B^0-$\bar{B}^0$ mixing at next-to-leading order

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(Received 4 July 2016; published 11 August 2016)

We compute the perturbative corrections to the heavy quark effective theory sum rules for the matrix element of the $\Delta B = 2$ operator that determines the mass difference of B^0, $\bar{B}^0$ states. Technically, we obtain analytically the nonfactorizable contributions at order α_s to the bag parameter that first appear at the three-loop level. Together with the known nonperturbative corrections due to vacuum condensates and $1/m_b$ corrections, the full next-to-leading order result is now available. We present a numerical value for the renormalization group invariant bag parameter that is phenomenologically relevant and compare it with recent lattice determinations.

DOI: 10.1103/PhysRevD.94.034024

I. INTRODUCTION

The mixing of states in the systems of neutral flavored mesons belongs to the most sensitive probes for effects from physics beyond the standard model (SM). While the mixing in the kaon and the charmed-meson systems has significant or even dominant long distance effect contribution, the mixing for the neutral B mesons is dominated by the top-quark contribution and hence is dominated by short-distance physics. Technically, this fact means that the still necessary nonperturbative input is given by a matrix element of a local operator with $\Delta B = 2$, even if physics beyond the SM is present.

Within the SM, the mixing frequency Δm of the $B^0 - \bar{B}^0$ oscillations is determined by the following expression:

$$\Delta m = \frac{G_F^2}{8\pi^2}(V_{td}^* V_{tb})^2 F(x_t) m_t^2 \eta_{\rm QCD}(\mu) \langle B^0|Q(\mu)|\bar{B}^0\rangle \quad (1.1)$$

where $x_t = m_t^2/m_W^2$, and

$$F(x) = \frac{1}{4}\left[1 + \frac{9}{1-x} - \frac{6}{(1-x)^2} - \frac{6x^2}{(1-x)^3}\log x\right]$$

is the Inami-Lim function [1] (as a review, see, e.g. [2–4]).

The mass difference Δm depends on the matrix element $\langle B^0|Q(\mu)|\bar{B}^0\rangle$ of the local four-quark operator

$$Q = J_\mu J^\mu = Z(\alpha_s^{(n_f)}(\mu))Q(\mu), \qquad J^\mu = \bar{d}_L \gamma^\mu b_L, \quad (1.2)$$

where b_L, d_L are the left-handed bare quark fields (see, e.g., [5,6]). The short-distance coefficient $\eta_{\rm QCD}(\mu)$ in (1.1) accounts for contributions of scales larger than the b-quark mass m_b. The dependence of $\eta_{\rm QCD}(\mu)$ on the renormalization point μ compensates the μ-dependence of the matrix element $\langle B^0|Q(\mu)|\bar{B}^0\rangle$ that is the main object of low energy (for the scales down of m_b) QCD analysis. The matrix element of the four quark operator is traditionally written as

$$\begin{aligned}\langle B^0|Q(\mu)|\bar{B}^0\rangle &= 2\left(1+\frac{1}{N_c}\right)\langle B^0|J_\mu|0\rangle \\ \cdot\langle 0|J^\mu|\bar{B}^0\rangle B(\mu) &= 2\left(1+\frac{1}{N_c}\right) f_B^2 M_B^2 B(\mu), \quad (1.3)\end{aligned}$$

where N_c is the number of colors, $N_c = 3$ in QCD, $B(\mu)$ is the bag parameter, and

$$\langle 0|J^\mu|\bar{B}^0(p)\rangle = -\frac{i}{2} f_B p^\mu \quad (1.4)$$

is given by the B meson decay constant f_B. Note that the decay constant f_B is a physical quantity which is independent of the renormalization point, and its numerical value is rather well known (as recent reviews, see, e.g. [7,8]). Hence the full μ dependence enters the bag parameter $B(\mu)$.

Setting $B(\mu) = 1$ corresponds to the naive factorization prescription for the matrix element (1.3) which would be true for the bare operator Q at tree level but is spoiled by the strong interactions for the "dressed" operator $Q(\mu)$. The hadronic parameter $B(\mu)$ can only be obtained by using some nonperturbative method, such as lattice simulations (see, e. g., [8–13]) or QCD sum rules [14–18]. While the naive factorization estimate $B(m_B) = 1$ is rather satisfactory even quantitatively, it is a kind of a model assumption, and a key issue in the precision phenomenological analysis of the processes of mixing is the determination of the deviation of $B(\mu)$ from unity. The matrix element appearing in (1.1) still depends on m_b which is a scale large compared to Λ_{QCD}. To evaluate this matrix element further, we perform a heavy quark expansion (HQE) for this quantity, resulting in a combined expansion in powers of $\alpha_s(m_b)$ and $\Lambda_{\mathrm{QCD}}/m_b$. The remaining matrix elements appearing in this expansion are defined in heavy quark effective theory (HQET) and may be estimated in an HQET sum rule.

In a previous paper [17], we have estimated the subleading terms of order $\Lambda_{\mathrm{QCD}}/m_b$ in such an expansion with a sum rule. However, in order to obtain the full next-to-leading order (NLO) result, we also need to estimate the perturbative contributions of order α_s. Within the framework of HQET sum rules this requires a calculation of three-loop diagrams. The relevant master integrals have been found in [19]. In the present paper we give the results of the calculation for the bag parameter. With this calculation the complete NLO terms are now known.

In the next section we collect some known perturbative results which are needed to set up the sum rule calculation discussed in Sec. III. Finally, we present a complete NLO result and discuss its implications for $B^0-\bar{B}^0$ mixing.

II. PERTURBATIVE CONTRIBUTIONS TO THE BAG PARAMETER

In this section we collect some perturbation theory results relevant for the analysis of mixing.

The μ dependence of the bag parameter at scales above the b quark mass is known to two loops [20]; the result reads

$$\begin{aligned} B(\mu) &= B(\mu_0)\left(\frac{\alpha_s^{(n_f)}(\mu)}{\alpha_s^{(n_f)}(\mu_0)}\right)^{\gamma_0/(2\beta_0^{(n_f)})}\left[1+\frac{\gamma_0}{2\beta_0^{(n_f)}}\left(\frac{\gamma_1}{\gamma_0}-\frac{\beta_1^{(n_f)}}{\beta_0^{(n_f)}}\right)\frac{\alpha_s^{(n_f)}(\mu)-\alpha_s^{(n_f)}(\mu_0)}{4\pi}+\mathcal{O}(\alpha_s^2)\right] \\ &= \hat{B}(\alpha_s^{(n_f)}(\mu))^{\gamma_0/(2\beta_0^{(n_f)})}\left[1+\frac{\gamma_0}{2\beta_0^{(n_f)}}\left(\frac{\gamma_1}{\gamma_0}-\frac{\beta_1^{(n_f)}}{\beta_0^{(n_f)}}\right)\frac{\alpha_s^{(n_f)}(\mu)}{4\pi}+\mathcal{O}(\alpha_s^2)\right], \end{aligned} \tag{2.1}$$

where the anomalous dimension of the operator Q in (1.2) is

$$\begin{aligned} \gamma(\alpha_s) &= \frac{d\log Z(\alpha_s(\mu))}{d\log\mu} = \gamma_0\frac{\alpha_s}{4\pi}+\gamma_1\left(\frac{\alpha_s}{4\pi}\right)^2+\mathcal{O}(\alpha_s^3), \\ \gamma_0 &= 6\frac{N_c-1}{N_c}, \qquad \gamma_1 = -\frac{N_c-1}{2N_c}\left(\frac{19}{3}N_c+21-\frac{57}{N_c}-\frac{4}{3}n_f\right) \end{aligned} \tag{2.2}$$

where n_f is the number of flavors including the b quark. The β-function coefficients are

$$\beta_0 = \frac{11}{3}N_c-\frac{2}{3}n_f, \qquad \beta_1 = \frac{34}{3}N_c^2-\left(\frac{13}{3}N_c-\frac{1}{N_c}\right)n_f. \tag{2.3}$$

In the physical quantity Δm (1.1), the μ dependence of $B(\mu)$ is compensated by the μ dependence of the Wilson coefficient $F(x_t)\eta_{\mathrm{QCD}}(\mu)$.

At scales μ below the b quark mass the QCD operators are expanded into a series in $\Lambda_{\mathrm{QCD}}/m_b$ by employing HQET; see e. g. [21–23]. In particular, the operator Q in (1.2) becomes [24,25]

$$Q(\mu) = 2\sum_{i=1}^{2}C_i(\mu)\tilde{Q}_i(\mu)+\mathcal{O}\left(\frac{1}{m_b}\right), \tag{2.4}$$

where the $1/m_b$ contributions have been discussed in [26]. The leading order part is

$$\tilde{Q}_1 = \tilde{J}_{1\mu}\tilde{J}_2^\mu, \qquad \tilde{J}_1^\mu = \bar{d}_L\gamma^\mu h_+, \qquad \tilde{J}_2^\mu = \bar{d}_L\gamma^\mu h_-, \tag{2.5}$$

$$\begin{aligned} \tilde{Q}_2 &= \tilde{Q}_2'+\frac{1}{4}\tilde{Q}_1, \qquad \tilde{Q}_2' = \tilde{J}_1\tilde{J}_2, \\ \tilde{J}_1 &= \bar{d}_L h_+, \qquad \tilde{J}_2 = \bar{d}_L h_-. \end{aligned} \tag{2.6}$$

The bare field h_+ annihilates the HQET heavy quark (moving with the four velocity v), and h_- creates the heavy antiquark (again moving with the four velocity v), which is a completely separate particle in HQET framework. The factor two in (2.4) comes from the fact that there are two b fields in Q, one of them becomes h_+ and the other one h_-. The HQET operators $\tilde{Q}_1, \tilde{Q}_2$ have opposite Fierz parities and hence do not mix under renormalization which is designed so to preserve Fierz transformations.

The matrix elements of the leading HQET operators in (2.5), (2.6) can be written as

$$\begin{aligned} \langle\mathbf{B}^0|\tilde{Q}_1(\mu)|\bar{\mathbf{B}}^0\rangle &= \left(1+\frac{1}{N_c}\right)\langle\mathbf{B}^0|\tilde{J}_{2\mu}(\mu)|0\rangle \\ &\quad\times\langle 0|\tilde{J}_1^\mu(\mu)|\bar{\mathbf{B}}^0\rangle\tilde{B}_1(\mu), \end{aligned} \tag{2.7}$$

$$\langle \mathbf{B}^0 | \tilde{Q}_2'(\mu) | \bar{\mathbf{B}}^0 \rangle = \left(1 - \frac{1}{2N_c}\right) \langle \mathbf{B}^0 | \tilde{J}_2(\mu) | 0 \rangle \langle 0 | \tilde{J}_1(\mu) | \bar{\mathbf{B}}^0 \rangle \tilde{B}_2'(\mu), \tag{2.8}$$

where the B meson states with a static b quark $|\mathbf{B}\rangle$ are normalized nonrelativistically

$$\langle \mathbf{B}(p') | \mathbf{B}(p) \rangle = (2\pi)^3 \delta(\vec{p}' - \vec{p}), \qquad |B(p)\rangle = \sqrt{2p^0} |\mathbf{B}(p)\rangle + \mathcal{O}(1/m_b),$$

and

$$\begin{aligned}
\langle 0 | \tilde{J}_1^\mu(\mu) | \bar{\mathbf{B}}^0 \rangle &= -\frac{1}{2} \langle 0 | \tilde{j}_1(\mu) | \bar{\mathbf{B}}^0 \rangle v^\mu, \qquad \langle 0 | \tilde{J}_1(\mu) | \bar{\mathbf{B}}^0 \rangle = -\frac{1}{2} \langle 0 | \tilde{j}_1(\mu) | \bar{\mathbf{B}}^0 \rangle, \\
\langle \mathbf{B}^0 | \tilde{J}_2^\mu(\mu) | 0 \rangle &= \frac{1}{2} \langle \mathbf{B}^0 | \tilde{j}_2(\mu) | 0 \rangle v^\mu, \qquad \langle \mathbf{B}^0 | \tilde{J}_2(\mu) | 0 \rangle = -\frac{1}{2} \langle \mathbf{B}^0 | \tilde{j}_2(\mu) | 0 \rangle, \\
\tilde{j}_1 &= \bar{d} \gamma_5 h_+, \qquad \tilde{j}_2 = \bar{d} \gamma_5 h_-, \\
\langle 0 | \tilde{j}_1(\mu) | \bar{\mathbf{B}}^0 \rangle &= i F(\mu), \qquad \langle \mathbf{B}^0 | \tilde{j}_2(\mu) | 0 \rangle = i F(\mu).
\end{aligned}$$

The B meson decay constant $\langle 0 | j^\mu | \bar{B}^0 \rangle = i f_B p_B^\mu$ (where $j^\mu = \bar{d} \gamma_5 \gamma^\mu b$) is

$$f_B = \sqrt{\frac{2}{m_B}} C(\mu) F(\mu) + \mathcal{O}\left(\frac{1}{m_b}\right), \tag{2.9}$$

where [27]

$$j^\mu v_\mu = C(\mu) \tilde{j}_1(\mu) + \mathcal{O}\left(\frac{1}{m_b}\right), \qquad C(m_b) = 1 - 2C_F \frac{\alpha_s(m_b)}{4\pi} + \mathcal{O}(\alpha_s^2) \tag{2.10}$$

$[C_F = (N_c^2 - 1)/(2N_c)]$. The anomalous dimension of the operators $\tilde{j}_{1,2}$ is [28–30][1]

$$\tilde{\gamma}(\alpha_s) = -3C_F \frac{\alpha_s}{4\pi} + C_F \left[\frac{2}{3}\pi^2 (C_A - 4C_F) + \frac{1}{2}\left(5C_F - \frac{49}{3} C_A\right) + \frac{5}{3} n_l\right] \left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3), \tag{2.11}$$

where $n_l = n_f - 1$ is the number of light flavors (now excluding b quark), and $C_A = N_c = 3$. In terms of these parameters, the anomalous dimension of the operator $\tilde{Q}_1$ in (2.5) [32] can be written as

$$\begin{aligned}
\tilde{\gamma}_1(\alpha_s) - 2\tilde{\gamma}(\alpha_s) &= \delta_{11} \left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3), \\
\delta_{11} &= \frac{N_c - 1}{3N_c} \left[2\pi^2 \left(3N_c - 2 - \frac{6}{N_c}\right) - 11N_c^2 - 15N_c - 12 + \frac{18}{N_c} + 2(N_c + 3) n_l\right].
\end{aligned} \tag{2.12}$$

Vanishing of the leading (linear in α_s) term in (2.12) reflects the (accidental) fact that at one loop and for scales below the b quark mass, the naive factorization of the four quark operator $\tilde{Q}_1$ into a product of two bilinear operators is scale independent, i.e. $\tilde{\gamma}_1 = 2\tilde{\gamma}$ [33,34]. Therefore the μ dependence of $\tilde{B}_1(\mu)$ is weak and contains no leading logarithms:

$$\tilde{B}_1(\mu) = \tilde{B}_1(\mu_0) \left[1 + \frac{\delta_{11}}{2\beta_0^{(n_l)}} \frac{\alpha_s^{(n_l)}(\mu) - \alpha_s^{(n_l)}(\mu_0)}{4\pi} + \mathcal{O}(\alpha_s^2)\right]. \tag{2.13}$$

The anomalous dimension of $\tilde{Q}_2$ is only known up to one-loop order [24,25]:

[1]The three-loop term is also known [31], but we do not need it.

$$\tilde{\gamma}_2(\alpha_s) - 2\tilde{\gamma}(\alpha_s) = \delta_{20}\frac{\alpha_s}{4\pi} + \mathcal{O}(\alpha_s^2), \qquad \delta_{20} = 4\frac{N_c+1}{N_c}, \tag{2.14}$$

and therefore

$$\begin{aligned}\tilde{B}_2(\mu) &\equiv -\left(1-\frac{1}{2N_c}\right)\tilde{B}_2'(\mu) + \frac{1}{4}\left(1+\frac{1}{N_c}\right)\tilde{B}_1(\mu)\\ &= \tilde{B}_2(\mu_0)\left(\frac{\alpha_s^{(n_l)}(\mu)}{\alpha_s^{(n_l)}(\mu_0)}\right)^{\delta_{20}/(2\beta_0^{(n_l)})}[1+\mathcal{O}(\alpha_s)].\end{aligned} \tag{2.15}$$

The matching to HQET is most conveniently performed at $\mu = m_b$, such that the matching coefficients contain no large logarithms:

$$\begin{aligned}Q(m_b) &= 2(C_1(m_b)\tilde{Q}_1(m_b) + C_2(m_b)\tilde{Q}_2'(m_b))\\ &\quad + \mathcal{O}\left(\frac{1}{m_b}\right),\end{aligned} \tag{2.16}$$

where [24,25,35]

$$\begin{aligned}C_1(m_b) &= 1 - \frac{8N_c^2+9N_c-15}{2N_c}\frac{\alpha_s^{(n_f)}(m_b)}{4\pi} + \mathcal{O}(\alpha_s^2),\\ C_2(m_b) &= -2(N_c+1)\frac{\alpha_s^{(n_f)}(m_b)}{4\pi} + \mathcal{O}(\alpha_s^2).\end{aligned} \tag{2.17}$$

Taking the matrix element of (2.16), using (1.3), (2.7), (2.8), and reexpressing f_B via $F(m_b)$ (2.9), we obtain

$$B(m_b) = \frac{C_1(m_b)}{C^2(m_b)}\tilde{B}_1(m_b) - \frac{N_c-\frac{1}{2}}{N_c+1}\frac{C_2(m_b)}{C^2(m_b)}\tilde{B}_2'(m_b). \tag{2.18}$$

Substituting $C_{1,2}(m_b)$ (2.17) and $C(m_b)$ (2.10), we arrive at

$$\begin{aligned}B(m_b) &= \left[1 - \frac{4N_c^2+9N_c-11}{2N_c}\frac{\alpha_s^{(n_f)}(m_b)}{4\pi}\right]\tilde{B}_1(m_b)\\ &\quad + (2N_c-1)\frac{\alpha_s^{(n_f)}(m_b)}{4\pi}\tilde{B}_2(m_b)\\ &\quad + \mathcal{O}\left(\alpha_s^2, \frac{1}{m_b}\right)\end{aligned} \tag{2.19}$$

where within the needed accuracy $\alpha_s^{(n_f)}(m_b) = \alpha_s^{(n_l)}(m_b)$. Consequently, in order to obtain the QCD bag parameter $B(\mu)$ with the NLO precision, we only need the leading order $\tilde{B}_2$; in particular, we do not need the two-loop anomalous dimension of the operator $\tilde{Q}_2$.

Dependence of $\tilde{B}_1(\mu)$ on μ is weak. $\tilde{B}_1(m_b)$ is related to $\tilde{B}_1(\mu)$ (where μ is a low normalization point used in the sum rules) by (2.13). Neglecting factorization breaking in the terms suppressed by α_s, i. e. setting $\tilde{B}_1(\mu) = \tilde{B}_2'(\mu) = 1$ in these terms, we obtain

$$B(m_b) = \tilde{B}_1(m_b) - \frac{11}{2}\left(1-\frac{1}{N_c}\right)\frac{\alpha_s(m_b)}{4\pi}. \tag{2.20}$$

There are two sources of factorization violation in the QCD bag parameter $B(m_b)$: the HQET bag parameter $\tilde{B}_1$ of the matrix element of the HQET operator $\tilde{Q}_1$ (which will be considered in Secs. III, IV) and the matching contribution (2.20). As expected, they are suppressed as $1/N_c$ in the large N_c limit.

This concludes the collection of necessary results concerning the renormalization of the matrix element of the four-quark operator and its matching to HQET at scales below the b quark mass. The remaining task is to evaluate the hadronic matrix element of the operator $\tilde{Q}_1$ in HQET, or the HQET bag parameter $\tilde{B}_1$, for which we perform a sum-rule analysis in HQET using operator product expansion (OPE).

III. OPE IN HQET FOR SUM RULES

In the following subsections we evaluate the matrix element of the four-quark operator $\tilde{Q}_1$ with HQET sum rules. We first consider the perturbative part of the sum rule, which requires a three-loop calculation of a suitably chosen correlator, and in a second step we study the quark-condensate contribution to the HQET sum rule.

A. Leading perturbative part

To evaluate the matrix element, we use a vertex (three-point) correlation function that has been first proposed for the analysis of the kaon mixing in [36]. This correlator reveals the factorizable structure of the matrix element more clearly than the two-point function but is significantly more difficult to compute at NLO in QCD compared to the calculation of the two-point function [37]. For the present analysis we however set up a three-point sum rule in HQET where the computational difficulties have been solved [19]. We consider the correlator

$$K = \int d^d x_1 d^d x_2 e^{ip_1x_1 - ip_2x_2}\langle 0|T\tilde{\jmath}_2(x_2)\tilde{Q}_1(0)\tilde{\jmath}_1(x_1)|0\rangle \tag{3.1}$$

of the operator $\tilde{Q}_1$ given in (2.5). Here we compute in dimensional regularization with $d = 4-2\varepsilon$ dimensions. The currents

FIG. 1. The leading perturbative contributions. The currents $\tilde{J}_1$, $\tilde{J}_2$ are shown slightly split.

FIG. 2. Some diagrams with corrections to the left loop. Of course, similar corrections to the right loop exist.

$$\tilde{\jmath}_1 = \bar{h}_+\gamma_5 d, \qquad \tilde{\jmath}_2 = \bar{h}_-\gamma_5 d \tag{3.2}$$

interpolate the ground state of a static B meson.

Both the HQET quark and the HQET antiquark propagate only forward in time $x \cdot v$, so that the product in (3.1) is nonzero only at $x_1 \cdot v < 0$, $x_2 \cdot v > 0$ and thus the time-ordered product coincides with the product.

The correlator K depends on two scalar quantities $\omega_{1,2} = p_{1,2} \cdot v$, $K = K(\omega_1, \omega_2)$ which correspond to the residual energies of the b quark and the anti-b quark respectively.

The perturbative diagrams for the correlator K can be subdivided into two classes. The factorizable diagrams include the leading contributions (Fig. 1) and those diagrams which contain corrections to the left loop and to the right one separately (e.g., Fig. 2). The right diagrams in Figs. 1 and 2 are equal to the corresponding left diagrams times the factor $(d-2)/(2N_c)$. This factor is obviously color suppressed $1/N_c$ at $d = 4$: there is one color loop (N_c) less, and the Dirac structures can be reduced to products (as in the left diagrams) by Fierz rearrangement. At $d \neq 4$ there is a contraction $\gamma_\mu \ldots \gamma^\mu$ within the same γ-matrix string in each right diagram, and it produces the factor $d-2$.

Nonfactorizable diagrams contain gluon exchanges between the left loop and the right one. They first appear at three loops (Fig. 3). Up to three loops, the results for the correlators $K(\omega_1, \omega_2)$ can be written as

$$K(\omega_1, \omega_2) = \left(1 + \frac{d-2}{2N_c}\right)\Pi(\omega_1)\Pi(\omega_2) + \Delta K(\omega_1, \omega_2), \tag{3.3}$$

where

$$\Pi(\omega) = \frac{N_c(-2\omega)^{2-2\varepsilon}}{(4\pi)^{d/2}}\left[I_1 - 2C_F \frac{g_0^2(-2\omega)^{-2\varepsilon}}{(4\pi)^{d/2}} \frac{d-2}{d-4} \times \left(I_1^2 - \frac{d(2d-5)}{d-4} I_2\right)\right] \tag{3.4}$$

is the correlator of $\tilde{\jmath}_1$ and $\tilde{J}_1$ [38–40], and

$$I_n = \Gamma(2n + 1 - nd)\Gamma^n\left(\frac{d}{2} - 1\right) \tag{3.5}$$

are the integrals corresponding to the "sunset" diagrams in HQET. The three-loop nonfactorizable contribution is

$$\Delta K(\omega_1, \omega_2) = N_c C_F \frac{g_0^2}{(4\pi)^{3d/2}} R(\omega_1, \omega_2). \tag{3.6}$$

We have reduced $R(\omega_1, \omega_2)$ to the master integrals investigated in [19] using the integration-by-parts program [41]

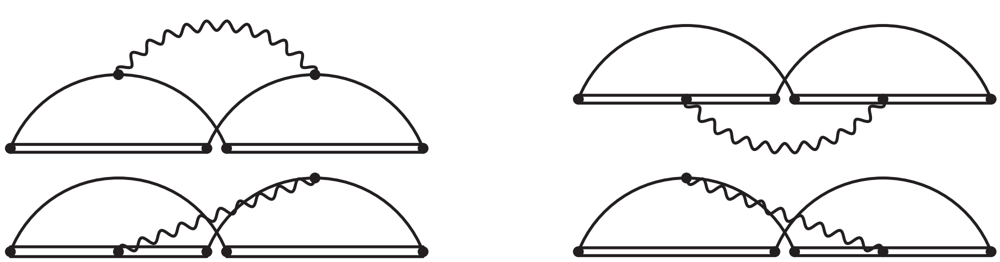


FIG. 3. Nonfactorizable diagrams.

$$\begin{aligned}R = &-\frac{(d-2)(3d-7)(d^2-16d+40)(\omega_1-2\omega_2)}{2(d-4)(3d-8)\omega_1(\omega_1-\omega_2)} I_3(-2\omega_1)^{3d-5} + (\omega_1 \leftrightarrow \omega_2)\\ &+\frac{(d-2)[(d-4)(3d-8)\omega_1-(d-2)(2d-5)\omega_2]}{(d-3)(d-4)\omega_1} I_1 I_2 (-2\omega_1)^{2d-4}(-2\omega_2)^{d-3} + (\omega_1 \leftrightarrow \omega_2)\\ &-\frac{(d-2)[(3d-8)(5d-14)\omega_1-2(d-4)(d^2-7d+11)\omega_2]}{(d-4)(3d-8)(\omega_1-\omega_2)} M_1(\omega_1,\omega_2) + (\omega_1 \leftrightarrow \omega_2)\\ &+\frac{(d-2)(2d^2-15d+26)}{2(d-3)} M_2(\omega_1,\omega_2) + \frac{(d-2)^2\omega_1\omega_2}{(d-3)^2} M_2'(\omega_1,\omega_2)\\ &+\frac{4(d-2)(d-3)(d^2-16d+40)\omega_1\omega_2}{(d-4)(3d-8)} M_3(\omega_1,\omega_2)\\ &-\frac{2(d-2)^2\omega_1}{d-4} M_4(\omega_1,\omega_2) + (\omega_1 \leftrightarrow \omega_2). \end{aligned} \tag{3.7}$$

The next step is to expand the master integrals around $d = 4$, i.e. in ε. The relevant technicalities are discussed in [19] and in the Appendix. We obtain

$$\Delta K(\omega_1,\omega_2) = N_c C_F \frac{g_0^2}{(4\pi)^{3d/2}} [\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)]^3 (-2\omega_1)^{2-3\varepsilon}(-2\omega_2)^{2-3\varepsilon} S(x), \tag{3.8}$$

where

$$x = \frac{\omega_2}{\omega_1}, \tag{3.9}$$

and $S(x) = S(x^{-1})$ is

$$\begin{aligned}S(x) = &\left[\frac{1}{48}(x^2+x^{-2}) - \frac{\pi^2}{3} + \frac{5}{4}\right]\frac{1}{3\varepsilon^2}\\ &+\left[-\frac{1}{16}(x^2-x^{-2})\log x + \frac{61}{288}(x^2+x^{-2}) + x + x^{-1} - 4\zeta_3 - \frac{4}{3}\pi^2 + \frac{41}{4}\right]\frac{1}{3\varepsilon}\\ &+\frac{1}{2}\left(\frac{1}{16}(x^2+x^{-2}) + \frac{\pi^2}{3} - \frac{5}{4}\right)\log^2 x - \left(\frac{61}{288}(x+x^{-1}) + 1\right)(x-x^{-1})\log x\\ &+\frac{1}{216}\left(\pi^2 + \frac{2519}{24}\right)(x^2+x^{-2}) - \frac{1}{3}\left(\frac{4}{9}\pi^2 - \frac{67}{4}\right)(x+x^{-1})\\ &-\frac{1}{3}\left(16\zeta_3 + \frac{4}{45}\pi^4 + \frac{25}{6}\pi^2 - \frac{193}{4}\right). \end{aligned} \tag{3.10}$$

The correlator $K(\omega_1,\omega_2)$ is analytic at $\omega_{1,2} < 0$. It has a cut in ω_1 from 0 to $+\infty$ with the discontinuity

$$\rho_1(\omega_1,\omega_2) = \frac{1}{2\pi i}[K(\omega_1+i0,\omega_2) - K(\omega_1-i0,\omega_2)] \tag{3.11}$$

if we keep $\omega_2 < 0$. The discontinuity $\rho_1(\omega_1,\omega_2)$ as a function of ω_2 (at some $\omega_1 > 0$) has a cut from 0 to $+\infty$ with the discontinuity in ω_2

$$\rho(\omega_1,\omega_2) = \frac{1}{2\pi i}[\rho_1(\omega_1,\omega_2+i0) - \rho_1(\omega_1,\omega_2-i0)]. \tag{3.12}$$

On dimensional grounds, the correlator at three loops has the form

$$K(\omega_1,\omega_2) = (-2\omega_1)^{2-3\varepsilon}(-2\omega_2)^{2-3\varepsilon} f(x), \tag{3.13}$$

where the function f can be gathered from the formulas given above. Looking at the spectral function $\rho_1(\omega_1,\omega_2)$, we first rotate ω_1: we set $\omega_1 = -\nu_1 e^{-i\alpha}$ ($\nu_1 > 0$) and vary α from 0 to $\pi - 0$ or $-\pi + 0$ (keeping $\omega_2 < 0$); this gives

$$\rho_1(\nu_1,\omega_2) = \frac{(2\nu_1)^{2-3\varepsilon}(-2\omega_2)^{2-3\varepsilon}}{2\pi i} \times \left[e^{3\pi i\varepsilon} f\left(-\frac{\omega_2}{\nu_1}e^{\pi i}\right) - e^{-3\pi i\varepsilon} f\left(-\frac{\omega_2}{\nu_1}e^{-\pi i}\right)\right], \tag{3.14}$$

where π means $\pi - 0$. Now we set $\omega_2 = -\nu_2 e^{-i\alpha}$ ($\nu_2 > 0$) and vary α from 0 to $\pi - 0$ or $-\pi + 0$:

$$\rho(\nu_1,\nu_2) = \frac{(2\nu_1)^{2-3\varepsilon}(2\nu_2)^{2-3\varepsilon}}{(2\pi i)^2} \times [(e^{6\pi i\varepsilon} + e^{-6\pi i\varepsilon}) f(x) - f(xe^{2\pi i}) - f(xe^{-2\pi i})], \quad x = \frac{\nu_2}{\nu_1}, \tag{3.15}$$

where $xe^{\pm 2\pi i}$ are at the Riemann sheets of the function $f(x)$ reached after crossing the cut in x from 0 to $-\infty$.

The bare double spectral density is

$$\rho(\omega_1,\omega_2) = \left(1 + \frac{1-\varepsilon}{N_c}\right)\rho(\omega_1)\rho(\omega_2) + \Delta\rho(\omega_1,\omega_2), \tag{3.16}$$

where [38–40]

$$\rho(\omega) = \frac{N_c(2\omega)^{2-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)}{1-2\varepsilon} \times \left[1 + C_F \frac{g_0^2(2\omega)^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(1+2\varepsilon)\Gamma(1-\varepsilon) \times \left(\frac{3}{\varepsilon} + \frac{4}{3}\pi^2 + 17\right)\right], \tag{3.17}$$

and

$$\Delta\rho(\omega_1,\omega_2) = N_c C_F \frac{g_0^2}{(4\pi)^{3d/2}} [\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)]^3 \times (2\omega_1)^{2-3\varepsilon}(2\omega_2)^{2-3\varepsilon} r(x), \tag{3.18}$$

where $r(x) = r(x^{-1})$. In the case of the operator $\tilde{Q}_1$ we have found that $r(x)$ does not, in fact, depend on x

$$r(x) = -\left(\frac{4}{3}\pi^2 - 5\right). \tag{3.19}$$

The expression for $r(x)$ is a key computational result of our paper.

The renormalized double spectral density $\rho_r(\omega_1,\omega_2) = \tilde{Z}_1^{-1}\tilde{Z}_j^{-2}\rho(\omega_1,\omega_2)$ is finite at the limit $\varepsilon \to 0$. This fact may be seen explicitly by using (with α_s accuracy) the relation $\tilde{Z}_1 = \tilde{Z}_j^2$ [see (2.12)]. Multiplying the factorizable part of (3.16) by $\tilde{Z}_1^{-1}\tilde{Z}_j^{-2} = \tilde{Z}_j^{-4}$ makes it finite separately. Therefore, also the nonfactorizable part has to become finite separately. At the limit $\varepsilon \to 0$ we obtain

$$\rho_r(\omega_1,\omega_2) = \left(1 + \frac{1}{N_c}\right)\rho_r(\omega_1)\rho_r(\omega_2) + \Delta\rho_r(\omega_1,\omega_2), \tag{3.20}$$

where [38–40]

$$\rho_r(\omega) = \frac{N_c(2\omega)^2}{(4\pi)^2}\left[1 + C_F\frac{\alpha_s}{4\pi}\left(-6\log\frac{2\omega}{\mu} + \frac{4}{3}\pi^2 + 17\right)\right] \tag{3.21}$$

and

$$\Delta\rho_r(\omega_1,\omega_2) = -N_c C_F \frac{\alpha_s}{(4\pi)^5}(2\omega_1)^2(2\omega_2)^2\left(\frac{4}{3}\pi^2 - 5\right). \tag{3.22}$$

We note again, that for the operator $\tilde{Q}_1$ as given in (2.5), $r(x)$ does not depend on x, i.e. on $\omega_{1,2}$; for other operators this is not necessarily so.

It is useful to rewrite the presentation (3.22) in the form

$$\Delta\rho_r(\omega_1,\omega_2) = -\frac{1}{N_c}C_F\frac{\alpha_s}{4\pi}\rho_r(\omega_1)\rho_r(\omega_2)\left(\frac{4}{3}\pi^2 - 5\right) \tag{3.23}$$

which is valid with $\mathcal{O}(a_s)$ accuracy. This form shows immediately the deviation from the factorization with correct relative normalization and can be used for the computation of corrections to the B parameter. Modifying the representation (3.23) even further one finds for the spectral density of three point correlator at NLO

$$\begin{aligned}\rho_r(\omega_1,\omega_2) &= \left(1 + \frac{1}{N_c}\right)\rho_r(\omega_1)\rho_r(\omega_2) + \Delta\rho_r(\omega_1,\omega_2)\\ &= \left(1 + \frac{1}{N_c}\right)\rho_r(\omega_1)\rho_r(\omega_2) \times \left(1 - \frac{\alpha_s}{4\pi}\frac{N_c - 1}{2N_c}\left(\frac{4}{3}\pi^2 - 5\right)\right)\end{aligned} \tag{3.24}$$

that is a master relation for the sum rules computation of "direct" contribution to ΔB.

In the next subsection we compute the contributions of the quark condensate to the correlator (3.1).

B. Quark condensate contribution

The power corrections to the sum rule discussed above are given in terms of quark and gluon condensates. The leading term is given by the quark condensate contributions to the correlator K. The diagrams contributing to these power corrections are shown in Figs. 4–6.

The leading order quark condensate contribution (Fig. 4) as well as some of the two-loop contributions (Fig. 5) are

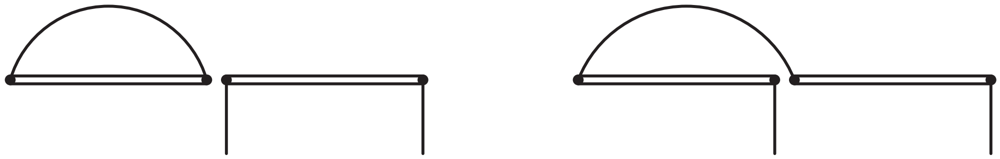


FIG. 4. The leading quark condensate contributions. Of course, the mirror-symmetric diagrams also exist.

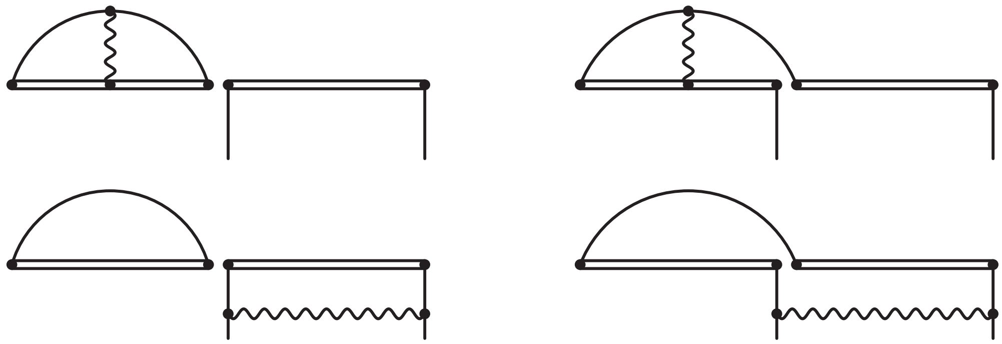


FIG. 5. Some of the factorizable contributions.

factorizable. They are contained in the product in (3.3), if we add the quark-condensate term [38]

$$\Pi_q(\omega) = \frac{1}{2}\frac{\langle\bar{d}d\rangle}{-2\omega}\left[1 + 2C_F\frac{g_0^2(-2\omega)^{-2\varepsilon}}{(4\pi)^{d/2}}(d-1)(d-4)I_1\right] \tag{3.25}$$

to the perturbative one (3.4).

The first nonfactorizable contributions due to quark condensate appear at the two-loop level as shown in Fig. 6. The contribution of these diagrams to the correlator becomes

$$\Delta K_q(\omega_1,\omega_2) = C_F\frac{g_0^2\langle\bar{d}d\rangle}{(4\pi)^d}R_q(\omega_1,\omega_2), \tag{3.26}$$

where

$$\begin{aligned} R_q = {}& \frac{4(\omega_1+\omega_2)[(d-2)(d-5)(\omega_1^2+\omega_2^2)-(d^3-10d^2+30d-30)\omega_1\omega_2]}{(d-4)(-2\omega_1)^{5-d}(-2\omega_2)^{5-d}}I_1^2 \\ & + \frac{2d-5}{2(d-3)(d-4)(d-5)\omega_2^2(\omega_1-\omega_2)} \\ & \times[(d-2)(d-5)^2\omega_1^3 + 2(d-2)(d-5)(2d-5)\omega_1^2\omega_2 - (d-3)(d^2-11d+6)\omega_1\omega_2^2 \\ & - 4(d-2)(d-3)\omega_2^3]I_2(-2\omega_1)^{2d-7} + (\omega_1\leftrightarrow\omega_2) \\ & + \frac{-(d-2)(d-5)\omega_1^3 - d\omega_1^2\omega_2 + (d-3)(d-8)\omega_1\omega_2^2 + (d-2)\omega_2^3}{4(d-4)\omega_1\omega_2^2(\omega_1-\omega_2)}M(\omega_1,\omega_2) + (\omega_1\leftrightarrow\omega_2) \end{aligned} \tag{3.27}$$

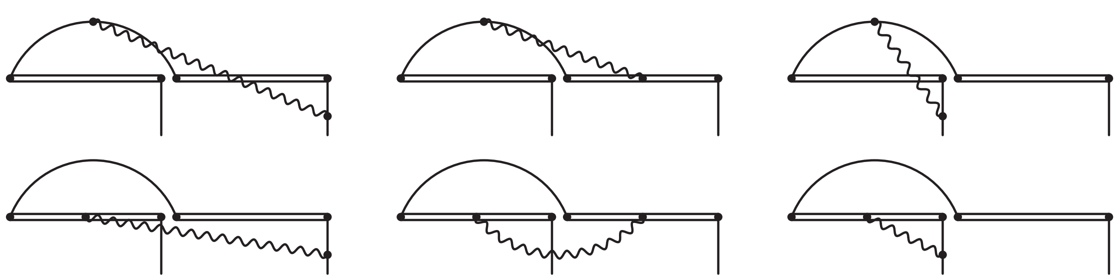


FIG. 6. Nonfactorizable contributions (the mirror-symmetric diagrams also exist).

where $M(\omega_1, \omega_2)$ is defined in (A1). Expanding in ε we obtain

$$\Delta K_q(\omega_1, \omega_2) = C_F \frac{g_0^2 \langle \bar{d}d \rangle}{(4\pi)^d} [\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)]^2 (-2\omega_1)^{\frac{1}{2}-2\varepsilon} (-2\omega_2)^{\frac{1}{2}-2\varepsilon} S_q(x), \tag{3.28}$$

where

$$\begin{aligned} S_q(x) = S_q(x^{-1}) = &-\frac{7}{16}\frac{x^{1/2}+x^{-1/2}}{\varepsilon^2} \\ &+ \left[\frac{7}{2}(x^{1/2}-x^{-1/2})\log x + (x^{1/2}+x^{-1/2})(x+x^{-1}-3)\frac{\pi^2}{3}\right. \\ &\left.-\frac{1}{4}(x^{1/2}+x^{-1/2})(5x+5x^{-1}+14)\right]\frac{1}{4\varepsilon} \\ &+ (x^{1/2}+x^{-1/2})(x+x^{-1}-3)[3\mathrm{Li}_3(1-x)+3\mathrm{Li}_3(1-x^{-1})-2L(x)\log x - 2\zeta_3] \\ &+ (x^{1/2}-x^{-1/2})(x+x^{-1})L(x) + \frac{1}{8}(x^{1/2}+x^{-1/2})(2x+2x^{-1}-7)\log^2 x \\ &+ (x^{1/2}+x^{-1/2})(10x+10x^{-1}-27)\frac{\pi^2}{24} \\ &+ \frac{1}{8}(x^{1/2}-x^{-1/2})(5x+5x^{-1}+32)\log x - \frac{1}{4}(x^{1/2}+x^{-1/2})(9x+9x^{-1}+11). \end{aligned} \tag{3.29}$$

Here the special function $L(x)$ is

$$L(x) = -L(x^{-1}) = \mathrm{Li}_2(1-x) + \frac{1}{4}\log^2 x.$$

Some useful properties of this function and relevant polylogarithms (Li_2, Li_3) are given in the Appendix.

Finally, the double discontinuity of the function $R_q(\omega_1, \omega_2)$ across the cuts $\omega_{1,2} > 0$ reads

$$\begin{aligned} \mathrm{disc}_2 R_q(\omega_1, \omega_2) = 2\Bigg[&\left(\frac{\pi^2}{3} - \frac{5}{4}\right)\omega_2^2 \delta(\omega_1) \\ &- (\omega_2+\omega_1)\left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} - 3\right)\log\left(1 - \frac{\omega_1}{\omega_2}\right)\Bigg]\theta(\omega_2 - \omega_1) \\ &+ (\omega_2 \leftrightarrow \omega_1). \end{aligned} \tag{3.30}$$

Note that the coefficient of the $\delta(\omega_1)$ is related (up to a proportionality factor) to that of a nonfactorizable perturbative correction in Eq. (3.19).

The spectral density of quark condensate contribution now reads

$$\begin{aligned} \Delta\rho_q(\omega_1, \omega_2) = C_F \frac{\alpha_s \langle \bar{d}d \rangle}{4\pi} \frac{2}{16\pi^2} \Bigg\{ &\left[\left(\frac{\pi^2}{3} - \frac{5}{4}\right)\omega_2^2 \delta(\omega_1) - (\omega_2+\omega_1)\left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} - 3\right)\log\left(1 - \frac{\omega_1}{\omega_2}\right)\right]\theta(\omega_2 - \omega_1) \\ &+ (\omega_2 \leftrightarrow \omega_1)\Bigg\}. \end{aligned} \tag{3.31}$$

The two-point correlator with the quark-condensate correction is given in (3.25).

IV. SUM RULES IN HQET

The sum rule is now set up by comparing the perturbatively computed correlator (3.24) with its hadronic representation. The hadronic spectral function is given by

$$\rho_H(\omega_1, \omega_2) = F^2 \langle B | \tilde{Q}_1 | \bar{B} \rangle \delta(\omega_1 - \bar{\Lambda}) \delta(\omega_2 - \bar{\Lambda}) + \rho_{\mathrm{cont}}(\omega_1, \omega_2) \tag{4.1}$$

where

$$\langle \bar{B}|\tilde{Q}_1|B\rangle = \left(1+\frac{1}{N_c}\right)\frac{1}{4}F(\mu)^2\tilde{B}_1$$
$$= (1+1/N_c)\frac{1}{4}F(\mu)^2(1+\Delta\tilde{B}_1) \quad (4.2)$$

and

$$\rho_{\rm cont}(\omega_1,\omega_2) = \rho_{\rm PT}(\omega_1,\omega_2)[1-\theta(\omega_c-\omega_1)\theta(\omega_c-\omega_2)]. \quad (4.3)$$

Here $\bar{\Lambda}$ is the B meson residual energy, $M_B - m_b = \bar{\Lambda}$ and ω_c is the continuum threshold. One sees that if one considers also the sum rules for two point correlators then the factorizable part of the matrix element disappears and one has the direct prediction for $\Delta\tilde{B}_1$.

The simplest way to extract $\Delta\tilde{B}_1$ is to use the finite energy sum rules (FESR) that equate the integrals over the square $0 < \omega_{1,2} < \omega_c$ of hadronic and OPE spectra. One obtains for the perturbation theory contribution the following expression

$$\Delta\tilde{B}_1(\mu) = -\frac{N_c-1}{2N_c}\left(\frac{4}{3}\pi^2-5\right)\frac{\alpha_s^{(n_l)}(\mu)}{4\pi}$$
$$\approx -0.68\frac{\alpha_s^{(n_l)}(\mu)}{\pi} = -2.72\frac{\alpha_s^{(n_l)}(\mu)}{4\pi}. \quad (4.4)$$

Here $n_l = 4$. Eq. (4.4) gives a direct contribution to the violation of factorization.

One can consider a more sophisticated analysis that controls power corrections as in the Borel modification of dispersion sum rules. In HQET, however, there is a nice way of solving the problem of controlling power corrections suggested by the structure of dispersion representation for the correlators in configuration space. Indeed, in coordinate-space, the renormalized correlator (3.1) at the parton level for Euclidean times $\tau_{1,2}$ ($\tau = it$) becomes

$$K_r(\tau_1,\tau_2) = \int_0^\infty d\omega_1 d\omega_2 e^{-\omega_1\tau_1-\omega_2\tau_2}\rho_r(\omega_1,\omega_2) + (\text{p.c.}), \quad (4.5)$$

where (p.c.) represents the power corrections proportional to vacuum condensates. The power corrections are important mainly for fixing the continuum threshold. We are not interested in the sum rules analysis on its own but in precise determination of $\Delta\tilde{B}_1$. Therefore we fix ω_c from all known sources [like $F(\mu)$ or f_B eventually] and use the knowledge about two-point sum rules where the main power correction is the quark condensate contribution.

The sum rule for the matrix element of the four-quark operator is obtained now from equating the OPE result to the hadronic expression for the correlator K with the spectral density (4.1)

$$K_{\rm had}(\tau_1,\tau_2) = \int_0^\infty d\omega_1 d\omega_2 e^{-\omega_1\tau_1-\omega_2\tau_2}\rho_{\rm had}(\omega_1,\omega_2) \quad (4.6)$$

which contains the desired matrix element (4.2). With the usual duality assumption for the excited states, we obtain the sum rule

$$F^2(\mu)\langle \mathbf{B}^0|\tilde{Q}_1(\mu)|\bar{\mathbf{B}}^0\rangle e^{-\bar{\Lambda}(\tau_1+\tau_2)}$$
$$= \int_0^{\omega_c} d\omega_1 \int_0^{\omega_c} d\omega_2 e^{-\omega_1\tau_1-\omega_2\tau_2}\rho_r(\omega_1,\omega_2) + (\text{p.c.}), \quad (4.7)$$

with the same parameters $\bar{\Lambda}$, $M_B - m_b = \bar{\Lambda}$ and the continuum threshold ω_c. The Euclidean times $\tau_{1,2}$ ($\tau = it$) play the role of suppressing-higher-states parameters ($1/\tau_{1,2}$ are the Borel parameters of the double Borel transform in $\omega_{1,2}$). One can study the stability of the result with respect to varying $\tau_{1,2}$. The version of sum rules in coordinate space in HQET is the most similar to the lattice treatment of the problem.

Dividing the sum rule (4.7) by two copies (product) of the two-point sum rules [38–40]

$$\frac{1}{2}F^2(\mu)e^{-\bar{\Lambda}\tau} = \int_0^{\omega_c} d\omega e^{-\omega\tau}\rho_r(\omega) + (\text{p.c.}), \quad (4.8)$$

we finally obtain the result for the bag factor

$$\tilde{B}_1(\mu) = 1 - \frac{N_c-1}{2N_c}\left(\frac{4}{3}\pi^2-5\right)\frac{\alpha_s^{(n_l)}(\mu)}{4\pi} + (\text{p.c.})$$
$$\approx 1 - 0.68\frac{\alpha_s^{(n_l)}(\mu)}{\pi} + (\text{p.c.}) \quad (4.9)$$

which coincides with that of the FESR approach. This result is valid at a low normalization scale $\mu \sim 1/\tau_{1,2}$ or, in fact, $\mu \sim \omega_c$. Also it assumes the same ω_c for both the two-point and three-point correlators [this is the reason why $\tilde{B}_1(\mu)$ is not explicitly dependent on ω_c]. Thus, Eq. (4.9) gives the most complicated contribution to the bag parameter directly coming from the three-loop correlation function [a "direct" violation of factorization to be contrasted with the violation in matching given in Eq. (2.20)].

There are still contributions originated from matching as given in Eq. (2.20) that should be added. Let us add them first neglecting higher order corrections due to different normalization points (running with NLO anomalous dimensions). They give the total violation of factorization in the form

$$-\frac{N_c-1}{2N_c}\left[11\frac{\alpha_s(m_b)}{4\pi}+\left(\frac{4}{3}\pi^2-5\right)\frac{\alpha_s(\mu)}{4\pi}\right]$$
$$\approx -(3.67+2.72)\frac{\alpha_s}{4\pi}. \tag{4.10}$$

where in the left-hand side we have still distinguished between the different scales of α_s which appear on the one hand in the matching and on the other hand in the QCD sum rule. However, μ is not fixed and can be chosen somewhere in the vicinity of ω_c such that $\mu > \omega_c$. In our numerical analysis below we choose the scale to be m_b and include the difference which is formally of order $\alpha_s(m_b)^2\log(m_b/\omega_c)$ in the uncertainty. Nevertheless, one sees that the direct violation [2.72 in Eq. (4.10)] is quantitatively important and is comparable in magnitude with the violation in matching [3.67 in Eq. (4.10)].

The deviation of $\tilde{B}_1(\mu)$ from unity that we have found so far measures the deviation from the naive factorization estimate due to perturbation theory contribution to the OPE. Now we account for the contribution of quark condensate that violates factorization. It can be important as its contribution to the two-point sum rule that determines $F(\mu)$ and eventually f_B is not small.

After integrating the $\rho_q(\omega_1,\omega_2)$ within the finite energy sum rules one finds

$$\int\rho_q(\omega_1,\omega_2)d\omega_1 d\omega_2 = C_F\frac{\alpha_s\langle\bar{q}q\rangle}{4\pi}\frac{2}{3}\frac{\omega_c^3}{(4\pi)^2}\left(\pi^2-\frac{149}{18}\right). \tag{4.11}$$

The two-point function sum rule (4.8) at $\tau=0$ (the finite-energy sum rule) gives

$$m_B f_B^2 = 2F^2 = N_c\frac{\omega_c^3}{3\pi^2} - \langle\bar{q}q\rangle;$$

we obtain

$$\Delta\tilde{B}_1|_q = \frac{N_c-1}{N_c}\frac{\langle\bar{q}q\rangle}{m_B f_B^2}\frac{\alpha_s}{4\pi}\left[1+\frac{\langle\bar{q}q\rangle}{m_B f_B^2}\right]\left(\pi^2-\frac{149}{18}\right). \tag{4.12}$$

Numerically one has

$$\frac{N_c-1}{N_c}\left(\pi^2-\frac{149}{18}\right)\approx 1.06$$

and

$$\frac{\langle\bar{q}q\rangle}{m_B f_B^2} = -0.07$$

for

$$\langle\bar{q}q\rangle = -(0.25\text{ GeV})^3,\quad m_B = 5.3\text{ GeV},\quad f_B = 200\text{ MeV}$$

that are typical values for the parameters. In our numerical analysis we neglect the quark condensate contribution in the square bracket in (4.12). One finds literally

$$\Delta\tilde{B}_1|_q = -0.08\frac{\alpha_s(m_b)}{4\pi} \tag{4.13}$$

and after adding uncertainties we finally write

$$\Delta\tilde{B}_1|_q = -(0.10\pm 0.04)\frac{\alpha_s(m_b)}{4\pi}. \tag{4.14}$$

The contribution is rather small. Note that this is, in fact, a numerical smallness. Indeed, the result is a difference of two large numbers (of order 10) $(\pi^2-\frac{149}{18})\approx 9.9-8.3=1.6$ that happens to be small (of order 1). Let us emphasize again that our estimates for the phenomenological parameters have very generous uncertainties. It is safe doing so because the contribution is rather small.

The non-PT terms (power corrections) have been analyzed in [14] and then extended and updated in [17]. The FESR estimate from the latter is

$$\Delta B_{\text{cond}} = -\frac{3\pi^2}{64}\left(\frac{1}{\omega_c^4}\left\langle\frac{\alpha_s}{\pi}GG\right\rangle - \frac{1}{\omega_c^5}\langle\bar{q}Gq\rangle\right)$$
$$= -\frac{3}{64}(0.06+0.1) = -0.008 \tag{4.15}$$

for standard values of gluon condensate $\langle\frac{\alpha_s}{\pi}GG\rangle$ [42] and mixed quark-gluon condensates $\langle\bar{q}Gq\rangle$ (e.g., see [43,44]). The final result after an accurate Borel SR analysis in HQET reads for the B_s meson [17]

$$\Delta B_{\text{cond}} = -0.006\pm 0.005, \tag{4.16}$$

and we use this estimate also for the B_d meson.

Because the values are very small they can be analyzed in linear approximation that means that the consideration of sum rules with included power corrections does not change the result for the parton part (no mutual influence).

Nonfactorizable $1/m_b$ corrections can only emerge in the α_s/m_b order (LO loops are completely factorized in QCD and this feature is reproduced in HQET as well). Therefore they are by factor $\Lambda/m_b = (0.5\text{ GeV})/(5\text{ GeV}) = 1/10$ smaller than those analyzed here and we simply include them in the uncertainty.

We discuss the final result in the next section where the comparison with lattice is also given.

V. RESULTS AND DISCUSSION

The main result of our analysis is the deviation ΔB from the value $B=1$ in factorization. In this section we collect all contributions and discuss the result.

The partonic result (i.e. the purely perturbative contribution) consists of three pieces originating from the matching, from the QCD sum rule analysis and from the running:

$$\Delta B|_{\mathrm{PT}} = -\frac{N_c - 1}{2N_c}\left[11\frac{\alpha_s(m_b)}{4\pi} + \left(\frac{4}{3}\pi^2 - 5\right)\frac{\alpha_s(\mu)}{4\pi}\right] + \frac{\delta_{11}}{2\beta_0^{(n_l)}}\frac{\alpha_s(m_b) - \alpha_s(\mu)}{4\pi} \approx -\left(\frac{4}{9}\pi^2 + 2\right)\frac{\alpha_s}{4\pi}.$$

As discussed after Eq. (4.10) we set for our numerical evaluation $\mu = m_b$ in the last step. Higher orders of $\alpha_s^2 \log(m_b/\omega_c)$ can be taken through NLO anomalous dimension but they are small and included as uncertainty in our analysis. To this end, we write

$$\Delta B|_{\mathrm{PT}} = -6.4\frac{\alpha_s(m_b)}{4\pi} \pm \left(X\frac{\alpha_s(m_b)}{4\pi}\right)\frac{\alpha_s(m_b)}{4\pi}$$

where X accounts for higher order terms. In order to estimate the uncertainty induced by such terms, we take a sizable value $X = 20$ for this parameter, and we obtain

$$\Delta B|_{\mathrm{PT}} = -6.4\frac{\alpha_s(m_b)}{4\pi} \pm 0.3\frac{\alpha_s(m_b)}{4\pi} = -(6.4 \pm 0.3)\frac{\alpha_s(m_b)}{4\pi}.$$

The choice of the value for the coupling constant is important for the absolute estimate. For the lattice estimates the reference value $\alpha_s(M_Z) = 0.1184$ from [45] is usually used [8]. Note that the estimate from the low energy τ decay data gives a close value [46]

$$\alpha_s(M_Z) = 0.1184 \pm 0.0007|_{\mathrm{exp}} \pm 0.0006|_{\mathrm{hq\ mass}}.$$

We stick, therefore, to the standard value

$$\alpha_s(m_b) = 0.20 \pm 0.02 \tag{5.1}$$

with rather generous uncertainty to account for possible systematic errors.

With the numerical value from (5.1) we obtain including systematic errors at the level of 30%

$$\Delta B_{\mathrm{PT}} = -0.10 \pm 0.02 \pm 0.03.$$

We now turn to the nonperturbative condensate terms. The quark-condensate term computed in this paper at order α_s gives

$$\Delta B_q = -(0.10 \pm 0.05)\frac{\alpha_s^{(n_l)}(m_b)}{4\pi} = -0.002 \pm 0.001. \tag{5.2}$$

In [17] the nonperturbative condensate terms that appear at tree level have been computed; see (4.15). Their numerical value is [17]

$$\Delta B_{\mathrm{nonPT}} = -0.006 \pm 0.005.$$

Including everything, we obtain the estimate

$$\Delta B = -0.11 \pm 0.04 \tag{5.3}$$

where we summed errors in quadrature.

In order to compare this to other calculations, it is useful to employ the translation factor to the renormalization group invariant parameter $\hat{B} = \hat{Z}B(m_b)$ is,

$$\hat{Z} = \alpha_s(m_b)^{-\frac{\gamma_0}{2\beta_0}}\left(1 + \frac{\alpha_s(m_b)}{4\pi}\left(\frac{\beta_1\gamma_0 - \beta_0\gamma_1}{2\beta_0^2}\right)\right)$$

with

$$\gamma_0 = 4, \qquad \gamma_1 = -7 + \frac{4}{9}n_f, \qquad n_f = 5,$$

which numerically is

$$\hat{Z} = 1.51$$

at $\alpha_s(m_b) = 0.2$ [12].

Applying this factor to our result

$$B(m_b)|_{\text{this paper}} = 1 - (0.11 \pm 0.04) \tag{5.4}$$

we obtain

$$\hat{B}|_{\text{this paper}} = 1.51\{1 - (0.11 \pm 0.04)\} = 1.34 \pm 0.06. \tag{5.5}$$

The main uncertainty comes from the choice of scale for $\alpha_s(\mu)$ between $\mu \sim \omega_c$ and m_b, higher orders in $\alpha_s(m_b)$, and the value of $\alpha_s(m_b)$. The uncertainties due to other sources (like NNLO matching, or systematics of sum rules) is difficult to quantify. For them we add some typical values known from the experience with similar correlation functions (see, e.g. [39,40]). More recent examples of uncertainty analysis within sum rules approach can be found in [7,17].

We note that the sum rule yields a quite precise prediction. This is due to the fact that the actual sum-rule calculation is performed for the deviation ΔB of the bag factor from unity. Although the calculation of ΔB suffers from the typical sum-rule uncertainty of tens of percents, the value obtained for $\hat{B}$ is quite precise since ΔB is small compared to unity.

This value has to be compared to lattice value results. The recent review [8] quotes the average

$$\hat{B}_{\rm latt} = 1.26(9)$$

for $n_f = 2 + 1$ flavors based on [9,10] and

$$\hat{B}_{\rm latt} = 1.30(6)$$

for $n_f = 2$ [11]. The recent result [12] is

$$\hat{B}_{\rm latt} = 1.38(12)(6). \tag{5.6}$$

The parameter B itself normalized at the b quark mass is given earlier as [13]

$$B_{\rm latt}(m_b) = 0.8 \pm 0.1$$

(unfortunately, the number is not given explicitly and the result is extracted from the figure only). At present, the progress in lattice computations is pretty fast and the results are going to further improve. Nevertheless, currently our sum rule estimate is competitive with the lattice calculations for the reasons discussed above.

A comment on the QCD computation of the bag parameter with the moments of the spectral density at the finite b-quark mass used in the analysis of Ref. [16] is in order here. The subtraction of divergences for the operator Q has been done in a way that is different from the scheme adopted for the computation of the coefficient functions of $\Delta B = 2$ Hamiltonian in [20]. Thus, the renormalized operator $Q(\mu)$ of [16] differs from the one given in [20] (and used in the present paper) by a finite amount of order α_s. We are going to convert the results of [16] to the canonical basis in a separate paper.

VI. SUMMARY

We have computed nonfactorizable corrections to the bag parameter for the $B_d^0 - \bar{B}_d^0$ mixing. The most complicated part is a "direct" contribution that requires an account for three-loop diagrams in HQET. The main result of phenomenological analysis is that these corrections are small, and factorization approximation is quantitatively valid. We have found

$$B(m_b) - 1 = -(0.11 \pm 0.04) \tag{6.1}$$

and

$$\hat{B}|_{\rm QCD} = 1.34 \pm 0.06 \tag{6.2}$$

for the B_d meson bag parameter.

The main advantage of our approach is that we classify the contributions (diagrams) at the level of OPE such that we can explicitly single out contributions that completely factorize. In that sense they can only produce unity in the bag parameter and do not require any computation if properly marked. Subtracting these terms at the level of OPE we keep only terms that explicitly violate factorization and use the sum rules for them. It happens that those terms are numerically small and even rather large uncertainties in their estimate still produce rather precise result for the matrix element itself.

ACKNOWLEDGMENTS

We thank Th. Feldmann for the interest in the work and discussion. A. G. is grateful to Siegen University for hospitality; his work has been partially supported by the Russian Ministry of Education and Science. This work is supported by the DFG Research Unit FOR 1873 "Quark Flavour Physics and Effective Theories."

APPENDIX: MASTER INTEGRALS

Expansions of the master integrals in ε up to finite terms have been obtained in [19] Appendix A. However, we have found that the coefficients of $M_{3,4}$ in the correlator are $\mathcal{O}(1/\varepsilon)$, and we need one more term in their expansions. The expansion of M_3 is given by (A.4) in [19]; the new additional term in the braces is

$$\begin{aligned}
+ \bigg[& 144(2x \log x - 1 + 19x - 3x^2)\mathrm{Li}_3(1-x) - 144(2x\log x + 3 - 19x + x^2)\mathrm{Li}_3(1-x^{-1}) \\
& + 288L^2(x) + 216(1-7x+x^2)L(x)\log x + 252(1-x^2)L(x) \\
& + \frac{81}{4}x\log^4 x + \frac{9}{2}(1-x^2)\log^3 x - \frac{9}{4}(19+70x+19x^2)\log^2 x + 18(1-x^2)\log x \\
& - 8\left(630\zeta_3 + \frac{71}{15}\pi^4 + 18\pi^2\right)x + 3(11 - 120x + 11x^2)\bigg]\varepsilon^4.
\end{aligned}$$

The expansion of M_4 is given by (A.5) in [19]; the new additional term in the braces is

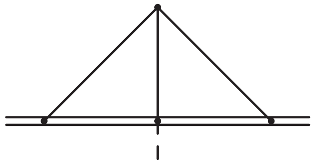


FIG. 7. Topology of two-loop integrals.

$$
\begin{aligned}
&-2\Big[144x^2L_4(x)-12x(2x\log x+3+18x-3x^2)\mathrm{Li}_3(1-x)\\
&+12x(2x\log x-1-18x+x^2)\mathrm{Li}_3(1-x^{-1})\\
&-24x^2L^2(x)+6x[4x\log^2x+18x\log x-5(1-x^2)]L(x)\\
&+3x(1-x^2)\log^3x+x[8\pi^2x+3(5-9x-5x^2)]\log^2x\\
&+3x[4(8\zeta_3+3\pi^2)x-1+x^2]\log x\\
&+2\left(270\zeta_3+\frac{28}{15}\pi^4+9\pi^2\right)x^2+2x(7+2x-x^2)\Big]\varepsilon^4,
\end{aligned}
$$

where the function

$$
\begin{aligned}
L_4(x)=-L_4(x^{-1})=\mathrm{Li}_4(x)+\frac{1}{6}\log^3x\log(1-x)\\
-\frac{1}{16}\log^4x-\frac{\pi^2}{12}\log^2x-\frac{\pi^4}{90}
\end{aligned}
$$

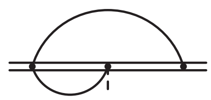
is analytical in $(0,+\infty)$ (no branching singularity at $x=1$). We have also checked that the expansions (A.2) and (A.3) of M_2, M_2' in [19] satisfy the identity

$$
M_2'=\frac{d-3}{(d-4)\omega_1^2\omega_2}\times\left[(\omega_1^2-\omega_2^2)\frac{\partial M_2}{\partial\omega_2}+\frac{1}{2}(3d-8)(\omega_1+2\omega_2)M_2\right]
$$

following from IBP.

For the calculation of two-loop diagrams in Sec. III B we need Feynman integrals shown in Fig. 7. Using LiteRed [41] we reduce them to 3 trivial master integrals $I_1^2(-2\omega_1)^{d-3}(-2\omega_2)^{d-3}$, $I_2(-2\omega_1)^{2d-5}$, $I_2(-2\omega_2)^{2d-5}$ and 2 nontrivial ones,

$$M(\omega_1,\omega_2)=$$



$$=I_1I(3-d,1,1;\omega_1,\omega_2) \tag{A1}$$

and $M(\omega_2,\omega_1)$. Expansion of $M(\omega_1,\omega_2)$ in ε is

$$
\begin{aligned}
M(\omega_1,\omega_2)=-\frac{\Gamma^2(1-\varepsilon)\Gamma(1+4\varepsilon)}{16\varepsilon^2(1-2\varepsilon)(1-4\varepsilon)(3-4\varepsilon)}\Big\{&x(x-1)-(4x^2-6x+1)\varepsilon\\
&-2[x(x-1)(4L(x)+\log^2x)-2(2x-1)\log x]\varepsilon^2\\
&+8\Big[x(x-1)\left(4\mathrm{Li}_3(1-x)+2\mathrm{Li}_3(1-x^{-1})-4L(x)\log x-\frac{1}{3}\log^3x+4L(x)\right)\\
&+(x^2+x-1)\log^2x\Big]\varepsilon^3+\cdots\Big\}\frac{(-2\omega_2)^{2-4\varepsilon}}{x^2}.
\end{aligned}
$$

For calculations of spectral densities we used

$$
\begin{aligned}
\mathrm{Li}_2(1-xe^{\pm2\pi i})&=\mathrm{Li}_2(1-x)\mp2\pi i[\log|x-1|\pm\pi i\theta(x-1)],\\
\mathrm{Li}_3(1-xe^{\pm2\pi i})&=\mathrm{Li}_3(1-x)\mp\pi i[\log|x-1|\pm\pi i\theta(x-1)]^2,\\
\mathrm{Li}_n(x+i0)-\mathrm{Li}_n(x-i0)&=\frac{2\pi i}{\Gamma(n)}\log^{n-1}x\quad\text{where } x>0
\end{aligned}
$$

(where $1-xe^{\pm2\pi i}$ are on the Riemann sheets reached after crossing the cut). We also used the identity

$$
\mathrm{Li}_3(x)+\mathrm{Li}_3(1-x)+\mathrm{Li}_3(1-x^{-1})=\frac{1}{6}\log^3x-\frac{1}{2}\log^2x\log(1-x)+\frac{\pi^2}{6}\log x+\zeta_3.
$$

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