

Comprehensive analysis of Yukawa hierarchies on T^2/Z_N with magnetic fluxes

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Based on the results of the classification by T.-h. Abe *et al.*, Nucl. Phys. **B894**, 374 (2015), we exhaustively investigate Yukawa sector of $U(8)$ model on magnetized orbifolds T^2/Z_2 , T^2/Z_3 , T^2/Z_4 and T^2/Z_6 by evaluating ratios of the mass eigenvalues of the three states in all the possible configurations with one and two Higgs pairs where three generations are realized in fermions. Because of the smearing effect via kinetic mixing, one can realize a hierarchy such as 10^{-2} – 10^{-3} , but it is very difficult to achieve the mass ratio between the up and top quarks ($m_{\text{up}}/m_{\text{top}} \sim 10^{-5}$) on the complicated magnetized orbifolds T^2/Z_N ($N = 3, 4, 6$).

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I. INTRODUCTION

Even after the completion of the standard model (SM) by the discovery of the Higgs boson [1,2], the origin of properties of the matter fields, especially the very hierarchical mass spectra of the quarks and the leptons, is still concealed. Quantized magnetic fluxes of a *unified* gauge group among extra directions of the spacetime provide us a fascinating guiding principle, where the fluxes trigger an explicit breaking of a unified gauge group down to the ones including the SM gauge group. Interestingly, the four-dimensional chiral fermions on such magnetized backgrounds are degenerated and their profiles are quasi-localized in the extra dimensions, where the former and latter properties give us reasonable answers to the mysteries of the three generations and the Yukawa hierarchies, respectively. A reasonable starting point is ten-dimensional (10D) super Yang-Mills (SYM) theory on magnetized tori [3–9], which could be an effective theory of superstring theories.¹ Note that various phenomenological aspects have

been pursued: Yukawa couplings [7], realization of quark/lepton masses and their mixing angles [10,11], higher order couplings [12], flavor symmetries [13–19], massive modes [20], and others [21–34]. Some related works in other stringy contexts (e.g., intersecting D-brane model and heterotic string theory) are found in [8,9,35–45].² Two-dimensional torus is the simplest choice in the space which the magnetic fluxes are turned on. Besides, various toroidal orbifolds can be candidates. Note that their geometrical aspects are discussed within the context of string theory [50–52] and higher-dimensional field theories [53–59].

Previously, a technical difficulty exists for analytical calculations of Yukawa couplings when the backgrounds are both magnetized and orbifolded. However, a progress was made in Ref. [60] in a *dual* description with operator formalism,³ which enables us to write down analytical forms of Yukawa couplings on such complicated geometries even though Wilson line phases and/or Scherk-Schwarz phases are also introduced (see [62] and [51,63–71]). Inspired by this achievement, all of the possible configurations were derived and classified when the gauge group is $U(8)$ in Ref. [72], which is the minimal choice in the case of $U(N)$ -type theories [10,11,29,73,74]

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¹In this paper, our setups will be discussed in a framework of supersymmetric Yang-Mills theories with $U(N)$ gauge groups. It would, however, be of great interest to embed our setups into superstring theories. Although it is known that $U(N)$ gauge groups are hard to be obtained in 10D type II string theory with D-branes, our results will be still useful because some properties with respect to zero-mode wave functions, their degeneracies and Yukawa couplings induced by magnetic fluxes are the same as those even in cases of other gauge groups.

²Another attractive direction is considering various boundary conditions of fields on point interactions (zero-thickness branes) in the bulk space of a five-dimensional theory on S^1 (or a line segment) [46–49].

³Note that only T^2 -related cases were discussed in [60]. But, addressing other higher-dimensional tori is possible in principle in a similar method. The case of shifted orbifolds with magnetic fluxes was done in [61].

(see also [24,25] and [43,44] for discussions in $E_{6,7,8}$ and $SO(32)$ groups, respectively.).

Based on the result of classification, we exhaustively analyze all the possible configurations with one and two Higgs pairs on orbifolds $T^2/Z_2, T^2/Z_3, T^2/Z_4, T^2/Z_6$ where three generations are realized in fermions by evaluating ratios of the mass eigenvalues of the three states.⁴ An important point is that we should realize the ratio $m_{\text{up}}/m_{\text{top}} \sim 10^{-5}$ in the up-quark sector. Whether or not this magnitude is achievable is a significant criterion for selecting type of configurations on the backgrounds, $T^2/Z_2, T^2/Z_3, T^2/Z_4, T^2/Z_6$.

This paper is organized as follows. In Sec. II, we briefly review basic properties of magnetized systems, which include the explicit form for Yukawa calculations on magnetized orbifolds based on the two-dimensional torus. In Sec. III, we calculate ratios of mass eigenvalues of fermion zero modes in all the configurations with three generations and one or two pairs of $SU(2)_L$ Higgs doublets. Section IV is the conclusion.

II. BRIEF REVIEW ON TEN-DIMENSIONAL SYM ON MAGNETIZED ORBIFOLDS

A. Setups

First of all, we briefly review the basics of the ten-dimensional super Yang-Mills theory on (generalized) magnetized orbifolds based on descriptions in [72], where we focus on the two-dimensional part determining flavor structure of T^2/Z_N ($N = 2, 3, 4$, or 6). We consider the $U(N)$ theory in the notation adopted in [72],

$$S = \int_{M^4} d^4x \int_{(T^2)^3} d^6z \times \left\{ -\frac{1}{4} \text{tr}(F_{MN}F^{MN}) + \frac{1}{2} \text{tr}(\bar{\lambda}\Gamma^M iD_M\lambda) \right\}, \quad (2.1)$$

which is defined on a product of four-dimensional Minkowski space and three factorizable 2-tori. The capital roman indices M, N run over $\mu (= 0, 1, 2, 3)$, $\{z_i, \bar{z}_i\}$, where the i -th ($i = 1, 2, 3$) 2-torus is described by the complex

coordinates $z_i = y_{2i+2} + iy_{2i+3}$ and its complex conjugate $\bar{z}_i = y_{2i+2} - iy_{2i+3}$ made by the two Cartesian coordinates representing the extra directions, y_{2i+2} and y_{2i+3} . We take each torus modulus parameter $\tau_i (\in \mathbb{C})$ as $\text{Im}\tau_i > 0$ for convenience. We use the short-hand notation d^6z meaning $\prod_{i=1}^3 dz_i d\bar{z}_i$. On T^2_i , the coordinate z_i is identified as $z_i \sim z_i + 1 \sim z_i + \tau_i$. The bulk $\mathcal{N} = 1$ supersymmetric (in ten-dimensional) theory contains the ten-dimensional vector fields A_M and the gaugino fields λ described by ten-dimensional Majorana-Weyl spinors.

The gaugino fields and the ten-dimensional vector fields are Kaluza-Klein (KK) decomposed as

$$\lambda(x, \{z_i, \bar{z}_i\}) = \sum_{l,m,n} \chi_{l,m,n}(x) \otimes \psi_l^{(1)}(z_1, \bar{z}_1) \otimes \psi_m^{(2)}(z_2, \bar{z}_2) \otimes \psi_n^{(3)}(z_3, \bar{z}_3), \quad (2.2)$$

$$A_M(x, \{z_i, \bar{z}_i\}) = \sum_{l,m,n} \varphi_{l,m,n;M}(x) \otimes \phi_{l,M}^{(1)}(z_1, \bar{z}_1) \otimes \phi_{m,M}^{(2)}(z_2, \bar{z}_2) \otimes \phi_{n,M}^{(3)}(z_3, \bar{z}_3), \quad (2.3)$$

where l, m, n are KK indices and $\psi_l^{(i)}$ is a two-dimensional spinor describing the l th KK mode on the i th T^2 , whose exact form is $\psi_l^{(i)} = (\psi_{l,+}^{(i)}, \psi_{l,-}^{(i)})^T$ and the corresponding two-dimensional chirality (+ or -) is denoted by s_i . We adopt the gamma matrices $\tilde{\Gamma}^m$ (identified by the Cartesian coordinates) corresponding to the i th torus as

$$\tilde{\Gamma}^{2i+2} = i\sigma_1, \quad \tilde{\Gamma}^{2i+3} = i\sigma_2, \quad (2.4)$$

where $\sigma_{1,2}$ are Pauli matrices. In the following part, we basically focus on the zero modes ($l = m = n = 0$) on the T^2/Z_N sector which describes flavor structure by omitting the KK and torus indices. We also skip to show coordinates $z_2, \bar{z}_2, z_3, \bar{z}_3$ in ten-dimensional fields to avoid clumsy descriptions.

We introduce factorizable Abelian magnetic fluxes on the three T^2 through the classical vector potential of A_M in the following forms,

$$A^{(b)}(z, \bar{z}) = \frac{\pi}{g\text{Im}\tau} \begin{pmatrix} M_1 \text{Im}[(\bar{z} + \bar{C}_1)dz] \mathbf{1}_{N_1 \times N_1} & & 0 \\ & \ddots & \\ 0 & & M_n \text{Im}[(\bar{z} + \bar{C}_n)dz] \mathbf{1}_{N_n \times N_n} \end{pmatrix} = \frac{1}{4i\text{Im}\tau} (\bar{\mathbf{B}}dz - \mathbf{B}d\bar{z}), \quad (2.5)$$

⁴A similar calculation was done in the context of gauge-Higgs unification scenario in the gauge group $G \times U(1)_X$ on T^2/Z_N ($N = 2, 3, 4, 6$), where G is a simple group including the part $SU(2)_L \times U(1)_Z$ in [75] by adopting the same method in [60,72]. Note that matter unification is not realized in this model.

where $C_j (j = 1, \dots, n)$ represent the corresponding Wilson line phases on T^2/Z_N , and $M_j (j = 1, \dots, n)$ should be integers because of Dirac's quantization condition on T^2/Z_N . On the magnetized T^2/Z_N , possible choices of C_j are limited. Since it was shown that this degrees of freedom can be gauged away by a large gauge transformations [62], we set $C_j = 0$. Under this background, the original gauge group $U(N)$ explicitly breaks down as $U(N) \rightarrow \prod_{a=1}^n U(N_a)$ with $N = \sum_{a=1}^n N_a$. We can derive the following relations,

$$A^{(b)}(z+1, \bar{z}+1) = A^{(b)}(z, \bar{z}) + d\xi_1(z), \quad (2.6)$$

$$A^{(b)}(z+\tau, \bar{z}+\bar{\tau}) = A^{(b)}(z, \bar{z}) + d\xi_\tau(z), \quad (2.7)$$

$$\xi_1(z) = \frac{1}{2\text{Im}\tau} \text{Im}[\mathbf{B}], \quad (2.8)$$

$$\xi_\tau(z) = \frac{1}{2\text{Im}\tau} \text{Im}[\bar{\tau}\mathbf{B}], \quad (2.9)$$

where d plays as an exterior derivative on T^2/Z_N . Here, the Lagrangian density in Eq. (2.1) should be single-valued under every torus identification $z \sim z+1 \sim z+\tau$, and then in the gaugino fields $\lambda(x, z, \bar{z})$, the following pseudoperiodic boundary conditions should be arranged,

$$\lambda(x, z+1, \bar{z}+1) = U_1(z)\lambda(x, z, \bar{z})U_1(z)^\dagger, \quad (2.10)$$

$$\lambda(x, z+\tau, \bar{z}+\bar{\tau}) = U_\tau(z)\lambda(x, z, \bar{z})U_\tau(z)^\dagger, \quad (2.11)$$

with

$$U_1(z) := e^{ig\xi_1(z)+2\pi i\alpha}, \quad U_\tau(z) := e^{ig\xi_\tau(z)+2\pi i\beta}, \quad (2.12)$$

$$\alpha := \begin{pmatrix} \alpha_1 \mathbf{1}_{N_1 \times N_1} & & 0 \\ & \ddots & \\ 0 & & \alpha_n \mathbf{1}_{N_n \times N_n} \end{pmatrix},$$

$$\beta := \begin{pmatrix} \beta_1 \mathbf{1}_{N_1 \times N_1} & & 0 \\ & \ddots & \\ 0 & & \beta_n \mathbf{1}_{N_n \times N_n} \end{pmatrix}, \quad (2.13)$$

where α_j and β_j ($j = 1, \dots, n$) describe Scherk-Schwarz phases and can take limited numbers.

Here, we exemplify the fermionic part in the specific case of $U(N) \rightarrow U(N_a) \times U(N_b)$. As two-dimensional spinors, the gaugino fields are decomposed as

$$\psi(z, \bar{z}) = \begin{pmatrix} \psi^{aa}(z, \bar{z}) & \psi^{ab}(z, \bar{z}) \\ \psi^{ba}(z, \bar{z}) & \psi^{bb}(z, \bar{z}) \end{pmatrix}. \quad (2.14)$$

The parts represented by ψ^{aa} and ψ^{bb} correspond to the representation under the unbroken gauge group $U(N_a) \times U(N_b)$, while ψ^{ab} and ψ^{ba} are the bifundamental matter fields as (N_a, \bar{N}_b) and (\bar{N}_a, N_b) , respectively. We obtain the zero-mode equations for these gaugino fields on the T^2/Z_N with the two-dimensional chirality (+ or -) as

$$\begin{pmatrix} \partial_{\bar{z}}\psi_+^{aa} & [\partial_{\bar{z}} + \frac{\pi}{2\text{Im}\tau}(M_{ab}z)]\psi_+^{ab} \\ [\partial_{\bar{z}} + \frac{\pi}{2\text{Im}\tau}(M_{ba}\bar{z})]\psi_+^{ba} & \partial_{\bar{z}}\psi_+^{bb} \end{pmatrix} = 0, \quad (2.15)$$

$$\begin{pmatrix} \partial_z\psi_-^{aa} & [\partial_z - \frac{\pi}{2\text{Im}\tau}(M_{ab}\bar{z})]\psi_-^{ab} \\ [\partial_z - \frac{\pi}{2\text{Im}\tau}(M_{ba}z)]\psi_-^{ba} & \partial_z\psi_-^{bb} \end{pmatrix} = 0, \quad (2.16)$$

with the short-hand notations $M_{ab} := M_a - M_b$. The effective boundary conditions of the fields are easily written down,

$$\begin{aligned} \psi_s^{ab}(z+1, \bar{z}+1) &= e^{i\frac{\pi\alpha}{\text{Im}\tau}\text{Im}[M_{ab}z]+2\pi i\alpha_{ab}}\psi_s^{ab}(z, \bar{z}), \\ \psi_s^{ba}(z+1, \bar{z}+1) &= e^{i\frac{\pi\alpha}{\text{Im}\tau}\text{Im}[M_{ba}\bar{z}]+2\pi i\alpha_{ba}}\psi_s^{ba}(z, \bar{z}), \\ \psi_s^{aa}(z+1, \bar{z}+1) &= \psi_s^{aa}(z, \bar{z}), \\ \psi_s^{bb}(z+1, \bar{z}+1) &= \psi_s^{bb}(z, \bar{z}), \end{aligned} \quad (2.17)$$

$$\begin{aligned} \psi_s^{ab}(z+\tau, \bar{z}+\bar{\tau}) &= e^{i\frac{\pi\beta}{\text{Im}\tau}\text{Im}[\bar{\tau}(M_{ab}z)]+2\pi i\beta_{ab}}\psi_s^{ab}(z, \bar{z}), \\ \psi_s^{ba}(z+\tau, \bar{z}+\bar{\tau}) &= e^{i\frac{\pi\beta}{\text{Im}\tau}\text{Im}[\bar{\tau}(M_{ba}\bar{z})]+2\pi i\beta_{ba}}\psi_s^{ba}(z, \bar{z}), \\ \psi_s^{aa}(z+\tau, \bar{z}+\bar{\tau}) &= \psi_s^{aa}(z, \bar{z}), \\ \psi_s^{bb}(z+\tau, \bar{z}+\bar{\tau}) &= \psi_s^{bb}(z, \bar{z}), \end{aligned} \quad (2.18)$$

with the short-hand notations $\alpha_{ab} := \alpha_a - \alpha_b$ and $\beta_{ab} := \beta_a - \beta_b$. We note that s shows the corresponding two-dimensional chirality.

On T^2 , possible twisted orbifolding is to impose the covariance on the fields under the rotation with the angle ω , $z \rightarrow \omega z$, where ω is $e^{2\pi i/N}$ with $N = 2, 3, 4, 6$. In other words, Z_2, Z_3, Z_4 and Z_6 (twisted) orbifoldings are realizable on T^2 . In non-Abelian gauge theories, a non-trivial gauge structure part P appears in the Z_N manipulation as

$$A_\mu(x, \omega z, \bar{\omega}\bar{z}) = PA_\mu(x, z, \bar{z})P^{-1}, \quad (2.19)$$

$$A_z(x, \omega z, \bar{\omega}\bar{z}) = \bar{\omega}PA_z(x, z, \bar{z})P^{-1}, \quad (2.20)$$

$$A_{\bar{z}}(x, \omega z, \bar{\omega}\bar{z}) = \omega PA_{\bar{z}}(x, z, \bar{z})P^{-1}, \quad (2.21)$$

$$\lambda_{s=+}(x, \omega z, \bar{\omega}\bar{z}) = P\lambda_{s=+}(x, z, \bar{z})P^{-1}, \quad (2.22)$$

$$\lambda_{s=-}(x, \omega z, \bar{\omega}\bar{z}) = \omega P\lambda_{s=-}(x, z, \bar{z})P^{-1}, \quad (2.23)$$

where P should satisfy the conditions $P \in U(N)$ and $P^N = \mathbf{1}_{N \times N}$. Here, to prevent an additional explicit gauge symmetry breaking via the orbifoldings, we should take the following form in P ,

$$P = \begin{pmatrix} \eta_1 \mathbf{1}_{N_1 \times N_1} & & 0 \\ & \ddots & \\ 0 & & \eta_n \mathbf{1}_{N_n \times N_n} \end{pmatrix}, \quad (2.24)$$

with $\eta_j = \{1, \omega, \dots, \omega^{N-1}\} (j \in 1, \dots, n)$. Within the concrete example of $U(N) \rightarrow U(N_a) \times U(N_b)$ discussed in the previous subsection, ψ_+^{aa} and ψ_+^{bb} have trivial Z_N parity irrespective of the values of η_a and η_b , while ψ_+^{ab} and ψ_+^{ba} can contain nontrivial values of $\eta_a \bar{\eta}_b$, $\bar{\eta}_a \eta_b$, respectively. The conditions for the two-dimensional gauginos with negative chirality are evaluated with ease by use of the relation in Eq. (2.23).

B. Yukawa coupling on magnetized T^2

Before we go for issues on magnetized T^2 with orbifolding, we summarize how to calculate Yukawa couplings in which orbifolding is not imposed.

When $M_{ab} > 0$, the fields ψ_+^{ab} and ψ_-^{ba} contain $|M_{ab}|$ normalizable zero modes, while the others ψ_+^{ba} and ψ_-^{ab} have no corresponding one. On the other hand, in $M_{ab} < 0$, $|M_{ab}|$ normalizable zero modes are generated from each of ψ_+^{ba} and ψ_-^{ab} , whereas there is nothing from ψ_+^{ab} and ψ_-^{ba} . In the case of $M_{ab} = 0$, like ψ_s^{aa} or ψ_s^{bb} , only one nonlocalized mode is generated from each of the all sectors and nothing of phenomenological interest occurs. When $M_{ab} > 0$, which is equal to $M_{ba} < 0$, the wave functions of ψ_+^{ab} and ψ_-^{ba} take the following forms:

$$\begin{aligned} \psi_+^{ab}(z) &= \sum_{I=0}^{|M_{ab}|-1} \begin{pmatrix} \Theta_{M_{ab}}^{(I+\alpha_{ab}, \beta_{ab})}(z, \tau) \\ 0 \end{pmatrix}, \\ \psi_-^{ba}(\bar{z}) &= \sum_{I=0}^{|M_{ba}|-1} \begin{pmatrix} 0 \\ \Theta_{M_{ba}}^{(I+\alpha_{ba}, \beta_{ba})}(\bar{z}, \bar{\tau}) \end{pmatrix}, \end{aligned} \quad (2.25)$$

$$\begin{aligned} \Theta_{M_{ab}}^{(I+\alpha_{ab}, \beta_{ab})}(z, \tau) &= \mathcal{N}_{|M_{ab}|} e^{i\pi M_{ab} z \frac{\text{Im}(z)}{\text{Im}\tau}} \\ &\cdot \vartheta \left[\begin{matrix} I+\alpha_{ab} \\ M_{ab} \\ -\beta_{ab} \end{matrix} \right] (M_{ab} z, M_{ab} \tau), \end{aligned} \quad (2.26)$$

$$\begin{aligned} \Theta_{M_{ba}}^{(I+\alpha_{ba}, \beta_{ba})}(\bar{z}, \bar{\tau}) &= \mathcal{N}_{|M_{ba}|} e^{i\pi M_{ba} \bar{z} \frac{\text{Im}(\bar{z})}{\text{Im}\bar{\tau}}} \\ &\cdot \vartheta \left[\begin{matrix} I+\alpha_{ba} \\ M_{ba} \\ -\beta_{ba} \end{matrix} \right] (M_{ba} \bar{z}, M_{ba} \bar{\tau}). \end{aligned} \quad (2.27)$$

Here, $I (= 0, \dots, |M_{ab}| - 1)$ discriminates the $|M_{ab}|$ -degenerated zero-mode states. The (generalized) ϑ function is defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c\nu, c\tau) = \sum_{l=-\infty}^{\infty} e^{i\pi(a+l)^2 c\tau} e^{2\pi i(a+l)(c\nu+b)}, \quad (2.28)$$

with the properties

$$\begin{aligned} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c(\nu+n), c\tau) &= e^{2\pi i a c n} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c\nu, c\tau), \\ \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c(\nu+n\tau), c\tau) &= e^{-i\pi c n^2 \tau - 2\pi i n(c\nu+b)} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c\nu, c\tau), \\ \vartheta \begin{bmatrix} a+m \\ b+n \end{bmatrix} (c\nu, c\tau) &= e^{2\pi i a n} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c\nu, c\tau), \\ \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (c\nu, c\tau) &= \vartheta \begin{bmatrix} a \\ 0 \end{bmatrix} (c\nu+b, c\tau), \end{aligned} \quad (2.29)$$

where a and b are real numbers, c, m and n are integers, and ν and τ are complex numbers with $\text{Im}\tau > 0$. The following orthonormality condition determines the normalization factor $\mathcal{N}_{|M_{ab}|}$,

$$\begin{aligned} \int_{T^2} d^2 z (\Theta_{M_{ab}}^{(I+\alpha_{ab}, \beta_{ab})}(z, \tau))^* (\Theta_{M_{ab}}^{(J+\alpha_{ab}, \beta_{ab})}(z, \tau)) &= \delta_{I,J} \\ (M_{ab} > 0), \\ \int_{T^2} d^2 \bar{z} (\Theta_{M_{ba}}^{(I+\alpha_{ba}, \beta_{ba})}(\bar{z}, \bar{\tau}))^* (\Theta_{M_{ba}}^{(J+\alpha_{ba}, \beta_{ba})}(\bar{z}, \bar{\tau})) &= \delta_{I,J} \\ (M_{ba} < 0), \end{aligned} \quad (2.30)$$

with $d^2 z := dz d\bar{z}$. An important relationship is easily derived (in the case of $M_{ab} > 0$),

$$(\Theta_{M_{ab}}^{(I+\alpha_{ab}, \beta_{ab})}(z, \tau))^* = \Theta_{M_{ba}}^{(-I+\alpha_{ba}, \beta_{ba})}(\bar{z}, \bar{\tau}), \quad (2.31)$$

where the index I is identified under the condition, $\text{mod } |M_{ab}|$, and we can always redefine $-I$ as $I' (= 0, \dots, |M_{ab}| - 1)$.

On a flux background, zero-mode profiles are not only split but also localized around points different from each other. Then we can expect that hierarchical values in Yukawa couplings are created via overlap integrals in the Yukawa sector of this model. The concrete form of the Yukawa couplings is as follows,

$$\begin{aligned} \lambda_{I,J,K} &= \int_{T^2} d^2 z \Theta_{M_I}^{(I+\alpha_I, \beta_I)}(z, \tau) \Theta_{M_J}^{(J+\alpha_J, \beta_J)}(z, \tau) \\ &\times (\Theta_{M_K}^{(K+\alpha_K, \beta_K)}(z, \tau))^*, \end{aligned} \quad (2.32)$$

where we drop a constant factor via gauge structure and the indices I, J, K discriminate degenerated states of three kinds of fields. M_i, α_i, β_i ($i = I, J, K$) represent corresponding magnetic fluxes, two kinds of Scherk-Schwarz phases. In a suitable symmetry breaking like the above examples, we can find conditions on the parameters,

$$M_I + M_J = M_K, \quad (2.33)$$

$$\alpha_I + \alpha_J = \alpha_K, \quad (2.34)$$

$$\beta_I + \beta_J = \beta_K, \quad (2.35)$$

where we implicitly use the rule in Eq. (2.33) and the relation in Eq. (2.31) when we write down the actual form in Eq. (2.32).

After some mathematical calculations, we can derive the analytical result of the Yukawa coupling in Eq. (2.32) as

$$\begin{aligned} \lambda_{I,J,K} &= \frac{\mathcal{N}_{M_I} \mathcal{N}_{M_J}}{\mathcal{N}_{M_K}} \\ &\times \sum_{m \in Z_{M_K}} \vartheta \left[\begin{matrix} M_I(I+\alpha_I) - M_J(J+\alpha_J) + m M_I M_J \\ M_I M_J M_K \\ 0 \end{matrix} \right] (X, Y) \\ &\times \delta_{I+\alpha_I+J+\alpha_J+mM_I, K+\alpha_K+\ell M_K}, \end{aligned} \quad (2.36)$$

with $X := M_I \beta_J - M_J \beta_I$, $Y := \tau M_I M_J M_K$ and possible choices of integers ℓ [7,72]. In other words, we consider the Kronecker's delta with the condition “mod $|M_K|$.”

C. Yukawa coupling on magnetized T^2/Z_N

Next we consider the case on magnetized T^2/Z_N . Here, the form of the fermion wave function with the Z_N parity η , the (two-dimensional) positive chirality and the state-discriminating index I on T^2 is

$$\psi_{+, \eta}(z) = \sum_{I=0}^{|M|-1} \psi_{+, \eta}^I(z), \quad \psi_{+, \eta}^I(z) = \begin{pmatrix} \tilde{\Theta}_{M; \eta}^{(I+\alpha, \beta)}(z, \tau) \\ 0 \end{pmatrix}, \quad (2.37)$$

where we assume that M is a positive integer.⁵ Constructing the concrete form of $\tilde{\Theta}_{M; \eta}^{(I+\alpha, \beta)}(z, \tau)$ itself can be done straightforwardly just following the general recipe as⁶

$$\tilde{\Theta}_{M; \eta}^{(I+\alpha, \beta)}(z, \tau) = \frac{1}{N} \sum_{x=0}^{N-1} (\bar{\eta})^x \Theta_M^{(I+\alpha, \beta)}(\omega^x z, \tau). \quad (2.38)$$

Naively, Yukawa couplings on T^2/Z_N seem to be formulated as

⁵Note that the correspondence to the negative chirality case is basically straightforward by the replacements $z \rightarrow \bar{z}$, $\tau \rightarrow \bar{\tau}$.

⁶Indeed, we have degrees of freedom of putting any terms on orbifold fixed points. However, for simplicity, we assume the absence of such terms.

$$\begin{aligned} \tilde{\lambda}_{I,J,K} &= \int_{T^2} d^2 z \tilde{\Theta}_{M_I; \eta_I}^{(I+\alpha_I, \beta_I)}(z, \tau) \tilde{\Theta}_{M_J; \eta_J}^{(J+\alpha_J, \beta_J)}(z, \tau) \\ &\times (\tilde{\Theta}_{M_K; \eta_K}^{(K+\alpha_K, \beta_K)}(z, \tau))^*, \end{aligned} \quad (2.39)$$

where we find the condition on the Z_N parities (via the invariance of the system),⁷

$$\eta_I \eta_J \bar{\eta}_K = 1. \quad (2.40)$$

However, this is not the end of the story. In general, the kinetic terms on T^2/Z_N , which are described as

$$\mathcal{K}_{IJ}^{(Z_N; \eta)} = \int_{T^2} d^2 z (\tilde{\Theta}_{M; \eta}^{(I+\alpha, \beta)}(z, \tau))^* \tilde{\Theta}_{M; \eta}^{(J+\alpha, \beta)}(z, \tau), \quad (2.41)$$

are no longer diagonal, where the number of independent physical states should be reduced as $\text{rank}[\mathcal{K}_{IJ}^{(Z_N; \eta)}] < |M|$. Thereby, in the physical eigenstates, after considering the correct normalization in the kinetic terms by the unitary transformation with the corresponding diagonalizing matrix $U^{(Z_N; \eta)}$, where $\mathcal{K}_{IJ}^{(Z_N; \eta)}$ is transformed as

$$\begin{aligned} \mathcal{K}^{(Z_N; \eta)} &\rightarrow (U^{(Z_N; \eta)})^\dagger \mathcal{K}^{(Z_N; \eta)} U^{(Z_N; \eta)} \\ &= \text{diag}(\underbrace{1, \dots, 1}_{\text{rank}[\mathcal{K}^{(Z_N; \eta)}]}, 0, \dots, 0). \end{aligned} \quad (2.42)$$

The mode function on T^2/Z_N should be

$$\tilde{\Theta}_{M; \eta}^{(I+\alpha, \beta)}(z, \tau) \rightarrow \sum_{I'=0}^{|M|-1} \tilde{\Theta}_{M; \eta}^{(I+\alpha, \beta)}(z, \tau) (U^{(Z_N; \eta)})_{II'}, \quad (2.43)$$

where I' is the index of physical eigenstates from zero to $\text{rank}[\mathcal{K}_{IJ}^{(Z_N; \eta)}] - 1$. The operator formalism helps us to evaluate explicit forms of $\tilde{\lambda}_{I,J,K}$ defined in Eq. (2.39) and the matrix $U^{(Z_N; \eta)}$ (see [60,72] for details.).

Taking into account the effects of the diagonalization, the final form of the Yukawa coupling is expressed as

$$\begin{aligned} \tilde{\lambda}'_{I', J', K'} &= \sum_{I=0}^{|M_I|-1} \sum_{J=0}^{|M_J|-1} \sum_{K=0}^{|M_K|-1} \tilde{\lambda}_{I,J,K} (U^{Z_N; \eta_I})_{I, I'} (U^{Z_N; \eta_I})_{J, J'} \\ &\times (U^{Z_N; \eta_K})_{K, K'}^*, \end{aligned} \quad (2.44)$$

where the indices for identifying kinetic eigenstates, I' , J' , K' , have $\text{rank}[\mathcal{K}^{(Z_N; \eta_I)}]$, $\text{rank}[\mathcal{K}^{(Z_N; \eta_J)}]$, $\text{rank}[\mathcal{K}^{(Z_N; \eta_K)}]$ numbers of nonzero configurations, respectively. In general, the mixing effect through $U^{Z_N; \eta_I}$ contributes to the physics.

⁷Note that a complex-conjugated state holds the corresponding complex-conjugated Z_N parity.

TABLE I. Numbers of possible configurations with three generations in the Z_2 case. ‘‘General cases’’ and ‘‘Trivial BCs only’’ means the cases with and without nontrivial Scherk-Schwarz phases, respectively. Corresponding numbers of the Higgs pairs (N_H) are also shown. The case indicated by 1_{trivial} means the one Higgs pair appears under the nonmagnetized background in the bc sector.

General cases		Trivial BCs only	
$M_{ab}, M_{ca} < 0$	$M_{ab} < 0, M_{ca} > 0$	$M_{ab}, M_{ca} < 0$	$M_{ab} < 0, M_{ca} > 0$
41($N_H = 5$)	16($N_H = 1_{\text{trivial}}$)	5($N_H = 5$)	4($N_H = 1_{\text{trivial}}$)
56($N_H = 6$)	65($N_H = 1$)	2($N_H = 6$)	5($N_H = 1$)
30($N_H = 7$)			
8($N_H = 8$)		2($N_H = 8$)	
1($N_H = 9$)		1($N_H = 9$)	
136 + 81 = 217 in total		10 + 9 = 19 in total	

D. Possible configurations of three generations in $U(8)$ model

We focus on the following pattern of the gauge symmetry breaking under the magnetic flux is $U(N) \rightarrow U(N_a) \times U(N_b) \times U(N_c)$ with $N = N_a + N_b + N_c$, where the corresponding 1-form potential is

$$A^{(b)}(z, \bar{z}) = \frac{\pi}{q \text{Im}\tau} \times \text{diag}(M_a \text{Im}[(\bar{z})dz] \mathbf{1}_{N_a \times N_a}, M_b \text{Im}[(\bar{z})dz] \mathbf{1}_{N_b \times N_b}, M_c \text{Im}[(\bar{z})dz] \mathbf{1}_{N_c \times N_c}). \quad (2.45)$$

We find six types of bifundamental matter fields under $U(N_a) \times U(N_b) \times U(N_c)$, λ^{ab} , λ^{bc} , λ^{ca} , λ^{ba} , λ^{cb} , λ^{ac} , whose gauge properties are $(N_a, \bar{N}_b, 1)$, $(1, N_b, \bar{N}_c)$, $(\bar{N}_a, 1, N_c)$, $(\bar{N}_a, N_b, 1)$, $(1, \bar{N}_b, N_c)$, $(N_a, 1, \bar{N}_c)$, respectively. When we adopt the choice $N_a = 4$, $N_b = 2$, $N_c = 2$, $U(4)_{PSC} \times U(2)_L \times U(2)_R$ gauge groups are realized from the $U(8)$ group up to $U(1)$ factors, where the subscripts PSC , L and R denote the Pati-Salam color, left- and right-electroweak gauge groups,⁸ respectively. In such a situation, when the actual chirality of the gaugino is left (negative), λ^{ab} corresponds to the left-handed quarks and leptons, and λ^{ca} accords with (charge-conjugated) right-handed quarks and leptons, respectively. When the magnetic fluxes are suitably assigned, the situation with three generations is materialized. Besides, λ^{bc} plays as up-type and down-type Higgsinos. After we assume that (four-dimensional $\mathcal{N} = 1$) supersymmetry is preserved at least

⁸From a phenomenological point of view, we can consider the following additional breakdowns originating from flux, $U(4)_{PSC} \rightarrow U(3)_C \times U(1)_1$ and $U(2)_R \rightarrow U(1)_2 \times U(1)_3$ (up to $U(1)$ factors), where $U(3)_C$ is the color gauge group (up to a $U(1)$ factor). Under the latter breaking, the up-type and down-type Higgsino/Higgs sectors can feel different magnetic fluxes individually. Consequently, the numbers of the two types of fields diverge. Some of the combinations of the $U(1)$ part would be anomalous. Then they could be massive and decoupled via the Green-Schwarz mechanism.

locally at the ab , bc and ca sectors, the corresponding Higgses via extra-dimensional components of the ten-dimensional vector fields are still massless under the fluxes and the number of the fields are the same with Higgsino fields. Also, no tachyonic mode is expected at the tree level. Here, in general, multiple Higgs fields appear from the bc sector. Interestingly, when λ^{ab} , λ^{bc} and λ^{ca} have zero modes, λ^{ba} , λ^{cb} and λ^{ac} cannot contain any zero mode, and thus no exotic particle arises from these fermionic sectors. In the case of the actual chirality being right (positive), we should flip the roles of the two categories. The $U(8)$ gauge group is the minimal group for matter unification within the $U(N)$ gauge theories.⁹ The following properties are observed,

$$\begin{aligned} M_{ab} + M_{bc} + M_{ca} &= 0, \\ \alpha_{ab} + \alpha_{bc} + \alpha_{ca} &= 0, \\ \beta_{ab} + \beta_{bc} + \beta_{ca} &= 0, \\ \eta_{ab}\eta_{bc}\eta_{ca} &= 1, \end{aligned} \quad (2.46)$$

where the above parameters are defined by the fundamental ones like $M_{ab} = M_a - M_b$ except the Z_N parities. The Z_N parities are described as $\eta_{ab} = \eta_a \bar{\eta}_b$, $\eta_{bc} = \eta_b \bar{\eta}_c$, $\eta_{ca} = \eta_c \bar{\eta}_a$.

We assume both nonvanishing magnetic fluxes and orbifold twists. Indeed, thanks to magnetic fluxes, there is the possibility that $\mathcal{N} = 4$ SUSY in four-dimensional spacetime is broken into $\mathcal{N} = 1, 2$, or 0 (non-SUSY case). Similarly, orbifold twists can break $\mathcal{N} = 4$ SUSY into $N = 1$ or 2. In this paper, we do not try to construct concrete full setups. However, we can construct $N = 1$

⁹In the previous models [10,11], flavor structures among quarks and leptons are characterized only on two dimensions of the six-dimensional compact space. In this paper, we focus on a two-dimensional toroidal orbifold, and our setup is expected to be embedded in $SO(32)$ SYM theory and six-dimensional/ten-dimensional other theories with $U(N)$ groups.

TABLE II. Numbers of possible configurations with three generations in the Z_3 case. The convention is the same in Table I.

General cases		Trivial BCs only	
$M_{ab}, M_{ca} < 0$	$M_{ab} < 0, M_{ca} > 0$	$M_{ab}, M_{ca} < 0$	$M_{ab} < 0, M_{ca} > 0$
11($N_H = 4$)	17($N_H = 1_{\text{trivial}}$)	1($N_H = 4$)	9($N_H = 1_{\text{trivial}}$)
83($N_H = 5$)	142($N_H = 1$)	6($N_H = 5$)	27($N_H = 1$)
190($N_H = 6$)	21($N_H = 2$)	7($N_H = 6$)	
83($N_H = 7$)		6($N_H = 7$)	
11($N_H = 8$)		1($N_H = 8$)	
378 + 180 = 558 in total		21 + 36 = 57 in total	

 TABLE III. Numbers of possible configurations with three generations in Z_4 case. The convention is the same in Table I.

General cases		Trivial BCs only	
$M_{ab}, M_{ca} < 0$	$M_{ab} < 0, M_{ca} > 0$	$M_{ab}, M_{ca} < 0$	$M_{ab} < 0, M_{ca} > 0$
9($N_H = 4$)	24($N_H = 1_{\text{trivial}}$)	3($N_H = 4$)	12($N_H = 1_{\text{trivial}}$)
128($N_H = 5$)	228($N_H = 1$)	37($N_H = 5$)	60($N_H = 1$)
254($N_H = 6$)	18($N_H = 2$)	59($N_H = 6$)	6($N_H = 2$)
120($N_H = 7$)		27($N_H = 7$)	
17($N_H = 8$)		10($N_H = 8$)	
528 + 270 = 798 in total		136 + 78 = 214 in total	

 TABLE IV. Numbers of possible configurations with three generations in Z_6 case. The convention is the same in Table I.

General cases		Trivial BCs only	
$M_{ab}, M_{ca} < 0$	$M_{ab} < 0, M_{ca} > 0$	$M_{ab}, M_{ca} < 0$	$M_{ab} < 0, M_{ca} > 0$
14($N_H = 4$)	24($N_H = 1_{\text{trivial}}$)	4($N_H = 4$)	12($N_H = 1_{\text{trivial}}$)
156($N_H = 5$)	282($N_H = 1$)	45($N_H = 5$)	73($N_H = 1$)
326($N_H = 6$)	27($N_H = 2$)	76($N_H = 6$)	8($N_H = 2$)
150($N_H = 7$)		36($N_H = 7$)	
20($N_H = 8$)		10($N_H = 8$)	
666 + 333 = 999 in total		171 + 93 = 264 in total	

SUSY models by selecting magnetic fluxes and orbifold twists suitably.¹⁰

All the possible configurations with three generations which fulfill the conditions in Eq. (2.46) were derived in Ref. [72].¹¹ Note that the constraints $\alpha_i = \beta_i$ ($i = ab, bc, ca$) are requested by symmetry in the cases of $T^2/Z_{3,4,6}$. We note that the first line of Eq. (2.46) tells us that at least one of the signs of the three fluxes should be different from the others, and it is enough that we focus on the two possibilities in the signs of the fluxes of the matter sectors M_{ab} and M_{ca} as

$$M_{ab} < 0, \quad M_{ca} < 0; \quad M_{ab} < 0, \quad M_{ca} > 0. \quad (2.47)$$

There is another possibility of $M_{ab} > 0, M_{ca} < 0$, but this case is physically the same as $M_{ab} < 0, M_{ca} > 0$. Besides, after ignoring the difference coming from the combinatorics, we can introduce the additional condition,

$$|M_{ab}| \leq |M_{ca}|. \quad (2.48)$$

Results of classification are shown in Tables I (Z_2), II (Z_3), III (Z_4), IV (Z_6), respectively.

III. RESULTS

Now we know that possible numbers of $SU(2)_L$ doublet Higgs boson pairs for up-type and down-type fields under

¹⁰In order to realize $N = 1$ SUSY models, we need to appropriately assign magnetic fluxes and/or boundary conditions of orbifolding. See, e.g., Ref. [10].

¹¹In our setup, R-parity-violating terms are prohibited by Pati-Salam and/or $U(1)$ gauge symmetries. For a review, see Ref. [10].

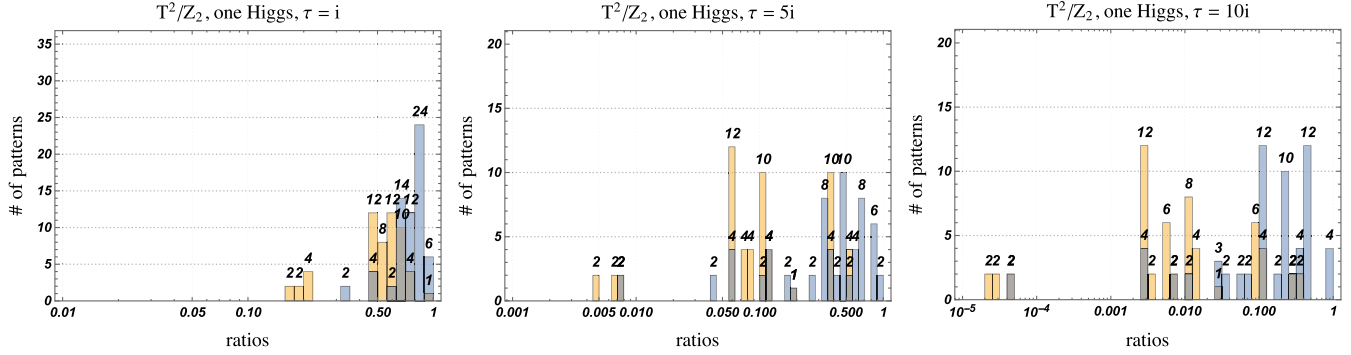


FIG. 1. Distributions of realized mass eigenvalues are shown in T^2/Z_2 when one Higgs boson appears in the three choices of the modulus parameter $\tau = i$ (left panel), $5i$ (center panel), $10i$ (right panel). The orange (blue) bars correspond to the mass ratio m_1/m_3 (m_2/m_3) under the ordering $m_1 \leq m_2 \leq m_3$. The total number of the possibilities with three generations is 65 (in each panel).

the presence of magnetic fluxes are one or two (when $M_{ab} < 0, M_{ca} > 0$) or from four to eight (when $M_{ab} < 0, M_{ca} < 0$), respectively. In the latter case, it had been investigated that when a suitable relation was fulfilled among the VEVs of the Higgs bosons, the so-called Gaussian Froggatt-Nielsen mechanism works and observed quark mass hierarchies and mixing angles are realized [11]. On the other hand, such cases are less predictive in the sense that various additional parameters with respect to scalar VEVs are required.

The situation in the former case is just the opposite. Here, only lower numbers of VEVs contribute to Yukawa hierarchies and are then more predictive, whereas less degrees of freedom can be used for realizations of the quark and lepton configurations in the SM. In this work, we only focus on the magnitude of realized fermion mass hierarchies in the former case to declare prospects in such simple possibilities exhaustively. We comment on the total numbers of configurations found in panels in Figs. 1–5 are in general less than the numbers of corresponding allowed configurations shown in Tables I–IV. This is because part of configurations in Tables I–IV results in mass matrices

with rank reduction (less than three), where such cases are apparently not suitable and skipped to be shown. The cancellations after summing up all the indices in Eq. (2.44) are a possible origin of this reduction.

A. One Higgs (pair) case

At first, we consider the case with one Higgs pair. We note that except for T^2/Z_2 , the modulus parameter is inevitably fixed by requirement in $Z_N (N = 3, 4, 6)$ orbifolds as $\tau = e^{2\pi i/N}$. In other words, one additional parameter exists only in T^2/Z_2 , where the magnitude of τ determines the degrees of quasilocalization of the mode functions.

In Fig. 1, distributions of realized mass eigenvalues are shown in T^2/Z_2 when one Higgs boson appears in the three choices of the modulus parameter $\tau = i$ (left panel), $5i$ (center panel), $10i$ (right panel). Here, the orange (blue) bars correspond to the mass ratio m_1/m_3 (m_2/m_3) under the ordering $m_1 \leq m_2 \leq m_3$. Digits on top of bars indicate how many configurations are stored in corresponding regions of m_1/m_3 or m_2/m_3 . Our result is consistent with

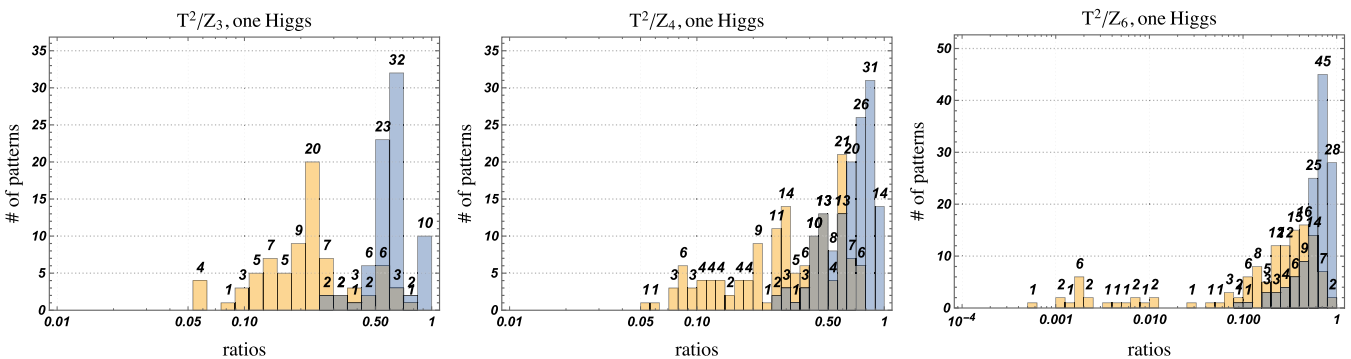


FIG. 2. Distributions of realized mass eigenvalues are shown when one Higgs boson appears in the cases of T^2/Z_3 (left panel), T^2/Z_4 (center panel), T^2/Z_6 (right panel). The orange (blue) bars correspond to the mass ratio m_1/m_3 (m_2/m_3) under the ordering $m_1 \leq m_2 \leq m_3$. The total numbers of the possibilities with three generations are 78 (T^2/Z_3), 144 (T^2/Z_4), 135 (T^2/Z_6), respectively.

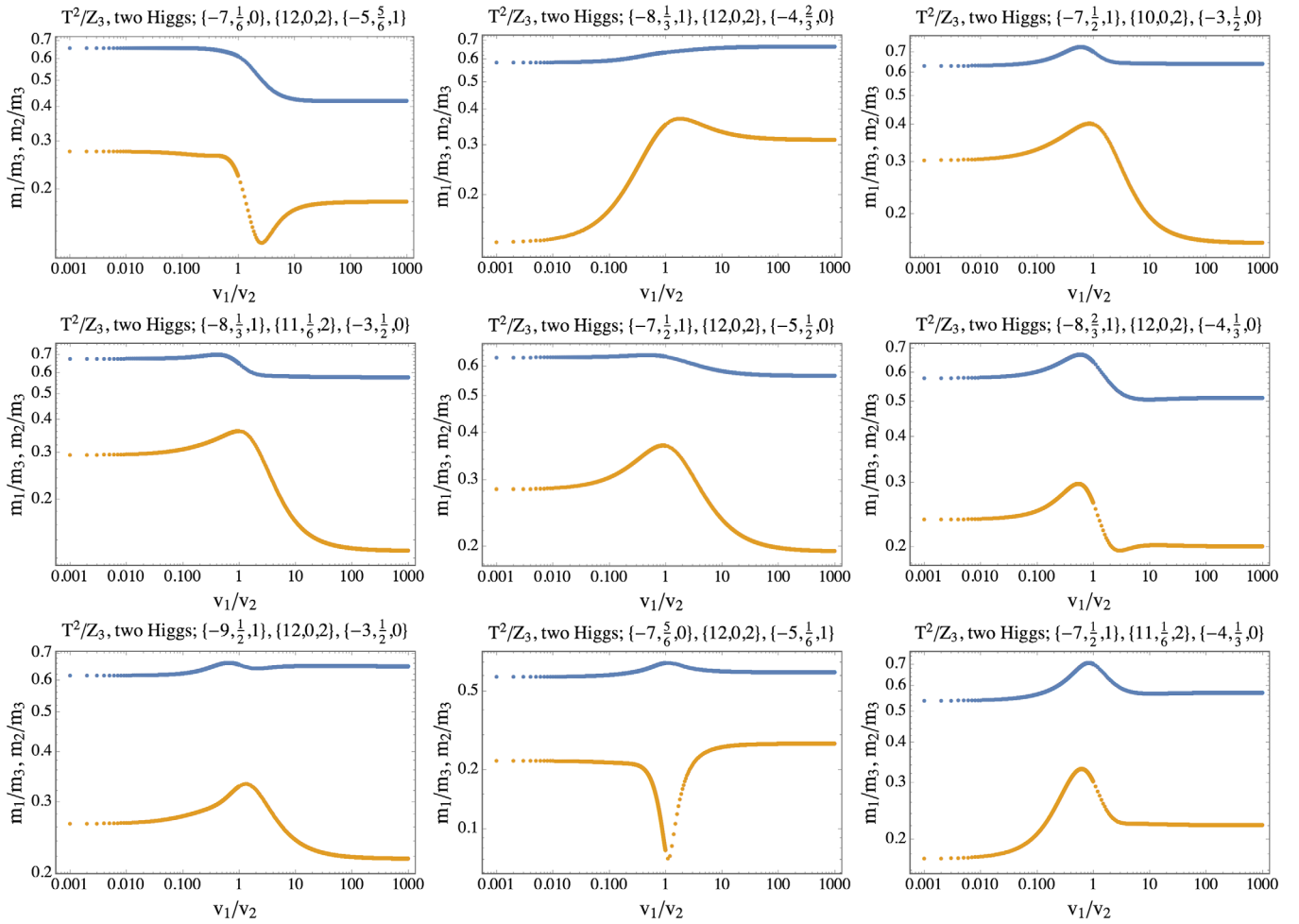


FIG. 3. Distributions of realized mass eigenvalues are shown when two Higgs bosons appear in the cases of T^2/Z_3 . We analyze all the cases (nine patterns in total) where rank-three mass matrices are realized. Configurations are summarized as $\{M_{ab}, \alpha_{ab}, s_{ab}\}$, $\{M_{ca}, \alpha_{ca}, s_{ca}\}$, $\{M_{bc}, \alpha_{bc}, s_{bc}\}$, where we define $\eta_{ab} \equiv e^{2\pi i s_{ab}/3}$. The same holds for the bc and ca sectors. The orange (blue) dots correspond to the mass ratio m_1/m_3 (m_2/m_3) under the ordering $m_1 \leq m_2 \leq m_3$.

those in the previous studies, e.g., in [72–74]. When we take $\text{Im}[\tau]$ as greater than around ten, the hierarchy of 10^{-5} is realizable in m_1/m_3 . Then we can realize the mass difference between the up quark and the top quark in this class by choosing the value of $\text{Im}[\tau]$ as ten or a bit more.

In Fig. 2, counterparts are depicted when one Higgs boson appears in the cases of T^2/Z_3 (left panel), T^2/Z_4 (center panel), T^2/Z_6 (right panel). Conventions are the same as those in Fig. 1. Note that no additional parameters are there in these cases and then possibilities with one Higgs pair on $T^2/Z_{3,4,6}$ are completed in the figures. Unfortunately, the minima of m_1/m_3 are around 5×10^{-2} (for $T^2/Z_{3,4}$) and 1×10^{-3} (for T^2/Z_6), where they are far from the target 10^{-5} . Thereby, the simplest case in the number of realized Higgs pairs ($N_H = 1$) is discarded on $T^2/Z_{3,4,6}$ completely as candidates for describing the SM fermion sector. Such a tendency is understandable via the formula in Eq. (2.44). Different from the Z_2 case, in more

than the Z_3 cases, the structure of the kinetic mixing described by the unitary matrix $U^{Z_N:n}$ becomes nontrivial and such mixings tend to smear the difference in the original T^2 basis, where at this stage, large hierarchies are expected since the magnitude of magnetic fluxes is larger compared with when the geometry is T^2/Z_2 or the simple T^2 . In the case of Z_6 , the highest magnitudes in magnetic fluxes are realized, where a bit more hierarchy is expected in spite of the smearing through the kinetic mixing. This would be the origin of why the minimum of m_1/m_3 is a bit smaller on T^2/Z_6 than on $T^2/Z_{3,4}$. Note that a similar conclusion was made in Ref. [75].

B. Two Higgs (pair) case

Next, we go for the case with two Higgs pairs ($N_H = 2$), where two Higgs VEVs contribute to ratios of mass eigenvalues. As far as we focus on the ratios of mass

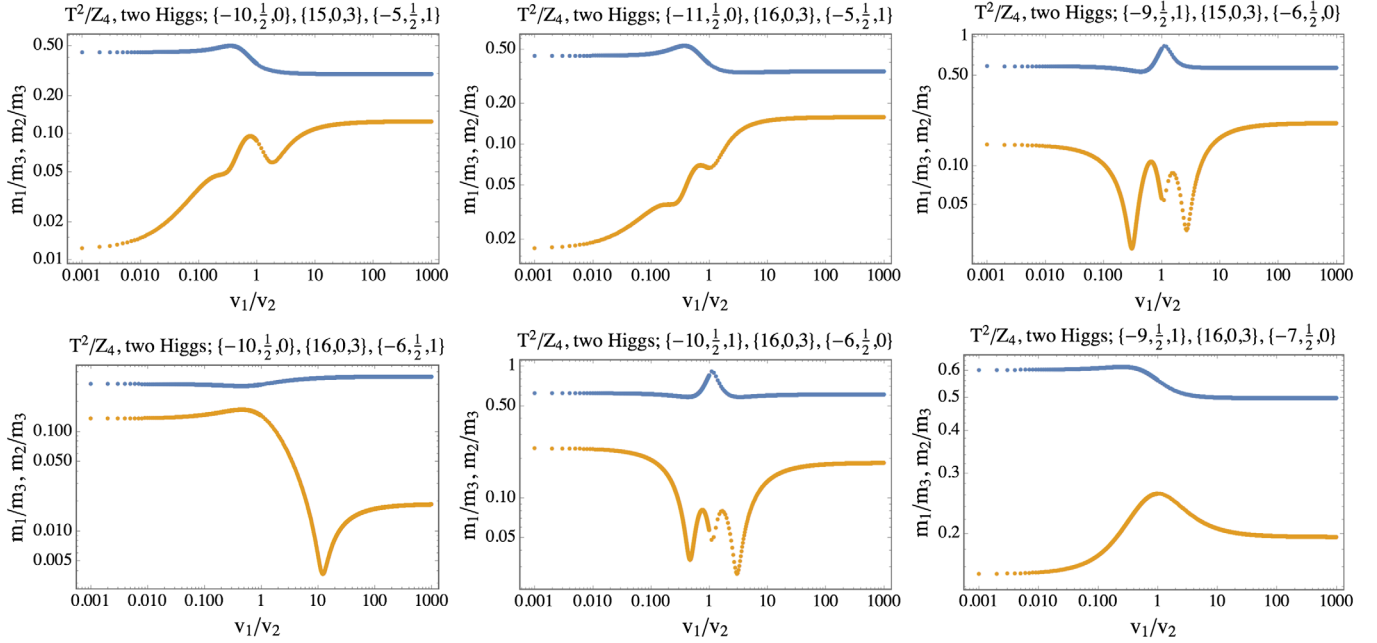


FIG. 4. Distributions of realized mass eigenvalues are shown when two Higgs bosons appear in the cases of T^2/Z_4 . We analyze all the cases (six patterns in total) that rank-three mass matrices are realized. Configurations are summarized as $\{M_{ab}, \alpha_{ab}, s_{ab}\}$, $\{M_{ca}, \alpha_{ca}, s_{ca}\}$, $\{M_{bc}, \alpha_{bc}, s_{bc}\}$, where we define $\eta_{ab} \equiv e^{2\pi i s_{ab}/4}$. The same holds for the bc and ca sectors. Conventions are the same as those adopted in Fig. 3.

eigenvalues, only the ratio of the two VEVs v_1/v_2 [$v_{1,2}$ corresponding to $J' = 0, 1$ in Eq. (2.44)] is relevant.¹²

We find that nine, six and seven configurations generate rank-three mass matrices on T^2/Z_3 , T^2/Z_4 and T^2/Z_6 , respectively. In each case, we examine patterns of m_1/m_3 and m_2/m_3 by plotting the ratios in various choices of v_1/v_2 . Note that no $N_H = 2$ example exists when the background is T^2/Z_2 as explicitly shown in Table I.

The results are summarized in Figs. 3 (T^2/Z_3), 4 (T^2/Z_4) and 5 (T^2/Z_6), where intervals of dots (showing choices of v_1/v_2) are 0.001 from 0.001 to 1, and 0.1 from 1 to 1000, respectively. Note that in the region where v_1/v_2 is more than 1000 or less than 0.001, either of the two mass matrices dominates and the two mass ratios are saturated. Roughly speaking, the minimum values of m_1/m_3 among possible configurations are around 5×10^{-2} for T^2/Z_3 and 5×10^{-3} for $T^2/Z_{4,6}$, which are, of course, far from the

required value 10^{-5} . Then, we conclude that it is hard to realize the ratio $m_{\text{up}}/m_{\text{top}}$ in every possibility with two Higgs pairs without fine-tuning in v_1/v_2 . Possibility would still remain when we accept fine-tuning in the ratio even through it is, at least to some extent, contradict to the basic motivation for considering such magnetized backgrounds in extra dimensions.¹³ However, if the mass hierarchy itself can become achievable under fine-tuning, we should justify the origin by dynamics.

We point out that the realized spectra are nonlinearly changed when we adjust the value of v_1/v_2 . Especially around $v_1/v_2 = 1$, values of m_1/m_3 and m_2/m_3 alter significantly since elements of two mass matrices are comparable due to the mixing effect. As an example, we show explicit forms of two mass matrices in T^2/Z_6 with quantum numbers $\{M_{ab}, \alpha_{ab}, s_{ab}\}$, $\{M_{ca}, \alpha_{ca}, s_{ca}\}$, $\{M_{bc}, \alpha_{bc}, s_{bc}\} = \{-15, 1/2, 0\}$, $\{24, 0, 5\}$, $\{-9, 1/2, 1\}$:

$$\mathcal{M}_1 = v_1 \begin{pmatrix} -0.204991 - 0.0796877i & 0.00942303 + 0.09068i & 0.0519518 + 0.0517703i \\ 0.0763198 + 0.0222452i & -0.00676905 - 0.017782i & -0.0077252 - 0.00374813i \\ -0.0507492 - 0.0180038i & 0.0683165 + 0.0433135i & -0.0157981 - 0.12252i \end{pmatrix}, \quad (3.1)$$

¹²We may suffer from large flavor-changing neutral currents (FCNCs) because two or more than two Higgs doublets emerge unless only one pair of Higgs doublets stays in the electroweak scale, while the others are decoupled. Even if sufficient Yukawa hierarchies are realized in our setup, we have to assume the Higgs sector including a specific Higgs mass matrix that can provide a light mass eigenstate and sufficiently heavy mass eigenstates of Higgs doublets for evading the FCNC effects. Also, it is still a challenging issue to obtain such a Higgs sector.

¹³In $N_H = 5$, the authors of [11] found a configuration where all of the VEV ratios are within a natural range [0.1, 10].

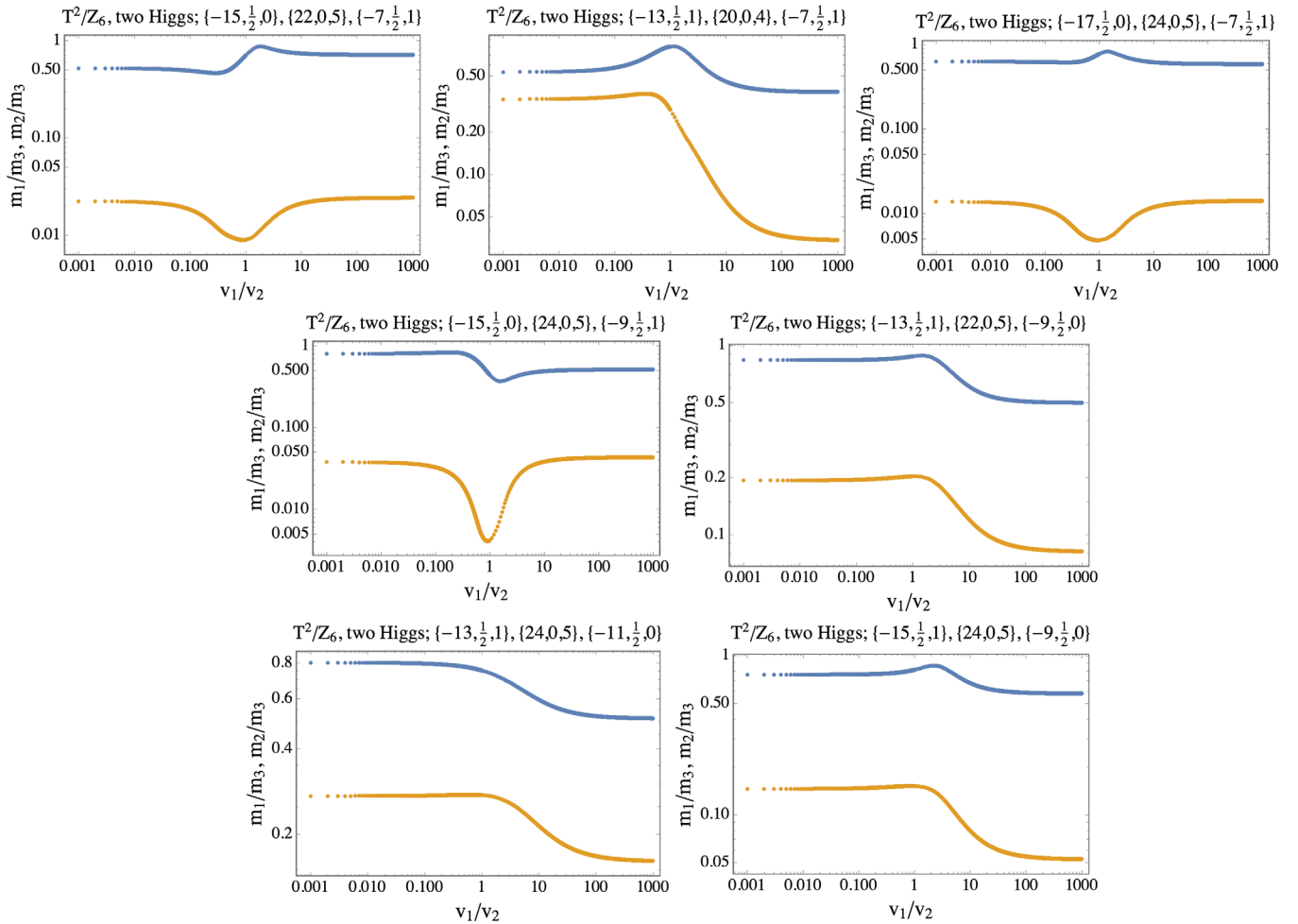


FIG. 5. Distributions of realized mass eigenvalues are shown when two Higgs bosons appear in the cases of T^2/Z_6 . We analyze all the cases (seven patterns in total) that rank-three mass matrices are realized. Configurations are summarized as $\{M_{ab}, \alpha_{ab}, s_{ab}\}$, $\{M_{ca}, \alpha_{ca}, s_{ca}\}$, $\{M_{bc}, \alpha_{bc}, s_{bc}\}$, where we define $\eta_{ab} \equiv e^{2\pi i s_{ab}/6}$. The same holds for the bc and ca sectors. Conventions are the same as those adopted in Fig. 3.

$$\mathcal{M}_2 = v_2 \begin{pmatrix} 0.00505039 + 0.0102538i & 0.00941751 - 0.180948i & -0.0776918 - 0.115464i \\ -0.00982989 - 0.00704602i & 0.00161259 + 0.0544036i & 0.0166123 + 0.0174854i \\ 0.0133678 - 0.0127981i & 0.0383659 - 0.114777i & -0.00590916 + 0.172257i \end{pmatrix}. \quad (3.2)$$

Here, we define Z_6 parity, $\eta_{ab} \equiv e^{2\pi i s_{ab}/6}$. The same holds for the bc and ca sectors. Note that elements in the mass matrix (3.1) as well as (3.2) have no strong hierarchy.¹⁴ That is due to the effects of kinetic mixing, which smears hierarchies. Obviously, the mass matrix (3.1) [(3.2)] is dominant when $v_1 \gg v_2$ [$v_1 \ll v_2$]. However, when $v_1 \sim v_2$, partial cancellations happen sizably in the calculation of mass eigenvalues and the ratio m_1/m_3 drops below 0.005 as shown in Fig. 5. The relatively hierarchical nature

¹⁴For simplicity in the calculation, we ignore the overall normalization factor ($\mathcal{N}_{M_i} \mathcal{N}_{M_j} / \mathcal{N}_{M_k}$) in Eq. (2.36), which manifestly does not affect the mass ratios.

in $T^2/Z_{4,6}$ compared with T^2/Z_3 would originate from the relatively large magnitude of magnetic fluxes.

IV. CONCLUSIONS

In this paper, we discussed how large mass hierarchy is realized on the magnetized extra dimension with orbifolding of T^2/Z_2 , T^2/Z_3 , T^2/Z_4 and T^2/Z_6 . We calculated realized mass eigenvalues in all the possibilities that predict three generations in zero mode fermions with one and two pairs of $SU(2)_L$ Higgs doublets (for up-type and down-type fermions). In $T^2/Z_{3,4,6}$, the effect of the kinetic mixings is nontrivial in the Yukawa calculation in Eq. (2.44) since it

smears hierarchies in the mass matrices. This feature brings us to the conclusion that it is very difficult to realize the mass ratio among the up quark and the top quark $m_{\text{up}}/m_{\text{top}} \sim 10^{-5}$ in all the configurations with one Higgs pair and with two Higgs pairs. In the two Higgs cases, interference effects among two mass matrices are maximized at around $v_1/v_2 = 1$ because elements of two matrices are comparable due to the mixing effect. Situations would be the same when more Higgs doublets are realized on $T^2/Z_{3,4,6}$ where kinetic mixing smearing hierarchies would prevent a large mass difference like

$m_{\text{up}}/m_{\text{top}} \sim 10^{-5}$. Such obstacles are absent on T^2 and T^2/Z_2 . This information is very useful when we try to construct an actual model on magnetized extra dimensions.

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