{abc}\hat{x}_c, \qquad \hat{x}_a\hat{x}_a = 1. \qquad (2.9)$$

Derivatives on $S_F^2(\ell)$ are given by the derivations on the matrix algebra, which are simply implemented by the adjoint action of $su(2)$ on $S_F^2$,

$$f \to adX_a^{(2\ell+1)}f \coloneqq [X_a^{(2\ell+1)}, f], \;\; f \in \mathrm{Mat}(2\ell + 1). \qquad (2.10)$$

In the commutative limit $\ell \to \infty$, $\hat{x}_a$ converge to the standard coordinates $x_a$ on $\mathbb{R}^3$, restricted to the unit sphere $x_a x_a = 1$, and the derivations $[X_a^{(2\ell+1)}, \cdot]$ become the vector fields $-i\mathcal{L}_a = \epsilon_{abc}x_b\partial_c$.

Fluctuations $A_a = A_a(y)$ about the vacuum (2.6) may be studied by writing

$$\Phi_a = X_a + A_a, \qquad (2.11)$$

where the shorthand notation $X_a^{(2\ell+1)} \otimes \mathbf{1}_n = X_a$ has been introduced. A short calculation yields that

$$F_{ab} = [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - \epsilon_{abc}A_c, \qquad (2.12)$$

which has the form of the curvature tensor for $U(n)$ gauge fields over $S_F^2$. This suggests that the model emerging after spontaneous symmetry breaking can be interpreted as a $U(n)$ gauge theory on $\mathcal{M} \times S_F^2$ with the gauge fields $A_M(y) = (A_\mu(y), A_a(y)) \in u(n) \otimes u(2\ell + 1)$ and the field strength tensor $F_{MN} = (F_{\mu\nu}, F_{a\mu}, F_{ab})$,

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \\ F_{\mu a} &= D_\mu\Phi_a = \partial_\mu\Phi_a + [A_\mu, \Phi_a] \\ &= \partial_\mu A_a - [X_a, A_\mu] + [A_\mu, A_a], \\ F_{ab} &= [\Phi_a, \Phi_b] - \epsilon_{abc}\Phi_c \\ &= [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - \epsilon_{abc}A_c. \end{aligned} \qquad (2.13)$$

For notational clarity, it is useful to indicate that, after the spontaneous breaking of the $SU(\mathcal{N})$ gauge symmetry and subsequent emergence of $U(n)$ gauge theory on $\mathcal{M} \times S_F^2$, roman indices $(a, b = 1, 2, 3)$ label the components of the coordinates of the fuzzy sphere and also naturally label the components of vectorial and tensorial quantities over $S_F^2$. Thus, for instance, $A_a(y)$ are those components of the gauge field $A_M(y)$ over $\mathcal{M} \times S_F^2$, transforming as components of vectors under rotations of $S_F^2$ and remaining invariant under the action of the symmetry group of the manifold $\mathcal{M}$. Also, after symmetry breaking, $A_\mu(y)$ become those components of $A_M(y)$, remaining invariant under rotations of $S_F^2$ and transforming as components of vectors under the symmetry group of the manifold $\mathcal{M}$ as can be observed from the considerations given above.

It is a well-known fact that on the fuzzy sphere there are three components of the gauge field $A_a$, which can only be disentangled from each other in the commutative limit. On $S^2$, there are only two degrees of freedom for the gauge field $A_a$ and the standard treatment is to impose the constraint $x_a A_a = 0$ to eliminate the normal component of $A_a$. Here the constraint term $V_2$ in (2.2) serves the purpose of suppressing the normal component of $A_a$ by giving it a large mass $\mathfrak{g}\sqrt{\ell(\ell+1)}$, as $\ell \to \infty$, [5,8]. In the discussion above we have worked with dimensionless $\Phi_a$. We can restore the dimensions by taking $\Phi_a \to \gamma\Phi_a$ where $\gamma$ has the mass dimensions $[m]^{d/2-1}$. Working with the dimensionful $\Phi_a$'s, we have the mass dimension of the couplings $g$ and $\tilde{g}$ are $[g] = [m]^{-d/2+2}$ and $[\tilde{g}] = [m]^{d/2-2}$. We also note that performing the scaling $\tilde{\Phi}_a = \sqrt{2}g\Phi_a$ and taking $g\tilde{g} = 1$, the part of the action without the constraint term, $V_2(\Phi)$, may be expressed as the $L^2$-norm of $F_{MN}$ and we may write

$$S = \frac{1}{4g^2}\int d^d y Tr_{n(2\ell+1)} F^{\dagger}_{MN} F^{MN} - \mathfrak{g}^2 V_2(\Phi). \quad (2.14)$$

A Kaluza-Klein mode expansion of the gauge fields over the fuzzy extra dimension given in [5], and an inspection of its low lying modes, supports the effective gauge theory interpretation. A complementary approach in the context is the equivariant parametrization technique which entails imposing proper symmetry conditions on the fields of the model so that they transform covariantly under the action of the symmetry group of the extra dimensions up to gauge transformations of the emergent model. As discussed in the introduction, we now take up the task of examining the $U(3)$ model on $\mathcal{M} \times S_F^2$ by employing this method.

## III. $SU(2)$-EQUIVARIANT GAUGE FIELDS FOR $U(3)$ GAUGE THEORY

Here, our initial aim is to construct the explicit form of $SU(2)$-equivariant gauge fields in the $U(3)$ theory. To be somewhat more precise, we determine those field configurations that are transforming as scalars and vectors under rotations of $S_F^2$ up to $U(3)$ gauge transformation. For this purpose, we introduce the infinitesimal symmetry generators $\omega_a$ as

$$\omega_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_3 - \mathbf{1}_{(2\ell+1)} \otimes i\Sigma_a, \quad (3.1)$$

where $\Sigma_a$ form the spin 1 irreducible representation of $SU(2) \subset SU(3)$: $(\Sigma_a)_{ij} = i\epsilon_{iaj}$ and $\omega_a$ satisfy the condition

$$[\omega_a, \omega_b] = \epsilon_{abc}\omega_c. \quad (3.2)$$

Clearly, the adjoint action $ad\omega_a \cdot = [\omega_a, \cdot]$ is composed of infinitesimal rotations over $S_F^2$ combined with those infinitesimal $SU(3)$ transformations, which are generated by $\Sigma_a$. The adjoint representation of $SU(3)$ decomposes to $SU(2)$ IRR's as

$$\underline{8} \to \underline{5} \oplus \underline{3}. \quad (3.3)$$

In this branching, $\Sigma_a$ generate the $\underline{3}$ (spin 1) IRR of $SU(2)$, while the remaining five generators of $SU(3)$ may be given in the form of the quadrupole tensor,

$$Q_{ab} = \frac{1}{2}\{\Sigma_a, \Sigma_b\} - \frac{2}{3}\delta_{ab}, \quad (3.4)$$

$$(Q_{ab})_{ij} = \delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi} - \frac{2}{3}\delta_{ab}\delta_{ij}, \quad (3.5)$$

carrying the spin 2 (i.e., $\underline{5}$) IRR of $SU(2)$. For each IRR of $SU(2)$ in the branching (3.3), we may expect to construct one rotational invariant under $ad\omega_a$ in addition to the identity matrix $\mathbf{1}_{(2\ell+1)3}$ and we at once proceed to see that this is indeed so.[1] These invariants may be simply taken as $X_a\Sigma_a$ and $X_a X_b Q_{ab}$; however, we prefer to express them as idempotent matrices, which turn out to be suitable for the subsequent construction of the equivariant vectors, as well as for clarity.

In order to find the $SU(2)$-equivariant gauge fields, we impose the following symmetry constraints,

$$[\omega_a, A_\mu] = 0, \qquad [\omega_a, A_b] = \epsilon_{abc}A_c, \quad (3.6)$$

which simply imply that, under the adjoint action of $\omega_a$, $A_\mu$ are rotational invariants and $A_a$ transform as vectors.

$SU(2)$ IRR content of $\omega_a$ may be found by the following tensor product,

$$\ell \otimes 1 = (\ell-1) \oplus \ell \oplus (\ell+1), \quad (3.7)$$

and therefore IRR decomposition of the adjoint action of $\omega_a$ is

$$[(\ell-1) \oplus \ell \oplus (\ell+1)] \otimes [(\ell-1) \oplus \ell \oplus (\ell+1)] = \mathbf{3}0 \oplus \mathbf{7}1 \oplus \cdots, \quad (3.8)$$

where the coefficients in bold denote the multiplicities of respective IRRs in front of which they appear. From this Clebsch-Gordan expansion, it can be seen that the set of solutions for $A_\mu$ is three dimensional. We span this space by the invariants $Q_1$, $Q_2$, as defined below and $\mathbf{1}_{(2\ell+1)3}$, and introduce the following explicit parametrization of $A_\mu$,

[1] Generalization of this construction to all $U(n)$ gauge theories on $\mathcal{M} \times S_F^2$ is discussed in Sec. VI.

$$A_\mu = -\frac{1}{2}a_\mu^{(1)}(y)Q_1 + \frac{1}{2}a_\mu^{(2)}(y)Q_2 + \frac{i}{2}\left(\frac{a_\mu^{(1)}(y) - a_\mu^{(2)}(y)}{3} + b_\mu(y)\right)\mathbf{1}, \tag{3.9}$$

where $a_\mu^{(1)}, a_\mu^{(2)}, b_\mu$ are Hermitian $U(1)$ gauge fields[2] on $\mathcal{M}$ and $Q_1$, $Q_2$ are anti-Hermitian idempotents given as [34],[3]

$$Q_1 = \frac{2(iX_a\Sigma_a + \ell + 1)(iX_b\Sigma_b + 1) - (\ell+1)(2\ell+1)\mathbf{1}}{i(\ell+1)(2\ell+1)},$$
$$Q_1^\dagger = -Q_1, \qquad Q_1^2 = -\mathbf{1}_{3(2\ell+1)},$$
$$Q_2 = \frac{2(iX_a\Sigma_a - \ell)(iX_b\Sigma_b + 1) - \ell(2\ell+1)\mathbf{1}}{i\ell(2\ell+1)},$$
$$Q_2^\dagger = -Q_2, \qquad Q_2^2 = -\mathbf{1}_{3(2\ell+1)}. \tag{3.10}$$

Thus, we see that $U(3)$ gauge symmetry is broken down to $U(1)\times U(1)\times U(1)$. Under the gauge transformation generated by $U = e^{-\frac{1}{2}\theta_1(y)Q_1}e^{\frac{1}{2}\theta_2(y)Q_2}e^{i(\frac{1}{6}\theta_1(y)-\frac{1}{6}\theta_2(y)+\frac{1}{2}\theta_3(y))\mathbf{1}}$, it is readily seen that $A_\mu \to A'_\mu$ with $a_\mu^{(i)\prime} = a_\mu^{(i)} + \partial_\mu\theta_i$ and $b'_\mu = b_\mu + \partial_\mu\theta_3$; hence the rotational symmetry of $A_\mu$ is preserved.

Equation (3.8) shows that the dimension of the set of solutions for $A_a$ is 7 and its parametrization may be chosen as follows:

$$A_a = \frac{1}{2}\varphi_1(y)[X_a, Q_1] + \frac{1}{2}\chi_1(y)[X_a, Q_2] - \frac{1}{2}(\varphi_2(y)+1)Q_1[X_a, Q_1] + \frac{1}{2}(\chi_2(y)-1)Q_2[X_a, Q_2] + \frac{i}{2}\frac{\varphi_3(y)}{2(\ell+1/2)}(\{X_a, Q_1\} - iQ_2[X_a, Q_2]) + \frac{i}{2}\frac{\chi_3(y)}{2(\ell+1/2)}(\{X_a, Q_2\} - iQ_1[X_a, Q_1]) + \frac{1}{2}\psi(y)\frac{\omega_a}{\ell+1/2}. \tag{3.11}$$

Let us digress for a moment and inspect (3.11) in some detail. Observe that we have essentially used commutators and anticommutators of $Q_1$ and $Q_2$ with $X_a$ to construct a suitable basis for vectors fulfilling (3.6). As coefficients of these vectors, we have introduced the real scalar fields $\varphi_1, \varphi_2, \varphi_3, \chi_1, \chi_2, \chi_3$ and $\psi$ on $\mathcal{M}$. We see shortly that some of these naturally combine to form complex scalars when the model is dimensionally reduced over $S_F^2$.

[2]The reason for this particular form of the coefficients of $Q_1$, $Q_2$ and 1 in (3.9) becomes clear as we proceed to perform the dimensional reduction over $S_F^2$ in the next section.

[3]In [34], these idempotents were introduced for the purpose of constructing the spin 1 Dirac operator on the fuzzy sphere.

In the commutative limit, $\ell\to\infty$ ($\frac{X_a}{\ell}\to\hat{x}_a, \hat{x}_a\hat{x}_a = 1$), we have

$$iQ_1 = q_1 = (\Sigma_a\hat{x}_a)^2 + (\Sigma_a\hat{x}_a) - 1,$$
$$iQ_2 = q_2 = (\Sigma_a\hat{x}_a)^2 - (\Sigma_a\hat{x}_a) - 1, \tag{3.12}$$

where $q_1^2 = q_2^2 = \mathbf{1}_3$. Another idempotent may be given as a linear combination of $q_1$ and $q_2$ and $\mathbf{1}_3$ as $q_3 = -(q_1+q_2) - \mathbf{1}_3$ [34]. Using (3.12), we find that the commutative limit of $A_a$ in (3.11) takes the form

$$A_a \underset{\ell\to\infty}{\to} -\frac{\varphi_1(y)}{2}\mathcal{L}_a q_1 - \frac{\chi_1(y)}{2}\mathcal{L}_a q_2 - i\frac{(\varphi_2(y)+1)}{2}q_1\mathcal{L}_a q_1 + i\frac{(\chi_2(y)-1)}{2}q_2\mathcal{L}_a q_2 + \frac{\varphi_3(y)}{2}\hat{x}_a q_1 + \frac{\chi_3(y)}{2}\hat{x}_a q_2 + \frac{\psi(y)}{2}\hat{x}_a. \tag{3.13}$$

Imposing the constraint $x_aA_a = 0$ eliminates the radial component of the gauge field. We see from (3.13) that this condition is satisfied if and only if we set $\varphi_3 = \chi_3 = \psi = 0$. The remaining terms of $A_a$ in (3.13) and the commutative limit of $A_\mu$ [apart from a $b_\mu$-field due to the $U(1)$ subgroup of $U(3)$, which decouples from the rest in the commutative limit, or is eliminated by solving its equation of motion in powers of $\frac{1}{\ell}$, as we see later on in Sec. V] are in agreement with the cylindrical symmetric ansatz for the $SU(3)$ Yang-Mills theory of Bais and Weldon [32].

## IV. DIMENSIONAL REDUCTION OF THE YANG-MILLS ACTION

In this section, we pursue the dimensional reduction of our model over $S_F^2$. We substitute our equivariant gauge fields $A_\mu$ and $A_a$ into the action (2.1), and then by tracing over the fuzzy sphere $S_F^2$, we obtain the reduced action on $\mathcal{M}$. The following identities are very useful to simplify the calculations,

$$[X_a, \{X_a, Q_i\}] = 0, \qquad [Q_i, \{X_a, Q_i\}] = 0,$$
$$\{X_a, [X_a, Q_i]\} = 0, \qquad \{Q_i, [X_a, Q_i]\} = 0, \tag{4.1}$$

where $i = 1, 2$ and the sum over only the repeated index "a" is implied.

Borrowing the notation of [8],

$$S = \int_{\mathcal{M}} d^dy\left(\mathcal{L}_F + \mathcal{L}_G + \frac{1}{\tilde{g}^2}V_1 + \mathfrak{g}^2V_2\right). \tag{4.2}$$

Now, we start to calculate each term in (4.2) separately. For the field strength term, the curvature $F_{\mu\nu}$ can be expressed in terms of the rotational invariants $Q_1$, $Q_2$ and $\mathbf{1}$ as

$$F_{\mu\nu} = -\frac{1}{2}f_{\mu\nu}^{(1)}Q_1 + \frac{1}{2}f_{\mu\nu}^{(2)}Q_2 + i\frac{1}{2}\left(\frac{f_{\mu\nu}^{(1)} - f_{\mu\nu}^{(2)}}{3} + h_{\mu\nu}\right)\mathbf{1}, \tag{4.3}$$

where we have introduced

$$f^{(1)}_{\mu\nu} \coloneqq \partial_\mu a^{(1)}_\nu - \partial_\nu a^{(1)}_\mu, \qquad f^{(2)}_{\mu\nu} \coloneqq \partial_\mu a^{(2)}_\nu - \partial_\nu a^{(2)}_\mu,$$
$$h_{\mu\nu} \coloneqq \partial_\mu b_\nu - \partial_\nu b_\mu. \tag{4.4}$$

Then, $\mathcal{L}_F$ takes the form

$$\begin{aligned}\mathcal{L}_F &\coloneqq \frac{1}{4g^2} Tr_{\mathcal{N}}(F^\dagger_{\mu\nu}F^{\mu\nu}) \\ &= \frac{1}{g^2}\left(\frac{\ell+1}{9(2\ell+1)} f^{(1)}_{\mu\nu} f^{(1)\mu\nu} + \frac{\ell}{9(2\ell+1)} f^{(2)}_{\mu\nu} f^{(2)\mu\nu} \right. \\ &\quad + \frac{1}{18} f^{(1)}_{\mu\nu} f^{(2)\mu\nu} + \frac{1}{16} h_{\mu\nu} h^{\mu\nu} + \frac{1}{6(2\ell+1)} f^{(1)}_{\mu\nu} h^{\mu\nu} \\ &\quad \left. + \frac{1}{6(2\ell+1)} f^{(2)}_{\mu\nu} h^{\mu\nu}\right).\end{aligned} \tag{4.5}$$

The covariant derivative term $D_\mu \Phi_a$ is calculated to be

$$\begin{aligned} D_\mu \Phi_a &= \frac{1}{2}(D_\mu \varphi_1)[X_a, Q_1] + \frac{1}{2}(D_\mu \chi_1)[X_a, Q_2] \\ &\quad - \frac{1}{2}(D_\mu \varphi_2) Q_1 [X_a, Q_1] + \frac{1}{2}(D_\mu \chi_2) Q_2 [X_a, Q_2] \\ &\quad + \frac{i}{4}\frac{\partial_\mu \varphi_3}{(\ell+1/2)}(\{X_a, Q_1\} - iQ_2[X_a, Q_2]) \\ &\quad + \frac{i}{4}\frac{\partial_\mu \chi_3}{(\ell+1/2)}(\{X_a, Q_2\} - iQ_1[X_a, Q_1]) \\ &\quad + \frac{1}{2(\ell+1/2)}(\partial_\mu \psi)\omega_a, \end{aligned} \tag{4.6}$$

where $D_\mu \varphi_i = \partial_\mu \varphi_i + \epsilon_{ji} a^{(1)}_\mu \varphi_j$ and $D_\mu \chi_i = \partial_\mu \chi_i + \epsilon_{ji} a^{(2)}_\mu \chi_j$. After tracing, the gradient term $\mathcal{L}_G$ reads

$$\mathcal{L}_G = \mathrm{Tr}((D_\mu \Phi_a)^\dagger D_\mu \Phi_a) \tag{4.7}$$

$$\begin{aligned} &= \frac{2\ell(2\ell+3)}{3(\ell+1)(2\ell+1)}((D_\mu\varphi_1)^2 + (D_\mu\varphi_2)^2) + \frac{2(2\ell-1)(\ell+1)}{3\ell(2\ell+1)}((D_\mu\chi_1)^2 + (D_\mu\chi_2)^2) \\ &\quad + \frac{6\ell^5 + 15\ell^4 + 4\ell^3 - 9\ell^2 + 2}{3\ell(\ell+1)(2\ell+1)^3}((\partial_\mu\varphi_3)^2 + (\partial_\mu\chi_3)^2) + \frac{\ell^2+\ell+2}{(2\ell+1)^2}(\partial_\mu\psi)^2 - \frac{2\ell(\ell+1)}{3(2\ell+1)^2}\partial_\mu\varphi_3\partial_\mu\chi_3 \\ &\quad - \frac{2\ell(2\ell^2 - 5\ell - 9)}{3(2\ell+1)^3}\partial_\mu\psi\partial_\mu\varphi_3 - \frac{2(2\ell^3 + 11\ell^2 + 7\ell - 2)}{3(2\ell+1)^3}\partial_\mu\chi_3\partial_\mu\psi. \end{aligned} \tag{4.8}$$

We note that $\varphi_1, \varphi_2$ and $\chi_1$, $\chi_2$ naturally combine to two complex scalar fields $\varphi \coloneqq \varphi_1 + i\varphi_2, \chi \coloneqq \chi_1 + i\chi_2$, with $D_\mu\varphi = (\partial_\mu + ia^{(1)}_\mu)\varphi$ and $D_\mu\chi = (\partial_\mu + ia^{(2)}_\mu)\chi$, which we make use of in the next section.

In order to calculate the potential term $V_1$, it is useful to work with the dual of the curvature $F_{ab}$. We find

$$\begin{aligned} \frac{1}{2}\epsilon_{abc}F_{ab} &= \Lambda_1 + \Lambda_2|\varphi|^2 + \Lambda_3|\chi|^2 + \Lambda_4(\varphi_3^2 + \chi_3^2) + \Lambda_5\varphi_3 + \Lambda_6\chi_3 + \Lambda_7\varphi_3\chi_3 + \Lambda_8\varphi_3\psi + \Lambda_9\chi_3\psi \\ &\quad + \Lambda_{10}(\varphi_1 + \varphi_2 Q_1)[X_a, Q_1] + \Lambda_{11}(\chi_1 + \chi_2 Q_2)[X_a, Q_2] + \Lambda_{12}\psi + \Lambda_{13}\psi^2, \end{aligned} \tag{4.9}$$

where $\Lambda_i, i = 1, \ldots, 11$ are the $3(2\ell+1) \times 3(2\ell+1)$-dimensional matrices that are listed in the appendix. Using (4.9), the potential term $V_1$ may be determined as

$$\begin{aligned} V_1 = Tr_{\mathcal{N}}(F^\dagger_{ab}F_{ab}) &= \alpha_1 - \alpha_2|\varphi|^2 - \alpha_3|\chi|^2 - \alpha_4\varphi_3^2 - \alpha_5\chi_3^2 - \alpha_6\varphi_3 + \alpha_7\chi_3 - \alpha_8\varphi_3\chi_3 - \alpha_9\varphi_3\psi - \alpha_{10}\chi_3\psi + \alpha_{11}\psi^2 + \beta_1|\varphi|^4 \\ &\quad - \beta_2|\varphi|^2|\chi|^2 + \beta_3|\varphi|^2\varphi_3^2 + \beta_4|\varphi|^2\chi_3^2 - \beta_5|\varphi|^2\varphi_3 + \beta_6|\varphi|^2\chi_3 - \beta_7|\varphi|^2\varphi_3\chi_3 + \beta_8|\varphi|^2\varphi_3\psi - \beta_9|\varphi|^2\chi_3\psi \\ &\quad + \beta_{10}|\varphi|^2\psi^2 + \gamma_1|\chi|^4 - \gamma_2|\chi|^2\varphi_3^2 + \gamma_3|\chi|^2\chi_3^2 + \gamma_4|\chi|^2\varphi_3 - \gamma_5|\chi|^2\chi_3 + \gamma_6|\chi|^2\varphi_3\chi_3 - \gamma_7|\chi|^2\varphi_3\psi \\ &\quad - \gamma_8|\chi|^2\chi_3\psi + \gamma_9|\chi|^2\psi^2 - \delta_1(\varphi_3^4 + \chi_3^4 + 6\varphi_3^2\chi_3^2) - \delta_2(\varphi_3^3 + 3\varphi_3\chi_3^2) - \delta_3(\chi_3^3 + 3\chi_3\varphi_3^2) \\ &\quad - \delta_4(\varphi_3^3\chi_3 + \chi_3^3\varphi_3) - \delta_5(\varphi_3^3\psi + 3\varphi_3\chi_3^2\psi) - \delta_6(\chi_3^3\psi + 3\chi_3\varphi_3^2\psi) + \delta_7(\varphi_3^2\psi + \chi_3^2\psi) \\ &\quad + \delta_8(\varphi_3^2\psi^2 + \chi_3^2\psi^2) - \delta_9\varphi_3\chi_3\psi - \delta_{10}\varphi_3\psi^2 - \delta_{11}\chi_3\psi^2 - \delta_{12}\varphi_3\chi_3\psi^2 - \delta_{13}\varphi_3\psi^2 \\ &\quad - \delta_{14}\chi_3\psi^3 - \delta_{15}\psi^3 - \delta_{16}\psi^4, \end{aligned} \tag{4.10}$$

where all the $\ell$-dependent constants $\alpha$, $\beta$, $\gamma$, $\delta$ are given in the appendix.

In the $\ell \to \infty$ limit we find

$$V_1(\Phi)|_{\ell\to\infty} = \frac{2}{3}(|\varphi|^2 + \varphi_3 - 1)^2 + \frac{2}{3}(|\chi|^2 - \chi_3 - 1)^2 + \frac{2}{3}(|\varphi|^2 - |\chi|^2)^2 + \frac{4}{3}|\varphi|^2\varphi_3^2 + \frac{4}{3}|\chi|^2\chi_3^2 - \frac{1}{6}\varphi_3^2 - \frac{1}{6}\chi_3^2 + \frac{1}{2}\psi^2 - \frac{1}{3}(\varphi_3\chi_3 + \varphi_3\psi + \chi_3\psi). \quad (4.11)$$

The potential $V_1(\Phi) = \mathrm{Tr}_{\mathcal{N}}(F_{ab}^\dagger F_{ab})$ is positive definite, although the rhs of (4.10) and (4.11) is not manifestly so. For the limiting case (4.11) we have determined that minima occur at the following configurations,

$$\text{i) } |\varphi|^2 = 0, \quad |\chi|^2 = 1, \quad \varphi_3 = \chi_3 = \psi = 0, \quad (4.12)$$

$$\text{ii) } |\varphi|^2 = 0, \quad |\chi|^2 = 0, \quad \varphi_3 = 1, \quad \chi_3 = -1, \quad \psi = 0, \quad (4.13)$$

$$\text{iii) } |\varphi|^2 = \frac{1}{\sqrt{2}}, \quad |\chi|^2 = 0, \quad \varphi_3 = 0, \quad \chi_3 = -\frac{3}{2}, \quad \psi = -\frac{1}{2}, \quad (4.14)$$

$$\text{iv) } |\varphi|^2 = 0, \quad |\chi|^2 = \frac{1}{\sqrt{2}}, \quad \varphi_3 = \frac{3}{2}, \quad \chi_3 = 0, \quad \psi = \frac{1}{2}. \quad (4.15)$$

For the computation of the last term in (4.2), we first obtain the expression

$$\Phi_a\Phi_a + \ell(\ell+1) = R_1 + R_2 iQ_1 + R_3 iQ_2, \quad (4.16)$$

where $R_1$, $R_2$, and $R_3$ are listed in the appendix. Then, the potential term $V_2$ is determined to be

$$V_2(\Phi) = \left(R_1^2 + R_2^2 + R_3^2 - \frac{2(2\ell-3)}{3(2\ell+1)}R_1R_2 - \frac{2(2\ell+5)}{3(2\ell+1)}R_1R_3 - \frac{2}{3}R_2R_3\right). \quad (4.17)$$

In the large $\ell$ limit we find

$$\begin{aligned}\mathfrak{g}^2 V_2(\Phi)|_{\ell\to\infty} &= \frac{1}{3}\mathfrak{g}^2((R_1 - R_2 - R_3)^2 + (-R_1 + R_2 - R_3)^2 + (-R_1 - R_2 + R_3)^2)|_{\ell\to\infty}, \\ &= \frac{1}{3}\mathfrak{g}^2\ell^2((-\psi + \varphi_3 + \chi_3)^2 + (\psi - \varphi_3 + \chi_3)^2 + (\psi + \varphi_3 - \chi_3)^2). \quad (4.18)\end{aligned}$$

In the next section we first consider the scaling limit $\mathfrak{g} \to 0$, $\ell \to \infty$, with $\mathfrak{g}\ell$ kept finite but small. Then, among the minima of the potential $V_1(\Phi)$ listed above, only (4.12) minimizes (4.18) as can easily be observed.

## V. VORTICES

In this section, we inspect the structure of the reduced action (4.2) on $\mathcal{M} \equiv \mathbb{R}^2$ and show that it has static vortex-type solutions. We are interested in exploring these in two different limits, namely, (i) $\ell \to \infty, \mathfrak{g} \to 0$ with $\mathfrak{g}\ell$ remaining finite but small and (ii) $\mathfrak{g} \to \infty$ with $\ell$ being large but finite. These limits are physically well motivated since in the absence of any canonical choices for the parameter $\mathfrak{g}$, they give the two extremes for handling the constraint term $V_2(\Phi)$.

### A. Case (i)

In this case the reduced action becomes

$$S = \int d^2y\left(\frac{1}{18g^2}(f_{\mu\nu}^{(1)}f^{(1)\mu\nu} + f_{\mu\nu}^{(2)}f^{(2)\mu\nu} + f_{\mu\nu}^{(1)}f^{(2)\mu\nu}) + \frac{1}{16g^2}h_{\mu\nu}h^{\mu\nu} + \frac{2}{3}(|D_\mu\varphi|^2 + |D_\mu\chi|^2) + \frac{1}{4}((\partial_\mu\varphi_3)^2 + (\partial_\mu\chi_3)^2 + (\partial_\mu\psi)^2) - \frac{1}{6}(\partial_\mu\varphi_3\partial_\mu\chi_3 + \partial_\mu\varphi_3\partial_\mu\psi + \partial_\mu\chi_3\partial_\mu\psi) + \frac{1}{\tilde{g}^2}V_1(\Phi)|_{\ell\to\infty}\right). \quad (5.1)$$

We observe that the gauge field $b_\mu$ decouples from the rest of the action, and does not play any role in the rest of this subsection. Thus we essentially have an Abelian Higgs-type model with $U(1)\times U(1)$ gauge symmetry. The vacuum configuration is given by (4.12) and has the structure of $T^2 = S^1 \times S^1$, with $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, indicating that the vortex solutions constructed below are characterized by two winding numbers, say $(N, M)$.

To search for vortex solutions, it is possible to work with the usual rotationally symmetric ansatz [35], which in our case may be written out as

$$\begin{gathered} a_r^{(1)} = a_r^{(2)} = 0, \quad a_\theta^1 \coloneqq a_\theta^{(1)}(r), \quad a_\theta^2 \coloneqq a_\theta^{(2)}(r), \\ \varphi = \zeta(r)e^{iN\theta}, \quad \chi = \eta(r)e^{iM\theta}, \quad \varphi_3 = \rho(r), \\ \chi_3 = \sigma(r), \quad \psi = \tau(r), \end{gathered} \quad (5.2)$$

where the Cartesian coordinates $(y_1, y_2)$ are replaced by the polar variables $(r, \theta)$. With this ansatz the action reads

$$S = 2\pi \int dr \Bigg( \frac{1}{9g^2 r} (a_\theta^{1\prime} a_\theta^{1\prime} + a_\theta^{2\prime} a_\theta^{2\prime} + a_\theta^{1\prime} a_\theta^{2\prime}) + \frac{2r}{3} (\zeta'^2 + \eta'^2) + \frac{2}{3r} (N + a_\theta^1)^2 \zeta^2 + \frac{2}{3r} (M + a_\theta^2)^2 \eta^2 + \frac{r}{4} (\rho'^2 + \sigma'^2 + \tau'^2) - \frac{r}{6} (\rho'\sigma' + \rho'\tau' + \sigma'\tau') + \frac{4r}{3\tilde{g}^2} \bigg( (1 - \zeta^2 - \eta^2) + \frac{3}{8} (\rho^2 + \sigma^2 + \tau^2) - \rho + \sigma - \frac{1}{4} (\rho\sigma + \rho\tau + \sigma\tau) + \zeta^4 + \eta^4 - \zeta^2\eta^2 + \zeta^2(\rho^2 + \rho) + \eta^2(\sigma^2 - \sigma) \bigg) \Bigg), \tag{5.3}$$

where primes are denoting the derivatives with respect to $r$.

Euler-Lagrange equations for the fields are

$$\begin{aligned}
\zeta'' + \frac{\zeta'}{r} - \left( \frac{1}{r^2} (N + a_\theta^1)^2 + \frac{2}{\tilde{g}^2} (-1 + 2\zeta^2 - \eta^2 + \rho^2 + \rho) \right) \zeta &= 0, \\
\eta'' + \frac{\eta'}{r} - \left( \frac{1}{r^2} (M + a_\theta^2)^2 + \frac{2}{\tilde{g}^2} (-1 + 2\eta^2 - \zeta^2 + \sigma^2 - \sigma) \right) \eta &= 0, \\
a_\theta^{1\prime\prime} - \frac{a_\theta^{1\prime}}{r} + \frac{1}{2} a_\theta^{2\prime\prime} - \frac{a_\theta^{2\prime}}{2r} - 6g^2 (N + a_\theta^1) \zeta^2 &= 0, \\
a_\theta^{2\prime\prime} - \frac{a_\theta^{2\prime}}{r} + \frac{1}{2} a_\theta^{1\prime\prime} - \frac{a_\theta^{1\prime}}{2r} - 6g^2 (M + a_\theta^2) \eta^2 &= 0, \\
\rho'' + \frac{\rho'}{r} - \frac{\sigma' + \tau'}{3r} - \frac{\sigma'' + \tau''}{3} - \frac{2\rho}{\tilde{g}^2} + \frac{8}{3\tilde{g}^2} + \frac{2}{3\tilde{g}^2} (\sigma + \tau) - \frac{8}{3\tilde{g}^2} \zeta^2 (2\rho + 1) &= 0, \\
\sigma'' + \frac{\sigma'}{r} - \frac{\rho' + \tau'}{3r} - \frac{\rho'' + \tau''}{3} - \frac{2\sigma}{\tilde{g}^2} - \frac{8}{3\tilde{g}^2} + \frac{2}{3\tilde{g}^2} (\rho + \tau) - \frac{8}{3\tilde{g}^2} \eta^2 (2\sigma - 1) &= 0, \\
\tau'' + \frac{\tau'}{r} - \frac{\rho' + \sigma'}{3r} - \frac{\rho'' + \sigma''}{3} - \frac{2\tau}{\tilde{g}^2} + \frac{2}{3\tilde{g}^2} (\rho + \sigma) &= 0.
\end{aligned} \tag{5.4}$$

We do not know any analytic solutions to these coupled nonlinear differential equations. However, we can construct the solution profiles for small and large $r$. For $r \to 0$, series solutions give

$$\begin{aligned}
&\zeta = \zeta_0 r^N + O(r^{N+2}), \quad \eta = \eta_0 r^M + O(r^{M+2}), \\
&a_\theta^1 = a_0^{(1)} r^2 + O(r^4), \quad a_\theta^2 = a_0^{(2)} r^2 + O(r^4) \\
&\rho = \rho_0 + O(r^2), \quad \sigma = \sigma_0 + O(r^2), \quad \tau = \tau_0 + O(r^2),
\end{aligned} \tag{5.5}$$

where $\zeta_0, \eta_0, a_0^{(1)}, a_0^{(2)}, \rho_0, \sigma_0, \tau_0$ are constants.

For large $r$, we first note that the asymptotic behavior of fields is enforced by the requirement of the finiteness of the action for the vortex-type solutions. We have $\zeta(r) \to 1, \eta(r) \to 1, a_\theta^1(r) \to N, a_\theta^2(r) \to M, \rho(r) \to 0, \sigma(r) \to 0, \tau(r) \to 0$ as $r \to \infty$, where the integers $N$ and $M$ are the winding numbers of the vortex configuration. In order to obtain the profiles for large $\ell$, we can consider the small fluctuations about these limiting values and write $\zeta = 1 - \delta\zeta, \eta = 1 - \delta\eta, a_\theta^1 = -N + \delta a^1, a_\theta^2 = -M + \delta a^2$. Assuming that $(\frac{\delta a_\theta^1}{r})^2$ and $(\frac{\delta a_\theta^2}{r})^2$ are subleading compared to $\delta\zeta, \delta\eta, \rho, \sigma, \tau$, the Euler-Lagrange equations (5.4) become

$$\begin{aligned}
&\delta\zeta'' + \frac{\delta\zeta'}{r} - \frac{2}{\tilde{g}^2} (4\delta\zeta - \rho - 2\delta\eta) = 0, \qquad \delta\eta'' + \frac{\delta\eta'}{r} - \frac{2}{\tilde{g}^2} (4\delta\eta + \sigma - 2\delta\zeta) = 0 \\
&\delta a^{1\prime\prime} - \frac{\delta a^{1\prime}}{r} + 4g^2 \delta a^2 - 8g^2 \delta a^1 = 0, \qquad \delta a^{2\prime\prime} - \frac{\delta a^{2\prime}}{r} + 4g^2 \delta a^1 - 8g^2 \delta a^2 = 0, \\
&\rho'' + \frac{\rho'}{r} - \frac{10}{\tilde{g}^2} \rho - \frac{4}{\tilde{g}^2} \sigma + \frac{8}{\tilde{g}^2} \delta\zeta - \frac{4}{\tilde{g}^2} \delta\eta = 0, \qquad \sigma'' + \frac{\sigma'}{r} - \frac{10}{\tilde{g}^2} \sigma - \frac{4}{\tilde{g}^2} \rho - \frac{8}{\tilde{g}^2} \delta\eta + \frac{4}{\tilde{g}^2} \delta\zeta = 0, \\
&\tau'' + \frac{\tau'}{r} - \frac{2}{\tilde{g}^2} \tau - \frac{4}{\tilde{g}^2} \rho - \frac{4}{\tilde{g}^2} \sigma - \frac{4}{\tilde{g}^2} \delta\eta + \frac{4}{\tilde{g}^2} \delta\zeta = 0.
\end{aligned} \tag{5.6}$$

We can solve these coupled linear differential equations in terms of the modified Bessel functions $K_\alpha$ and find

$$\begin{aligned}
\delta\zeta &= A_1 K_0\left(\frac{2\sqrt{2}r}{\tilde{g}}\right) + A_2 K_0\left(\frac{\sqrt{2}r}{\tilde{g}}\right) - A_3 K_0\left(\frac{3\sqrt{2}r}{\tilde{g}}\right),\\
\delta\eta &= A_2 K_0\left(\frac{\sqrt{2}r}{\tilde{g}}\right) + A_3 K_0\left(\frac{3\sqrt{2}r}{\tilde{g}}\right) + A_4 K_0\left(\frac{2\sqrt{2}r}{\tilde{g}}\right),\\
\rho &= A_2 K_0\left(\frac{\sqrt{2}r}{\tilde{g}}\right) + 3A_3 K_0\left(\frac{3\sqrt{2}r}{\tilde{g}}\right) - 2A_4 K_0\left(\frac{2\sqrt{2}r}{\tilde{g}}\right),\\
\sigma &= 2A_1 K_0\left(\frac{2\sqrt{2}r}{\tilde{g}}\right) - A_2 K_0\left(\frac{\sqrt{2}r}{\tilde{g}}\right) + 3A_3 K_0\left(\frac{3\sqrt{2}r}{\tilde{g}}\right),\\
\tau &= \frac{2}{3}(A_1 - A_4) K_0\left(\frac{2\sqrt{2}r}{\tilde{g}}\right) + 2A_3 K_0\left(\frac{3\sqrt{2}r}{\tilde{g}}\right)\\
&\quad + A_5 K_0\left(\frac{\sqrt{2}r}{\tilde{g}}\right),\\
\delta a^1 &= C_1 r K_1(2gr) + C_2 r K_1(2\sqrt{3}gr),\\
\delta a^2 &= C_1 r K_1(2gr) - C_2 r K_1(2\sqrt{3}gr),
\end{aligned} \tag{5.7}$$

where $A_i, i = 1\ldots,5$ and $C_j, j = 1,2$ are constants, which can only be determined numerically. It is easy to see that our assumption that $(\frac{\delta a_\theta^1}{r})^2$ and $(\frac{\delta a_\theta^2}{r})^2$ are subleading to $\delta\zeta, \delta\eta, \rho, \sigma, \tau$ can be fulfilled if we take $4g > \sqrt{2}/\tilde{g}$. A well-known fact is that the field strength and scalars are, respectively, responsible for the repulsive and attractive character of forces between vortices [35]. We find from (5.7) that the field strengths $B^1 := f^1_{12} = \frac{1}{r} f^1_{r\theta} = \frac{1}{r}\partial_r a^1_\theta$ and $B^2 := f^2_{12} = \frac{1}{r} f^2_{r\theta} = \frac{1}{r}\partial_r a^2_\theta$ are proportional to $\propto \frac{1}{\sqrt{r}} e^{-2gr}$ while the scalar fields $\delta\zeta, \delta\eta, \rho, \sigma$ and $\tau$ decay like $\frac{1}{\sqrt{r}} e^{-\frac{\sqrt{2}}{\tilde{g}} r}$ asymptotically. Thus these vortices attract for $g\tilde{g} > \frac{\sqrt{2}}{2}$ and particularly for the case $g\tilde{g} = 1$ needed for the standard Yang-Mills (2.14), and they repel in the parameter interval $\frac{\sqrt{2}}{4} < g\tilde{g} < \frac{\sqrt{2}}{2}$. From the asymptotic profiles of the fields, we cannot immediately conclude the presence of BPS solutions at the point $g\tilde{g} = \frac{\sqrt{2}}{2}$ of the parameter space, where there appears to be a change between attractive and repulsive nature of forces between vortices. In fact, we do not find any BPS equations from (5.1) at this point of the parameter space, while as we see in the next subsection, $g\tilde{g} = 1$ is a critical point at which BPS vortices are found as $\ell \to \infty$ and $\mathfrak{g} \to \infty$.

### B. Case (ii)

Taking the limit $\mathfrak{g} \to \infty$ is equivalent to enforcing the constraint $\Phi_a \Phi_a + \ell(\ell+1) = 0$. It can be easily seen from (4.16) that this constraint can only be fulfilled by setting $R_1 = 0$, $R_2 = 0$, and $R_3 = 0$. Using these three conditions, we can solve $\varphi_3, \chi_3$ and $\psi$ in terms of $|\varphi|$ and $|\chi|$ in powers of $\frac{1}{\ell}$. Substituting back into the action should then give us a reduced action with only two complex scalars $\varphi$ and $\chi$. To leading nonvanishing order in powers of $\frac{1}{\ell}$, we find that

$$\begin{aligned}
\psi &= \frac{1}{2\ell}(1 - |\varphi|^2) + \frac{1}{2\ell}(1 - |\chi|^2) + O\left(\frac{1}{\ell^2}\right),\\
\varphi_3 &= -\frac{3}{4\ell^2}(1 - |\varphi|^2) - \frac{2\ell+1}{4\ell^2}(1 - |\chi|^2) + O\left(\frac{1}{\ell^3}\right),\\
\chi_3 &= \frac{1}{4\ell^2}(1 - |\chi|^2) - \frac{2\ell+1}{4\ell^2}(1 - |\varphi|^2) + O\left(\frac{1}{\ell^3}\right).
\end{aligned} \tag{5.8}$$

Substituting from (5.8) for $\varphi_3, \chi_3, \psi$, expanding the $\ell$-dependent coefficients to order $\frac{1}{\ell^2}$, the action (4.2) takes the form

$$\begin{aligned}
S = \int d^2y \Bigg( &\frac{1}{18g^2}\left(1 + \frac{1}{2\ell} - \frac{3}{4\ell^2}\right) f^{(1)}_{\mu\nu} f^{(1)\mu\nu} + \frac{1}{18g^2}\left(1 - \frac{1}{2\ell} - \frac{1}{4\ell^2}\right) f^{(2)}_{\mu\nu} f^{(2)\mu\nu} + \frac{1}{18g^2}\left(1 - \frac{1}{\ell^2}\right) f^{(1)}_{\mu\nu} f^{(2)\mu\nu}\\
&+ \frac{2}{3}\left(1 - \frac{1}{2\ell^2}\right)(|D_\mu\varphi|^2 + |D_\mu\chi|^2) + \frac{1}{6\ell^2}((\partial_\mu|\varphi|^2)^2 + (\partial_\mu|\chi|^2)^2 + \partial_\mu|\varphi|^2\partial_\mu|\chi|^2)\\
&+ \frac{1}{\tilde{g}^2}\Bigg(\frac{4}{3}\left(1 + \frac{1}{4\ell^2}\right) - \frac{4}{3}\left(1 - \frac{1}{\ell} + \frac{1}{\ell^2}\right)|\varphi|^2 - \frac{4}{3}\left(1 + \frac{1}{\ell} - \frac{1}{\ell^2}\right)|\chi|^2 - \frac{4}{3}\left(1 + \frac{3}{4\ell^2}\right)|\varphi|^2|\chi|^2 + \frac{4}{3}\left(1 - \frac{1}{2\ell} + \frac{1}{2\ell^2}\right)|\varphi|^4\\
&+ \frac{4}{3}\left(1 + \frac{1}{2\ell} - \frac{1}{2\ell^2}\right)|\chi|^4 + \frac{1}{3\ell^2}(|\varphi|^4|\chi|^2 + |\chi|^4|\varphi|^2)\Bigg)\Bigg),
\end{aligned} \tag{5.9}$$

where we wrote

$$h_{\mu\nu} = -\frac{2}{3}\left(\frac{1}{\ell} - \frac{1}{2\ell^2}\right)(f^{(1)}_{\mu\nu} + f^{(2)}_{\mu\nu}), \tag{5.10}$$

which follows from the equation of motion of $b_\mu$ at the $\frac{1}{\ell^2}$ order.

For this case too, we make the rotationally symmetric vortex solution ansatz (5.2) and find the action to take the form

$$\begin{aligned} S = 2\pi \int dr \bigg( & \frac{1}{9g^2 r}\left(1+\frac{1}{2\ell}-\frac{3}{4\ell^2}\right) a_\theta^{1\prime} a_\theta^{1\prime} + \frac{1}{9g^2 r}\left(1-\frac{1}{2\ell}-\frac{1}{4\ell^2}\right) a_\theta^{2\prime} a_\theta^{2\prime} + \frac{1}{9g^2 r}\left(1-\frac{1}{\ell^2}\right) a_\theta^{1\prime} a_\theta^{2\prime} \\ & + \frac{2}{3}\left(1-\frac{1}{2\ell^2}\right)\left(r\zeta'^2 + \frac{(N+a_\theta^1)^2}{r}\zeta^2 + r\eta'^2 + \frac{(M+a_\theta^2)^2}{r}\eta^2\right) + \frac{r}{6\ell^2}(4\zeta'^2\zeta^2 + 4\eta'^2\eta^2 + 4\zeta'\zeta\eta'\eta) \\ & + \frac{1}{\tilde{g}^2}\left(\frac{4r}{3}\left(1+\frac{1}{4\ell^2}\right) - \frac{4r}{3}\left(1-\frac{1}{\ell}+\frac{1}{\ell^2}\right)\zeta^2 - \frac{4r}{3}\left(1+\frac{1}{\ell}-\frac{1}{\ell^2}\right)\eta^2 - \frac{4r}{3}\left(1+\frac{3}{4\ell^2}\right)\zeta^2\eta^2 \right. \\ & \left. + \frac{4r}{3}\left(1-\frac{1}{2\ell}+\frac{1}{2\ell^2}\right)\zeta^4 + \frac{4r}{3}\left(1+\frac{1}{2\ell}-\frac{1}{2\ell^2}\right)\eta^4 + \frac{r}{3\ell^2}(\zeta^4\eta^2 + \eta^4\zeta^2)\right)\bigg). \end{aligned} \tag{5.11}$$

Equations of motion for the fields $\zeta, \eta, a_\theta^1, a_\theta^2$ follow after a straightforward calculation and are given in the appendix. Profiles of these fields around $r = 0$ are the same as in the previous case (5.5).

For large $r$, it is easy to find the linearized equations for the fluctuations about the vacuum values. We write as before $\zeta = 1 - \delta\zeta, \eta = 1 - \delta\eta, a_\theta^1 = -N + \delta a^1, a_\theta^2 = -M + \delta a^2$ and we obtain the equations

$$\begin{aligned} \delta\zeta'' + \frac{\delta\zeta'}{r} - \frac{2}{\tilde{g}^2}\left(4-\frac{2}{\ell}+\frac{2}{\ell^2}\right)\zeta + \frac{2}{\tilde{g}^2}\left(2+\frac{1}{2\ell^2}\right)\eta = 0, \\ \delta\eta'' + \frac{\delta\eta'}{r} - \frac{2}{\tilde{g}^2}\left(4+\frac{2}{\ell}-\frac{2}{\ell^2}\right)\eta + \frac{2}{\tilde{g}^2}\left(2+\frac{1}{2\ell^2}\right)\zeta = 0, \\ \delta a^{1\prime\prime} - \frac{\delta a^{1\prime}}{r} - 2g^2\left(4-\frac{2}{\ell}+\frac{1}{\ell^2}\right)\delta a^1 + 2g^2\left(2-\frac{1}{\ell^2}\right)\delta a^2 = 0, \\ \delta a^{2\prime\prime} - \frac{\delta a^{2\prime}}{r} - 2g^2\left(4+\frac{2}{\ell}-\frac{1}{\ell^2}\right)\delta a^2 + 2g^2\left(2-\frac{1}{\ell^2}\right)\delta a^1 = 0. \end{aligned} \tag{5.12}$$

Solutions for these equations are given in terms of modified Bessel functions $K_n$,

$$\begin{aligned} \delta\zeta &= E_1\left(-1+\frac{1}{\ell}+\frac{3}{2\ell^2}\right)K_0\left(\frac{\sqrt{12+3/\ell^2}\,r}{\tilde{g}}\right) + E_2\left(1+\frac{1}{\ell}-\frac{1}{2\ell^2}\right)K_0\left(\frac{\sqrt{4-3/\ell^2}\,r}{\tilde{g}}\right), \\ \delta\eta &= E_1 K_0\left(\frac{\sqrt{12+3/\ell^2}\,r}{\tilde{g}}\right) + E_2 K_0\left(\frac{\sqrt{4-3/\ell^2}\,r}{\tilde{g}}\right), \\ \delta a^1 &= F_1\left(-1+\frac{1}{\ell}-\frac{1}{\ell^2}\right) r K_1(2\sqrt{3}gr) + F_2\left(1+\frac{1}{\ell}\right) r K_1(2gr), \\ \delta a^2 &= F_1 r K_1(2\sqrt{3}gr) + F_2 r K_1(2gr), \end{aligned} \tag{5.13}$$

where $E_1, E_2, F_1, F_2$ are constants. Here, we can also define the parameter intervals for the attractive and repulsive behavior of forces between the vortices. It is easy to see that for $g\tilde{g} > \frac{\sqrt{4-3/\ell^2}}{2}$, the field strengths decay faster than the scalar fields, so we have attractive vortices. On the other hand, for $\frac{\sqrt{4-3/\ell^2}}{4} < g\tilde{g} < \frac{\sqrt{4-3/\ell^2}}{2}$ we have repulsive forces between the vortices.

As $\ell \to \infty$ the action (5.9) at the critical point $g\tilde{g} = 1$ becomes

$$\begin{aligned} S = \int d^2y \frac{1}{18g^2}(f_{\mu\nu}^{(1)} f^{(1)\mu\nu} + f_{\mu\nu}^{(2)} f^{(2)\mu\nu} + f_{\mu\nu}^{(1)} f^{(2)\mu\nu}) + \frac{2}{3}(|D_\mu\varphi|^2 + |D_\mu\chi|^2) + \frac{2}{3}g^2((|\varphi|^2 + \varphi_3 - 1)^2 \\ + (|\chi|^2 - \chi_3 - 1)^2 + (|\varphi|^2 - |\chi|^2)^2). \end{aligned} \tag{5.14}$$

In this case we may express the action in the form

$$\begin{aligned} S = \int d^2y \frac{1}{18g^2}(B^1 + 2g^2(2|\varphi|^2 - |\chi|^2 - 1))^2 + \frac{1}{18g^2}(B^2 + 2g^2(2|\chi|^2 - |\varphi|^2 - 1))^2 \\ + \frac{1}{18g^2}(B^1 + B^2 + 2g^2(|\varphi|^2 + |\chi|^2 - 2))^2 + \frac{2}{3}(\overline{D_1\varphi} - i\overline{D_2\varphi})(D_1\varphi + iD_2\varphi) \\ + \frac{2}{3}(\overline{D_1\chi} - i\overline{D_2\chi})(D_1\chi + iD_2\chi) + \frac{2}{3}(B^1 + B^2) - \frac{2i}{3}(\partial_1(\bar\varphi D_2\varphi) - \partial_2(\bar\varphi D_1\varphi)) - \frac{2i}{3}(\partial_1(\bar\chi D_2\chi) - \partial_2(\bar\chi D_1\chi)), \end{aligned} \tag{5.15}$$

where $B^1 = f^1_{12}, B^2 = f^2_{12}$ as we have noted previously. The last two terms in (5.15) vanish as they can be expressed as line integrals around a circle at infinity. Noting that the fluxes of $B^1$ and $B^2$ are $2\pi N$ and $2\pi M$, respectively, $N$, $M$ being the winding numbers of the vortex configuration, we see that the action is bounded from below with $S \geq \frac{4}{3}\pi(N + M)$. This bound is saturated when the fields satisfy the BPS equations,

$$\begin{aligned} D_1\varphi + iD_2\varphi = 0, \quad B^1 + 2g^2(2|\varphi|^2 - |\chi|^2 - 1) = 0, \\ D_1\chi + iD_2\chi = 0, \quad B^2 + 2g^2(2|\chi|^2 - |\varphi|^2 - 1) = 0. \end{aligned} \tag{5.16}$$

These equations give a particular generalization of the BPS equations for the Abelian Higgs model [35]. In fact, these equation appear to be formally the same as the self-dual instanton equations for the $SU(3)$ Yang-Mills theory with cylindrical symmetry studied by Bais and Weldon [32]. There is a clear distinction between the two however; the latter are in the context of Yang-Mills theories over $\mathbb{R}^4$ and the cylindrically symmetric ansatz essentially dimensionally reduces that theory to an Abelian Higgs-type model over $\mathbb{H}^2$, with the $SU(3)$ instanton solutions being characterized by a Pontryagin index, which is given as the sum of the two winding numbers of the Abelian Higgs-type model over $\mathbb{H}^2$ with $U(1) \times U(1)$ gauge symmetry, while our BPS equations are obtained for the $U(1) \times U(1)$ Abelian Higgs-type model over $\mathbb{R}^2$.

## VI. GENERALIZATION OF $SU(2)$-EQUIVARIANT GAUGE FIELDS FOR $U(n)$ GAUGE THEORY

Now, we briefly indicate how the results of Sec. III generalize to $U(n)$ gauge theories over $\mathcal{M} \times S^2_F$. For this purpose we write the symmetry generators $\omega_a$,

$$\omega_a = X_a^{(2\ell+1)} \otimes \mathbf{1}_n - \mathbf{1}^{(2\ell+1)} \otimes i\tilde\Sigma^k_a, \tag{6.1}$$

where $\tilde\Sigma^k_a$ form the spin $k$ irreducible representation of $SU(2)$ with $n = 2k + 1$. Thus, the $SU(2)$ IRR content of $\omega_a$ is

$$\ell \otimes k = (\ell + k) \oplus (\ell + k - 1) \oplus \cdots \oplus |\ell - k|, \tag{6.2}$$

and the IRR content of the adjoint action of $\omega_a$ can be found to be

$$[\ell \otimes k]^{\otimes 2} = (\mathbf{2k+1})0 \oplus (\mathbf{6k+1})1 \oplus \cdots. \tag{6.3}$$

This decomposition means that under the adjoint action of $\omega_a$, there are $(2k + 1)$ scalars and $(6k + 1)$ vectors. It indicates that with our symmetry constraints (3.6), the set of solutions to $A_\mu$ should be $(2k + 1)$ dimensional while the set of the solutions to $A_a$ should be $(6k + 1)$ dimensional. It is possible to find the parametrization of $A_\mu$ by using the following rotational invariants:

$$\begin{aligned} \mathbf{1}_{(2\ell+1)(2k+1)}, \quad \tilde\Sigma^k_a X_a, \quad (\tilde\Sigma^k_a X_a)^2, \\ (\tilde\Sigma^k_a X_a)^3, \quad \ldots, \quad (\tilde\Sigma^k_a X_a)^{2k}. \end{aligned} \tag{6.4}$$

We may recall that the adjoint representation of $SU(n)$ is $n^2 - 1$ dimensional and decomposes under the $SU(2)$ IRRs as

$$n^2 - 1 = \oplus \sum_{j=1}^{n-1} (2j + 1). \tag{6.5}$$

This is a multipole expansion starting with the dipole term and going up to the $(n - 1)^{\text{th}}$-pole term. Thus, considering that we may construct one rotational invariant per multipole term, together with the identity we have $n = 2k + 1$ rotational invariants as we have already inferred from (6.3). The invariants listed in (6.4) may be expressed in terms of the appropriate multipole tensors and can further be combined into idempotents as given in (3.10) for the case of $k = 1$; and the vectors can be obtained subsequently.

## VII. EQUIVARIANT FIELD MODES OVER OTHER VACUUM CONFIGURATIONS

It is possible to investigate the structure of equivariant fields over other fuzzy vacuum configurations. One such case of particular interest is the vacuum configuration

$$S_F^{2\,\text{Int}} := S^2_F(\ell) \oplus S^2_F(\ell) \oplus S^2_F\left(\ell + \frac{1}{2}\right) \oplus S^2_F\left(\ell - \frac{1}{2}\right), \tag{7.1}$$

studied in [11],[4] where it was also noted that the $S_F^{2\,\text{Int}}$ vacuum forms the bosonic part of the fuzzy supersphere

[4]Note that in this case the $V_2(\Phi)$ term is omitted from the action (2.1). Nevertheless, it is possible to impose it as a constraint as discussed in [11].

with $OSP(2,2)$ supersymmetry [36–38]. There, the structure of this vacuum was revealed by performing the field redefinition

$$\Phi_a = \phi_a + \Gamma_a, \qquad \Gamma_a = -\frac{i}{2}\Psi^\dagger \tilde{\tau}_a \Psi, \tag{7.2}$$

where

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \Psi_\alpha \in \mathrm{Mat}(\mathcal{N}), \quad \alpha = 1, 2 \tag{7.3}$$

is a doublet of the global $SU(2)$ symmetry of the action (2.1). In (7.2) and (7.3), $\phi_a$, $\Psi_\alpha$, and $\Gamma_a$ are all transforming adjointly under $SU(\mathcal{N})$ and $\tilde{\tau}_a = \tau_a \otimes 1_{\mathcal{N}}$ with $\tau_a$ being the Pauli matrices. We note that $\phi_a, (a = 1, 2, 3)$ have $3N^2$ real degrees of freedom while $\Psi$ has $4N^2$ real degrees of freedom in total. However, what enters into the definition of $\Gamma_a$ are the equivalence classes $\Psi \sim U\Psi$, $U \in SU(\mathcal{N})$, as it can readily be observed that $\Gamma_a$ are invariant under the left action $U\Psi$ of $SU(\mathcal{N})$ on $\Psi$. It is thus clear that $\Gamma_a (a = 1, 2, 3)$ have in total $4N^2 - N^2 = 3N^2$ degrees of freedom as $\phi_a$'s do and (7.2) is indeed a reparametrization of the fields $\Phi_a$ [12].

Using (7.2), we see that up to gauge transformations (2.3) the vacuum configuration is given as

$$\Phi_a = (X_a^{(2\ell+1)} \otimes \mathbf{1}_4 \otimes \mathbf{1}_n) + (\mathbf{1}_{2\ell+1} \otimes \Gamma_a^0 \otimes \mathbf{1}_n), \tag{7.4}$$

where $\Gamma_a^0 = -\frac{i}{2}\psi^\dagger \tau_a \psi$ are $4 \times 4$ matrices and the two-component spinor $\Psi^0 \equiv \psi$ is taken as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \coloneqq \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{7.5}$$

and where $b_\alpha, b_\alpha^\dagger$ are two sets of fermionic annihilation-creation operators that span the four-dimensional Hilbert space with the basis vectors

$$|n_1, n_2\rangle \equiv (b_1^\dagger)^{n_1} (b_2^\dagger)^{n_2} |0, 0\rangle, \qquad n_1, n_2 = 0, 1. \tag{7.6}$$

$SU(2)$ IRR content of $\Gamma_a^0$ is

$$0_{\mathbf{0}} \oplus 0_{\mathbf{2}} \oplus \frac{1}{2}, \tag{7.7}$$

where $0_{\mathbf{0}}, 0_{\mathbf{2}}$ stand for the two inequivalent singlets. These two singlets are distinguished by the eigenvalues of the number operator $N = b_\alpha^\dagger b_\alpha$ that take the values 0 and 2, respectively. It is easy to see that the projections to the singlet and doublet subspaces respectively may be found on these representations as

$$\begin{aligned} P_0 &= 1 - N + 2N_1 N_2, \\ P_{0_0} &= -\frac{1}{2}(N-2)P_0 = 1 - N + N_1 N_2, \\ P_{0_2} &= \frac{1}{2} N P_0 = N_1 N_2 = -\frac{1}{2} N + \frac{1}{2} P_{\frac{1}{2}}, \\ P_{\frac{1}{2}} &= N - 2N_1 N_2, \end{aligned} \tag{7.8}$$

where $N = N_1 + N_2, N_1 = b_1^\dagger b_1, N_2 = b_2^\dagger b_2$.

$SU(2)$ IRR content of vacuum configuration (7.4) can be derived from the Clebsch-Gordan decomposition as

$$\ell \otimes \left(0_{\mathbf{0}} \oplus 0_{\mathbf{2}} \oplus \frac{1}{2}\right) \equiv \ell \oplus \ell \oplus \left(\ell + \frac{1}{2}\right) \oplus \left(\ell - \frac{1}{2}\right), \quad \ell \neq 0. \tag{7.9}$$

This indicates that the vacuum configuration (7.1) can be interpreted as a direct sum of four concentric fuzzy spheres as it has already been discussed in [11]. In that article low energy structure of $U(2)$ gauge theory over $\mathcal{M} \times S_F^{2\,\mathrm{Int}}$ was investigated in detail. Here, our aim is to consider the $U(3)$ gauge theory over $\mathcal{M} \times S_F^{2\,\mathrm{Int}}$ and construct the $SU(2)$ equivariant gauge fields characterizing its low energy behavior. In order to determine the latter, we choose the $SU(2)$ symmetry generators $\omega_a$ as

$$\begin{aligned} \omega_a &= (X_a^{(2\ell+1)} \otimes \mathbf{1}_4 \otimes \mathbf{1}_3) + (\mathbf{1}_{2\ell+1} \otimes \Gamma_a^0 \otimes \mathbf{1}_3) \\ &\quad - (\mathbf{1}_{2\ell+1} \otimes \mathbf{1}_4 \otimes i\Sigma_a) \\ &\eqqcolon X_a + \Gamma_a^0 - i\Sigma_a \\ &\eqqcolon D_a - i\Sigma_a, \omega_a \in u(2\ell+1) \otimes u(4) \otimes u(3), \end{aligned} \tag{7.10}$$

and they satisfy (3.2). $\omega_a$ carries a direct sum of IRRs of $SU(2)$, which is given as

$$\begin{aligned} &\left(\ell \oplus \ell \oplus \left(\ell + \frac{1}{2}\right) \oplus \left(\ell - \frac{1}{2}\right)\right) \otimes 1 \\ &\equiv \mathbf{2}((\ell-1) \oplus \ell \oplus (\ell+1)) \oplus \mathbf{2}\left(\left(\ell + \frac{1}{2}\right) \oplus \left(\ell - \frac{1}{2}\right)\right) \\ &\oplus \left(\ell - \frac{3}{2}\right) \oplus \left(\ell + \frac{3}{2}\right). \end{aligned} \tag{7.11}$$

Projections to the representations appearing in the rhs of (7.11) are given in Table 1, where

$$Q_I = \frac{i}{\frac{1}{2}(\ell + \frac{1}{2})}\left(X_a \Gamma_a - \frac{1}{4}\Pi_{\frac{1}{2}}\right), \qquad Q_I^2 = -\Pi_{\frac{1}{2}}. \tag{7.12}$$

$SU(2)$-equivariant gauge fields can be obtained by imposing the symmetry constraints in (3.6) and the additional constraint

$$[\omega_a, \Psi_\alpha] = \frac{i}{2}(\tilde{\tau}_a)_{\alpha\beta}\Psi_\beta. \tag{7.13}$$

TABLE I. Projections to the representations appearing in the rhs of (7.11).

| Projector | Representation |
|---|---|
| $\Pi_{0_0} = \mathbf{1}_{2\ell+1} \otimes P_{0_0} \otimes \mathbf{1}_3$ | $(\ell-1) \oplus \ell \oplus (\ell+1)$ |
| $\Pi_{0_2} = \mathbf{1}_{2\ell+1} \otimes P_{0_2} \otimes \mathbf{1}_3$ | $(\ell-1) \oplus \ell \oplus (\ell+1)$ |
| $\Pi_+ = \frac{1}{2}(iQ_I + \Pi_{\frac{1}{2}})$ | $(\ell-\frac{1}{2}) \oplus (\ell+\frac{1}{2}) \oplus (\ell+\frac{3}{2})$ |
| $\Pi_- = \frac{1}{2}(-iQ_I + \Pi_{\frac{1}{2}})$ | $(\ell-\frac{3}{2}) \oplus (\ell-\frac{1}{2}) \oplus (\ell+\frac{1}{2})$ |
| $\Pi_0 = \Pi_{0_0} + \Pi_{0_2} = \mathbf{1}_{2\ell+1} \otimes P_0 \otimes \mathbf{1}_3$ | $\mathbf{2}((\ell-1) \oplus \ell \oplus (\ell+1))$ |
| $\Pi_{\frac{1}{2}} = \Pi_+ + \Pi_- = \mathbf{1}_{2\ell+1} \otimes P_{\frac{1}{2}} \otimes \mathbf{1}_3$ | $\mathbf{2}((\ell+\frac{1}{2}) \oplus (\ell-\frac{1}{2})) \oplus (\ell-\frac{3}{2}) \oplus (\ell+\frac{3}{2})$ |

The dimensions of solution spaces for $A_\mu$, $A_a$ and $\Psi_\alpha$ can be derived by the Clebsch-Gordan decomposition of the adjoint action of $\omega_a$. The relevant part of this decomposition is

$$\left[\mathbf{2}((\ell-1) \oplus \ell \oplus (\ell+1)) \oplus \mathbf{2}\left(\left(\ell+\frac{1}{2}\right) \oplus \left(\ell-\frac{1}{2}\right)\right) \oplus \left(\ell-\frac{3}{2}\right) \oplus \left(\ell+\frac{3}{2}\right)\right]^{\otimes 2} \equiv \mathbf{22}\, 0 \oplus \mathbf{40}\, \frac{1}{2} \oplus \mathbf{54}\, 1 \oplus \cdots. \tag{7.14}$$

This simply means that there are 22 rotational invariants and $A_\mu$ may be parametrized by these invariants. A suitable set may be listed as the following projectors and idempotents (in the subspace they belong to),

$$\begin{aligned}
&\Pi_{0_0}, \quad \Pi_{0_2}\Pi_+, \quad \Pi_- iS_1, \quad iS_2, \quad Q^1_{0_0} = \Pi_{0_0} Q_1, \quad Q^2_{0_0} = \Pi_{0_0} Q_2, \\
&Q^1_{0_2} = \Pi_{0_2} Q_1, \quad Q^2_{0_2} = \Pi_{0_2} Q_2, \quad Q^1_-, \quad Q^2_-, \quad Q^1_+, \quad Q^2_+, \quad Q^1_{+-}, \quad Q^2_{-+}, \\
&Q_{S11} = S_1 Q_1, \quad Q_{S12} = S_1 Q_2, \quad Q_{S21} = S_2 Q_1, \quad Q_{S22} = S_2 Q_2, \quad Q_F, \quad Q_H,
\end{aligned} \tag{7.15}$$

where

$$\begin{aligned}
Q^1_- &= \frac{1}{\ell(2\ell+3)}((2\ell+1)(\ell+1)\Pi_- Q_1 \Pi_- - i\Pi_-), \\
Q^2_- &= \frac{\ell(2\ell+1)}{(\ell+1)(2\ell-1)}\Pi_- Q_2 \Pi_- + \frac{(2\ell+1)}{\ell(2\ell-1)(2\ell+3)}\Pi_- Q_1 \Pi_- - \frac{i}{\ell(\ell+1)(2\ell-1)(2\ell+3)}\Pi_-, \\
Q^1_+ &= \frac{(2\ell+1)(\ell+1)}{\ell(2\ell+3)}\Pi_+ Q_1 \Pi_+ + \frac{(2\ell+1)^2}{(2\ell-1)(2\ell+3)}\Pi_+ Q_2 \Pi_+ - i\frac{(4\ell^3+4\ell^2-\ell+1)}{\ell(2\ell-1)(2\ell+3)}\Pi_+, \\
Q^2_+ &= \frac{1}{(\ell+1)(2\ell-1)}(\ell(2\ell+1)\Pi_+ Q_2 \Pi_+ - i\Pi_+), \\
Q^1_{+-} &= \Pi_+ Q_1 \Pi_- - i\Pi_{\frac{1}{2}} + 2i\Pi_+, \qquad Q^2_{-+} = \Pi_- Q_2 \Pi_+ - i\Pi_{\frac{1}{2}} + 2i\Pi_-, \\
S_i &= \mathbf{1}_{2\ell+1} \otimes s_i \otimes \mathbf{1}_2, \qquad s_i = \begin{pmatrix} \sigma_i & 0_2 \\ 0_2 & 0_2 \end{pmatrix}, \qquad i = 1, 2,
\end{aligned} \tag{7.16}$$

and

$$\begin{aligned}
Q_F &= \frac{1}{3}\Gamma_a\Sigma_a - 2i(\Gamma_a\Sigma_a)^2 - i\frac{4}{3}\Pi_{\frac{1}{2}}, \\
Q_H &= \frac{4(2\ell+1)}{6\ell^2+11\ell+1}Q' - \frac{4(2\ell^2+3\ell)}{6\ell^2+11\ell+1}Q'' - i\frac{(2\ell-1)(\ell+1)}{6\ell^2+11\ell+1}\Pi_+ - i\frac{3(2\ell-1)(\ell+1)}{6\ell^2+11\ell+1}\Pi_- \\
&\quad + i\frac{4\sqrt{4\ell^2+10\ell+2}}{6\ell^2+11\ell+1}\epsilon_{abc}X_a\Gamma_b\Sigma_c + i\frac{16}{6\ell^2+11\ell+1}(\epsilon_{abc}X_a\Gamma_b\Sigma_c)^2, \\
Q' &= \frac{\ell(2\ell+1)}{(\ell+1)(2\ell-1)}\Pi_- Q_2 \Pi_- + \frac{(2\ell+1)^2}{(2\ell-1)(2\ell+3)}\Pi_- Q_1 \Pi_- - i\frac{4\ell^3+8\ell^2+3\ell-2}{(\ell+1)(2\ell-1)(2\ell+3)}\Pi_-, \\
Q'' &= \frac{(2\ell+1)}{(\ell+1)(2\ell-1)(2\ell+3)}\Pi_+ Q_2 \Pi_+ + \frac{(2\ell+1)(\ell+1)}{\ell(2\ell+3)}\Pi_+ Q_1 \Pi_+ - i\frac{1}{\ell(\ell+1)(2\ell-1)(2\ell+3)}\Pi_+.
\end{aligned} \tag{7.17}$$

Using *Mathematica* it is easy to verify that

$$(iS_i)^2 = -\Pi_0, \quad (Q^i_{0_0})^2 = -\Pi^i_{0_0}, \quad (Q^i_{0_2})^2 = -\Pi^i_{0_2}, \quad (Q^i_{\pm})^2 = -\Pi_{\pm}, \quad (Q^1_{+-})^2 = -\Pi_{\frac{1}{2}},$$
$$(Q^2_{-+})^2 = -\Pi_{\frac{1}{2}}, \quad (Q_{Sij})^2 = -\Pi_0, \quad Q^2_F = -\Pi_{\frac{1}{2}}, \quad Q^2_H = -\Pi_{\frac{1}{2}}, \quad Q'^2 = -\Pi_-, \quad Q'^2 = -\Pi_+. \tag{7.18}$$

In Eq. (7.14), it is seen that under the adjoint action of $\omega_a$, there are 54 objects that transform as vectors. Using the rotational invariant in (7.15), we can construct these as follows:

$$\begin{aligned}
&[D_a, Q^i_{0_0}], \quad Q^i_{0_0}[D_a, Q^i_{0_0}], \quad \{D_a, Q^i_{0_0}\},\\
&[D_a, Q^i_{0_2}], \quad Q^i_{0_2}[D_a, Q^i_{0_2}], \quad \{D_a, Q^i_{0_2}\},\\
&[D_a, Q^i_-], \quad Q^i_-[D_a, Q^i_-], \quad \{D_a, Q^i_-\},\\
&[D_a, Q^i_+], \quad Q^i_+[D_a, Q^i_+], \quad \{D_a, Q^i_+\},\\
&[D_a, Q_H], \quad Q_H[D_a, Q_H], \quad \{D_a, Q_H\},\\
&[D_a, Q_F], \quad Q_F[D_a, Q_F], \quad \{D_a, Q_F\},\\
&[D_a, Q_{S11}], \quad Q^1_0[D_a, Q_{S11}], \quad \{D_a, Q_{S11}\},\\
&[D_a, Q_{S12}], \quad Q^2_0[D_a, Q_{S12}], \quad \{D_a, Q_{S12}\},\\
&[D_a, Q_{S21}], \quad Q^1_0[D_a, Q_{S21}], \quad \{D_a, Q_{S21}\},\\
&[D_a, Q_{S22}], \quad Q^2_0[D_a, Q_{S22}], \quad \{D_a, Q_{S22}\},\\
&[D_a, Q^1_{+-}], \quad Q^1_{\frac{1}{2}}[D_a, Q^1_{+-}], \quad \{D_a, Q^1_{+-}\},\\
&[D_a, Q^2_{-+}], \quad Q^2_{\frac{1}{2}}[D_a, Q^2_{-+}], \quad \{D_a, Q^2_{-+}\},\\
&\Pi_{0_0}\omega_a, \quad \Pi_{02}\omega_a, \quad \Pi_-\omega_a, \Pi_+\omega_a, \quad S_1\omega_a, S_2\omega_a.
\end{aligned} \tag{7.19}$$

Here $Q^1_0 = \Pi_0 Q_1, Q^2_0 = \Pi_0 Q_2, Q^1_{\frac{1}{2}} = \Pi_{\frac{1}{2}} Q_1, Q^2_{\frac{1}{2}} = \Pi_{\frac{1}{2}} Q_2$, and no sum over repeated indices is implied. It is possible to parametrize $A_a$ in terms of these 54 objects. For the 40 objects that transform as spinors under the adjoint action of $\omega_a$, we can, for instance, take

$$\begin{aligned}
&\Pi_{0_0}\beta_\alpha Q_{-+}, \quad Q^1_{0_0}\beta_\alpha\Pi_-, \quad Q^2_{0_0}\beta_\alpha\Pi_-, \quad \Pi_{0_0}\beta_\alpha Q_{+-}, \quad Q^1_{0_0}\beta_\alpha\Pi_+, \quad Q^2_{0_0}\beta_\alpha\Pi_+,\\
&Q^1_{0_0}\beta_\alpha Q_{+-}, \quad Q^2_{0_0}\beta_\alpha Q_{-+}, \quad \Pi_-\beta_\alpha Q^1_{0_2}, \quad \Pi_-\beta_\alpha Q^2_{0_2}, \quad \Pi_+\beta_\alpha Q^1_{0_2}, \quad \Pi_+\beta_\alpha Q^2_{0_2},\\
&Q^1_{+-}\beta_\alpha\Pi_{0_2}, \quad Q^2_{-+}\beta_\alpha\Pi_{0_2}, \quad Q^1_{+-}\beta_\alpha Q^1_{0_2}, \quad Q^2_{-+}\beta_\alpha Q^2_{0_2}, \quad S_1\beta_\alpha\Pi_+, \quad S_1\beta_\alpha\Pi_-,\\
&\Pi_-\beta_\alpha S_2, \quad \Pi_+\beta_\alpha S_2, \quad Q_{S11}\beta_\alpha\Pi_+, \quad Q_{S11}\beta_\alpha\Pi_-, \quad Q_{S12}\beta_\alpha\Pi_+, \quad Q_{S12}\beta_\alpha\Pi_-,\\
&\Pi_-\beta_\alpha Q_{S21}, \quad \Pi_-\beta_\alpha Q_{S22}, \quad \Pi_+\beta_\alpha Q_{S12}, \quad \Pi_+\beta_\alpha Q_{S22}, \quad Q_{S11}\beta_\alpha Q^1_{+-}, \quad Q_{S12}\beta_\alpha Q^2_{-+},\\
&Q^1_{+-}\beta_\alpha Q_{S21}, \quad Q^2_{-+}\beta_\alpha Q_{S22}, \quad \Pi_{0_0}\beta_\alpha Q^1_+, \quad \Pi_{0_0}\beta_\alpha Q^2_-, \quad Q_{S11}\beta_\alpha Q^1_+, \quad Q_{S12}\beta_\alpha Q^2_-,\\
&Q^1_+\beta_\alpha Q_{S21}, \quad Q^2_-\beta_\alpha Q_{S22}, \quad Q^1_+\beta_\alpha\Pi_{0_2}, \quad Q^2_-\beta_\alpha\Pi_{0_2}.
\end{aligned} \tag{7.20}$$

Thus, we have determined all the equivariant low energy degrees of freedom for the $U(3)$ gauge theory over $\mathcal{M} \times S_F^{2\,\mathrm{Int}}$. A few remarks are now in order. First, we emphasize once again that, from a geometrical point of view the vacuum $S_F^{2\,\mathrm{Int}}$ may be interpreted as stacks of concentric D2-branes with magnetic monopole fluxes and due to this fact it is possible to think of the equivariant gauge field modes that we have found as the modes of the gauge fields living on the world volume of these D-branes. Let us also stress that the equivariant spinors given above do not constitute independent degrees of freedom in the $U(3)$ effective gauge theory over $\mathcal{M} \times S_F^{2\,\mathrm{Int}}$. Their bilinears, however, may be constructed to yield the equivariant scalars and vectors. In other words, it is possible to use these equivariant spinor modes to express the "square roots" of the equivariant gauge field modes.

It is possible to explore the dimensional reduction of the $U(3)$ gauge theory over $S_F^{2\,\mathrm{Int}}$ or over its projections, such as the monopole bundles $S_F^{2\pm} = S_F^2(\ell) \oplus S_F^2(\ell \pm \frac{1}{2})$ with winding numbers $\pm 1$. In this latter case, it is easy to observe that the reduced model yields two decoupled Abelian Higgs-type models, each carrying $U(1)^{\otimes 3}$ as found in Sec. IV and the vortex solutions determined in Sec. V are valid within each sector. Dimensional reduction over $S_F^{2\,\mathrm{Int}}$ is quite tedious calculationwise and is not considered here.

## VIII. CONCLUSIONS AND OUTLOOK

Let us briefly summarize and discuss the results of our article, state our conclusions, and indicate some directions that we aim to explore in the near future. As we mentioned in the introduction, a large amount of investigations themed on exploring several aspects of SSB and mass deformed $SU(\mathcal{N})$ $N = 4$ SYM, as well as $SU(\mathcal{N})$ YM theories coupled to a triplet of adjoint matter fields, has recently been accumulating mainly due to strong motivations emanating from string theory and M-theory. An essential common feature of the models under investigation is that they possess a fuzzy sphere, the direct product of two fuzzy spheres, or the direct sums of these fuzzy spaces as vacuum configurations. Such fuzzy vacua appear via the spontaneous breaking of the $SU(\mathcal{N})$ gauge symmetry of the models down to a smaller gauge group, $U(n)$ $(n < \mathcal{N})$, and analysis of the fluctuations around these vacuum configurations reveals that the latter have the structure of gauge fields over either $\mathcal{S}_F^2$ or $\mathcal{S}_F^2 \times \mathcal{S}_F^2$. KK-type mode expansion of the gauge fields or their equivariant parametrization provide complementary approaches in understanding and interpreting the emerging models after symmetry breaking as effective gauge theories with fuzzy spaces as extra dimensions.

In this paper, we have analyzed the low energy structure of the $U(3)$ gauge theory on $\mathcal{M} \times S_F^2$. We have determined the equivariant fields transforming invariantly and as vectors under the combined adjoint action of $SU(2)$ rotations over the fuzzy sphere and those $U(3)$ gauge transformations generated by $SU(2) \subset U(3)$ carrying the spin 1 IRR of $SU(2)$, when the $SU(2)$ subgroup is maximally embedded in $SU(3)$. Our results reveal that the dipole and quadrupole terms, which appear in the branching of the adjoint representation of $SU(3)$ as $\underline{8} \to \underline{5} \oplus \underline{3}$ under $SU(2)$ are the useful objects in constructing the equivariant scalars and this generalizes in $U(n)$ theories over $\mathcal{M} \times S_F^2$ to employing the $n-1$ multipole terms in the branching of the adjoint representation of $SU(n)$ under $SU(2)$ as we have determined in Sec. VI. Results of Sec. III also indicate that the equivariance conditions that we have imposed on the fields break the $U(3)$ gauge symmetry down $U(1) \times U(1) \times U(1)$. In Sec. IV, we determined the LEA emanating from the equivariant parametrization of the fields and found that it consists of two complex scalars, each coupling to one of the gauge fields $a_\mu^i$, $(i = 1, 2)$ only, and three real scalars coupling to the complex fields and to each other through a quartic potential. We have seen that in the $\ell \to \infty$ limit, gauge field $b_\mu$ either decouples completely from the rest of the LEA or it is eliminated by solving its equation of motion in powers of $\frac{1}{\ell}$. By determining the vacuum structure of the effective potential for the scalars, we were able to give two different vortex solutions for the LEA on $\mathbb{R}^2$, both of which are characterized by two winding numbers in each case. We have also made clear how the commutative limit of our results relates to the instanton solutions in self-dual $SU(3)$ Yang-Mills theory for cylindrically symmetric gauge fields of Bais and Weldon [32] and indicated the apparent connection between the BPS vortices that we obtain in a certain commutative limit in Sec. V and the instanton solution in [32]. In the penultimate section of our article we have provided a complete analysis of the $U(3)$-equivariant fields over $\mathcal{M} \times S_F^{2\,\mathrm{Int}}$ and determined the equivariant field modes characterizing the low energy behavior of the effective $U(3)$ theory on $\mathcal{M} \times S_F^{2\,\mathrm{Int}}$ in terms of suitable idempotents and projection operators. The reason for our interest in $S_F^{2\,\mathrm{Int}}$ was explained previously; we only stress once again that $S_F^{2\,\mathrm{Int}}$ may be seen as stacks of concentric fuzzy D-branes carrying magnetic monopole fluxes from a stringy viewpoint, and consequently equivaraint gauge field modes found in Sec. VII may be interpreted as those living on the world volume of these D-branes, and may potentially be useful in bridging the effective gauge theory and the string theoretic perspectives.

In our future work, we plan to apply the techniques used in this article to explore the low energy structure of $\mathbb{Z}_3$ orbifold projected $\mathcal{N} = 4$ SYM theories in four dimensions which are deformed by the addition of cubic SSB and mass terms in the scalar fields [25]. These models are known to have orbifold twisted fuzzy spheres as vacuum configurations and we are initially interested in revealing the physically interesting vacuum configurations made up of direct sums of orbifold twisted fuzzy spheres, and subsequently aim to analyze how effective gauge theories with twisted fuzzy spheres as extra dimensions emerge by determining the explicit parametrization of gauge fields fulfilling certain well-motivated symmetry conditions. We hope to report on the developments on these ideas elsewhere.

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## APPENDIX A: DETAILS OF THE DIMENSIONAL REDUCTION OVER $S_F^2$

$$\begin{aligned}
\Lambda_1 &:= -\frac{2\ell^4+6\ell^3+4\ell^2-\ell-2}{4\ell(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 + \frac{2\ell^4+2\ell^3-2\ell^2-\ell-1}{4(\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2 + \frac{\omega_c}{2\ell^2+2\ell+1},\\
\Lambda_2 &:= -\frac{4\ell^4+8\ell^3+5\ell^2}{4(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2 - \frac{8\ell^5+18\ell^4+11\ell^3+3\ell^2}{4(\ell+1)(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2 + \frac{\ell\omega_c}{2\ell^2+2\ell+1},\\
\Lambda_3 &:= \frac{(\ell+1)(8\ell^4+14\ell^3+5\ell^2-3\ell-2)}{4\ell(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 + \frac{(\ell+1)(4\ell^3+4\ell^2+\ell+1)}{4(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2 - \frac{(\ell+1)\omega_c}{2\ell^2+2\ell+1},\\
\Lambda_4 &:= -\frac{4\ell^4+10\ell^3+4\ell^2-\ell-2}{4\ell(2\ell+1)^2(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 + \frac{4\ell^4+6\ell^3-2\ell^2-5\ell-3}{4(2\ell+1)^2(\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2 + \frac{\omega_c}{(2\ell+1)^2},\\
\Lambda_5 &:= \frac{2\ell^5+10\ell^4+14\ell^3+3\ell^2-3\ell-2}{2\ell(\ell+1)(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 - \frac{2\ell^4+2\ell^3-\ell^2-\ell-2}{2(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2 - \frac{2\omega_c}{2\ell^2+2\ell+1},\\
\Lambda_6 &:= -\frac{2\ell^4+6\ell^3+5\ell^2+\ell-2}{2(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_1 - \frac{2\ell^5-6\ell^3-\ell^2+3\ell+2}{2\ell(\ell+1)(2\ell+1)(2\ell^4+4\ell^3+\ell^2-\ell-1)}P_2 + \frac{2\omega_c}{2\ell^2+2\ell+1},\\
\Lambda_7 &:= \frac{2\ell^3+6\ell^2+3\ell-3}{2(\ell+1)(2\ell+1)^2(\ell^2+\ell-1)}P_1 + \frac{2\ell^3-3\ell+2}{2\ell(2\ell+1)^2(\ell^2+\ell-1)}P_2 - \frac{2\omega_c}{(2\ell+1)^2},\\
\Lambda_8 &:= \Lambda_9 := -\Lambda_{13} := -\frac{1}{(2\ell+1)^2}, \qquad \Lambda_{10} := \frac{2\ell^2+3\ell-1}{2(\ell+1)(2\ell+1)}\varphi_3 - \frac{1}{2(2\ell+1)}\chi_3 + \frac{1}{2\ell+1}\psi,\\
\Lambda_{11} &:= -\frac{2\ell^2+\ell-2}{2\ell(2\ell+1)}\chi_3 - \frac{1}{2(2\ell+1)}\varphi_3 + \frac{1}{2\ell+1}\psi,\\
\Lambda_{12} &:= \frac{1}{2\ell+1}(-Q_1[X_c,Q_1] - Q_2[X_c,Q_2] - \omega_c + 2X_c),
\end{aligned} \tag{A1}$$

where $P_1 := -Q_1[X_c,Q_1] - i\{X_c,Q_2\}$ and $P_2 := -Q_2[X_c,Q_2] - i\{X_c,Q_1\}$.

$$\begin{aligned}
\alpha_1 &= \frac{4(\ell^2+\ell-1)^2(\ell^2+\ell+1)}{3\ell^3(\ell+1)^3}, \qquad \alpha_2 = \frac{4(2\ell^4+5\ell^3+\ell^2-\ell+3)}{3(\ell+1)^3(2\ell+1)},\\
\alpha_3 &= \frac{4(2\ell^4+3\ell^3-2\ell^2-4\ell+2)}{3\ell^3(2\ell+1)}, \qquad \alpha_4 = \alpha_5\\
\alpha_5 &= \frac{2(-3\ell^8-12\ell^7-14\ell^6+13\ell^4+12\ell^3+16\ell^2+12\ell-12)}{3\ell^3(\ell+1)^3(2\ell+1)^2},\\
\alpha_6 &= \frac{4(4\ell^7+10\ell^6+2\ell^5-2\ell^4-3\ell^3-15\ell^2+4)}{3\ell^3(\ell+1)^2(2\ell+1)^2},\\
\alpha_7 &= \frac{4(4\ell^7+18\ell^6+26\ell^5+2\ell^4-35\ell^3-28\ell^2+7\ell+6)}{3\ell^2(\ell+1)^3(2\ell+1)^2},\\
\alpha_8 &= \frac{4(\ell^6+3\ell^5+15\ell^4+25\ell^3-30\ell^2-42\ell+24)}{3\ell^2(\ell+1)^2(2\ell+1)^2},\\
\alpha_9 &= \frac{4(2\ell^6+23\ell^5+43\ell^4-11\ell^3-45\ell^2+6\ell+6)}{3\ell(\ell+1)^2(2\ell+1)^3},\\
\alpha_{10} &= \frac{4(2\ell^6-11\ell^5-42\ell^4-7\ell^3+46\ell^2+6\ell-12)}{3\ell^2(\ell+1)(2\ell+1)^3},\\
\alpha_{11} &= \frac{2(\ell^4+2\ell^3-5\ell^2-6\ell+4)}{\ell(\ell+1)(2\ell+1)^2}.
\end{aligned} \tag{A2}$$

$$\begin{aligned}
\beta_1 &= \frac{4\ell^2(4\ell^3+14\ell^2+14\ell+3)}{3(\ell+1)^3(2\ell+1)^2}, \qquad \beta_2 = \frac{4(4\ell^2+4\ell-3)}{3(2\ell+1)^2},\\
\beta_3 &= \frac{4(8\ell^6+36\ell^5+46\ell^4+5\ell^3-9\ell^2+7\ell-3)}{3(\ell+1)^3(2\ell+1)^3},\\
\beta_4 &= \frac{4(2\ell^4+9\ell^3+15\ell^2+7\ell-3)}{3(\ell+1)^3(2\ell+1)^3}, \qquad \beta_5 = \frac{4(-4\ell^4-8\ell^3+7\ell^2+11\ell-6)}{3(\ell+1)^2(2\ell+1)^2},\\
\beta_6 &= \frac{4(\ell-1)^2(2\ell^2+7\ell+6)}{3(\ell+1)^3(2\ell+1)^2}, \qquad \beta_7 = \frac{8(8\ell^4+22\ell^3+7\ell^2-10\ell+3)}{3(\ell+1)^2(2\ell+1)^3},\\
\beta_8 &= \frac{8\ell(4\ell^3+12\ell^2+7\ell-3)}{(\ell+1)^2(2\ell+1)^3}, \qquad \beta_9 = \beta_{10} = \frac{8\ell(2\ell+3)}{(\ell+1)(2\ell+1)^3},
\end{aligned} \tag{A3}$$

$$\begin{aligned}
\gamma_1 &= \frac{4(\ell+1)^2(4\ell^3-2\ell^2-2\ell+1)}{3\ell^3(2\ell+1)^2}, \qquad \gamma_2 = \frac{2(-4\ell^4+2\ell^3-8\ell+4)}{3\ell^3(2\ell+1)^3},\\
\gamma_3 &= \frac{4(8\ell^6+12\ell^5-14\ell^4-21\ell^3+12\ell^2+12\ell-6)}{3\ell^3(2\ell+1)^3},\\
\gamma_4 &= \frac{4(\ell+2)^2(2\ell^2-3\ell+1)}{3\ell^3(2\ell+1)^2}, \qquad \gamma_5 = \frac{4(4\ell^4+8\ell^3-7\ell^2-11\ell+6)}{3\ell^2(2\ell+1)^2},\\
\gamma_6 &= \frac{8(8\ell^4+10\ell^3-11\ell^2-10\ell+6)}{3\ell^2(2\ell+1)^3}, \qquad \gamma_7 = \gamma_9 = \frac{8(\ell+1)(2\ell-1)}{\ell(2\ell+1)^3},\\
\gamma_8 &= \frac{8(4\ell^4+4\ell^3-5\ell^2-3\ell+2)}{\ell^2(2\ell+1)^3},
\end{aligned} \tag{A4}$$

$$\begin{aligned}
\delta_1 &= \frac{2(-3\ell^8-12\ell^7-12\ell^6+6\ell^5+13\ell^4+2\ell^3+2\ell-2)}{3\ell^3(\ell+1)^3(2\ell+1)^4},\\
\delta_2 &= \frac{4(2\ell^8+15\ell^7+23\ell^6-11\ell^5-23\ell^4+\ell^3-11\ell^2+4)}{3\ell^3(\ell+1)^2(2\ell+1)^4},\\
\delta_3 &= \frac{4(2\ell^8+\ell^7-26\ell^6-54\ell^5-8\ell^4+64\ell^3+44\ell^2-13\ell-10)}{3\ell^2(\ell+1)^3(2\ell+1)^4},\\
\delta_4 &= \frac{8(\ell^6+3\ell^5+5\ell^4+5\ell^3-8\ell^2-10\ell+6)}{3\ell^2(\ell+1)^2(2\ell+1)^4}, \qquad \delta_5 = \frac{8(2\ell^6+7\ell^5+3\ell^4-15\ell^3-15\ell^2+3\ell+3)}{3\ell(\ell+1)^2(2\ell+1)^5},\\
\delta_6 &= \frac{8(2\ell^6+5\ell^5-2\ell^4-3\ell^3+8\ell^2+\ell-2)}{3\ell^2(\ell+1)(2\ell+1)^5}, \qquad \delta_7 = \frac{4(3\ell^4+6\ell^3-5\ell^2-8\ell+4)}{\ell(\ell+1)(2\ell+1)^3},\\
\delta_8 &= \frac{4(3\ell^4+6\ell^3-\ell^2-4\ell+2)}{\ell(\ell+1)(2\ell+1)^4}, \qquad \delta_9 = \frac{8(\ell^6+3\ell^5+3\ell^4+\ell^3-6\ell^2-6\ell+4)}{\ell^2(\ell+1)^2(2\ell+1)^3},\\
\delta_{10} &= \frac{4(2\ell^4+3\ell^3-5\ell^2-4\ell+4)}{\ell(2\ell+1)^4}, \qquad \delta_{11} = \frac{4(2\ell^4+5\ell^3-2\ell^2-7\ell+2)}{(\ell+1)(2\ell+1)^4},\\
\delta_{12} &= \frac{8\ell(\ell+1)}{(2\ell+1)^4}, \qquad \delta_{13} = \frac{8\ell(2\ell^2-5\ell-9)}{3(2\ell+1)^5}, \qquad \delta_{14} = \frac{8(2\ell^3+11\ell^2+7\ell-2)}{3(2\ell+1)^5},\\
\delta_{15} &= \frac{4(-\ell^2-\ell+2)}{(2\ell+1)^3}, \qquad \delta_{16} = \frac{2(-\ell^2-\ell-2)}{(2\ell+1)^4}.
\end{aligned} \tag{A5}$$

$$R_1 = -\frac{\ell}{2(\ell+1)}(|\varphi|^2-1) - \frac{\ell+1}{2\ell}(|\chi|^2-1) + \frac{1}{\ell^2+\ell}(\chi_3-\varphi_3) - \frac{2\ell^4+4\ell^3-2\ell-1}{2(2\ell+1)^2(\ell^2+\ell)}(\chi_3-\varphi_3)^2$$
$$-\frac{2\ell^2+2\ell-1}{2\ell+1}\psi + \frac{1}{2\ell+1}(\chi_3-\varphi_3)\psi - \frac{\ell^2+\ell+1}{(2\ell+1)^2}\psi^2, \tag{A6}$$

$$R_2 = \frac{\ell}{2\ell^2+3\ell+1}(|\varphi|^2-1) + \frac{2\ell^2+\ell-1}{2(2\ell^2+1)}(|\chi|^2-1) + \frac{\ell^2+2\ell-1}{(2\ell+1)(\ell^2+\ell)}\left(\chi_3 - \frac{\chi_3^2+\varphi_3^2}{2(2\ell+1)}\right)$$
$$-\frac{2\ell^3+2\ell^2-3\ell+1}{\ell(2\ell+1)}\left(\varphi_3 - \frac{\varphi_3\chi_3}{2\ell+1}\right) - \frac{\ell+1}{2\ell+1}\left(\psi + \frac{\psi^2}{2\ell+1}\right) - \frac{2\ell^2+3\ell-1}{(2\ell+1)^2}\varphi_3\psi$$
$$+\frac{\ell+1}{(2\ell+1)^2}\chi_3\psi, \tag{A7}$$

$$R_3 = \frac{2\ell^2+3\ell}{2(2\ell^2+3\ell+1)}(|\varphi|^2-1) - \frac{\ell+1}{2\ell^2+\ell}(|\chi|^2-1) + \frac{\ell^2-2}{(2\ell+1)(\ell^2+\ell)}\left(\varphi_3 - \frac{\varphi_3^2+\chi_3^2}{2(2\ell+1)}\right)$$
$$-\frac{2\ell^3+4\ell^2-\ell-4}{(2\ell+1)(\ell+1)}\left(\chi_3 - \frac{\chi_3\varphi_3}{2\ell+1}\right) - \frac{\ell}{2\ell+1}\left(\psi + \frac{\psi^2}{2\ell+1}\right) - \frac{\ell}{(2\ell+1)^2}\varphi_3\psi$$
$$-\frac{2\ell^2+\ell-2}{(2\ell+1)^2}\chi_3\psi. \tag{A8}$$

Equations of motion that follow from the variations of the action (5.11) are

$$\left(1-\frac{1}{\ell^2}+\frac{1}{\ell^2}(\zeta^2+\eta^2)\right)\left(\zeta''+\frac{\zeta'}{r}\right) - \left(-\eta^2\left(1+\frac{3}{4\ell^2}+\frac{(M+a_\theta^2)^2}{2\ell^2r^2}-\frac{(N+a_\theta^1)^2}{\ell^2r^2}\right)+\frac{3}{\ell^2}\zeta^2\eta^2\right.$$
$$\left.-\frac{7}{4\ell^2}\eta^4+\left(1-\frac{1}{\ell^2}\right)\frac{(N+a_\theta^1)^2}{r^2}-\frac{1}{\ell^2}\zeta'^2-\frac{1}{2\ell^2}\eta'^2-\left(1-\frac{1}{\ell}+\frac{1}{2\ell^2}\right)+\left(2-\frac{1}{\ell}\right)\zeta^2\right)\zeta=0,$$
$$\left(1-\frac{1}{\ell^2}+\frac{1}{\ell^2}(\zeta^2+\eta^2)\right)\left(\eta''+\frac{\eta'}{r}\right) - \left(-\zeta^2\left(1+\frac{3}{4\ell^2}+\frac{(N+a_\theta^1)^2}{2\ell^2r^2}-\frac{(M+a_\theta^2)^2}{\ell^2r^2}\right)+\frac{3}{\ell^2}\zeta^2\eta^2\right.$$
$$\left.-\frac{7}{4\ell^2}\zeta^4+\left(1-\frac{1}{\ell^2}\right)\frac{(M+a_\theta^2)^2}{r^2}-\frac{1}{\ell^2}\eta'^2-\frac{1}{2\ell^2}\zeta'^2-\left(1+\frac{1}{\ell}-\frac{3}{2\ell^2}\right)+\left(2-\frac{1}{\ell}-\frac{2}{\ell^2}\right)\eta^2\right)\eta=0,$$
$$a_\theta^{1\prime\prime}-\frac{a_\theta^{1\prime}}{r}+\left(2-\frac{1}{\ell^2}\right)(M+a_\theta^2)\eta^2-\left(4-\frac{2}{\ell}+\frac{1}{\ell^2}\right)(N+a_\theta^1)\zeta^2=0,$$
$$a_\theta^{2\prime\prime}-\frac{a_\theta^{2\prime}}{r}+\left(2-\frac{1}{\ell^2}\right)(N+a_\theta^1)\zeta^2-\left(4+\frac{2}{\ell}-\frac{1}{\ell^2}\right)(M+a_\theta^2)\eta^2=0. \tag{A9}$$


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