Classifying *CP* transformations according to their texture zeros: Theory and implications

Peng Chen,[1,*] Gui-Jun Ding,[1,†] Felix Gonzalez-Canales,[2,3,‡] and J. W. F. Valle[2,§]

[1]*Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

[2]*AHEP Group, Institut de Física Corpuscular—C.S.I.C./Universitat de València, Parc Cientific de Paterna. C/Catedratico José Beltrán, 2 E-46980 Paterna (València), Spain*

[3]*Departamento de Física, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Apartado Postal 14-740, 07000 México D.F., Mexico*

(Received 30 April 2016; published 4 August 2016)

We provide a classification of generalized *CP* symmetries preserved by the neutrino mass matrix according to the number of zero entries in the associated transformation matrix. We determine the corresponding constrained form of the lepton mixing matrix, characterized by correlations between the mixing angles and the *CP* violating phases. We compare with the corresponding restrictions from current neutrino oscillation global fits and show that, in some cases, the Dirac *CP* phase characterizing oscillations is highly constrained. Implications for current and upcoming long baseline neutrino oscillation experiments T2K, NOνA, and DUNE, as well as neutrinoless double beta decay experiments are discussed. We also study the cosmological implications of such schemes for leptogenesis.

DOI: 10.1103/PhysRevD.94.033002

I. INTRODUCTION

Non-Abelian symmetries provide an attractive framework in terms of which to tackle the longstanding flavor problem in particle physics [1–5]. Assuming light neutrinos to be Majorana particles, as suggested on general grounds [6], we consider the case where the neutrino and charged lepton mass matrices have remnant symmetries, both of flavor and *CP* types. These would ultimately reflect some unspecified flavor symmetry of the underlying gauge theory, providing also a more general framework to describe mu-tau flavor symmetry in neutrino physics [7,8]. One can show that flavor symmetries can be generated by performing two successive *CP* transformations [9,10]. Conversely the remnant *CP* transformation of the lepton mass matrices can constrain the lepton flavor mixing in a quite efficient way. In particular, the Majorana and Dirac leptonic *CP* violating phases can be predicted [7,9,10]. Notice that the Dirac *CP* phase is loosely constrained while the Majorana phases are unconstrained if the neutrino mass matrix is invariant under the action of a remnant Z_2 flavor symmetry [11–13]. Such *CP* symmetries were first explored in the context of gauge theories [14] and their early application to neutrino physics led to the well-known $\mu-\tau$ reflection [15–17] which predicts maximal atmospheric mixing angle and maximal Dirac *CP* phase. In the present work, if one generic remnant *CP* transformation $\mathbf{X}$ is preserved by the neutrino mass matrix the $\mathbf{X}$ matrix should be symmetric and unitary. The light neutrino masses would be degenerate [7,9,10] if $\mathbf{X}$ is nonsymmetric. By performing the Takagi factorization we have $\mathbf{X}=\Sigma\Sigma^T$, where Σ is a unitary matrix. Without loss of generality, we shall work in the charged lepton diagonal basis so that the lepton flavor mixing completely arises from the neutrino sector. The invariance of the neutrino mass matrix under the action of $\mathbf{X}$ implies that the lepton mixing matrix $\mathbf{U}$ should fulfill [7,9,10]

$$\mathbf{U}^{-1}\mathbf{X}\mathbf{U}^{*}=\mathrm{diag}(\pm1,\pm1,\pm1). \tag{1}$$

Then the lepton mixing matrix is determined to be of the form

$$\mathbf{U}=\Sigma\mathbf{O}_{3\times3}\hat{X}^{-1/2}, \tag{2}$$

where $\hat{X}^{-1/2}$ is a diagonal matrix with nonvanishing entries equal to ±1 or $\pm i$, which is necessary for making neutrino masses positive [6]. Without loss of generality, this matrix can be parametrized as

$$\hat{X}^{-1/2}=\mathrm{diag}(1,i^{k_1},i^{k_2}), \tag{3}$$

with $k_{1,2}=0,\ 1,\ 2,\ 3$. The Majorana phases may be changed by π due to $\hat{\mathbf{X}}^{-1/2}$. On the other hand, the $\mathbf{O}_{3\times3}$ is a generic three-dimensional real orthogonal matrix, and it can be parametrized as

[*]pche@mail.ustc.edu.cn
[†]dinggj@ustc.edu.cn
[‡]felix.gonzalez@ific.uv.es
[§]valle@ific.uv.es
http://astroparticles.es/

$$\mathbf{O}_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

where $\theta_{1,2,3}$ are real parameters, and a possible overall minus sign of $\mathbf{O}_{3\times3}$ is omitted since it is irrelevant to flavor mixing parameters. Therefore the lepton mixing matrix is predicted to depend on three free parameters $\theta_{1,2,3}$ besides the parameters characterizing the residual CP transformation $\mathbf{X}$. Notice that if Σ is a Takagi factorization matrix of $\mathbf{X}$, $\Sigma\mathbf{O}'_{3\times3}$ is also a valid Takagi factorization matrix, where $\mathbf{O}'_{3\times3}$ is an arbitrary real orthogonal matrix which can be absorbed into $\mathbf{O}_{3\times3}$ by parameter redefinition. As a result, the prediction for the lepton mixing matrix in Eq. (2) remains valid. In the following, we shall classify all possible forms of the remnant CP transformations according to the number of zero elements in $\mathbf{X}$ and investigate the corresponding predictions for the lepton flavor mixing parameters and their implications for the lepton CP violation in conventional neutrino oscillations, neutrinoless double beta decay as well as leptogenesis.

A. The angle-phase parametrization

For three light Majorana neutrinos the leptonic mixing matrix can be expressed in terms of three rotation angles and three CP violation phases [6]. Here we will take these six independent parameters expressed within the PDG prescription given as [18][1]

$$\mathbf{U}_{\mathrm{PDG}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{CP}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{CP}} & c_{13}c_{23} \end{pmatrix}\mathcal{K}, \tag{5}$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$ and $\mathcal{K}$ is a diagonal matrix of phases chosen as $\mathcal{K} = \mathrm{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2})$. The δ_{CP} is the Dirac CP violation phase and α_{21}, α_{31} are two Majorana CP violation phases. In this parametrization the mixing angles and magnitudes of the entries of the $\mathbf{U}_{\mathrm{PDG}}$ matrix are related as

$$\sin^2\theta_{13} = |(\mathbf{U}_{\mathrm{PDG}})_{13}|^2, \qquad \sin^2\theta_{12} = \frac{|(\mathbf{U}_{\mathrm{PDG}})_{12}|^2}{1 - |(\mathbf{U}_{\mathrm{PDG}})_{13}|^2}$$

$$\text{and} \quad \sin^2\theta_{23} = \frac{|(\mathbf{U}_{\mathrm{PDG}})_{23}|^2}{1 - |(\mathbf{U}_{\mathrm{PDG}})_{13}|^2}. \tag{6}$$

The well-known Jarlskog-like invariant is defined as $J_{CP} = \Im\{(\mathbf{U}_{\mathrm{PDG}})^*_{11}(\mathbf{U}_{\mathrm{PDG}})^*_{23}(\mathbf{U}_{\mathrm{PDG}})_{13}(\mathbf{U}_{\mathrm{PDG}})_{21}\}$. It describes CP violation in conventional neutrino oscillations, and takes the form

$$J_{CP} = \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta_{CP}. \tag{7}$$

This expression gives a very transparent interpretation of the Dirac leptonic CP invariant. There are two additional rephase invariants $I_1 = \Im\{(\mathbf{U}_{\mathrm{PDG}})^2_{12}(\mathbf{U}_{\mathrm{PDG}})^{*2}_{11}\}$ and $I_2 = \Im\{(\mathbf{U}_{\mathrm{PDG}})^2_{13}(\mathbf{U}_{\mathrm{PDG}})^{*2}_{11}\}$; associated with the Majorana phases [20–22] they take the following form:

$$I_1 = \frac{1}{4}\sin^2 2\theta_{12}\cos^4\theta_{13}\sin\alpha_{12} \quad \text{and}$$

$$I_2 = \frac{1}{4}\sin^2 2\theta_{13}\cos^2\theta_{12}\sin\alpha'_{13}, \tag{8}$$

where $\alpha'_{13} = \alpha_{13} - 2\delta_{CP}$. These invariants appear in lepton number violating processes such as neutrinoless double beta decay which do not depend, as expected, on the Dirac invariant J_{CP}.

II. TEXTURE ZEROS OF THE REMNANT CP TRANSFORMATIONS

Due to the presence of zero entries the $\mu - \tau$ reflection is rather predictive and can lead to maximal θ_{23} and δ_{CP} [15–17]. Moreover, in the widely discussed approach of combining discrete flavor with generalized CP symmetries [23–40], the most general CP transformations quite often contain zero element(s). In the present work, we shall perform a comprehensive study of residual CP transformation according to the number of texture zeros. In order to classify the remnant CP transformations in terms of texture zeros, it is necessary to establish the way of counting these zero entries in the

[1]For a description in the original symmetric form of the lepton mixing matrix see [7]. As discussed in [19] the symmetric presentation is more transparent in describing $0\nu\beta\beta$ while the equivalent PDG prescription is more convenient for describing neutrino oscillations.

TABLE I. The *CP* transformation matrices with four texture zeros and the corresponding Σ matrices, where α, β, and γ are real. The resulting lepton mixing matrix is obtained through the Eq. (2). Type I is the conventional $\mu-\tau$ reflection while for types II and III Σ matrices are related by a permutation of the second and third row. The experimentally measured values of the three mixing angles θ_{12}, θ_{23}, and θ_{13} cannot be reproduced.

Four texture zeros			
Type	$\mathbf{X}$	Σ	Mixing parameters
I	$\begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & 0 & e^{i\beta} \\ 0 & e^{i\beta} & 0 \end{pmatrix}$	$\begin{pmatrix} e^{i\alpha/2} & 0 & 0 \\ 0 & \frac{e^{i\beta/2}}{\sqrt{2}} & i\frac{e^{i\beta/2}}{\sqrt{2}} \\ 0 & \frac{e^{i\beta/2}}{\sqrt{2}} & -i\frac{e^{i\beta/2}}{\sqrt{2}} \end{pmatrix}$	$\sin^2\theta_{13}=\sin^2\theta_2,\ \sin^2\theta_{23}=\frac{1}{2},\ \sin^2\theta_{12}=\sin^2\theta_3,$ $\sin\delta_{CP}=\mathrm{sign}(\sin\theta_2\sin 2\theta_3),\ \alpha_{21}=k_1\pi,\ \alpha'_{31}=k_2\pi$
II	$\begin{pmatrix} 0 & 0 & e^{i\beta} \\ 0 & e^{i\alpha} & 0 \\ e^{i\beta} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{e^{i\beta/2}}{\sqrt{2}} & i\frac{e^{i\beta/2}}{\sqrt{2}} \\ e^{i\alpha/2} & 0 & 0 \\ 0 & \frac{e^{i\beta/2}}{\sqrt{2}} & -i\frac{e^{i\beta/2}}{\sqrt{2}} \end{pmatrix}$	$\sin^2\theta_{13}=\frac{1}{2}\cos^2\theta_2,\ \sin^2\theta_{23}=\frac{2-2\cos 2\theta_2}{3-\cos 2\theta_2},\ \sin^2\theta_{12}=\frac{1}{2}+\frac{\cos^2\theta_2\cos 2\theta_3}{3-\cos 2\theta_2},$ $J_{CP}=-\frac{1}{4}\sin\theta_2\sin 2\theta_3\cos^2\theta_2,\ I_1=\frac{1}{4}(-1)^{k_1+1}\sin\theta_2\sin 2\theta_3\cos^2\theta_2,$ $I_2=\frac{1}{4}(-1)^{k_2}\sin\theta_2\sin 2\theta_3\cos^2\theta_2.$
III	$\begin{pmatrix} 0 & e^{i\beta} & 0 \\ e^{i\beta} & 0 & 0 \\ 0 & 0 & e^{i\alpha} \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{e^{i\beta/2}}{\sqrt{2}} & i\frac{e^{i\beta/2}}{\sqrt{2}} \\ 0 & \frac{e^{i\beta/2}}{\sqrt{2}} & -i\frac{e^{i\beta/2}}{\sqrt{2}} \\ e^{i\alpha/2} & 0 & 0 \end{pmatrix}$	$\sin^2\theta_{13}=\frac{1}{2}\cos^2\theta_2,\ \sin^2\theta_{23}=\frac{1+\cos 2\theta_2}{3-\cos 2\theta_2},\ \sin^2\theta_{12}=\frac{1}{2}+\frac{\cos^2\theta_2\cos 2\theta_3}{3-\cos 2\theta_2},$ $J_{CP}=\frac{1}{4}\sin\theta_2\sin 2\theta_3\cos^2\theta_2,\ I_1=\frac{1}{4}(-1)^{k_1+1}\sin\theta_2\sin 2\theta_3\cos^2\theta_2,$ $I_2=\frac{1}{4}(-1)^{k_2}\sin\theta_2\sin 2\theta_3\cos^2\theta_2.$

$\mathbf{X}$ matrix: two zeros off-diagonal counts as one texture zero, while one zero on the diagonal counts as one texture zero [41]. The *CP* transformations are unitary-symmetric matrices and consequently have nonzero determinant. Hence, the $\mathbf{X}$ matrix cannot be represented by a matrix with six or five zero elements. In other words, the maximum number of zero entries in the *CP* transformation matrices must be four. The *CP* transformation matrices with four texture zeros and the corresponding Σ matrices are given in Table I. For this case the lepton mixing matrix is obtained through Eq. (2), and the mixing parameters are given in the fourth column of Table I.

The type-I $\mathbf{X}$ matrix with four texture zeros corresponds to the so-called $\mu-\tau$ reflection [8,15–17]. It is remarkable that both the atmospheric mixing angle θ_{23} and the *CP* violation phase δ_{CP} are predicted to be maximal for any values of the free parameters θ_i while Majorana phases take on *CP*–conserving values. For the type-II $\mathbf{X}$ matrix one sees that reactor and atmospheric angles are strongly correlated with each other,

$$\sin^2\theta_{23}=1-\tan^2\theta_{13},\qquad \sin^2\theta_{12}=\frac{1}{2}(1+\tan^2\theta_{13}\cos 2\theta_3). \tag{9}$$

Given the measured value of reactor angle θ_{13}, we have $\sin^2\theta_{12}\simeq\frac{1}{2}$ and $\sin^2\theta_{23}\simeq 1$. The measured values of the solar and atmospheric mixing angles can not be achieved in this case.

For the residual *CP* transformation with four texture zeros type-III, the lepton mixing matrix is related to the type-II case through the exchange of the second and third rows,

$$\Sigma_{\mathrm{III}}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\Sigma_{\mathrm{II}}. \tag{10}$$

As a consequence, except for the atmospheric angle θ_{23} and Jarlskog J_{CP}, the expressions for the rephasing invariants and mixing angles are the same as those obtained in the previous case. For the measured value of θ_{13}, the other two angles are $\sin^2\theta_{12}\simeq\frac{1}{2}$ and $\sin^2\theta_{23}\simeq 0$. The measured values of the three mixing angles θ_{12}, θ_{13} and θ_{23} cannot be accommodated simultaneously. There is only one *CP* transformation matrix with three texture zeros that we will denote as type IV *CP* transformation matrix and whose explicit form is

$$\text{type IV:}\ \mathbf{X}=\mathrm{diag}(e^{i\alpha},e^{i\beta},e^{i\gamma}), \tag{11}$$

with $\alpha,\ \beta,\ \gamma\in\mathbb{R}$. Its Takagi factorization reads as $\Sigma=\mathrm{diag}(e^{i\alpha/2},e^{i\beta/2},e^{i\gamma/2})$. The lepton mixing parameters are given by $\sin^2\theta_{13}=\sin^2\theta_2$, $\sin^2\theta_{12}=\sin^2\theta_3$, $\sin^2\theta_{23}=\sin^2\theta_1$, and $\sin\delta_{CP}=\sin\alpha_{21}=\sin\alpha_{31}=0$. We see that all three *CP* phases are zero or π in this case while the lepton mixing angles are unconstrained.

The *CP* transformation matrices with two texture zeros and the corresponding Σ matrices are given in Table II. The explicit form of the lepton matrix can be obtained from Eq. (2) and the mixing parameters are given in the fourth column of Table II. The type V *CP* transformation with two texture zeros is exactly the generalized $\mu-\tau$ reflection symmetry [7]. The atmospheric angle and the Dirac *CP* phase δ_{CP} only depend on two parameters θ_1 and Θ, as they are related by

TABLE II. The CP transformation matrices with two texture zeros and the corresponding Σ matrices. The predictions for the lepton mixing parameters are displayed in the last column. The lepton mixing matrices for types VI and VII are related through the exchange of the second and third rows.

Two texture zeros			
Type	$\mathbf{X}$	Σ	Mixing parameters
V	$\begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta}\cos\Theta & ie^{i(\beta+\gamma)/2}\sin\Theta \\ 0 & ie^{i(\beta+\gamma)/2}\sin\Theta & e^{i\gamma}\cos\Theta \end{pmatrix}$	$\begin{pmatrix} e^{\frac{i\alpha}{2}} & 0 & 0 \\ 0 & e^{\frac{i\beta}{2}}\cos\frac{\Theta}{2} & ie^{\frac{i\beta}{2}}\sin\frac{\Theta}{2} \\ 0 & ie^{\frac{i\gamma}{2}}\sin\frac{\Theta}{2} & e^{\frac{i\gamma}{2}}\cos\frac{\Theta}{2} \end{pmatrix}$	$\sin^2\theta_{13}=\sin^2\theta_2, \sin^2\theta_{12}=\sin^2\theta_3$, $\sin^2\theta_{23}=\frac{1}{2}(1-\cos\Theta\cos 2\theta_1)$, $\sin\delta_{CP}=\frac{\mathrm{sign}(\sin\theta_2\sin 2\theta_3)\sin\Theta}{\sqrt{1-\cos^2\Theta\cos^2 2\theta_1}}$, $\alpha_{21}=k_1\pi$, $\alpha'_{31}=k_2\pi$
VI	$\begin{pmatrix} e^{i\alpha}\cos\Theta & 0 & ie^{i(\alpha+\gamma)/2}\sin\Theta \\ 0 & e^{i\beta} & 0 \\ ie^{(\alpha+\gamma)/2}\sin\Theta & 0 & e^{i\gamma}\cos\Theta \end{pmatrix}$	$\begin{pmatrix} 0 & e^{\frac{i\alpha}{2}}\cos\frac{\Theta}{2} & ie^{\frac{i\alpha}{2}}\sin\frac{\Theta}{2} \\ e^{\frac{i\beta}{2}} & 0 & 0 \\ 0 & ie^{\frac{i\gamma}{2}}\sin\frac{\Theta}{2} & e^{\frac{i\gamma}{2}}\cos\frac{\Theta}{2} \end{pmatrix}$	See Eqs. (13), (15)
VII	$\begin{pmatrix} e^{i\alpha}\cos\Theta & ie^{i(\alpha+\beta)/2}\sin\Theta & 0 \\ ie^{i(\alpha+\beta)/2}\sin\Theta & e^{i\beta}\cos\Theta & 0 \\ 0 & 0 & e^{i\gamma} \end{pmatrix}$	$\begin{pmatrix} 0 & e^{\frac{i\alpha}{2}}\cos\frac{\Theta}{2} & ie^{\frac{i\alpha}{2}}\sin\frac{\Theta}{2} \\ 0 & ie^{\frac{i\beta}{2}}\sin\frac{\Theta}{2} & e^{\frac{i\beta}{2}}\cos\frac{\Theta}{2} \\ e^{\frac{i\gamma}{2}} & 0 & 0 \end{pmatrix}$	...

$$\sin^2\delta_{CP}\sin^2 2\theta_{23}=\sin^2\Theta. \tag{12}$$

Moreover the Majorana phases α_{21} and α_{31} take on CP–conserving values. The phenomenological implications of this interesting mixing pattern for neutrinoless double beta decay as well as conventional neutrino oscillations have been discussed in detail in Ref. [7].

For the $\mathbf{X}$ matrix with two texture zeros of type VI, the mixing angles are given by

$$\begin{aligned}
\sin^2\theta_{13} &= \frac{1}{2}(1-\cos\Theta\cos 2\theta_1)\cos^2\theta_2,\\
\sin^2\theta_{23} &= \frac{2\sin^2\theta_2}{2-(1-\cos\Theta\cos 2\theta_1)\cos^2\theta_2},\\
\sin^2\theta_{12} &= \frac{(1+\cos\Theta\cos 2\theta_1)\cos^2\theta_3+\sin\theta_2[(1-\cos\Theta\cos 2\theta_1)\sin\theta_2\sin^2\theta_3-\cos\Theta\sin 2\theta_1\sin 2\theta_3]}{2-(1-\cos\Theta\cos 2\theta_1)\cos^2\theta_2}.
\end{aligned} \tag{13}$$

We easily see that the following sum rules are fulfilled:

$$\begin{aligned}
\sin^2\theta_{23}\cos^2\theta_{13} &= \sin^2\theta_2,\\
\frac{\cos^2\theta_{13}\cos^2\theta_{23}-\sin^2\theta_{13}}{1-\sin^2\theta_{23}\cos^2\theta_{13}} &= \cos\Theta\cos 2\theta_1,\\
\sin^2\theta_{12}(1-\sin^2\theta_{23}\cos^2\theta_{13})-\cos^2\theta_{23}\cos^2\theta_3-\sin^2\theta_{13}\sin^2\theta_{23}\sin^2\theta_3 &= \pm\frac{1}{2}\sqrt{1-\sin^2\theta_{23}\cos^2\theta_{13}}\left(\frac{\cos 2\theta_{13}}{\cos^2\theta_{13}}-\sin^2\theta_{23}\right)\\
&\quad\times\tan 2\theta_1\sin 2\theta_3.
\end{aligned} \tag{14}$$

The Jarlskog-like invariant and the invariants associated with the Majorana phases take the following form:

$$\begin{aligned}
J_{CP} &= -\frac{1}{4}\sin\Theta\sin\theta_2\cos^2\theta_2\sin 2\theta_3,\\
I_1 &= \frac{1}{4}(-1)^{k_1+1}\sin\Theta\sin\theta_2[2\cos\Theta\sin 2\theta_1\sin\theta_2\cos 2\theta_3+(\cos^2\theta_2+\cos\Theta\cos 2\theta_1(1+\sin^2\theta_2))\sin 2\theta_3],\\
I_2 &= \frac{1}{2}(-1)^{k_2}\sin\Theta\cos^2\theta_2\sin\theta_3[(1-\cos\Theta\cos 2\theta_1)\sin\theta_2\cos\theta_3+\cos\Theta\sin 2\theta_1\sin\theta_3].
\end{aligned} \tag{15}$$

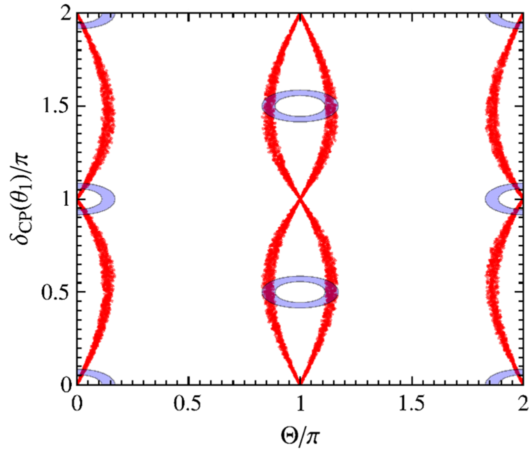

FIG. 1. The correlations between the parameters Θ, δ_{CP} (red points) and θ_1 (blue rings) for the case of type VI *CP* transformation, where the three lepton mixing angles are required to lie in the experimentally preferred 3σ regions.

Notice that the three phases α, β, and γ can be absorbed into the charged lepton fields; therefore, they are unphysical and hence do not appear in the above mixing parameters. The correlations between the mixing parameters Θ, δ_{CP}, and θ_1 are displayed in Fig. 1. The blue rings are allowed parameter regions of Θ and θ_1 predicted by Eq. (14), where θ_{13} and θ_{23} are compatible with the preferred values from the neutrino oscillations global fit in [42] at the 3σ level. The red points are obtained from a random numerical scan over the parameters Θ and $\theta_{1,2,3}$. We can see the allowed regions of Θ from the two different approaches are compatible with each other.

The lepton mixing matrix corresponding to the type VII $\mathbf{X}$ matrix with two texture zeros is related to the one of type VI by the exchange of the second and the third rows. As a result, the lepton mixing parameters for the type VII case are the same as those of type VI except that θ_{23} and δ_{CP} become $\pi/2-\theta_{23}$ and $\pi+\delta_{CP}$, respectively. A detailed analysis of theoretical predictions of the *CP* transformation matrices with two texture zeros will be given in the next section.

Finally, the *CP* transformation matrices with one texture zero and the corresponding Σ matrices are given in Table III. The corresponding expressions for the mixing matrix are obtained by using Eq. (2) and the mixing parameters can be extracted straightforwardly. For the type VIII *CP* transformation with one zero element, the mixing angles are given by

$$\begin{aligned}
\sin^2\theta_{13} &= \frac{1}{8}(3-\cos 2\theta_1 - 2\cos^2\theta_1\cos 2\theta_2),\\
\sin^2\theta_{12} &= \frac{4\cos^2\theta_1\cos^2\theta_3 - 2\sin 2\theta_1\sin\theta_2\sin 2\theta_3 + 4(\sin^2\theta_1\sin^2\theta_2+\cos^2\theta_2)\sin^2\theta_3}{5+\cos 2\theta_1 + 2\cos^2\theta_1\cos 2\theta_2},\\
\sin^2\theta_{23} &= \frac{4\cos^2\Theta(\sin^2\theta_2+\sin^2\theta_1\cos^2\theta_2) + 8\cos^2\theta_1\cos^2\theta_2\sin^2\Theta + 2\sqrt{2}\sin 2\theta_1\cos^2\theta_2\sin 2\Theta}{5+\cos 2\theta_1 + 2\cos^2\theta_1\cos 2\theta_2}.
\end{aligned} \tag{16}$$

The smallness of θ_{13} requires $\theta_1 \simeq 0, \pi$, and $\theta_2 \simeq 0, \pi$. Consequently, the solar mixing angles would be $\sin^2\theta_{12} \simeq \frac{1}{2}$ which is outside the experimentally allowed ranges [42]. As a result, the measured values of θ_{12} and θ_{13} cannot be accommodated simultaneously in this case, and this mixing pattern is not viable. This observation is indeed confirmed in our numerical analysis.

TABLE III. The *CP* transformation matrices with one texture zeros and the corresponding Σ matrices with $c_\Theta \equiv \cos\Theta$ and $s_\Theta \equiv \sin\Theta$. The lepton mixing matrix is obtained through Eq. (2). The lepton mixing matrix for type IX and type X are related by the exchange of the second and third rows.

One texture zero		
Type	$\mathbf{X}$	Σ
VIII	$\begin{pmatrix} 0 & e^{i\alpha}c_\Theta & e^{i\beta}s_\Theta \\ e^{i\alpha}c_\Theta & e^{i\gamma}s_\Theta^2 & -e^{i(-\alpha+\beta+\gamma)}c_\Theta s_\Theta \\ e^{i\beta}s_\Theta & -e^{i(-\alpha+\beta+\gamma)}c_\Theta s_\Theta & e^{i(-2\alpha+2\beta+\gamma)}c_\Theta^2 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} -ie^{i(\alpha-\gamma/2)} & e^{i(\alpha-\gamma/2)} & 0 \\ ie^{i\gamma/2}c_\Theta & e^{i\gamma/2}c_\Theta & \sqrt{2}e^{i\gamma/2}s_\Theta \\ ie^{i(-\alpha+\beta+\gamma/2)}s_\Theta & e^{i(-\alpha+\beta+\gamma/2)}s_\Theta & -\sqrt{2}e^{i(-\alpha+\beta+\gamma/2)}c_\Theta \end{pmatrix}$
IX	$\begin{pmatrix} e^{i\alpha}c_\Theta^2 & e^{i\beta}s_\Theta & e^{i\gamma}c_\Theta s_\Theta \\ e^{i\beta}s_\Theta & 0 & -e^{i(-\alpha+\beta+\gamma)}c_\Theta \\ e^{i\gamma}c_\Theta s_\Theta & -e^{i(-\alpha+\beta+\gamma)}c_\Theta & e^{i(-\alpha+2\gamma)}s_\Theta^2 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} -ie^{i\alpha/2}s_\Theta & -e^{i\alpha/2}s_\Theta & \sqrt{2}e^{i\alpha/2}c_\Theta \\ ie^{i(\beta-\alpha/2)} & -e^{i(\beta-\alpha/2)} & 0 \\ ie^{i(\gamma-\alpha/2)}c_\Theta & e^{i(\gamma-\alpha/2)}c_\Theta & \sqrt{2}e^{i(\gamma-\alpha/2)}s_\Theta \end{pmatrix}$
X	$\begin{pmatrix} e^{i\alpha}c_\Theta^2 & e^{i\gamma}c_\Theta s_\Theta & e^{i\beta}s_\Theta \\ e^{i\gamma}c_\Theta s_\Theta & e^{i(-\alpha+2\gamma)}s_\Theta^2 & -e^{i(-\alpha+\beta+\gamma)}c_\Theta \\ e^{i\beta}s_\Theta & -e^{i(-\alpha+\beta+\gamma)}c_\Theta & 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} -ie^{i\alpha/2}s_\Theta & -e^{i\alpha/2}s_\Theta & \sqrt{2}e^{i\alpha/2}c_\Theta \\ ie^{i(\gamma-\alpha/2)}c_\Theta & e^{i(\gamma-\alpha/2)}c_\Theta & \sqrt{2}e^{i(\gamma-\alpha/2)}s_\Theta \\ ie^{i(\beta-\alpha/2)} & -e^{i(\beta-\alpha/2)} & 0 \end{pmatrix}$

For the type IX remnant CP transformation with one zero, we find the lepton mixing angles are

$$\begin{aligned}
\sin^2\theta_{13} &= \frac{1}{4}[1+\cos^2\theta_1\cos^2\theta_2+(3\cos^2\theta_1\cos^2\theta_2-1)\cos 2\Theta-\sqrt{2}\sin 2\theta_1\cos^2\theta_2\sin 2\Theta],\\
\sin^2\theta_{12} &= 2[(\sqrt{2}\cos\Theta(\sin\theta_1\cos\theta_3+\cos\theta_1\sin\theta_2\sin\theta_3)+\sin\Theta(\cos\theta_1\cos\theta_3-\sin\theta_1\sin\theta_2\sin\theta_3))^2\\
&\quad+\cos^2\theta_2\sin^2\theta_3\sin^2\Theta]/[3-\cos^2\theta_1\cos^2\theta_2+(1-3\cos^2\theta_1\cos^2\theta_2)\cos 2\Theta+\sqrt{2}\sin 2\theta_1\cos^2\theta_2\sin 2\Theta],\\
\sin^2\theta_{23} &= \frac{2(\sin^2\theta_2+\sin^2\theta_1\cos^2\theta_2)}{3-\cos^2\theta_1\cos^2\theta_2+(1-3\cos^2\theta_1\cos^2\theta_2)\cos 2\Theta+\sqrt{2}\sin 2\theta_1\cos^2\theta_2\sin 2\Theta}. \qquad (17)
\end{aligned}$$

Also, we see that the mixing angles are correlated as follows:

$$\begin{aligned}
&2\cos^2\theta_{13}\cos^2\theta_{23}=1-\cos^2\theta_1\cos^2\theta_2,\\
&\cos 2\theta_{13}=2\cos^2\theta_{13}\cos^2\theta_{23}\cos^2\Theta+(1-2\cos^2\theta_{13}\cos^2\theta_{23})(\sqrt{2}\tan\theta_1\sin 2\Theta-\cos 2\Theta). \qquad (18)
\end{aligned}$$

The Jarlskog-like invariant associated with the Dirac CP phase has the form

$$\begin{aligned}
J_{CP} &= \frac{1}{16}\cos\theta_1\cos\theta_2[4\sin 2\theta_1\sin\theta_2\cos 2\theta_3-\sin 2\theta_3(1-3\cos 2\theta_1+2\cos^2\theta_1\cos 2\theta_2)]\cos 2\Theta\\
&\quad+\frac{1}{128\sqrt{2}}[(12\cos^2\theta_1\sin\theta_1\cos 3\theta_2+(\sin\theta_1-15\sin 3\theta_1)\cos\theta_2)\sin 2\theta_3\\
&\quad+4(\cos\theta_1+3\cos 3\theta_1)\sin 2\theta_2\cos 2\theta_3]\sin 2\Theta, \qquad (19)
\end{aligned}$$

while the invariants associated with the Majorana phases are

$$\begin{aligned}
I_1 &= \left\{\frac{1}{64\sqrt{2}}[4(3\cos 3\theta_1-7\cos\theta_1)\sin\theta_2\cos 2\theta_3+(31\sin\theta_1-9\sin 3\theta_1+6\cos\theta_1\sin 2\theta_1\cos 2\theta_2)\sin 2\theta_3]\right.\\
&\quad\times\cos\theta_2\cos\Theta\sin\Theta+\frac{1}{64\sqrt{2}}[(9\sin\theta_1-15\sin 3\theta_1+10\cos\theta_1\sin 2\theta_1\cos 2\theta_2)\sin 2\theta_3\\
&\quad-4(\cos\theta_1-5\cos 3\theta_1)\sin\theta_2\cos 2\theta_3]\cos\theta_2\cos 3\Theta\sin\Theta+\frac{1}{128}[4(5\sin\theta_1-3\sin 3\theta_1)\sin\theta_2\cos 2\theta_3\\
&\quad+(13\cos\theta_1-9\cos 3\theta_1-2(5-3\cos 2\theta_1)\cos\theta_1\cos 2\theta_2)\sin 2\theta_3]\cos\theta_2\sin^2\Theta\\
&\quad+\frac{1}{128}[(9\cos\theta_1-21\cos 3\theta_1-2(1-7\cos 2\theta_1)\cos\theta_1\cos 2\theta_2)\sin 2\theta_3\\
&\quad\left.+4(\sin\theta_1-7\sin 3\theta_1)\sin\theta_2\cos 2\theta_3]\cos\theta_2\sin\Theta\sin 3\Theta\right\}\cos(k_1\pi), \qquad (20)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \left\{\frac{1}{128\sqrt{2}}[((31\sin\theta_1+15\sin 3\theta_1)\cos\theta_2-12\cos^2\theta_1\sin\theta_1\cos 3\theta_2)\sin 2\theta_3\right.\\
&\quad-8(6+(1+3\cos 2\theta_1)\cos 2\theta_3)\cos\theta_1\sin 2\theta_2]\cos\Theta\sin\Theta\\
&\quad+\frac{1}{128\sqrt{2}}[((9\sin\theta_1+25\sin 3\theta_1)\cos\theta_2-20\cos^2\theta_1\sin\theta_1\cos 3\theta_2)\sin 2\theta_3\\
&\quad-8(2-(1-5\cos 2\theta_1)\cos 2\theta_3)\cos\theta_1\sin 2\theta_2]\cos 3\Theta\sin\Theta\\
&\quad+\frac{1}{256}[[2(-1+15\cos 2\theta_1)\cos\theta_2+(10-6\cos 2\theta_1)\cos 3\theta_2]\cos\theta_1\sin 2\theta_3\\
&\quad+8(2+(1+3\cos 2\theta_1)\cos 2\theta_3)\sin\theta_1\sin 2\theta_2]\sin^2\Theta\\
&\quad+\frac{1}{128}[((-13+35\cos 2\theta_1)\cos\theta_2+(1-7\cos 2\theta_1)\cos 3\theta_2)\cos\theta_1\sin 2\theta_3\\
&\quad\left.+4(2+(5+7\cos 2\theta_1)\cos 2\theta_3)\sin\theta_1\sin 2\theta_2]\sin\Theta\sin 3\Theta\right\}\cos(k_2\pi). \qquad (21)
\end{aligned}$$

For the type X remnant CP transformation with one zero, we find that the resulting lepton mixing matrix is related with the previous case through the exchange of the second and third rows. The expressions for the reactor and solar mixing angles, as well as for the I_1 and I_2 invariants, are the same as those of Eqs. (17), (20), (21), while the atmospheric mixing angle $\sin^2\theta_{23}$ becomes $1-\sin^2\theta_{23}$. The Dirac CP phase δ_{CP} becomes $\pi+\delta_{CP}$ so that the overall sign of the Jarlskog invariant is reversed.

III. DEMOCRATIC CP SYMMETRY AS AN EXAMPLE WITHOUT ZERO ELEMENTS

If no entry of the residual CP transformation $\mathbf{X}$ vanishes, the explicit form of $\mathbf{X}$ cannot be fixed uniquely. For illustration, in this section we shall study a particular CP symmetry whose elements have the same absolute value. That is to say, the absolute value of each element of $\mathbf{X}$ is equal to $1/\sqrt{3}$. In what follows it will be dubbed as democratic CP symmetry. In this case, the most general form of $\mathbf{X}$ can be written as

$$\mathbf{X}=\frac{1}{\sqrt{3}}\begin{pmatrix} e^{i\alpha} & e^{i(\frac{\alpha+\beta}{2}+\beta_3)} & e^{i(\frac{\alpha+\gamma}{2}+\beta_2)} \\ e^{i(\frac{\alpha+\beta}{2}+\beta_3)} & e^{i\beta} & e^{i(\frac{\beta+\gamma}{2}+\beta_1)} \\ e^{i(\frac{\alpha+\gamma}{2}+\beta_2)} & e^{i(\frac{\beta+\gamma}{2}+\beta_1)} & e^{i\gamma} \end{pmatrix}. \tag{22}$$

The unitary condition of $\mathbf{X}$ implies that β_1, β_2, and β_3 should satisfy the following equalities: $e^{i\beta_1}+e^{-i\beta_1}+e^{i(\beta_2-\beta_3)}=0$, $e^{i\beta_2}+e^{-i\beta_2}+e^{i(\beta_3-\beta_1)}=0$, and $e^{i\beta_3}+e^{-i\beta_3}+e^{i(\beta_1-\beta_2)}=0$. It can be easily checked that these equations have four pairs of solutions,

$$\begin{aligned}
&\beta_1=\beta_2=\beta_3=\pm\frac{2\pi}{3}, \qquad \mathbf{X}_1^{\pm}=\frac{1}{\sqrt{3}}\begin{pmatrix} e^{i\alpha} & e^{i(\frac{\alpha+\beta}{2}\pm\frac{2\pi}{3})} & e^{i(\frac{\alpha+\gamma}{2}\pm\frac{2\pi}{3})} \\ e^{i(\frac{\alpha+\beta}{2}\pm\frac{2\pi}{3})} & e^{i\beta} & e^{i(\frac{\beta+\gamma}{2}\pm\frac{2\pi}{3})} \\ e^{i(\frac{\alpha+\gamma}{2}\pm\frac{2\pi}{3})} & e^{i(\frac{\beta+\gamma}{2}\pm\frac{2\pi}{3})} & e^{i\gamma} \end{pmatrix},\\
&\beta_1=\pm\frac{2\pi}{3}, \qquad \beta_2=\beta_3=\mp\frac{\pi}{3}, \qquad \mathbf{X}_2^{\pm}=\frac{1}{\sqrt{3}}\begin{pmatrix} e^{i\alpha} & e^{i(\frac{\alpha+\beta}{2}\mp\frac{\pi}{3})} & e^{i(\frac{\alpha+\gamma}{2}\mp\frac{\pi}{3})} \\ e^{i(\frac{\alpha+\beta}{2}\mp\frac{\pi}{3})} & e^{i\beta} & e^{i(\frac{\beta+\gamma}{2}\pm\frac{2\pi}{3})} \\ e^{i(\frac{\alpha+\gamma}{2}\mp\frac{\pi}{3})} & e^{i(\frac{\beta+\gamma}{2}\pm\frac{2\pi}{3})} & e^{i\gamma} \end{pmatrix},\\
&\beta_2=\pm\frac{2\pi}{3}, \qquad \beta_1=\beta_3=\mp\frac{\pi}{3}, \qquad \mathbf{X}_3^{\pm}=\frac{1}{\sqrt{3}}\begin{pmatrix} e^{i\alpha} & e^{i(\frac{\alpha+\beta}{2}\mp\frac{\pi}{3})} & e^{i(\frac{\alpha+\gamma}{2}\pm\frac{2\pi}{3})} \\ e^{i(\frac{\alpha+\beta}{2}\mp\frac{\pi}{3})} & e^{i\beta} & e^{i(\frac{\beta+\gamma}{2}\mp\frac{\pi}{3})} \\ e^{i(\frac{\alpha+\gamma}{2}\pm\frac{2\pi}{3})} & e^{i(\frac{\beta+\gamma}{2}\mp\frac{\pi}{3})} & e^{i\gamma} \end{pmatrix},\\
&\beta_3=\pm\frac{2\pi}{3}, \qquad \beta_1=\beta_2=\mp\frac{\pi}{3}, \qquad \mathbf{X}_4^{\pm}=\frac{1}{\sqrt{3}}\begin{pmatrix} e^{i\alpha} & e^{i(\frac{\alpha+\beta}{2}\pm\frac{2\pi}{3})} & e^{i(\frac{\alpha+\gamma}{2}\mp\frac{\pi}{3})} \\ e^{i(\frac{\alpha+\beta}{2}\pm\frac{2\pi}{3})} & e^{i\beta} & e^{i(\frac{\beta+\gamma}{2}\mp\frac{\pi}{3})} \\ e^{i(\frac{\alpha+\gamma}{2}\mp\frac{\pi}{3})} & e^{i(\frac{\beta+\gamma}{2}\mp\frac{\pi}{3})} & e^{i\gamma} \end{pmatrix}.
\end{aligned} \tag{23}$$

We can see that the four admissible CP transformations $\mathbf{X}_1^{\pm}$, $\mathbf{X}_2^{\pm}$, $\mathbf{X}_3^{\pm}$, and $\mathbf{X}_4^{\pm}$ are related to each other as follows: $\mathbf{X}_1^{\pm}=\mathrm{diag}(1,-1,-1)\mathbf{X}_2^{\pm}\mathrm{diag}(1,-1,-1)=\mathrm{diag}(-1,1,-1)\mathbf{X}_3^{\pm}\mathrm{diag}(-1,1,-1)=\mathrm{diag}(-1,-1,1)\mathbf{X}_4^{\pm}\times\mathrm{diag}(-1,-1,1)$. Therefore the Takagi factorization matrix $\Sigma_i^{\pm}$ for $\mathbf{X}_i^{\pm}$ ($i=1,2,3,4$) are related with each other as well, $\Sigma_1^{\pm}=\mathrm{diag}(1,-1,-1)\Sigma_2^{\pm}=\mathrm{diag}(-1,1,-1)\Sigma_3^{\pm}=\mathrm{diag}(-1,-1,1)\Sigma_4^{\pm}$. As a result, we conclude that the four CP transformations $\mathbf{X}_1^{\pm}$, $\mathbf{X}_2^{\pm}$, $\mathbf{X}_3^{\pm}$, and $\mathbf{X}_4^{\pm}$ give rise to the same lepton mixing matrix up to a phase factor which can be absorbed by redefining the charged lepton fields. Furthermore, it can be easily checked that Σ_i^{+} and Σ_i^{-} can be related by $\Sigma_i^{-}=\mathrm{diag}(e^{i\alpha},e^{i\beta},e^{i\gamma})\Sigma_i^{+*}$. Therefore the predicted PMNS matrix by $\mathbf{X}_i^{+}$ and $\mathbf{X}_i^{-}$ are complex conjugate of each other up to the phase factor $\mathrm{diag}(e^{i\alpha},e^{i\beta},e^{i\gamma})$ which can also be absorbed by the charged leptons. Hence it is sufficient to only discuss the CP transformation $\mathbf{X}_{\mathrm{XI}}\equiv\mathbf{X}_1^{+}$ which corresponds to $\beta_1=\beta_2=\beta_3=2\pi/3$ with

$$\mathbf{X}_{\mathrm{XI}}=\frac{1}{\sqrt{3}}\begin{pmatrix} e^{i\alpha} & e^{i(\frac{\alpha+\beta}{2}+\frac{2\pi}{3})} & e^{i(\frac{\alpha+\gamma}{2}+\frac{2\pi}{3})} \\ e^{i(\frac{\alpha+\beta}{2}+\frac{2\pi}{3})} & e^{i\beta} & e^{i(\frac{\beta+\gamma}{2}+\frac{2\pi}{3})} \\ e^{i(\frac{\alpha+\gamma}{2}+\frac{2\pi}{3})} & e^{i(\frac{\beta+\gamma}{2}+\frac{2\pi}{3})} & e^{i\gamma} \end{pmatrix}. \tag{24}$$

The corresponding Takagi factorization and the prediction for the PMNS matrix can be straightforwardly obtained

$$\begin{aligned}
\Sigma_{\mathrm{XI}}&=\mathrm{diag}(e^{i\alpha},e^{i\beta},e^{i\gamma})e^{-\frac{i\pi}{12}}\begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{pmatrix}\mathrm{diag}(1,e^{i\pi/3},1),\\
\mathbf{U}&=\Sigma_{\mathrm{XI}}\mathbf{O}_{3\times3}\hat{\mathbf{X}}_{\nu}^{-1/2}.
\end{aligned} \tag{25}$$

We can read out the lepton mixing angles as

$$\begin{aligned}
\sin^2\theta_{13} &= \frac{1}{6}[4\sin^2\theta_2 + \sqrt{2}\sin 2\theta_2 \sin\theta_1 + 2\sin^2\theta_1\cos^2\theta_2],\\
\sin^2\theta_{12} &= \{[\sin\theta_1(\sqrt{2}\sin 2\theta_2 - 2\sin\theta_1\sin^2\theta_2) - 4\cos^2\theta_2]\sin^2\theta_3 - 2\cos^2\theta_1\cos^2\theta_3 + [\sin 2\theta_1\sin\theta_2\\
&\quad - \sqrt{2}\cos\theta_1\cos\theta_2]\sin 2\theta_3\}/(4\sin^2\theta_2 + \sqrt{2}\sin 2\theta_2\sin\theta_1 + 2\sin^2\theta_1\cos^2\theta_2 - 6),\\
\sin^2\theta_{23} &= \frac{\sin 2\theta_2(\sqrt{2}\sin\theta_1 + 2\sqrt{3}\cos\theta_1) - 2\sin^2\theta_2 - \cos^2\theta_2(\sqrt{6}\sin 2\theta_1 + \cos 2\theta_1 + 5)}{2(4\sin^2\theta_2 + \sqrt{2}\sin 2\theta_2\sin\theta_1 + 2\sin^2\theta_1\cos^2\theta_2 - 6)}.
\end{aligned} \tag{26}$$

For the CP invariants we get

$$\begin{aligned}
J_{CP} &= \frac{-1}{48\sqrt{2}}\{[4\sqrt{2}\sin 2\theta_2\sin^2\theta_1\cos\theta_1 + 4\sin 2\theta_1\cos 2\theta_2]\cos 2\theta_3 + [5\sin\theta_2\sin^2\theta_1\\
&\quad + \sqrt{2}\sin\theta_1(5\cos^2\theta_1 - 1)\cos\theta_2 + (3\cos^2\theta_1 + 1)\sin 3\theta_2 + \sqrt{2}\sin^3\theta_1\cos 3\theta_2]\sin 2\theta_3\},\\
I_1 &= \frac{(-1)^{k_1}}{12\sqrt{3}}\cos\theta_1\cos\theta_2\{[3\sqrt{2}\sin^2\theta_1 - 2\sin 2\theta_2\sin\theta_1 + \sqrt{2}(\cos^2\theta_1 + 1)\cos 2\theta_2]\sin 2\theta_3\\
&\quad + 2[2\cos\theta_1\cos\theta_2 - \sqrt{2}\sin 2\theta_1\sin\theta_2]\cos 2\theta_3\},\\
I_2 &= \frac{(-1)^{k_2}}{6\sqrt{3}}(\sin\theta_1\cos\theta_3 + \sin\theta_2\sin\theta_3\cos\theta_1)\{[2\sin\theta_1\cos 2\theta_2 + \sqrt{2}\sin 2\theta_2(\cos^2\theta_1 + 1)]\cos\theta_3\\
&\quad - [\sqrt{2}\sin 2\theta_1\cos\theta_2 + 2\sin\theta_2\cos\theta_1]\sin\theta_3\}.
\end{aligned} \tag{27}$$

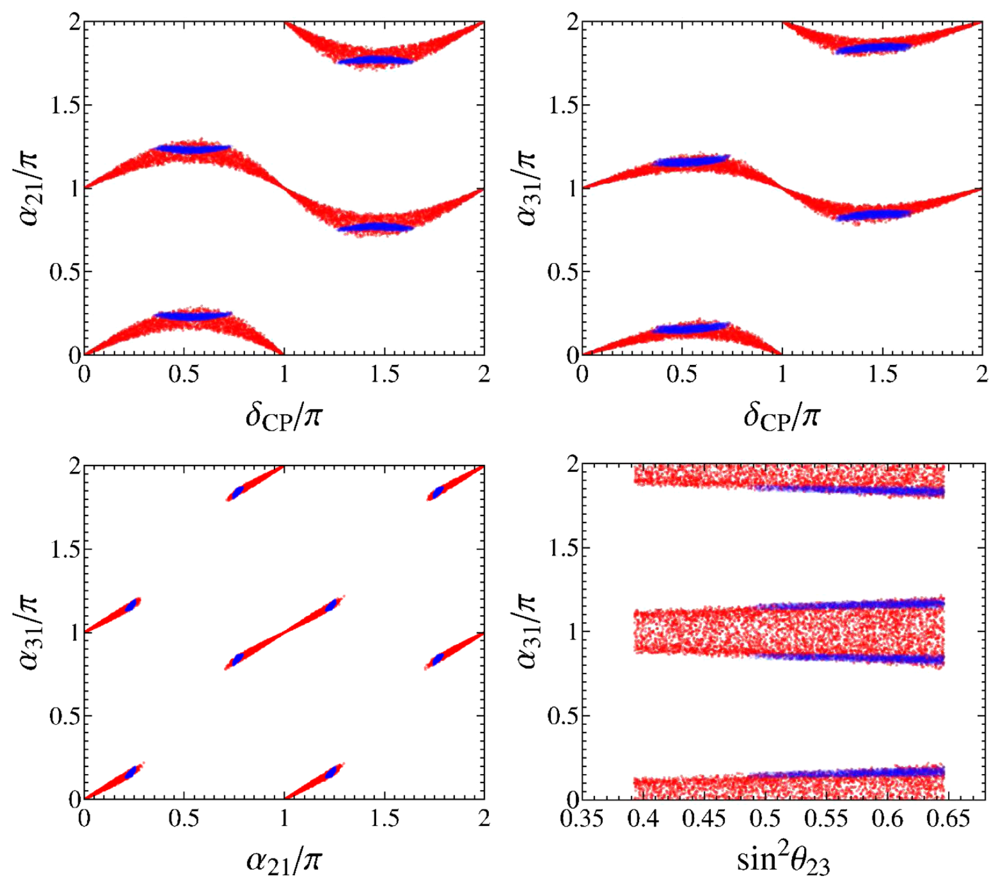

FIG. 2. The correlation between the mixing parameters predicted in the case of type VI residual CP transformation. All the parameters $\theta_{1,2,3}$ and Θ are taken to be random numbers in the interval of $[0, 2\pi]$. The three mixing angles θ_{12}, θ_{13}, and θ_{23} are required to be within the experimentally preferred 3σ intervals [42]. The blue points indicate the numerical results obtained by fixing $\Theta = \frac{\pi}{7}$ as a benchmark example.

IV. NUMERICAL ANALYSIS

Summarizing the above discussion, we see that types I, IV, V, VI, VII, IX, X and residual CP transformations with zero elements can accommodate the current experimental neutrino oscillation data [42]. In all these cases, the three mixing angles θ_{12}, θ_{13}, and θ_{23} as well as three CP phases δ_{CP}, α_{21}, and α_{31} are found to depend on just four free independent parameters Θ, θ_1, θ_2, and θ_3, where Θ characterizes the shape of the residual CP transformations. This characterizes the degree of predictivity of our present framework. For example, the type I CP transformation corresponds to the widely studied $\mu-\tau$ reflection, and leads to $\theta_{23}=45°$, $\delta_{CP}=\pm 90°$ and $\alpha_{21}, \alpha_{31}=0$ or π while the solar and reactor mixing angles are not constrained. On the other hand, the type IV CP transformation with three texture zeros is diagonal, and corresponds to the conventional CP transformation. As expected, in this case all three CP phases are predicted to vanish. The CP transformation of type V is the same as the generalized $\mu-\tau$ reflection which has been discussed by us in Ref. [7]. For the case of $\Theta=\pi/2$, our generalized $\mu-\tau$ reflection reduces to the standard $\mu-\tau$ reflection symmetry. This would provide an interesting new starting point for model building if either θ_{23} or δ_{CP} were established to be nonmaximal by future neutrino oscillation experiments.

As we already mentioned, the lepton mixing matrices for the case of two–zero texture type VI and type VII are related by the exchange of the second and the third rows. The mixing angles and CP invariants are given in Eqs. (13) and (15), respectively. In order to visualize the theoretical predictions in a more clear way, we perform a numerical analysis where the free parameters $\theta_{1,2,3}$ and the CP parameter Θ are scanned over the range of $[0, 2\pi]$, while the mixing parameters are calculated for each point, retaining only points that agree at 3σ level with experimentally determined mixing angles [42]. The correlations between the mixing parameters and distributions of the mixing parameters are plotted in Figs. 2 and 3.

One sees that the three CP phases are strongly correlated with each other, the Majorana phase α_{21} around $\pi/5$, $4\pi/5$, $6\pi/5$, and $9\pi/5$ is preferred, and the Majorana phase α_{31} around $3\pi/20$, $17\pi/20$, $23\pi/23$, and $37\pi/20$ is favored. If we set a value to the CP parameter Θ then the explicit form of the CP transformation $\mathbf{X}$ is fixed, so that definite predictions for the CP phases are obtained. As examples, we consider the case that the parameter Θ takes some specific values $\frac{\pi}{9}$, $\frac{2\pi}{17}$, $\frac{\pi}{8}$, $\frac{2\pi}{15}$, and $\frac{\pi}{7}$, the values of the parameters

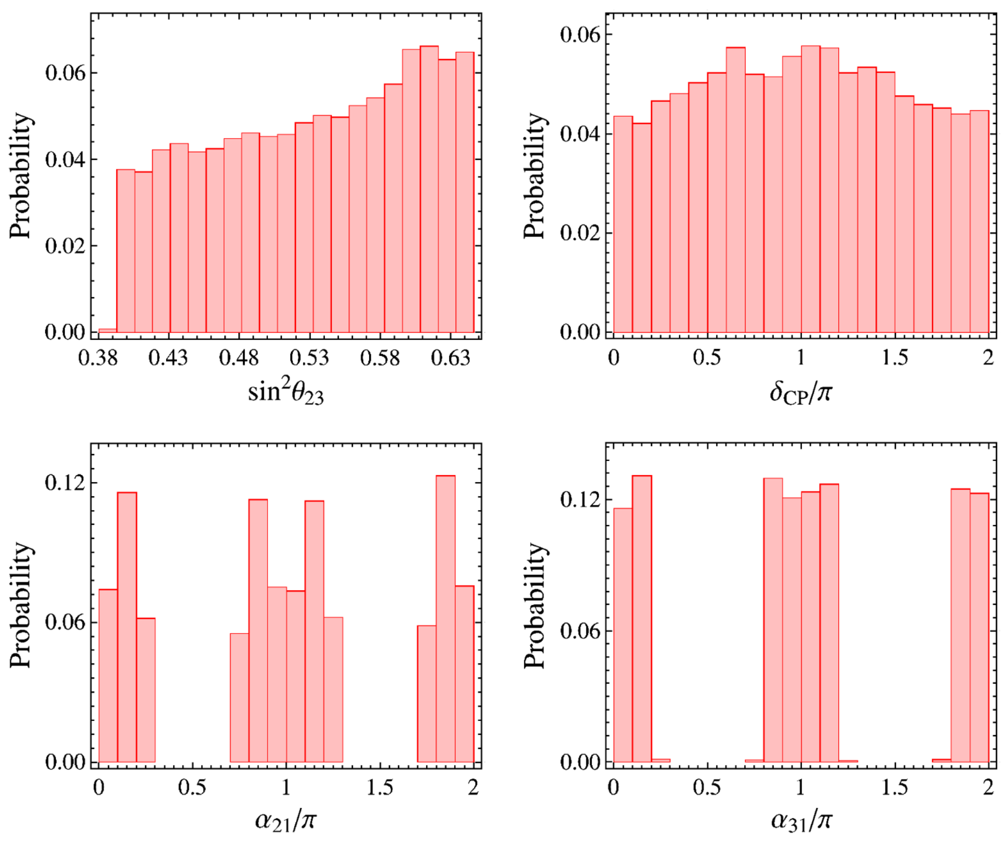

FIG. 3. The probability distribution of the atmospheric mixing angle $\sin^2\theta_{23}$, and the CP-violating phases δ_{CP}, α_{21}, and α_{31} predicted for the case of type- I residual CP transformation. All the parameters $\theta_{1,2,3}$ and Θ are taken to be random numbers in the range $[0, 2\pi]$, and the three lepton mixing angles are required to be compatible with experimental data at 3σ level [42].

TABLE IV. The predictions for the Dirac and Majorana CP phases in the case of type VI residual CP transformation. The parameter Θ is set to the representative values of $\pi/9$, $2\pi/17$, $\pi/8$, $2\pi/15$, and $\pi/7$. The parameters θ_1, θ_2, and θ_3 are fixed by the requirement of reproducing the best fit values of the three lepton mixing angles for NH neutrino mass spectrum [42].

Type VI, NH						
Θ	θ_1	θ_2	θ_3	δ_{CP}	α_{21} (mod π)	α_{31} (mod π)
$\frac{\pi}{9}$	8.413°	48.167° or 131.833°	49.591°	304.258°	147.618°	159.573°
	8.413°	48.167° or 131.833°	118.010°	136.032°	32.382°	23.728°
	171.587°	48.167° or 131.833°	61.990°	223.968°	147.618°	156.272°
	171.587°	48.167° or 131.833°	130.409°	55.742°	32.382°	20.427°
$\frac{2\pi}{17}$	7.645°	48.167° or 131.833°	50.218°	299.581°	145.744°	158.210°
	7.645°	48.167° or 131.833°	118.541°	132.071°	34.256°	24.956°
	172.355°	48.167° or 131.833°	61.459°	227.929°	145.744°	155.044°
	172.355°	48.167° or 131.833°	129.782°	60.419°	34.256°	21.790°
$\frac{\pi}{8}$	6.601°	48.167° or 131.833°	51.057°	293.664°	143.642°	156.616°
	6.601°	48.167° or 131.833°	119.265°	126.967°	36.358°	26.276°
	173.399°	48.167° or 131.833°	60.735°	233.033°	143.642°	153.724°
	173.399°	48.167° or 131.833°	128.943°	66.336°	36.358°	23.384°
$\frac{2\pi}{15}$	5.036°	48.167° or 131.833°	52.287°	285.502°	141.268°	154.690°
	5.036°	48.167° or 131.833°	120.355°	119.731°	38.732°	27.651°
	174.964°	48.167° or 131.833°	59.645°	240.269°	141.268°	152.349°
	174.964°	48.167° or 131.833°	127.713°	74.498°	38.732°	25.310°
$\frac{\pi}{7}$	1.655°	48.167° or 131.833°	54.848°	269.598°	138.569°	152.045°
	1.655°	48.167° or 131.833°	122.745°	104.895°	41.431°	28.774°
	178.345°	48.167° or 131.833°	57.255°	255.105°	138.569°	151.226°
	178.345°	48.167° or 131.833°	125.152°	90.402°	41.431°	27.955°

TABLE V. The predictions for the Dirac and Majorana CP phases in the case of type VI residual CP transformation. The parameter Θ is set to the representative values of $\pi/9$, $2\pi/17$, $\pi/8$, $2\pi/15$, and $\pi/7$. The parameters θ_1, θ_2, and θ_3 are fixed by the requirement of reproducing the best fit values of the three lepton mixing angles for IH neutrino mass spectrum [42].

Type VI, IH						
Θ	θ_1	θ_2	θ_3	δ_{CP}	α_{21} (mod π)	α_{31} (mod π)
$\frac{\pi}{9}$	8.769°	48.442° or 131.558°	49.303°	305.441°	147.458°	159.550°
	8.769°	48.442° or 131.558°	117.713°	137.276°	32.542°	23.926°
	171.231°	48.442° or 131.558°	62.287°	222.724°	147.458°	156.074°
	171.231°	48.442° or 131.558°	130.697°	54.559°	32.542°	20.450°
$\frac{2\pi}{17}$	8.037°	48.442° or 131.558°	49.907°	300.956°	145.574°	158.190°
	8.037°	48.442° or 131.558°	118.220°	133.513°	34.426°	25.172°
	171.963°	48.442° or 131.558°	61.780°	226.487°	145.574°	154.828°
	171.963°	48.442° or 131.558°	130.093°	59.044°	34.426°	21.810°
$\frac{\pi}{8}$	7.054°	48.442° or 131.558°	50.706°	295.348°	143.460°	156.605°
	7.054°	48.442° or 131.558°	118.903°	128.721°	36.540°	26.517°
	172.946°	48.442° or 131.558°	61.097°	231.279°	143.460°	153.483°
	172.946°	48.442° or 131.558°	129.294°	64.652°	36.540°	23.395°
$\frac{2\pi}{15}$	5.621°	48.442° or 131.558°	51.846°	287.814°	141.074°	154.704°
	5.621°	48.442° or 131.558°	119.902°	122.120°	38.926°	27.936°
	174.379°	48.442° or 131.558°	60.098°	237.880°	141.074°	152.064°
	174.379°	48.442° or 131.558°	128.154°	72.186°	38.926°	25.296°
$\frac{\pi}{7}$	3.005°	48.442° or 131.558°	53.862°	275.327°	138.360°	152.241°
	3.005°	48.442° or 131.558°	121.745°	110.708°	41.640°	29.262°
	176.995°	48.442° or 131.558°	58.255°	249.292°	138.360°	150.738°
	176.995°	48.442° or 131.558°	126.138°	84.673°	41.640°	27.759°

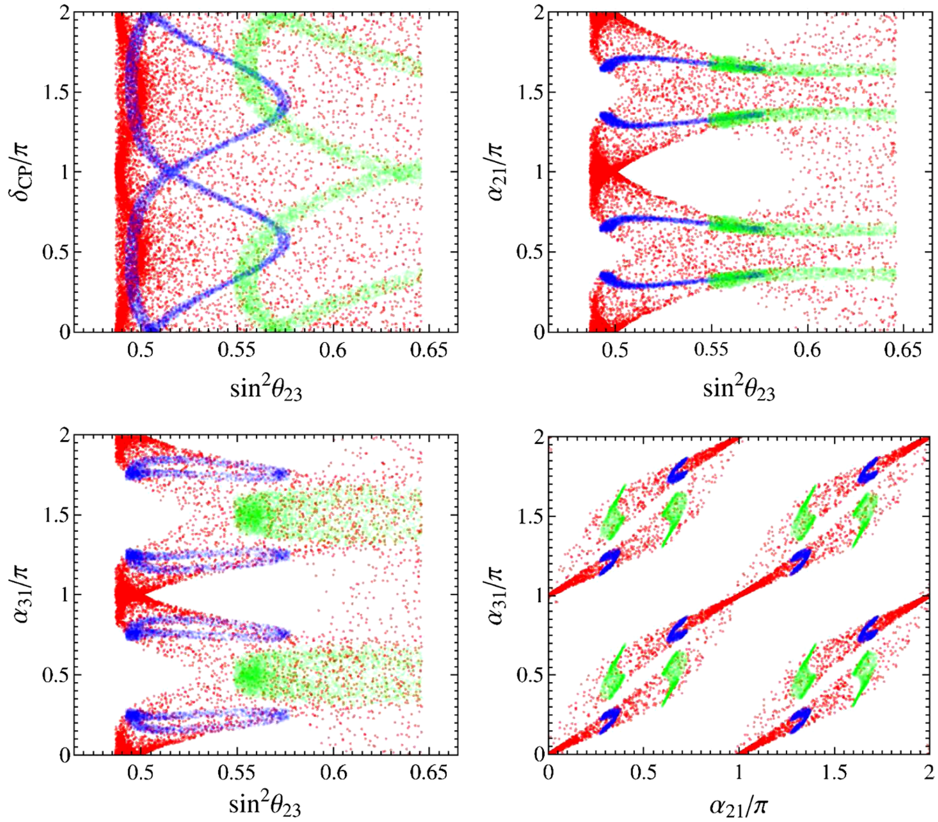

FIG. 4. The correlation between the mixing parameters predicted in the case of type-X residual CP transformation. All the parameters $\theta_{1,2,3}$ and Θ are varied randomly in the range $[0, 2\pi]$. The three mixing angles θ_{12}, θ_{13}, and θ_{23} are required to be within the 3σ allowed ranges [42]. The blue (green) points are obtained by fixing $\Theta = \frac{2\pi}{17}\left(\frac{2\pi}{9}\right)$ as a benchmark example.

θ_1, θ_2, and θ_3 are determined by the experimental best fit values of the lepton mixing angles from [42]. As a consequence, the lepton mixing matrix is fully fixed up to the factor $\hat{X}^{-1/2}$, and the values of the CP violating phases can be predicted, as are shown in Table IV for normal hierarchy (NH) and Table V for inverted hierarchy (IH). We can see that different values of the Dirac CP phase δ_{CP} can be achieved. Note in particular that, for certain values of Θ, θ_1, θ_2, and θ_3, the magnitude of δ_{CP} can be quite close to 270° which is weakly favored by present data [43].

The lepton mixing matrices for the one zero textures type IX and X differ by a permutation of the second and the third rows. The expressions for the mixing angles and CP invariants are given in Eqs. (17), (19), (20), (21). The numerical results for the correlation among the mixing parameters and probability distributions of the mixing parameters are displayed in Figs. 4 and 5. The strong correlations between different CP phases emerge once the value of the parameter Θ is fixed.

We see that the atmospheric angle can only be in the range $0.487 \leq \sin^2\theta_{23} \leq 0.646$, with θ_{23} close to maximal mixing favored for the type X texture, while both Majorana phases α_{21} and α_{31} tend to be close to 0, π, and 2π. There appears to be no preferred δ_{CP} phase within the viable parameter space. Furthermore, we study some concrete benchmark cases in which the parameters Θ take on certain representative values. The value of the parameters θ_1, θ_2, and θ_3 are fixed by the best fit value of the lepton mixing angles. In this way the CP violating phases can be predicted as listed in Tables VI and VII for NH and IH mass spectrums, respectively. Future long baseline facilities DUNE [44], LBNO [45], T2HK [46] can bring us increased precision on the Dirac phase δ_{CP} and θ_{23}. If θ_{23} was measured to lie outside the range [0.487, 0.646], the present proposal would be disfavored.

We perform a numerical analysis by treating the parameters $\theta_{1,2,3}$ as random real numbers scanned over the range $[0, 2\pi]$, with the three mixing angles calculated for each point in the parameter space. Subsequently only points which simultaneously are compatible with experimental data [42] are retained and from these points the CP violating phases are calculated. The predicted distributions of the lepton mixing parameters are shown in Fig. 6.

Because no specific values of θ_{12}, and θ_{13} are favored within 3σ, they are not shown in the figure. We see that the atmospheric mixing angle θ_{23} can be either the first octant or the second octant. Regarding the CP phases, there appears to be a slight preference for $\delta_{CP} \sim \pi/2$ and

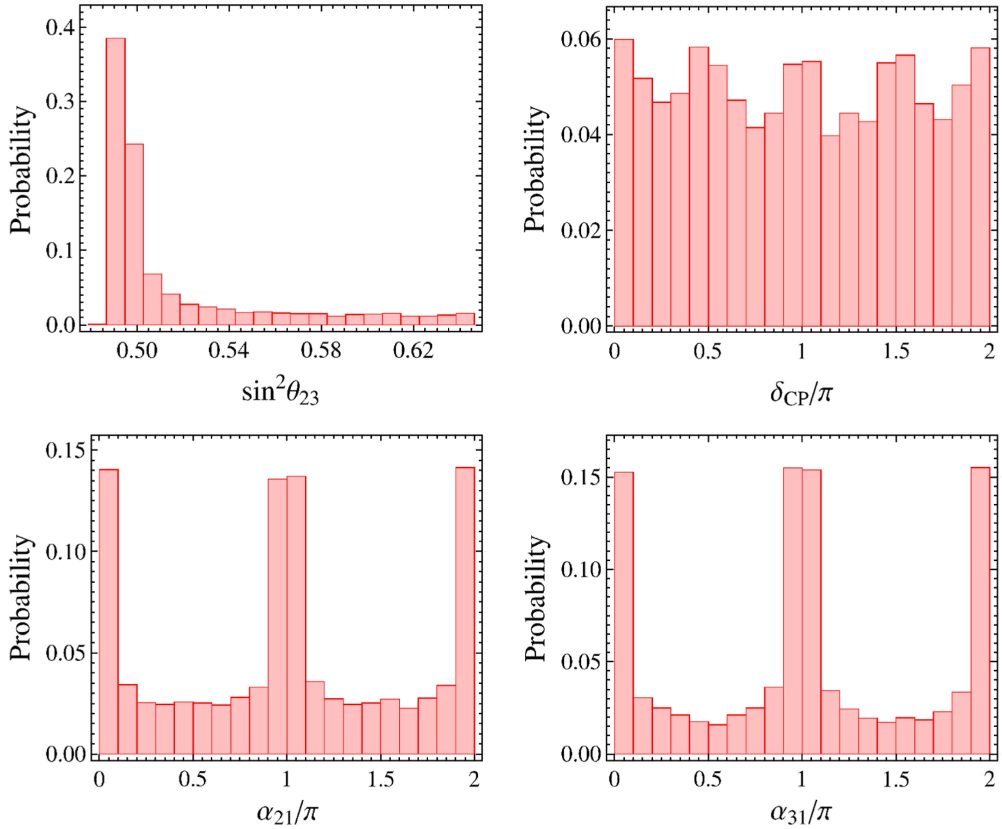

FIG. 5. The probability distribution of the lepton mixing parameters $\sin^2\theta_{23}$, δ_{CP}, α_{21}, and α_{31} predicted for the case of type-X residual CP transformation. All the parameters $\theta_{1,2,3}$ and Θ are taken to be random numbers in the range $[0, 2\pi]$, and the three lepton mixing angles are required to be compatible with experimental data at 3σ level [42].

$\delta_{CP} \sim 3\pi/2$, and the Majorana phase α_{21} around $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$ are favored while the values of α_{31} around 0 and π are preferred. For certain values of $\theta_{1,2,3}$, the best fit values of the mixing angles can be reproduced, and the corresponding predictions for CP phases are listed in Table VIII.

V. PHENOMENOLOGICAL IMPLICATIONS

Implications of the generalized $\mu-\tau$ reflection symmetry have already been discussed in Ref. [7]. In this section, we shall consider the phenomenological implications of the residual CP transformations as we have classified above in Tables I, II, and III, focusing on the case of "neutrino appearance" oscillation experiments and neutrinoless double beta decay. The cosmological implications for leptogenesis will be studied as well.

A. *CP* violation in conventional neutrino oscillations

The existence of leptonic CP violation would manifest itself as the differences in the oscillation probabilities involving neutrinos and antineutrinos in vacuum [47]:

$$\begin{aligned}\Delta P_{\alpha\beta} &\equiv P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)\\ &= -16 J_{\alpha\beta} \sin\Delta_{21} \sin\Delta_{23} \sin\Delta_{31},\end{aligned}$$

where we have adopted standard definitions $\Delta_{kj} \equiv \Delta m^2_{kj} L/(4E)$ and $\Delta m^2_{kj} = m^2_k - m^2_j$, L is the baseline and E stands for the energy of neutrino beam. The Jarlskog invariant is identified as

$$J_{\alpha\beta} = \Im(U_{\alpha 1} U_{\beta 2} U^*_{\alpha 2} U^*_{\beta 1}) = \pm J_{CP}, \tag{28}$$

where the positive (negative) sign holds for (anti)cyclic permutation of the flavor indices e, μ, and τ. For the oscillation between electron and muon neutrinos, the transition probability of $\nu_\mu \to \nu_e$ in vacuum is given by [47]

$$\begin{aligned}P(\nu_\mu \to \nu_e) \simeq{}& P_{\rm atm} + 2\sqrt{P_{\rm atm}}\sqrt{P_{\rm sol}}\cos(\Delta_{32} + \delta_{CP})\\ &+ P_{\rm sol},\end{aligned} \tag{29}$$

where $\sqrt{P_{\rm atm}} = \sin\theta_{23} \sin 2\theta_{13} \sin\Delta_{31}$, and $\sqrt{P_{\rm sol}} = \cos\theta_{23}\cos\theta_{13}\sin 2\theta_{12}\sin\Delta_{21}$. As a result, the oscillation probability asymmetry between neutrinos and antineutrinos in vacuum is of the form

TABLE VI. The predictions for the Dirac and Majorana *CP* phases in the case of type X residual *CP* transformation. The parameter Θ is set to the representative values of $\pi/9$. $2\pi/17$, $2\pi/9$, and $3\pi/13$. The parameters θ_1, θ_2, and θ_3 are fixed by the requirement of reproducing the best fit values of the three lepton mixing angles for NH neutrino mass spectrum [42].

Type X, NH						
Θ	θ_1	θ_2	θ_3	δ_{CP}	α_{21} (mod π)	α_{31} (mod π)
$\frac{\pi}{9}$	66.743°	4.29.0°	57.595°	108.232°	60.800°	40.840°
	66.743°	4.290°	123.826°	264.695°	119.200°	136.503°
	66.743°	175.710°	57.595°	251.768°	119.200°	139.160°
	66.743°	175.710°	123.826°	95.305°	60.800°	43.497°
$\frac{2\pi}{17}$	66.215°	12.491°	58.980°	122.397°	62.920°	39.600°
	66.215°	12.491°	124.976°	277.881°	117.080°	132.164°
	66.215°	167.509°	58.980°	237.603°	117.080°	140.400°
	66.215°	167.509°	124.976°	82.119°	62.920°	47.836°
$\frac{2\pi}{9}$	66.060°	13.984°	60.881°	245.654°	117.249°	76.422°
	66.060°	13.984°	114.766°	5.228°	62.751°	78.747°
	66.060°	166.016°	60.881°	114.346°	62.751°	103.578°
	66.060°	166.016°	114.766°	354.772°	117.249°	101.253°
$\frac{3\pi}{13}$	66.499°	9.096°	62.706°	268.830°	124.491°	87.338°
	66.499°	9.096°	113.502°	20.806°	55.509°	74.823°
	66.499°	170.904°	62.706°	91.170°	55.509°	92.662°
	66.499°	170.904°	113.502°	339.194°	124.491°	105.177°

TABLE VII. The predictions for the Dirac and Majorana *CP* phases in the case of type X residual *CP* transformation. The parameter Θ is set to the representative values of $2\pi/17$, $\pi/8$, $2\pi/15$, $3\pi/13$, and $4\pi/17$. The parameters θ_1, θ_2, and θ_3 are fixed by the requirement of reproducing the best fit values of the three lepton mixing angles for IH neutrino mass spectrum [42].

Type X, IH						
Θ	θ_1	θ_2	θ_3	δ_{CP}	α_{21} (mod π)	α_{31} (mod π)
$\frac{2\pi}{17}$	65.802°	6.854°	58.218°	113.294°	64.070°	42.230°
	65.802°	6.854°	124.081°	268.239°	115.930°	133.106°
	65.802°	173.146°	58.218°	246.706°	115.930°	137.770°
	65.802°	173.146°	124.081°	91.761°	64.070°	46.894°
$\frac{\pi}{8}$	65.244°	13.626°	59.379°	126.150°	66.506°	41.401°
	65.244°	13.626°	124.950°	279.935°	113.494°	128.680°
	65.244°	166.374°	59.379°	233.850°	113.494°	138.599°
	65.244°	166.374°	124.950°	80.065°	66.506°	51.320°
$\frac{2\pi}{15}$	64.729°	17.577°	59.985°	136.166°	69.451°	41.655°
	64.729°	17.577°	125.180°	288.512°	110.549°	124.550°
	64.729°	162.423°	59.985°	223.834°	110.549°	138.345°
	64.729°	162.423°	125.180°	71.488°	69.451°	55.450°
$\frac{3\pi}{13}$	65.352°	12.624°	61.950°	253.073°	122.680°	82.112°
	65.352°	12.624°	113.570°	7.050°	57.320°	73.409°
	65.352°	167.376°	61.950°	106.927°	57.320°	97.888°
	65.352°	167.376°	113.570°	352.950°	122.680°	106.591°
$\frac{4\pi}{17}$	65.587°	10.048°	63.026°	265.635°	126.593°	88.250°
	65.587°	10.048°	112.818°	15.195°	53.407°	71.217°
	65.587°	169.952°	63.026°	94.365°	53.407°	91.750°
	65.587°	169.952°	112.818°	344.805°	126.593°	108.783°

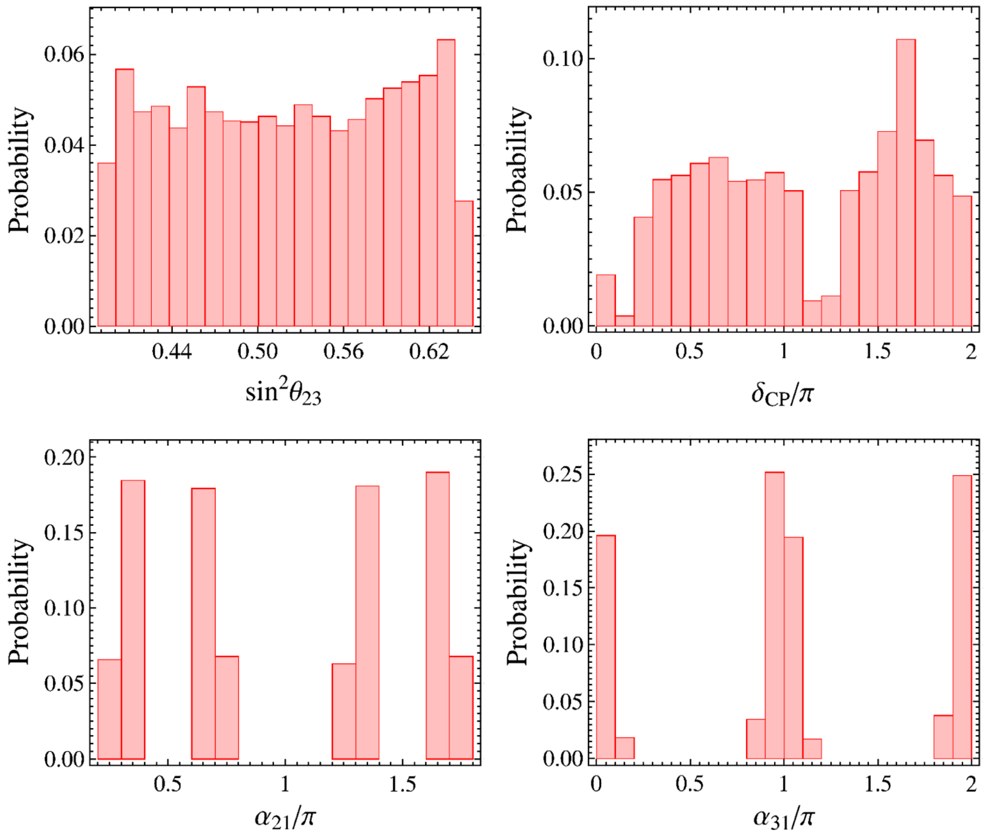

FIG. 6. The probability distribution of the lepton mixing parameters $\sin^2\theta_{23}$, δ_{CP}, α_{21}, and α_{31} predicted for the case of democratic CP symmetry. The parameters $\theta_{1,2,3}$ are taken to be random numbers in the range $[0, 2\pi]$, and the three lepton mixing angles are required to be compatible with experimental data at 3σ level [42].

$$A_{\mu e} = \frac{P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e)}{P(\nu_\mu \to \nu_e) + P(\bar{\nu}_\mu \to \bar{\nu}_e)} = -\frac{2\sqrt{P_{\text{atm}}}\sqrt{P_{\text{sol}}}\sin\Delta_{32}\sin\delta_{CP}}{P_{\text{atm}} + 2\sqrt{P_{\text{atm}}}\sqrt{P_{\text{sol}}}\cos\Delta_{32}\cos\delta_{CP} + P_{\text{sol}}}. \quad (30)$$

In order to accurately describe realistic long baseline neutrino oscillation experiments such as T2K, NOνA, or the DUNE proposal, it is important to include the matter effect associated with neutrino propagation inside the Earth. Indeed the latter could induce a fake CP violation effect. In this case the expressions for $\sqrt{P_{\text{atm}}}$ and $\sqrt{P_{\text{sol}}}$ in matter take the form [47]

$$\sqrt{P_{\text{atm}}} = \sin\theta_{23}\sin 2\theta_{13}\frac{\sin(\Delta_{31} - aL)}{\Delta_{31} - aL}\Delta_{31},$$
$$\sqrt{P_{\text{sol}}} = \cos\theta_{23}\sin 2\theta_{12}\frac{\sin(aL)}{aL}\Delta_{21}, \quad (31)$$

TABLE VIII. The predictions for the Dirac and Majorana CP phases for the democratic residual CP transformation. The values of θ_1, θ_2, and θ_3 are fixed by the requirement of accommodating the best fit values of the three lepton mixing angles [42].

Type XI						
	θ_1	θ_2	θ_3	δ_{CP}	α_{21} (mod π)	α_{31} (mod π)
NH	13.015°	2.354°	127.433°	65.801°	59.150°	172.892°
	13.015°	2.354°	179.849°	309.873°	120.850°	177.814°
	176.556°	9.122°	2.170°	172.589°	123.407°	6.067°
	176.556°	9.122°	53.464°	285.771°	56.593°	175.842°
IH	13.424°	2.113°	127.462°	64.401°	59.347°	172.721°
	13.424°	2.113°	179.962°	308.266°	120.653°	177.239°
	176.886°	9.403°	2.220°	173.932°	123.420°	6.171°
	176.886°	9.403°	53.508°	287.100°	56.580°	175.919°

where $a = G_F N_e/\sqrt{2}$, G_F is the Fermi constant and N_e is the density of electrons. The parameter a is approximately equal to $(3500\ \mathrm{km})^{-1}$ for $\rho Y_e = 1.5\,\mathrm{g/cm^3}$, where Y_e is the electron fraction (the electron number per atomic number) and we use the mean value $Y_e \simeq 0.5$ in the Earth [47]. Therefore the oscillation probability $P(\nu_\mu \to \nu_e)$ in matter to leading order is given by

$$\begin{aligned} P(\nu_\mu \to \nu_e) &= \sin^2\theta_{23}\sin^2 2\theta_{13}\frac{\sin^2(\Delta_{31}-aL)}{(\Delta_{31}-aL)^2}\Delta_{31}^2 \\ &\quad + \cos^2\theta_{23}\sin^2 2\theta_{12}\frac{\sin^2(aL)}{(aL)^2}\Delta_{21}^2 \\ &\quad + \sin 2\theta_{23}\sin 2\theta_{13}\sin 2\theta_{12}\frac{\sin(\Delta_{31}-aL)}{\Delta_{31}-aL} \\ &\quad \times \Delta_{31}\frac{\sin(aL)}{aL}\Delta_{21}\cos(\Delta_{32}+\delta_{CP}). \qquad (32) \end{aligned}$$

The antineutrino oscillation probability $P(\bar{\nu}_\mu \to \bar{\nu}_e)$ can be obtained from $P(\nu_\mu \to \nu_e)$ by $\delta_{CP} \to -\delta_{CP}$ and $a \to -a$,

$$\begin{aligned} P(\bar{\nu}_\mu \to \bar{\nu}_e) &= \sin^2\theta_{23}\sin^2 2\theta_{13}\frac{\sin^2(\Delta_{31}+aL)}{(\Delta_{31}+aL)^2}\Delta_{31}^2 \\ &\quad + \cos^2\theta_{23}\sin^2 2\theta_{12}\frac{\sin^2(aL)}{(aL)^2}\Delta_{21}^2 \\ &\quad + \sin 2\theta_{23}\sin 2\theta_{13}\sin 2\theta_{12}\frac{\sin(\Delta_{31}+aL)}{\Delta_{31}+aL} \\ &\quad \times \Delta_{31}\frac{\sin(aL)}{aL}\Delta_{21}\cos(\Delta_{32}-\delta_{CP}). \qquad (33) \end{aligned}$$

In Fig. 7 we show the $\nu_\mu \to \nu_e$ transition probability as well as the neutrino-antineutrino asymmetry in matter, when the residual *CP* transformation matrix $\mathbf{X}$ is assumed to be type VI. In this figure we require the oscillation mixing angles lie within their currently allowed 3σ region [42].

In Figs. 8 and 9 we show the corresponding behavior of the transition probability $P(\nu_\mu \to \nu_e)$ in terms of neutrino energy E, as well as of the *CP* parameter Θ describing our approach, when the *CP* symmetry matrix $\mathbf{X}$ is type VI, for baseline values 295 and 810 km, which correspond to the current T2K and NOνA experiments, respectively. One sees that the allowed values of the *CP* parameter Θ describing our approach are quite restricted.

B. Neutrinoless double decay

The rare decay $(A,Z) \to (A,Z+2) + e^- + e^-$ is the lepton number violating process "par excellence." Its observation would establish the Majorana nature of neutrinos irrespective of their underlying mass generation mechanism [48,49]. Within the simplest "long-range" light neutrino exchange mechanism its amplitude is sensitive to the Majorana phases. As discussed in [19] the most convenient parametrization of the lepton mixing matrix for the description of neutrinoless double decay is the fully symmetric one [6]. However, instead of using the "symmetrical" description as in [7], here we stick to the PDG form [18].

Up to relatively uncertain nuclear matrix elements [50] as well as experimental factors [51,52] the decay amplitude is proportional to the effective mass parameter

$$\begin{aligned} |m_{ee}| &= |m_1\cos^2\theta_{12}\cos^2\theta_{13} + m_2\sin^2\theta_{12}\cos^2\theta_{13}e^{i\alpha_{21}} \\ &\quad + m_3\sin^2\theta_{13}e^{i(\alpha_{31}-2\delta_{CP})}|. \qquad (34) \end{aligned}$$

Notice that both Majorana phases α_{21} and α_{31} can be shifted by π by the matrix $\hat{X}^{-1/2}$ in Eq. (3). Under the transformation $k_1 \to k_1 + 1$ $(k_2 \to k_2 + 1)$, we have

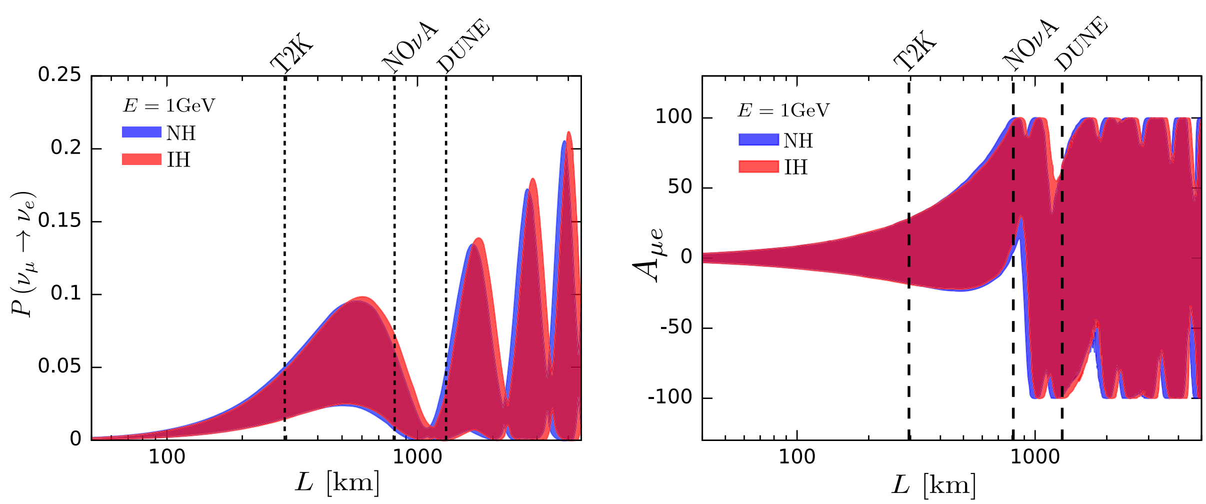

FIG. 7. In the left panel we show the $\nu_\mu \to \nu_e$ transition probability in matter for a neutrino energy of $E = 1$ GeV. The right panel displays the neutrino-anti-neutrino asymmetry $\mathcal{A}_{\mu e}$ in matter. The oscillation parameters are taken within their currently allowed 3σ regions [42]. The plot corresponds to type VI residual *CP* symmetry.

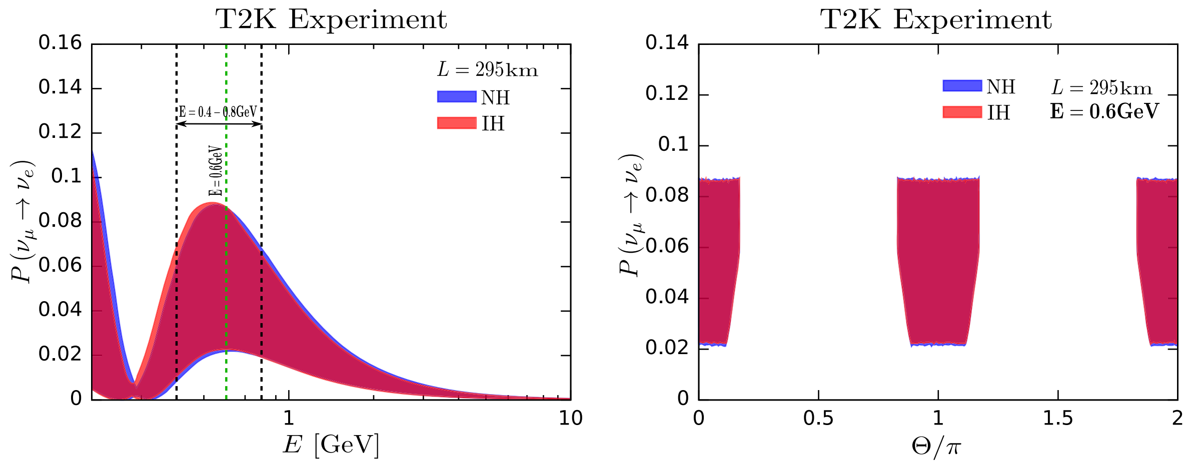

FIG. 8. The transition probability $P(\nu_\mu \to \nu_e)$ at a baseline of 295 km which corresponds to the T2K experiment. The neutrino oscillation parameters are taken within the currently allowed 3σ regions [42]. The plot corresponds to the case of type VI residual *CP* symmetry.

$\alpha_{21} \to \alpha_{21} + \pi$ $(\alpha_{31} \to \alpha_{31} + \pi)$. Hence without loss of generality, we can focus on four different cases $(k_1, k_2) = (0, 0)$, (0,1), (1,0), and (1,1).

We illustrate our results for the effective $0\nu\beta\beta$ mass parameter $|m_{ee}|$ by considering the type I *CP* symmetric scheme, given in Fig. 10, the type VI case, given in Fig. 11, as well as the results for type IX given in Fig. 12. The residual *CP* transformations of type V, type VII, and type X do not lead to new results for the effective mass $|m_{ee}|$ since θ_{23} is not involved in Eq. (34). The experimental errors on the mass-squared splittings are not considered, and the best fit values from [42] are used with $\Delta m^2_{21} = 7.60 \times 10^{-5}$ eV^2 and $|\Delta m^2_{31}| = 2.48 \times 10^{-3}$ eV^2 for normal ordering and $|\Delta m^2_{31}| = 2.38 \times 10^{-3}$ eV^2 for inverted ordering. Notice that the red and blue dashed lines (for inverted and normal neutrino mass ordering, respectively) denote the regions allowed at 3σ level by current neutrino oscillation data [42] for a generic model, without any special residual *CP* symmetry. For comparison we display the results for various *CP* symmetric cases. We also indicate the disfavored band associated with the most stringent upper bound

$$|m_{ee}| < 0.120 \text{ eV} \tag{35}$$

which follows from the EXO-200 experiment [53,54] in combination with results from the first phase of the KamLAND-ZEN experiment [55]. On the other hand, within the "base ΛCDM model," the cosmological upper limit on the total light neutrino mass from the latest Planck result is given by

$$\sum_i m_i < 0.230 \text{ eV}$$

at the 95% confidence level [56].

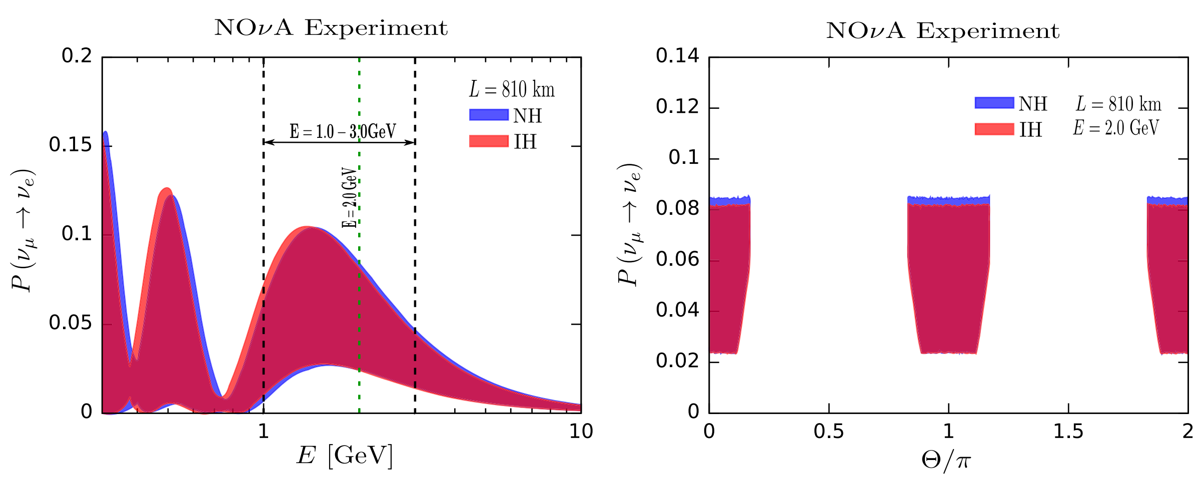

FIG. 9. The transition probability $P(\nu_\mu \to \nu_e)$ for type VI residual *CP* symmetry case at a baseline of 810 km which corresponds to the NOνA experiment. The neutrino oscillation parameters are taken within their currently allowed 3σ regions [42].

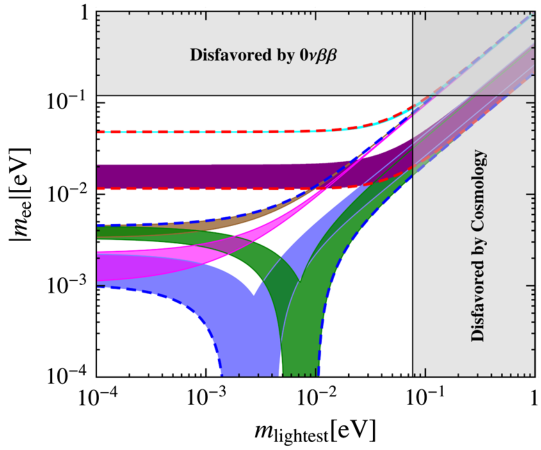

FIG. 10. The effective mass $|m_{ee}|$ describing neutrinoless double beta decay for type I *CP* symmetry. Both reactor and solar mixing angles are required to be within the experimental 3σ interval [42], while the atmospheric mixing angle is predicted to be maximal with $\theta_{23} = \pi/4$. For the inverted neutrino mass ordering, the cyan region corresponds to $(k_1, k_2) = (0, 0), (0, 1)$, and the purple area corresponds to $(k_1, k_2) = (1, 0)$, (1,1). For the normal ordering, the brown, magenta, blue, and dark green regions correspond to $(k_1, k_2) = (0, 0)$, (0,1), (1,0), and (1,1), respectively. The red and blue dashed lines indicate the 3σ boundaries allowed by current neutrino oscillation data [42] for inverted and normal neutrino mass ordering, respectively. For comparison we show also the most stringent upper bound from $0\nu\beta\beta$ searches, as well as current Planck sensitivity.

The results of this section are summarized in Figs. 10, 11, and 12 corresponding to the schemes based on type I, type VI, and type IX remnant *CP* symmetries, respectively. They clearly show that the attainable values for the effective mass parameter $|m_{ee}|$ cover more restrictive ranges than those expected in generic, non-*CP* symmetric schemes.

In particular, as illustrated by the green regions in the lower panels of Figs. 11 and 12, the generalized *CP* symmetry assumption may prevent the destructive interference amongst individual neutrino contributions. This leads to lower bounds for the $0\nu\beta\beta$ decay rates even for normal hierarchical neutrino mass spectra. This behavior is reminiscent of situations already encountered in the framework of specific flavor symmetry based models [57–60].

In the case of Σ matrix type XI, the predictions for the effective mass of the neutrinoless double beta decay are shown in Fig. 13. As one can read off from this figure, the effective mass $|m_{ee}|$ is around 0.026 eV or 0.040 eV for IH neutrino mass spectrum, which are within the sensitivity of planned $0\nu\beta\beta$ decay experiments. In the case of the NH spectrum, the value of $|m_{ee}|$ is bounded from below: $|m_{ee}| \geq 0.00065$ eV for $(k_1, k_2) = (0, 0)$, $|m_{ee}| \geq 0.00056$ eV for $(k_1, k_2) = (0, 1)$, and $|m_{ee}| \geq 0.0011$ eV for $(k_1, k_2) = (1, 0), (1, 1)$.

C. Leptogenesis

The origin of matter-antimatter asymmetry in the Universe is a puzzling and unexplained phenomenon. Although Sakharov discovered that *CP* violation is a necessary condition for explaining the matter-antimatter asymmetry of the Universe [61], the observed quark *CP* violation is insufficient for this purpose [62]. The idea that the generation of a primordial lepton-antilepton asymmetry early in the history of the Universe induces the observed cosmological baryon asymmetry has been studied extensively in recent years [63–65]. Although such "leptogenesis paradigm" is closely related with *CP* violation, the relation between generalized *CP* symmetries and leptogenesis remains, to a large extent, an open research topic. The scenario of two residual *CP* transformations preserved by the neutrino mass term has been analyzed in Ref. [66], and all the leptogenesis *CP* asymmetries are found to only depend on one single real parameter besides the light neutrino masses and the parameters characterizing the remnant *CP* transformations. In this section, we shall study the phenomenological consequence for leptogenesis if there is only one residual *CP* transformation in the neutrino sector. We shall consider the classical scenario of leptogenesis from the lightest right-handed (RH) neutrino N_1 decay in the type I seesaw model. In the RH neutrino and charged lepton mass basis, the type I seesaw Lagrangian can be written as

$$-\mathcal{L} = y_\alpha \bar{L}_\alpha H l_{\alpha R} + \bar{N}_{iR} \lambda_{i\alpha} \tilde{H}^\dagger L_\alpha + \frac{1}{2} M_i \bar{N}_{iR} N^c_{iR} + \text{H.c.}, \tag{36}$$

where L_α and $l_{\alpha R}$ denote the standard model left-handed (LH) lepton doublet and RH lepton singlet fields with $\alpha = e, \mu, \tau$ and H is the Higgs doublet field with the vacuum expectation value $v = \langle H^0 \rangle = 174$ GeV. The light neutrino mass matrix is given by the well-known seesaw formula

$$m_\nu = v^2 \lambda^T M^{-1} \lambda = \mathbf{U}^* m \mathbf{U}^\dagger, \tag{37}$$

where we denote $M = \text{diag}(M_1, M_2, M_3)$ and $m = \text{diag}(m_1, m_2, m_3)$, and m_i are the light neutrino mass eigenvalues. The most general neutrino Yukawa coupling matrix compatible with the low energy data is given by [67]

$$\lambda = \sqrt{M} \mathbf{R} \sqrt{m} \mathbf{U}^\dagger / v, \tag{38}$$

where R is generally a complex orthogonal matrix fulfilling $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T \mathbf{R} = 1$.

The temperature of the Universe at the very early time was extremely high and the lightest RH neutrino N_1 is in thermal equilibrium. As the temperature drops down to M_1, the N_1 decay process $N_1 \to H l_\alpha (\bar{H}\,\bar{l}_\alpha)$ and its inverse

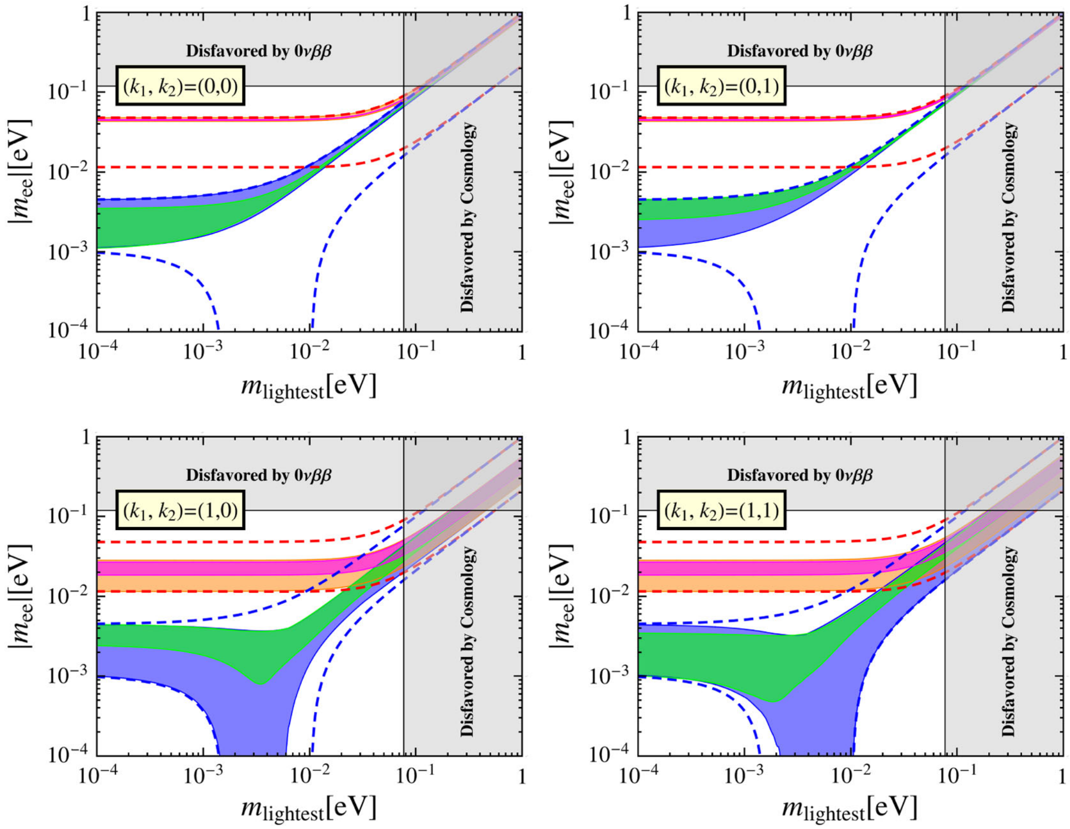

FIG. 11. The effective mass $|m_{ee}|$ describing the neutrinoless double beta decay amplitude. Using the current neutrino oscillation parameters at 3σ [42] one obtains the regions delimited by the red and blue dashed lines for inverted and normal neutrino mass ordering, respectively. In contrast to such generic case, the blue and orange regions correspond to letting Θ and $\theta_{1,2,3}$ as free parameters in the type VI case, while the green and magenta regions correspond to $\Theta = \frac{\pi}{7}$ with $\theta_{1,2,3}$ free.

process start to go out of equilibrium and an asymmetry between leptons and antileptons is induced accordingly. As the temperature of the Universe goes down to the critical temperature of the electroweak phase transition, the sphaleron interactions convert lepton asymmetry to baryon asymmetry. One can define the *CP* asymmetry generated by N_1 decays as [68–72]

$$\epsilon_\alpha \equiv \frac{\Gamma(N_1 \to H l_\alpha) - \Gamma(N_1 \to \bar{H}\bar{l}_\alpha)}{\sum_\alpha \Gamma(N_1 \to H l_\alpha) + \Gamma(N_1 \to \bar{H}\bar{l}_\alpha)} \tag{39}$$

$$= \frac{1}{8\pi(\lambda\lambda^\dagger)_{11}} \sum_{j\neq 1} \left\{ \mathrm{Im}[(\lambda\lambda^\dagger)_{1j}\lambda_{1\alpha}\lambda^*_{j\alpha}] g(x_j) + \mathrm{Im}[(\lambda\lambda^\dagger)_{j1}\lambda_{1\alpha}\lambda^*_{j\alpha}] \frac{1}{1-x_j} \right\}, \tag{40}$$

where $x_j = M_j^2/M_1^2$ and $g(x)$ is the loop function with

$$g(x) = \sqrt{x}\left[\frac{1}{1-x} + 1 - (1+x)\ln\left(\frac{1+x}{x}\right)\right] \overset{x\gg 1}{\to} -\frac{3}{2\sqrt{x}}. \tag{41}$$

As usual, we assume a hierarchical RH neutrinos mass spectrum $M_1 \ll M_2 \ll M_3$ which implies $x_j \gg 1$. As a consequence, the flavored *CP* asymmetries are approximately given by [65,72–77]

$$\epsilon_\alpha = -\frac{3M_1}{16\pi v^2} \frac{\Im(\sum_{ij}\sqrt{m_i m_j}\, m_j \mathbf{R}_{1i}\mathbf{R}_{1j}\mathbf{U}^*_{\alpha i}\mathbf{U}_{\alpha j})}{\sum_j m_j |\mathbf{R}_{1j}|^2}. \tag{42}$$

Besides the *CP* parameter ϵ_α, the final baryon asymmetry depends on the flavor-dependent washout mass parameters,

$$\tilde{m}_\alpha = \frac{|\lambda_{1\alpha}|^2 v^2}{M_1} = \left|\sum_j m_j^{1/2} \mathbf{R}_{1j}\mathbf{U}^*_{\alpha j}\right|^2. \tag{43}$$

At temperatures $T \sim M_1 > 10^{12}$ GeV where all lepton flavors are out of equilibrium, the total lepton asymmetry ϵ_1 is the sum of the ϵ_α,

$$\epsilon_1 = \sum_\alpha \epsilon_\alpha = -\frac{3M_1}{16\pi v^2} \frac{\sum_i m_i^2 \Im(\mathbf{R}^2_{1i})}{\sum_j m_j |\mathbf{R}_{1j}|^2}, \tag{44}$$

which is exactly the standard one-flavor result [64,65]. In the present work we shall be concerned with temperatures

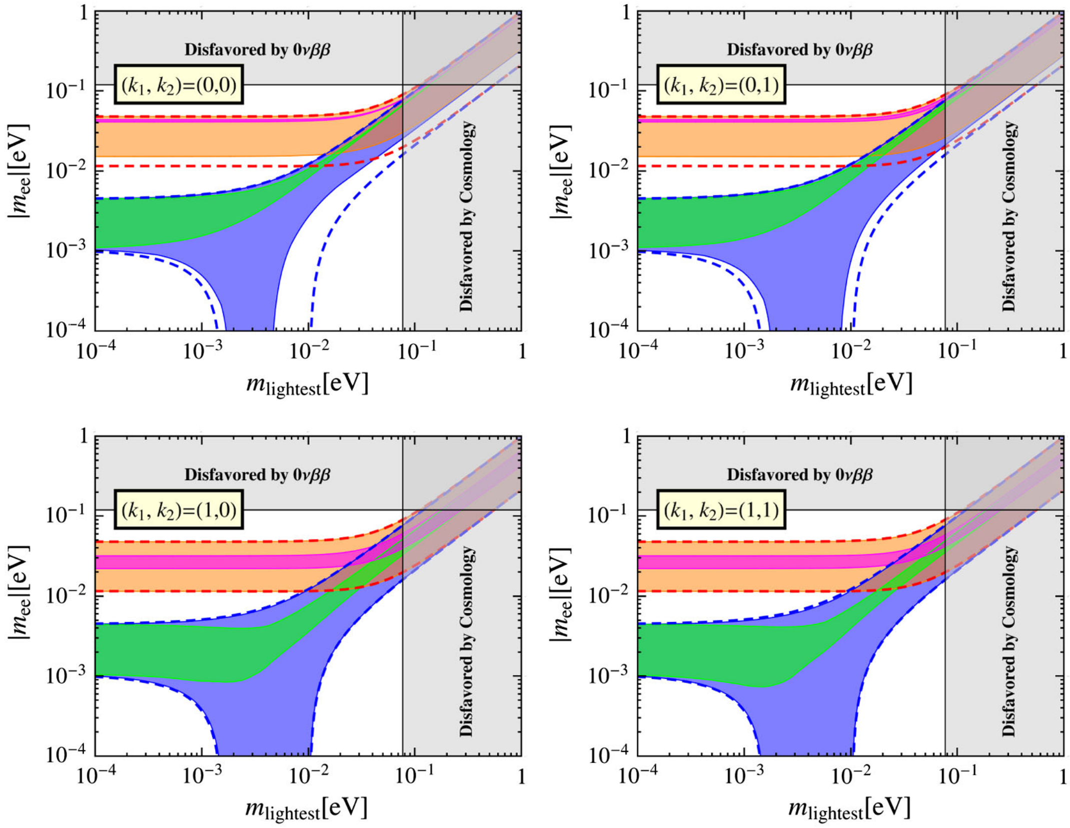

FIG. 12. The effective mass $|m_{ee}|$ describing the neutrinoless double beta decay amplitude. Using the current neutrino oscillation parameters at 3σ [42] one obtains the regions delimited by the red and blue dashed lines for inverted and normal neutrino mass ordering, respectively. In contrast to such generic case, the blue and orange regions correspond to the type IX case, with Θ and $\theta_{1,2,3}$ taken as free parameters, while the green and magenta regions correspond to $\Theta = \frac{\pi}{7}$ with $\theta_{1,2,3}$ free.

$(10^9 \leq T \sim M_1 \leq 10^{12})$ GeV. In this mass window only the interactions mediated by the τ Yukawa coupling are in equilibrium and the final baryon asymmetry is well approximated by [70,74]

$$Y_B \simeq -\frac{12}{37g^*}\left[\epsilon_2\eta\left(\frac{417}{589}\tilde{m}_2\right) + \epsilon_\tau\eta\left(\frac{390}{589}\tilde{m}_\tau\right)\right], \quad (45)$$

where the number of relativistic degrees of freedom g^* is taken to be $g^* = 106.75$ as in the standard model. The combined asymmetry $\epsilon_2 = \epsilon_e + \epsilon_\mu$ comes from the indistinguishable e and μ flavored leptons and $\tilde{m}_2 = \tilde{m}_e + \tilde{m}_\mu$. The efficiency factor $\eta(\tilde{m}_\alpha)$ accounts for the washing out of the total lepton asymmetry due to inverse decays,

$$\eta(\tilde{m}_\alpha) \simeq \left[\left(\frac{\tilde{m}_\alpha}{8.25\times 10^{-3}\text{ eV}}\right)^{-1} + \left(\frac{0.2\times 10^{-3}\text{ eV}}{\tilde{m}_\alpha}\right)^{-1.16}\right]^{-1}. \quad (46)$$

For the mass range of $M_1 < 10^9$ GeV, all three flavors are distinguishable. and the final value of the baryon asymmetry can be approximated by [70]

$$Y_B \simeq -\frac{12}{37g^*}\left[\epsilon_e\eta\left(\frac{151}{179}\tilde{m}_e\right) + \epsilon_\mu\eta\left(\frac{344}{537}\tilde{m}_\mu\right) + \epsilon_\tau\eta\left(\frac{344}{537}\tilde{m}_\tau\right)\right]. \quad (47)$$

The baryon asymmetry will be typically too small to account for the observed value in this case.

In the same fashion as studying lepton flavor mixing in previous sections, we assume that the seesaw Lagrangian of Eq. (36) is invariant under a CP transformation. We suppose that the Lagrangian in Eq. (36) is invariant under the following CP transformation:

$$\text{CP}:\ \nu_L \mapsto i\mathbf{X}\gamma_0\nu_L^c, \qquad N_R \mapsto i\hat{X}_N\gamma_0 N_R^c. \quad (48)$$

For the symmetry to hold, the neutrino Yukawa coupling matrix λ and the RH neutrino mass matrix M have to fulfill

$$\hat{X}_N^\dagger \lambda \mathbf{X} = \lambda^*, \qquad \hat{X}_N^\dagger M \hat{X}_N^* = M^*. \quad (49)$$

One can immediately see that $\hat{X}_N$ should be a diagonal matrix with elements equal to ± 1, i.e. $\hat{X}_N = \text{diag}(\pm 1, \pm 1, \pm 1)$, where the $\pm$ signs can be chosen

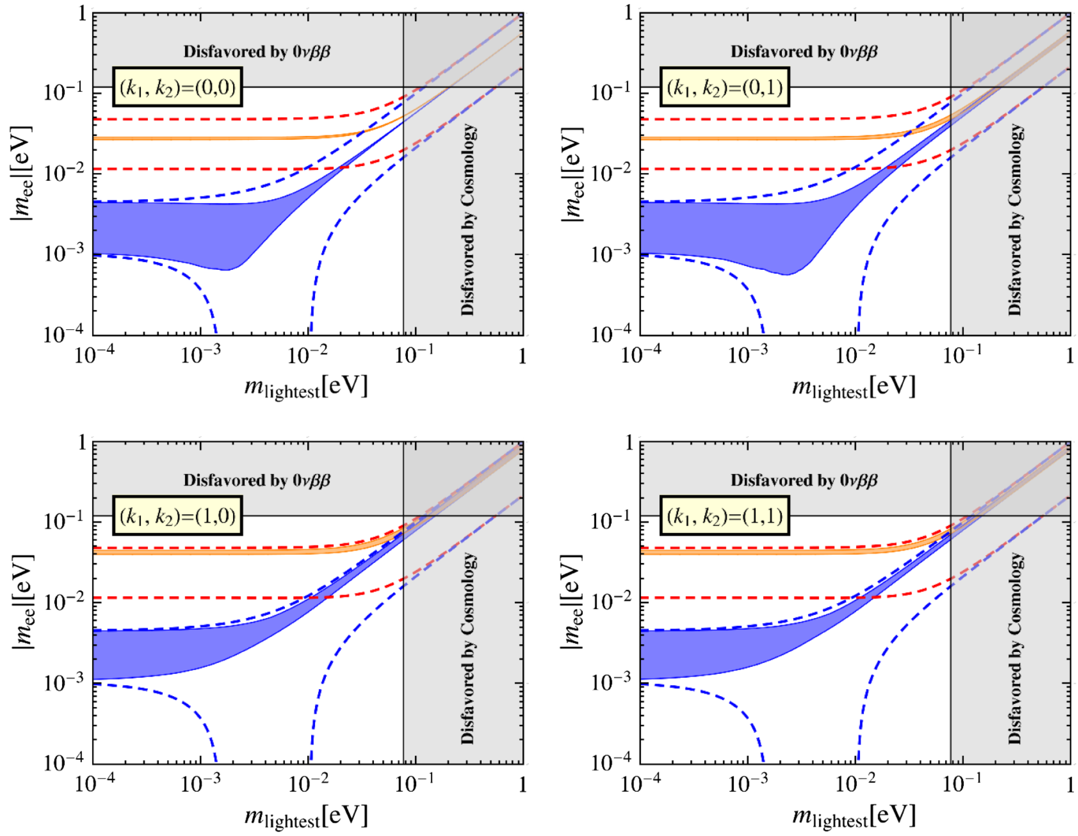

FIG. 13. The effective mass $|m_{ee}|$ describing neutrinoless double beta decay for the democratic *CP* symmetry. The red and blue dashed lines indicate the 3σ regions allowed by current neutrino oscillation data [42] for inverted and normal neutrino mass ordering, respectively. The blue and orange areas denote the possible values of $|m_{ee}|$ where $\theta_{1,2,3}$ are treated as free parameters and the three mixing angles are in the experimentally preferred 3σ range.

independently. From Eq. (49) we can derive that the postulated residual *CP* symmetry leads to the following constraints on the $\mathbf{R}$ matrix and lepton mixing matrix $\mathbf{U}$ [66]:

$$\mathbf{R}^* = \hat{X}_N \mathbf{R} \hat{X}, \qquad \mathbf{U}^\dagger \mathbf{X} \mathbf{U}^* = \hat{X}, \tag{50}$$

where $\hat{X} = \mathrm{diag}(\pm1, \pm1, \pm1)$. Note that the same constraint on the $\mathbf{R}$ matrix from the *CP* invariance was found in Refs. [74,78] As a consequence, the lepton mixing matrix is determined up to an orthogonal matrix $\mathbf{U} = \mathbf{\Sigma} \mathbf{O}_{3\times3} \hat{X}^{-1/2}$ as shown in Eq. (2). On the other hand, depending on the values of $\hat{X}_N$ and $\hat{X}$, each element of the $\mathbf{R}$ matrix satisfies $\mathbf{R}^*_{ij} = \pm \mathbf{R}_{ij}$ so that $\mathbf{R}_{ij}$ is either real or pure imaginary while $\mathbf{R}^2_{ij}$ must be real. Hence the total lepton asymmetry ϵ_1 is always predicted to be vanishing $\epsilon_1 = 0$, no matter what form the residual *CP* $\mathbf{X}$ takes. Thus, at temperatures where all lepton flavors are out of equilibrium and the one–flavor approximation is valid, no baryon asymmetry can be generated in the present framework. In the rest of the paper, we shall focus on the flavor dependent leptogenesis, with M_1 having a value in the interval of interest 10^9 GeV $\le M_1 \le 10^{12}$ GeV.

Notice that both diagonal matrices $\hat{X}$ and $\hat{X}_N$ are not constrained by the symmetry. In order to classify different possible cases in a concise and systematical way, we shall separate out $\hat{X}$ and $\hat{X}_N$ explicitly and introduce the following notations:

$$\mathbf{U}' = \mathbf{U} \hat{X}^{1/2}, \qquad \mathbf{R}' = \hat{X}_N^{1/2} \mathbf{R} \hat{X}^{1/2},$$
$$K_j = (\hat{X}_N)_{11} (\hat{X})_{jj}, \quad \text{with} \quad j = 1, 2, 3. \tag{51}$$

TABLE IX. The parametrization of the first row of the $\mathbf{R}'$ matrix for the possible values of K_1, K_2, and K_3, where both φ and ρ are real parameters.

(K_1, K_2, K_3)	$(\mathbf{R}'_{11}, \mathbf{R}'_{12}, \mathbf{R}'_{13})$
$(+,+,+)$	$(\cos\rho\cos\varphi, \cos\rho\sin\varphi, \sin\rho)$
$(-,+,+)$	$(\sinh\rho, \cosh\rho\cos\varphi, \cosh\rho\sin\varphi)$
$(+,-,+)$	$(\cosh\rho\sin\varphi, \sinh\rho, \cosh\rho\cos\varphi)$
$(+,+,-)$	$(\cosh\rho\cos\varphi, \cosh\rho\sin\varphi, \sinh\rho)$
$(+,-,-)$	$(\cosh\rho, \sinh\rho\cos\varphi, \sinh\rho\sin\varphi)$
$(-,+,-)$	$(\sinh\rho\sin\varphi, \cosh\rho, \sinh\rho\cos\varphi)$
$(-,-,+)$	$(\sinh\rho\cos\varphi, \sinh\rho\sin\varphi, \cosh\rho)$

Then $\mathbf{U}'$ would be a real matrix, and K_j is either $+1$ or -1. Furthermore, the *CP* asymmetry ϵ_α and the washout mass parameter $\tilde{m}_\alpha$ can be written as

$$\epsilon_\alpha = -\frac{3M_1}{16\pi v^2}\frac{\Im(\sum_{ij}\sqrt{m_i m_j}\, m_j \mathbf{R}'_{1i}\mathbf{R}'_{1j}\mathbf{U}'^{*}_{\alpha i}\mathbf{U}'_{\alpha j}K_j)}{\sum_j m_j(\mathbf{R}'_{1j})^2},$$
$$\tilde{m}_\alpha = \left|\sum_j m_j^{1/2}\mathbf{R}'_{1j}\mathbf{U}'^{*}_{\alpha j}\right|^2. \tag{52}$$

Obviously only the elements $\mathbf{R}'_{1i}$ of the first row of $\mathbf{R}'$ are relevant to ϵ_α and $\tilde{m}_\alpha$. The orthogonal condition $\mathbf{R}\mathbf{R}^T = 1$ gives rise to

$$\mathbf{R}'^2_{11}K_1 + \mathbf{R}'^2_{12}K_2 + \mathbf{R}'^2_{13}K_3 = 1. \tag{53}$$

The most general parametrization of the elements $\mathbf{R}'_{11}$, $\mathbf{R}'_{12}$, and $\mathbf{R}'_{13}$ for different possible values of K_1, K_2, K_3 are listed in Table IX. Note that the values $(K_1, K_2, K_3) = (-,-,-)$ are not admissible since the constraint in Eq. (53) cannot be fulfilled in that case.

In the present formalism, we show that in general the lepton mixing angles and *CP* phases depend on three parameters $\theta_{1,2,3}$, and two more parameters ρ and φ are involved in prediction for the baryon asymmetry. In what follows, we shall apply the above general results to the

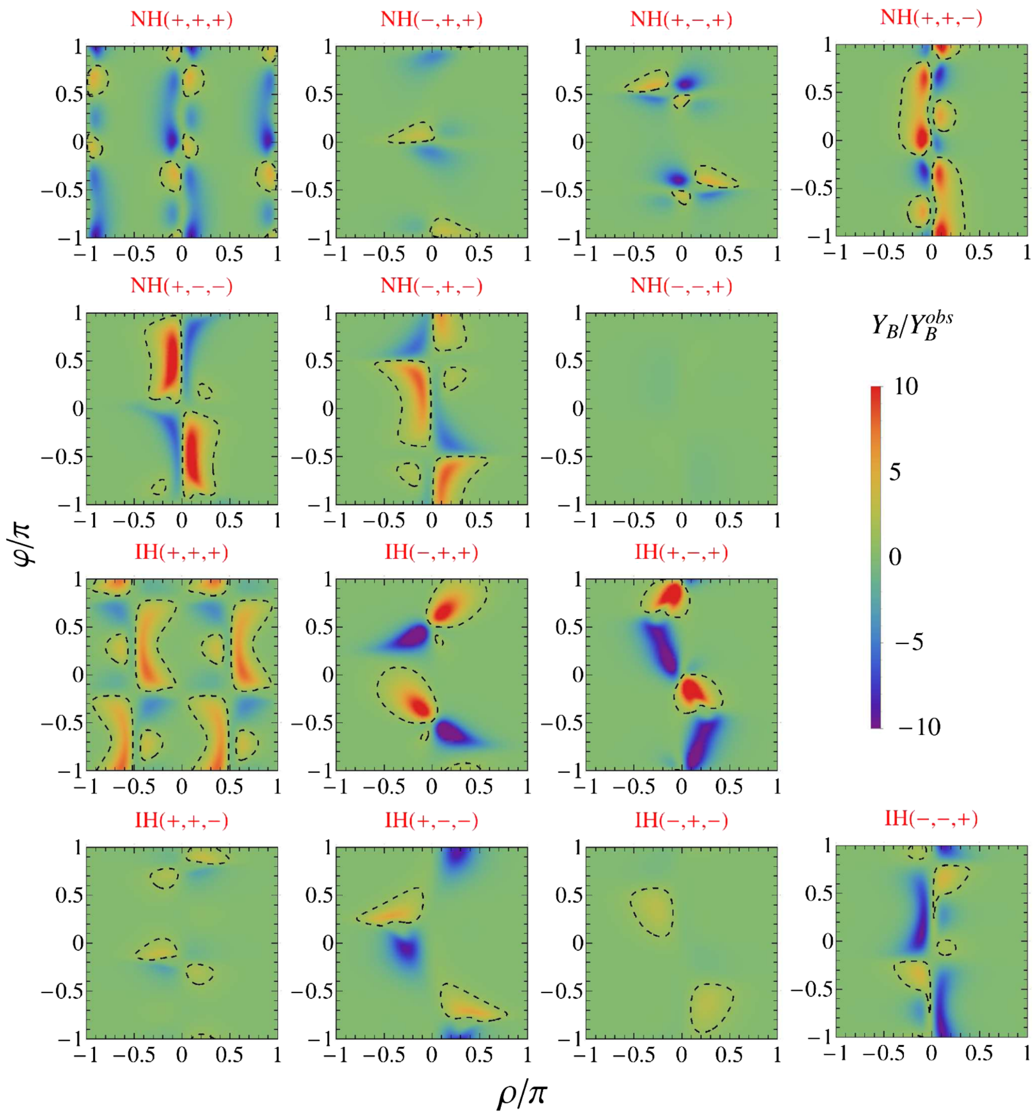

FIG. 14. Predictions for $Y_B/Y_B^{\rm obs}$ as a function of ρ and φ in the case of type V residual *CP* transformation with $\Theta = 2\pi/5$. We have chosen $M_1 = 5\times 10^{11}$ GeV, m_1(or m_3) $= 0.01$ eV. The mass-squared differences Δm^2_{21} and $|\Delta m^2_{31}|$ are taken to be the best fit values [42]. We set $\theta_1 = 58.026°[59.106°]$, $\theta_2 = 8.60°[8.70°]$, and $\theta_3 = 145.4°[145.4°]$ to reproduce the best fitting values of the mixing angles [42]. The dashed lines denote the precisely measured value of the baryon asymmetry $Y_B^{\rm obs} = 8.66\times 10^{-11}$ [79]. Note that successful leptogenesis is not possible for NH neutrino masses with $(K_1, K_2, K_3) = (-,-,+)$.

cases of type V and type VI residual CP symmetries with $\Theta = \frac{2\pi}{5}$ and $\frac{\pi}{7}$, respectively. The values of $\theta_{1,2,3}$ are determined to reproduce the best fit values of the three lepton mixing angles [42]. As a typical example, we choose the RH neutrino mass $M_1 = 5 \times 10^{11}$ GeV, the lightest neutrino mass is taken to be $m_1(\text{or } m_3) = 0.01$ eV, and the mass-squared splittings Δm^2_{21} and $|\Delta m^2_{31}|$ are fixed at their best fit values [42].

(i) Type V CP symmetry with $\Theta = \frac{2\pi}{5}$

It is the so-called generalized $\mu - \tau$ reflection symmetry [7]. The explicit form of the $\mathbf{X}$ matrix, its Takagi factorization matrix Σ and the corresponding predictions for mixing parameters are collected in Table II. From Eq. (50) we know that the $\mathbf{R}$ matrix and the mixing matrix $\mathbf{U}$ have the following properties:

$$\mathbf{R}_{1j} = \mathbf{R}^*_{1j} K_j, \qquad \mathbf{U}_{1j} = e^{i\alpha} \mathbf{U}^*_{1j} (\hat{X})_{jj}. \tag{54}$$

It follows that the CP asymmetry ϵ_e is vanishing $\epsilon_e = 0$ independent of the value of Θ in this case. The remaining two CP asymmetries ϵ_μ and ϵ_τ are related as $\epsilon_\mu = -\epsilon_\tau$ which is inferred from the general prediction $\epsilon_1 = \sum_\alpha \epsilon_\alpha = 0$. In order to accommodate the best fit values of the mixing angles [42], we take $\theta_1 = 58.026°[59.106°]$,

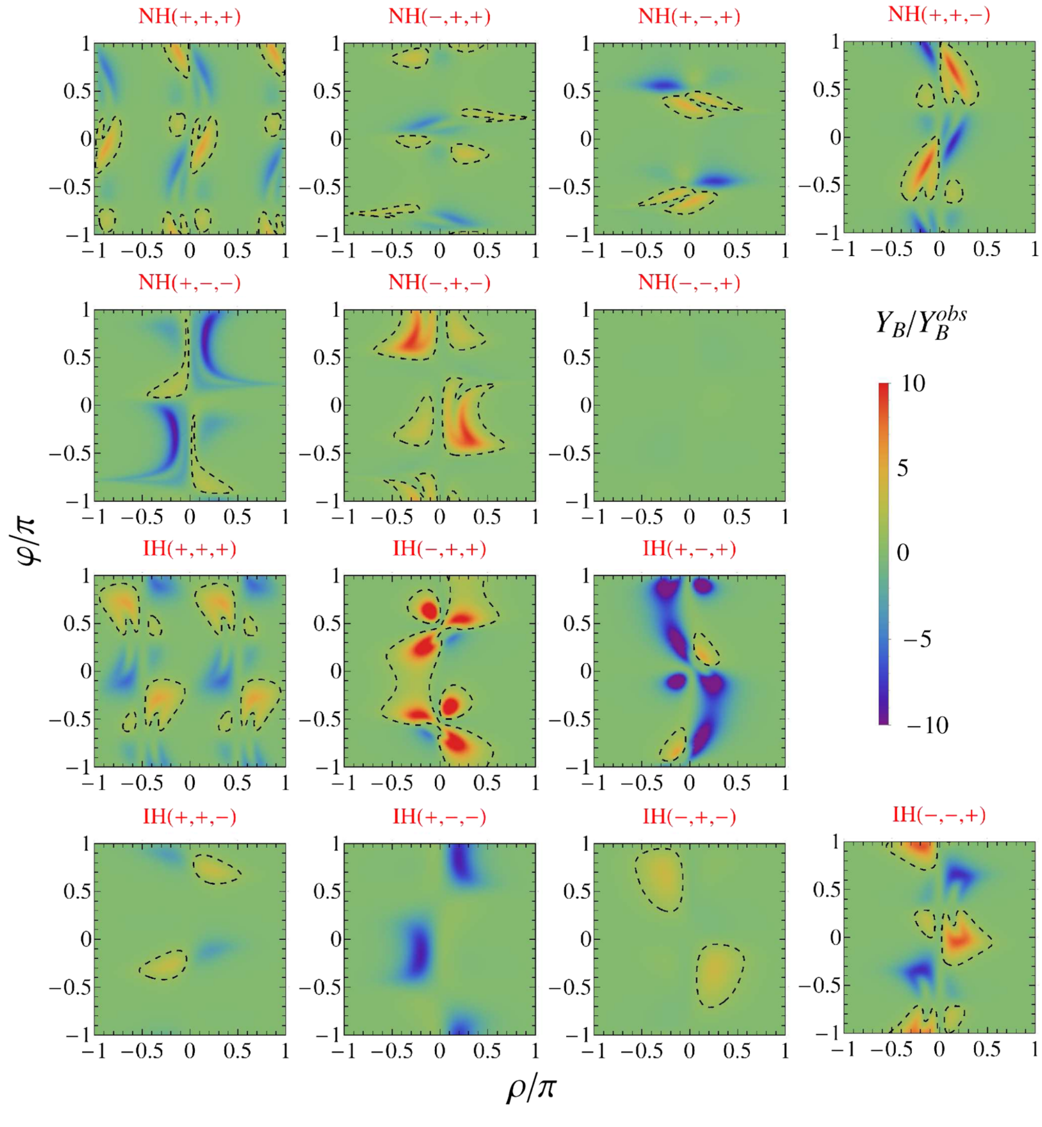

FIG. 15. Predictions for Y_B/Y_B^{obs} as a function of ρ and φ in the case of type VI residual CP transformation with $\Theta = \pi/7$. We have chosen $M_1 = 5 \times 10^{11}$ GeV, $m_1(\text{or } m_3) = 0.01$ eV. The mass-squared differences Δm^2_{21} and $|\Delta m^2_{31}|$ are taken to be the best fit values [42]. We set $\theta_1 = 178.345°[176.995°]$, $\theta_2 = 48.167°[48.442°]$, and $\theta_3 = 57.255°[58.255°]$ to reproduce the best fitting values of the mixing angles [42]. The dashed lines denote the precisely measured value of the baryon asymmetry $Y_B^{\text{obs}} = 8.66 \times 10^{-11}$ [79]. Note that successful leptogenesis is not possible for NH neutrino masses with $(K_1, K_2, K_3) = (-,-,+)$ and IH case with $(K_1, K_2, K_3) = (+,-,-)$.

$\theta_2 = 8.60°[8.70°]$, and $\theta_3 = 145.4°[145.4°]$ for NH and in square brackets for IH of the neutrino masses, respectively. Then one can predict the *CP* violating phases $\delta_{CP} = 253.727°[254.022°]$, $\alpha_{21}(\mathrm{mod}\,\pi) = 0°[0°]$, and $\alpha_{31}(\mathrm{mod}\,\pi) = 147.454°[148.045°]$. Note that the Dirac phase δ_{CP} is rather close to its present best fit value [42], although the statistical significance is quite low. Since the baryon asymmetry Y_B depends on ρ and φ, we display the contour regions of Y_B/Y_B^{obs} in the plane φ versus ρ in Fig. 14. We see that successful leptogenesis can happen except for NH neutrino mass spectrum with $(K_1, K_2, K_3) = (-, -, +)$.

(ii) Type VI *CP* symmetry with $\Theta = \frac{\pi}{7}$

In this case, we take $\theta_1 = 178.345°[176.995°]$, $\theta_2 = 48.167°[48.442°]$, and $\theta_3 = 57.255°[58.255°]$ so that the best fitting values of the lepton mixing angles are reproduced exactly. Accordingly, the *CP* phases are determined to be $\delta_{CP} = 255.105°[249.292°]$, $\alpha_{21}(\mathrm{mod}\pi) = 138.569°[138.360°]$, and $\alpha_{31}(\mathrm{mod}\,\pi) = 151.226°[150.738°]$. We plot the contour regions for Y_B/Y_B^{obs} in the $\rho - \varphi$ plane in Fig. 15. As can be seen, we can have successful leptogenesis except for the cases of NH neutrino masses with $(K_1, K_2, K_3) = (-, -, +)$ and IH with $(K_1, K_2, K_3) = (+, -, -)$.

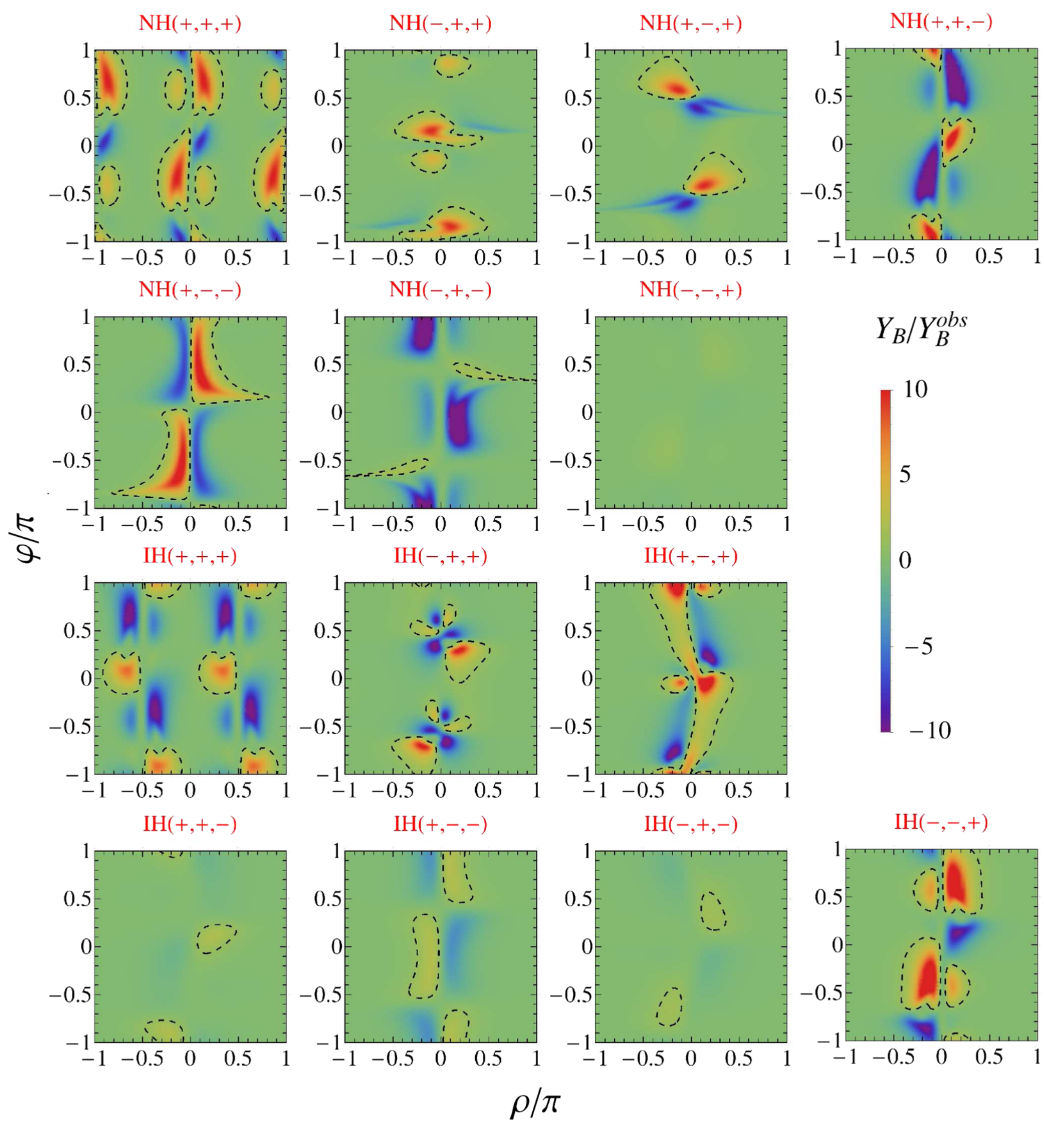

FIG. 16. Predictions for Y_B/Y_B^{obs} as a function of ρ and φ for the case of democratic *CP* transformation. We have chosen $M_1 = 5 \times 10^{11}$ GeV, $m_1(\text{or } m_3) = 0.01$ eV. The mass-squared differences Δm_{21}^2 and $|\Delta m_{31}^2|$ are taken to be the best fit values [42]. We set $\theta_1 = 176.556°[176.886°]$, $\theta_2 = 9.122°[9.403°]$, and $\theta_3 = 53.464°[53.508°]$ so as to reproduce the best fit values of the neutrinos mixing angles [42]. The dashed lines denote the precisely measured value of the baryon asymmetry $Y_B^{\mathrm{obs}} = 8.66 \times 10^{-11}$ [79]. Note that successful leptogenesis is not possible for NH neutrino masses with $(K_1, K_2, K_3) = (-, -, +)$.

We also study the predictions for leptogenesis in the type XI case. As an example, we choose $\theta_1 = 176.556°[176.886°]$, $\theta_2 = 9.122°[9.403°]$, and $\theta_3 = 53.464°[53.508°]$ to reproduce the best fit values of the neutrino mixing angles. Then the CP violating phases can be predicted as $\delta_{CP} = 285.771°[287.100°]$, $\alpha_{21}(\bmod \pi) = 56.593°[56.580°]$, and $\alpha_{31}(\bmod \pi) = 175.842°[175.919°]$. The contour region for $Y_B/Y_B^{\rm obs}$ in the plane φ versus ρ is shown in Fig. 16. The existing matter-antimatter asymmetry can be reproduced for appropriate values of ρ and φ except the case of NH with $(K_1, K_2, K_3) = (-,-,+)$.

VI. CONCLUSIONS

In this paper we have given a full classification of generalized CP symmetries preserved by the neutrino mass matrix, taking as basis the number of zero entries in the transformation matrix. We have determined the corresponding constrained form of the lepton mixing matrix. We have shown how this results in correlations between the lepton mixing angles and the Majorana and Dirac CP violating phases. We have also mapped out the corresponding restrictions that follow from current neutrino oscillation global fits and found that, in some cases, the Dirac CP violating phase characterizing neutrino oscillations is highly constrained. Focusing on the expected CP asymmetries for the "golden" oscillation channel we have derived implications for current long baseline neutrino oscillation experiments T2K, NOνA, forecasting also the corresponding results for the upcoming long baseline DUNE experiment. We have also discussed the predicted ranges for the effective neutrino mass parameter characterizing the neutrinoless double beta decay rates. Finally we have also studied the cosmological implications of such schemes for leptogenesis.

The results of this paper are quite general in the sense that they are independent of how the assumed residual CP symmetry is dynamically achieved. If the residual CP symmetry $\mathbf{X}$ originates from the breaking of the generalized CP symmetry compatible with a finite flavor symmetry group G_f, the admissible form of the residual CP transformation would be strongly constrained to satisfy the consistency condition. If $\mathbf{X}$ has at least one zero entry, it would belong to one of the cases studied in the present work. The corresponding prediction for lepton mixing matrix could be straightforwardly obtained by exploiting the master formula Eq. (2). In this paper we have discussed the possible mixing patterns which can be achieved in this method, and the resulting phenomenological predictions for neutrino oscillation, double beta decay, and leptogenesis. By comparing with the extensively studied scenarios with two residual CP transformations preserved in the neutrino sector [23–40], one expects to obtain new phenomenologically viable mixing patterns and new predictions for the CP violation phases.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grants No. 11275188, No. 11179007, and No. 11522546 (P. C. and G. J. D.); by the Spanish Grants FPA2014-58183-P, Multidark CSD2009-00064, and SEV-2014-0398 (MINECO), and PROMETEOII/2014/084 (Generalitat Valenciana) (J. W. F. V. and F. G. C.), and the Mexican Grants CONACYT 236394 (F. G. C.).

APPENDIX DEFINITION DOMAIN OF $\mathbf{O}_{3\times3}$

In this appendix, we would like to discuss the domain of the parameters θ_1, θ_2, and θ_3 in the $\mathbf{O}_{3\times3}$ matrix. In Eq. (4), $\mathbf{O}_{3\times3}$ is parametrized as

$$\mathbf{O}_{3\times3}(\theta_1,\theta_2,\theta_3) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \times \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \times \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{(A1)}$$

where θ_1, θ_2, and θ_3 can freely vary in the range of $[0, 2\pi)$. Notice that $\mathbf{O}_{3\times3}$ has the following properties:

$$\begin{aligned} \mathbf{O}_{3\times3}(\theta_1,\theta_2,\theta_3+\pi) &= \mathbf{O}_{3\times3}(\theta_1,\theta_2,\theta_3)\mathrm{diag}(-1,-1,1), \\ \mathbf{O}_{3\times3}(\theta_1,\theta_2+\pi,\theta_3) &= \mathbf{O}_{3\times3}(\theta_1,\theta_2,\pi-\theta_3)\mathrm{diag}(1,-1,-1), \\ \mathbf{O}_{3\times3}(\theta_1+\pi,\theta_2,\theta_3) &= \mathbf{O}_{3\times3}(\theta_1,\pi-\theta_2,\theta_3)\mathrm{diag}(-1,-1,1), \end{aligned} \quad \text{(A2)}$$

where the diagonal matrices can be absorbed into the matrix $\hat{X}_\nu^{-1/2}$. As a result, the fundamental interval of the parameters $\theta_{1,2,3}$ can be taken to be $[0, \pi)$.

[1] G. Altarelli and F. Feruglio, Discrete flavor symmetries and models of neutrino mixing, Rev. Mod. Phys. **82**, 2701 (2010).

[2] H. Ishimori *et al.*, *An Introduction to Non-Abelian Discrete Symmetries for Particle Physicists*, Lecture Notes in Physics (Springer, New York, 2012).

[3] S. F. King and C. Luhn, Neutrino mass and mixing with discrete symmetry, Rep. Prog. Phys. **76**, 056201 (2013).

[4] S. F. King, A. Merle, S. Morisi, Y. Shimizu, and M. Tanimoto, Neutrino Mass and Mixing: from Theory to Experiment, New J. Phys. **16**, 045018 (2014).

[5] S. F. King, Models of neutrino mass, mixing and CP violation, J. Phys. G **42**, 123001 (2015).

[6] J. Schechter and J. Valle, Neutrino masses in $SU(2)\times U(1)$ theories, Phys. Rev. D **22**, 2227 (1980).

[7] P. Chen, G.-J. Ding, F. Gonzalez-Canales, and J. W. F. Valle, Generalized $\mu-\tau$ reflection symmetry and leptonic CP violation, Phys. Lett. B **753**, 644 (2016).

[8] Z.-z. Xing and Z.-h. Zhao, A review of mu-tau flavor symmetry in neutrino physics, Rep. Prog. Phys. **79**, 076201 (2016).

[9] P. Chen, C.-C. Li, and G.-J. Ding, Lepton flavor mixing and CP symmetry, Phys. Rev. D **91**, 033003 (2015).

[10] P. Chen, C.-Y. Yao, and G.-J. Ding, Neutrino mixing from CP symmetry, Phys. Rev. D **92**, 073002 (2015).

[11] D. Hernandez and A. Y. Smirnov, Lepton mixing and discrete symmetries, Phys. Rev. D **86**, 053014 (2012).

[12] D. Hernandez and A. Y. Smirnov, Discrete symmetries and model-independent patterns of lepton mixing, Phys. Rev. D **87**, 053005 (2013).

[13] A. D. Hanlon, S. F. Ge, and W. W. Repko, Phenomenological consequences of residual $\mathbb{Z}_2^s$ and $\bar{\mathbb{Z}}_2^s$ symmetries, Phys. Lett. B **729**, 185 (2014).

[14] G. Ecker, W. Grimus, and W. Konetschny, Quark mass matrices in left-right symmetric gauge theories, Nucl. Phys. **B191**, 465 (1981); G. Ecker, W. Grimus, and H. Neufeld, Spontaneous CP violation in left-right symmetric gauge theories, Nucl. Phys. **B247**, 70 (1984); J. Bernabeu, G. C. Branco, and M. Gronau, CP restrictions on quark mass matrices, Phys. Lett. **169B**, 243 (1986); G. Ecker, W. Grimus, and H. Neufeld, A standard form for generalized CP transformations, J. Phys. A **20**, L807 (1987); H. Neufeld, W. Grimu, and G. Ecker, Generalized CP invariance, neutral flavor conservation and the structure of the mixing matrix, Int. J. Mod. Phys. A **03**, 603 (1988); W. Grimus and M. N. Rebelo, Automorphisms in gauge theories and the definition of CP and P, Phys. Rep. **281**, 239 (1997).

[15] P. F. Harrison and W. G. Scott, Symmetries and generalisations of tri-bimaximal neutrino mixing, Phys. Lett. B **535**, 163 (2002).

[16] W. Grimus and L. Lavoura, A nonstandard CP transformation leading to maximal atmospheric neutrino mixing, Phys. Lett. B **579**, 113 (2004).

[17] Y. Farzan and A. Yu. Smirnov, Leptonic CP violation: Zero, maximal or between the two extremes, J. High Energy Phys. 01 (2007) 059.

[18] K. Olive *et al.* (Particle Data Group Collaboration), Review of particle physics, Chin. Phys. C **38**, 090001 (2014).

[19] W. Rodejohann and J. W. F. Valle, Symmetrical parametrizations of the lepton mixing matrix, Phys. Rev. D **84**, 073011 (2011).

[20] G. C. Branco, L. Lavoura, and M. N. Rebelo, Majorana neutrinos and CP violation in the leptonic sector, Phys. Lett. B **180**, 264 (1986).

[21] E. E. Jenkins and A. V. Manohar, Rephasing invariants of quark and lepton mixing matrices, Nucl. Phys. **B792**, 187 (2008).

[22] G. C. Branco, R. G. Felipe, and F. R. Joaquim, Leptonic CP violation, Rev. Mod. Phys. **84**, 515 (2012).

[23] F. Feruglio, C. Hagedorn, and R. Ziegler, Lepton mixing parameters from discrete and CP symmetries, J. High Energy Phys. 07 (2013) 027.

[24] G.-J. Ding, S. F. King, C. Luhn, and A. J. Stuart, Spontaneous CP violation from vacuum alignment in S_4 models of leptons, J. High Energy Phys. 05 (2013) 084.

[25] F. Feruglio, C. Hagedorn, and R. Ziegler, A realistic pattern of lepton mixing and masses from S_4 and CP, Eur. Phys. J. C **74**, 2753 (2014).

[26] G.-J. Ding, S. F. King, and A. J. Stuart, Generalized CP and A_4 family symmetry, J. High Energy Phys. 12 (2013) 006.

[27] C.-C. Li and G.-J. Ding, Generalized CP and trimaximal TM_1 lepton mixing in S_4 family symmetry, Nucl. Phys. **B881**, 206 (2014).

[28] G.-J. Ding and Y.-L. Zhou, Predicting lepton flavor mixing from $\Delta(48)$ and generalized CP symmetries, Chin. Phys. C **39**, 021001 (2015).

[29] G.-J. Ding and Y.-L. Zhou, Lepton mixing parameters from $\Delta(48)$ family symmetry and generalized CP, J. High Energy Phys. 06 (2014) 023.

[30] G.-J. Ding and S. F. King, Generalized CP and $\Delta(96)$ family symmetry, Phys. Rev. D **89**, 093020 (2014).

[31] C.-C. Li and G.-J. Ding, Deviation from bimaximal mixing and leptonic CP phases in S_4 family symmetry and generalized CP, J. High Energy Phys. 08 (2015) 017.

[32] C.-C. Li and G.-J. Ding, Lepton Mixing in A_5 family symmetry and generalized CP, J. High Energy Phys. 05 (2015) 100.

[33] A. D. Iura, C. Hagedorn, and D. Meloni, Lepton mixing from the interplay of the alternating group A_5 and CP, J. High Energy Phys. 08 (2015) 037.

[34] P. Ballett, S. Pascoli, and J. Turner, Mixing angle and phase correlations from A5 with generalized CP and their prospects for discovery, Phys. Rev. D **92**, 093008 (2015).

[35] G. C. Branco, I. de Medeiros Varzielas, and S. F. King, Invariant approach to CP in unbroken $\Delta(27)$, Nucl. Phys. **B899**, 14 (2015); Invariant approach to CP in family symmetry models, Phys. Rev. D **92**, 036007 (2015).

[36] J. Turner, Predictions for leptonic mixing angle correlations and nontrivial Dirac CP violation from A_5 with generalized CP symmetry, Phys. Rev. D **92**, 116007 (2015).

[37] C. Hagedorn, A. Meroni, and E. Molinaro, Lepton mixing from $\Delta(3n^2)$ and $\Delta(6n^2)$ and CP, Nucl. Phys. **B891**, 499 (2015).

[38] G.-J. Ding, S. F. King, and T. Neder, Generalized CP and $\Delta(6n^2)$ family symmetry in semi-direct models of leptons, J. High Energy Phys. 12 (2014) 007.

[39] G.-J. Ding and S. F. King, Generalized CP and $\Delta(3n^2)$ family symmetry for semidirect predictions of the PMNS matrix, Phys. Rev. D **93** 025013 (2016).
[40] C.-C. Li, C.-Y. Yao, and G.-J. Ding, Lepton mixing predictions from infinite group series $D^{(1)}_{9n,3n}$ with generalized CP, J. High Energy Phys. 05 (2016) 007.
[41] H. Fritzsch and Z.-z. Xing, Mass and flavor mixing schemes of quarks and leptons, Prog. Part. Nucl. Phys. **45**, 1 (2000).
[42] D. Forero, M. Tortola, and J. Valle, Neutrino oscillations refitted, Phys. Rev. D **90**, 093006 (2014).
[43] K. Abe *et al.* (T2K Collaboration), Measurements of neutrino oscillation in appearance and disappearance channels by the T2K experiment with 6.6×10^{20} protons on target, Phys. Rev. D **91**, 072010 (2015).
[44] C. Adams *et al.* (LBNE Collaboration), The long-baseline neutrino experiment: Exploring fundamental symmetries of the Universe, arXiv:1307.7335.
[45] S. K. Agarwalla *et al.* (LAGUNA-LBNO Collaboration), The mass-hierarchy and CP-violation discovery reach of the LBNO long-baseline neutrino experiment, J. High Energy Phys. 05 (2014) 094.
[46] H.-K. W. Group (Hyper-Kamiokande Working Group Collaboration), Hyper-kamiokande physics opportunities, arXiv:1309.0184.
[47] H. Nunokawa, S. Parke, and J. W. Valle, CP violation and neutrino oscillations, Prog. Part. Nucl. Phys. **60**, 338 (2008).
[48] J. Schechter and J. Valle, Neutrinoless double beta decay in $SU(2) \times U(1)$ theories, Phys. Rev. D **25**, 2951 (1982).
[49] M. Duerr, M. Lindner, and A. Merle, On the quantitative impact of the Schechter-Valle theorem, J. High Energy Phys. 06 (2011) 091.
[50] F. Simkovic, S. M. Bilenky, A. Faessler, and T. Gutsche, Possibility of measuring the CP Majorana phases in $0\nu\beta\beta$ decay, Phys. Rev. D **87**, 073002 (2013).
[51] F. T. III Avignone, S. R. Elliott, and J. Engel, Double beta decay, majorana neutrinos, and neutrino mass, Rev. Mod. Phys. **80**, 481 (2008).
[52] A. Barabash, 75 years of double beta decay: Yesterday, today and tomorrow, arXiv:1101.4502.
[53] M. Auger *et al.* (EXO Collaboration), Search for Neutrinoless Double-Beta Decay in ^{136}Xe with EXO-200, Phys. Rev. Lett. **109**, 032505 (2012).
[54] J. B. Albert *et al.* (EXO-200 Collaboration), Search for Majorana neutrinos with the first two years of EXO-200 data, Nature (London) **510**, 229 (2014).
[55] A. Gando *et al.* (KamLAND-Zen Collaboration), Limit on Neutrinoless $\beta\beta$ Decay of ^{136}Xe from the First Phase of KamLAND-Zen and Comparison with the Positive Claim in ^{76}Ge, Phys. Rev. Lett. **110**, 062502 (2013).
[56] P. Ade *et al.* (Planck Collaboration), Planck 2015 results. XIII. Cosmological parameters, arXiv:1502.01589.
[57] L. Dorame, D. Meloni, S. Morisi, E. Peinado, and J. W. F. Valle, Constraining neutrinoless double beta decay, Nucl. Phys. **B861**, 259 (2012).
[58] L. Dorame, S. Morisi, E. Peinado, J. W. F. Valle, and A. D. Rojas, A new neutrino mass sum rule from inverse seesaw, Phys. Rev. D **86**, 056001 (2012).
[59] S. King, S. Morisi, E. Peinado, and J. W. F. Valle, Quark-lepton mass relation in a realistic A4 extension of the standard model, Phys. Lett. B **724**, 68 (2013).
[60] C. Bonilla, S. Morisi, E. Peinado, and J. W. F. Valle, Relating quarks and leptons with the T_7 flavor group, Phys. Lett. B **742**, 99 (2015).
[61] A. Sakharov, Violation of CP invariance, C asymmetry, and baryon asymmetry of the Universe, Pis'ma Zh. Eksp. Teor. Fiz. **5**, 32 (1967).
[62] V. A. Kuzmin, V. A. Rubakov, and M. E. Shaposhnikov, On the anomalous electroweak baryon number nonconservation in the early universe, Phys. Lett. B **155**, 36 (1985).
[63] M. Fukugita and T. Yanagida, Baryogenesis without grand unification, Phys. Lett. B **174**, 45 (1986).
[64] W. Buchmuller, R. D. Peccei, and T. Yanagida, Leptogenesis as the origin of matter, Annu. Rev. Nucl. Part. Sci. **55**, 311 (2005).
[65] S. Davidson, E. Nardi, and Y. Nir, Leptogenesis, Phys. Rep. **466**, 105 (2008).
[66] P. Chen, G.-J. Ding, and S. F. King, Leptogenesis and residual CP symmetry, J. High Energy Phys. 03 (2016) 206.
[67] J. A. Casas and A. Ibarra, Oscillating neutrinos and $\mu \to e, \gamma$, Nucl. Phys. **B618**, 171 (2001).
[68] L. Covi, E. Roulet, and F. Vissani, CP violating decays in leptogenesis scenarios, Phys. Lett. B **384**, 169 (1996).
[69] T. Endoh, T. Morozumi, and Z. Xiong, Primordial lepton family asymmtries in seesaw model, Prog. Theor. Phys. **111**, 123 (2004).
[70] A. Abada, S. Davidson, A. Ibarra, F.-X. Josse-Michaux, M. Losada, and A. Riotto, Flavour matters in leptogenesis, J. High Energy Phys. 09 (2006) 010.
[71] A. Abada, S. Davidson, F.-X. Josse-Michaux, M. Losada, and A. Riotto, Flavor issues in leptogenesis, J. Cosmol. Astropart. Phys. 04 (2006) 004.
[72] C. S. Fong, E. Nardi, and A. Riotto, Leptogenesis in the Universe, Adv. High Energy Phys. **2012**, 158303 (2012).
[73] S. Pascoli, S. T. Petcov, and A. Riotto, Connecting low energy leptonic CP-violation to leptogenesis, Phys. Rev. D **75**, 083511 (2007).
[74] S. Pascoli, S. Petcov, and A. Riotto, Leptogenesis and low energy CP violation in neutrino physics, Nucl. Phys. **B774**, 1 (2007).
[75] G. C. Branco, R. Gonzalez Felipe, and F. R. Joaquim, A new bridge between leptonic CP violation and leptogenesis, Phys. Lett. B **645**, 432 (2007).
[76] S. Davidson, J. Garayoa, F. Palorini, and N. Rius, Insensitivity of flavored leptogenesis to low energy CP violation, Phys. Rev. Lett. **99**, 161801 (2007).
[77] S. Blanchet and P. D. Bari, The minimal scenario of leptogenesis, New J. Phys. **14**, 125012 (2012).
[78] R. N. Mohapatra and C. C. Nishi, Implications of $\mu - \tau$ flavored CP symmetry of leptons, J. High Energy Phys. 08 (2015) 092.
[79] P. A. R. Ade *et al.* (Planck Collaboration), Planck 2015 results. XIII. Cosmological parameters, arXiv:1502.01589.