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How to resum perturbative series in 3d $\mathcal{N}=2$ Chern-Simons matter theories

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Continuing the work of Honda [Phys. Rev. Lett. **116**, 211601 (2016)], we study the perturbative series in general 3d $\mathcal{N}=2$ supersymmetric Chern-Simons matter theory with $U(1)_R$ symmetry, which is given by a power series expansion of inverse Chern-Simons levels. We find that the perturbative series is usually non-Borel summable along a positive real axis for various observables. Alternatively, we prove that the perturbative series is always Borel summable along a negative (positive) imaginary axis for positive (negative) Chern-Simons levels. It turns out that the Borel resummations along this direction are the same as the exact results and, therefore, are correct ways of resumming the perturbative series.

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I. INTRODUCTION

Chern-Simons (CS) theories coupled to matter play an important role in high-energy physics and condensed matter physics. When CS levels are finite, the theories are strongly coupled and systematic analysis is restricted. While one can always set up perturbation theories in the CS theories by expanding observables around infinite CS levels, the perturbative series is usually divergent as in a typical interacting quantum field theory (QFT) [1]. Therefore, it is generically hard to obtain information on the strongly coupled systems from the perturbative series. In this paper, we address this problem and discuss how we obtain exact results by appropriately resumming the perturbative series in 3d $\mathcal{N}=2$ supersymmetric (SUSY) CS matter theories.

One of the standard methods to resum the divergent series is Borel resummation. Given a perturbative series $\sum_{\ell=0}^{\infty} c_\ell g^{a+\ell}$ of a quantity $I(g)$, its Borel resummation along the direction θ is defined by

$$\mathcal{S}_\theta I(g) = \int_0^{\infty e^{i\theta}} dt\, e^{-\frac{t}{g}} \mathcal{B}I(t). \tag{1}$$

Here $\mathcal{B}I(t)$ is the analytic continuation of the formal Borel transformation $\sum_{\ell=0}^{\infty} \frac{c_\ell}{\Gamma(a+\ell)} t^{a+\ell-1}$ after performing the summation. While the perturbative series in typical interacting QFT is expected to be non-Borel summable along a positive real axis due to singularities in $\mathcal{B}I(t)$ [2], it is natural to ask when the perturbative series is Borel summable along $\mathbb{R}_+$, and if it is non-Borel summable, what is the correct way to resum the perturbative series. This is not just a technical question but a physically fundamental question since this is related to how to define interacting QFTs.

In [3], the author began to address this question. We have proven that the perturbative series in 4d $\mathcal{N}=2$ and 5d $\mathcal{N}=1$ SUSY gauge theories with Lagrangians is Borel summable along a positive real axis for various observables [4]. This result for the 4d $\mathcal{N}=2$ theories is expected from a recent proposal on a semiclassical realization of infrared renormalons [7], where the semiclassical solution does not exist in the $\mathcal{N}=2$ theories (see also [8]). Then it is natural to apply the technique in [3] to another class of theories. In this paper, we study the perturbative series in general three-dimensional $\mathcal{N}=2$ SUSY CS matter theories with $U(1)_R$ symmetry in terms of the inverse CS levels [9] (see [5] for studies of the three-dimensional $\mathcal{N}=6$ case). We apply the technique in [3] to the localization formula [11] for various observables in three-dimensional $\mathcal{N}=2$ CS matter theories.

Nevertheless, we find highly different results from the 4d $\mathcal{N}=2$ and 5d $\mathcal{N}=1$ theories. First of all, we find that the perturbative series is usually *not* Borel summable along $\mathbb{R}_+$ for various observables. Alternatively, we prove that the perturbative series is always Borel summable along a negative imaginary axis for positive CS levels and a positive imaginary axis for negative CS levels. We also prove that the Borel resummations along this direction are the same as the exact results [12]. Our main result is schematically written as [a more precise statement is (20)]

$$\mathcal{O}(g) = \mathcal{S}_{\frac{-\pi \mathrm{sgn}(k)}{2}} \mathcal{O}(g) = \int_0^{-i\mathrm{sgn}(k)\infty} dt\, e^{-\frac{t}{g}} \mathcal{B}\mathcal{O}(t), \tag{2}$$

where $g \propto 1/|k|$ with the CS level k and $\mathcal{B}\mathcal{O}(t)$ is the Borel transformation [14] of the small-g expansion of the observable $\mathcal{O}(g)$. This means that exact results are given by the Borel resummations along the direction $\theta = -\pi \mathrm{sgn}(k)/2$. In Sec. II, we prove the results (2) for the S^3 partition function, SUSY Wilson loops, bremsstrahlung function, two-point function of $U(1)$ flavor symmetry currents,

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partition function on squashed lens space, and two-point function of the stress tensor.

Our result (2) is quite surprising for the following reason. When the perturbative series is not Borel summable along $\mathbb{R}_+$, we usually consider the possibility of cancellations of the perturbative ambiguities by contributions from other saddle points such as instantons or perform a more complicated analysis such as median resummation to find a correct integral contour. We find that we can skip the complicated analyses and directly find the correct integral contour by understanding from the usual analyses should be important. We expect that our result is very important also for understanding non-SUSY CS matter theories. While we do not know if the perturbative series in the non-SUSY theories is Borel summable along the contour in (2), it is natural to expect that this choice of the contour makes analysis highly simplified [15].

II. DERIVATION OF RESULTS

A. Partition function on S^3

We suppose a three-dimensional $\mathcal{N}=2$ CS matter theory with a semisimple gauge group $G=G_1\times\cdots\times G_n$, which is coupled to chiral multiplets of representations $(\mathbf{R_1},\ldots,\mathbf{R_{N_f}})$ with R charges $(\Delta_1,\ldots,\Delta_{N_f})$. Applying the localization method [11], the S^3 partition function of this theory is given by [16]

$$Z_{S^3}(g)=\int_{-\infty}^{\infty}d^{|G|}\sigma Z_{\rm cl}(\sigma)Z_{\rm 1loop}(\sigma), \quad (3)$$

where [17]

$$Z_{\rm cl}(\sigma)=\exp\left[\sum_{p=1}^{n}\frac{i{\rm sgn}(k_p)}{g_p}{\rm tr}(\sigma^{(p)})^2\right],$$
$$Z_{\rm 1loop}(\sigma)=\frac{\prod_{\alpha\in{\rm root}_+}4{\rm sinh}^2(\pi\alpha\cdot\sigma)}{\prod_{m=1}^{N_f}\prod_{\rho_m\in R_m}s_1(\rho_m\cdot\sigma-i(1-\Delta_m))},$$
$$s_b(z)=\prod_{m=0}^{\infty}\prod_{n=0}^{\infty}\frac{mb+nb^{-1}+Q/2-iz}{mb+nb^{-1}+Q/2+iz}. \quad (4)$$

The parameter g_p is proportional to $1/|k|$. Now we are interested in small-g_p expansion of $Z_{S^3}(g)$:

$$Z_{S^3}(g)=\sum_{\{\ell_p\}=0}^{\infty}c_{\ell_1,\ldots,\ell_n}\prod_{p=1}^{n}g_p^{\frac{{\rm dim}(G_p)}{2}+\ell_p}. \quad (5)$$

We will see that the perturbative series is usually non-Borel summable along $\mathbb{R}_+$ but always Borel summable along a negative (positive) imaginary axis for $k_p>0$ $(k_p<0)$.

1. $U(N)_k$ adjoint SQCD

For simplicity of explanation, we begin with the three-dimensional $\mathcal{N}=2$ $U(N)_k$ SQCD with N_f fundamental (R charge Δ_f), $\bar{N}_f$ antifundamental (R charge $\bar{\Delta}_f$) and N_a adjoint chiral multiplets (R charge Δ_a). We will discuss the general case later. The S^3 partition function of this theory is

$$Z_{\rm SQCD}=\int_{-\infty}^{\infty}d^N\sigma\prod_{j=1}^{N}e^{\frac{i{\rm sgn}(k)}{g}\sigma_j^2}\frac{s_1^{\bar{N}_f}(\sigma_j+i(1-\bar{\Delta}_f))}{s_1^{N_f}(\sigma_j-i(1-\Delta_f))}$$
$$\times\frac{\prod_{i<j}4{\rm sinh}^2(\pi(\sigma_i-\sigma_j))}{\prod_{i,j}s_1^{N_a}(\sigma_i-\sigma_j-i(1-\Delta_a))}. \quad (6)$$

Now we apply the technique in [3] to this and investigate properties of the small-g expansion of $Z_{\rm SQCD}$. To do this, let us make the following change of variables,

$$\sigma_i=\sqrt{\tau}\hat{x}_i, \quad (7)$$

where $\hat{\mathbf{x}}=(\hat{x}_1,\ldots,\hat{x}_N)$ is the unit vector spanning unit S^{N-1}. Then we rewrite the partition function as

$$Z_{\rm SQCD}=\int_0^{\infty}d\tau\, e^{\frac{i{\rm sgn}(k)}{g}\tau}f(\tau)$$
$$=i{\rm sgn}(k)\int_0^{-i{\rm sgn}(k)\infty}dt\, e^{-\frac{t}{g}}f(i{\rm sgn}(k)t), \quad (8)$$

where

$$f(\tau)=\frac{\tau^{\frac{N^2}{2}-1}}{2}\int_{S^{N-1}}d^{N-1}\hat{x}\,h(\tau,\hat{x}),$$
$$h(\tau,\hat{x})=\frac{Z_{\rm VdM}(\hat{x})Z_{\rm 1loop}(\sqrt{\tau}\hat{x})}{Z_{\rm VdM}(\sqrt{\tau}\hat{x})},$$
$$Z_{\rm VdM}(\sigma)=\prod_{\alpha\in{\rm root}_+}(\pi\alpha\cdot\sigma)^2. \quad (9)$$

Note that (8) is similar to the Borel resummation formula (1) with the direction $\theta=-\pi{\rm sgn}(k)/2$. Therefore, one might wonder whether $f(\tau)$ is related to the Borel transformation of the original perturbative series. This question is technically equivalent to whether $f(\tau)$ consists purely of the convergent power series of τ and is very nontrivial in general.

Nevertheless, we can indeed prove in a similar way to [3] that $f(\tau)$ has the following relation to the Borel transformation,

$$i{\rm sgn}(k)f(\tau)=\mathcal{B}Z_{\rm SQCD}(-i{\rm sgn}(k)\tau), \quad (10)$$

where $\mathcal{B}Z_{\rm SQCD}(t)$ is the Borel transformation of the small-g expansion of $Z_{\rm SQCD}$. Here we just write down an outline of the proof (see Appendix for details): (i) We show uniform convergence of the small-τ expansion of $h(\tau,\hat{x})$. (ii) The uniform convergence tells us that $h(\tau,\hat{x})$ is the same as the analytic continuation of the convergent series and we can exchange the order of the power series expansion of $h(\tau,\hat{x})$ and the integration over $\hat{x}$. (iii) The integral transformation (8) guarantees that the coefficient of the perturbative series of $f(\tau)$ at $\mathcal{O}(\tau^{\frac{N^2}{2}+\ell-1})$ is given by $(-i\mathrm{sgn}(k))^\ell c_\ell/\Gamma(\frac{N^2+\ell}{2})$ [18]. Thus, we conclude

$$Z_{\mathrm{SQCD}} = \int_0^{-i\mathrm{sgn}(k)\infty} dt\, e^{-\frac{t}{g}} \mathcal{B}Z_{\mathrm{SQCD}}(t). \tag{11}$$

Since the Borel transformation does not have singularities along the integral contour [19], the small-g expansion of Z_{SQCD} is Borel summable along the direction $\theta = -\mathrm{sgn}(k)\pi/2$. Equation (11) also tells us that the Borel resummation with this direction gives the exact result.

When does the pertubative series become Borel summable along $\mathbb{R}_+$? Since $t \in \mathbb{R}_+$ corresponds to $\sigma_j \in (-e^{\frac{\pi i}{4}\mathrm{sgn}(k)}\infty, e^{\frac{\pi i}{4}\mathrm{sgn}(k)}\infty)$ in (6), a sufficient condition for this is absence of singularities in the one-loop determinant along this line, namely, $N_a = 0$ (or $\Delta_a = 1$). Next, we ask, when the perturbative series is Borel summable along $\mathbb{R}_+$, how is this related to the exact result? To answer this question, we need to change the integral contour to $\mathbb{R}_+$ as in Fig. 1. There is a subtlety in this, which is related to the CS level shift coming from integration over the massive fermions (see e.g. [20]). When the integral variables σ are very large, the contribution from chiral multiplet becomes

$$\frac{1}{s_1(\sigma - i(1-\Delta))} = \exp\left[\frac{i\pi\mathrm{sgn}(\sigma)}{2}\sigma^2 + \mathcal{O}(|\sigma|)\right]. \tag{12}$$

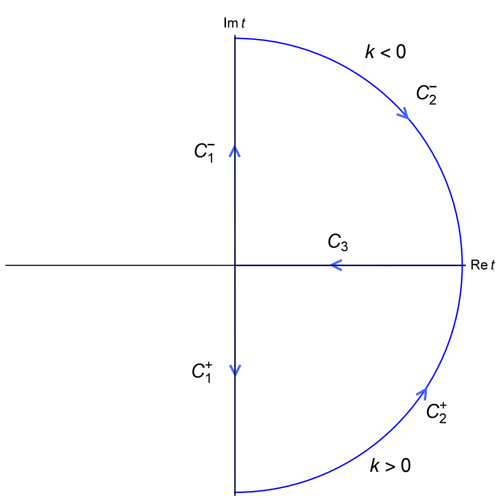

FIG. 1. The integral contour, which relates the Borel resummation along the imaginary axis to the one along the real positive axis.

This effectively shifts the CS level by $\mathrm{sgn}(\sigma)/2$ and the shift in the adjoint SQCD is totally $\mathrm{sgn}(\sigma_j)(N_f - \bar{N}_f)/2$. Hence, the contribution from $C_2^\pm$ disappears for $|k| > |N_f - \bar{N}_f|/2$. If we consider this region, then we find

$$Z_{\mathrm{SQCD}} = \left(\int_0^\infty dt + \oint_C dt\right) e^{-\frac{t}{g}}\mathcal{B}Z_{\mathrm{SQCD}}(t), \tag{13}$$

where the integral contour C is $C = C_1^+ + C_2^+ + C_3$ for $k > 0$ and $C = C_1^- + C_2^- + C_3$ for $k < 0$. Thus, the Borel resummation along $\mathbb{R}_+$ gives the exact result when the second term is zero. A sufficient condition for this is again $N_a = 0$.

It is worth looking at the $N_f = \bar{N}_f = N_a = 0$ case, which corresponds to the $\mathcal{N} = 2$ CS theory without chiral multiplets and is the same as the pure CS theory up to level shift. Since Z_{1loop} does not have poles for this case, the Borel transformation also does not have any poles. This reflects the fact that the perturbative series in the pure CS theory is convergent.

2. General three-dimensional $\mathcal{N} = 2$ CS matter theory

Extension to general three-dimensonal $\mathcal{N} = 2$ CS matter theory is straightforward. First, we insert the delta function constraint $\Delta(\sigma)$ to the integrand [21] such that the following coordinate spans a sphere with radius $\sqrt{\tau_p}$

$$\sigma_i^{(p)} = \sqrt{\tau_p}\hat{x}_i^{(p)}. \tag{14}$$

Then the partition function again takes the form of (8) extended to multivariables,

$$\begin{aligned} Z_{S^3} &= \int_0^\infty d^n\tau e^{\sum_{p=1}^n \frac{i\mathrm{sgn}(k_p)}{g_p}\tau_p} f(\tau) \\ &= \left[\prod_{p=1}^n i\mathrm{sgn}(k_p)\int_0^{-i\mathrm{sgn}(k_p)\infty} dt_p e^{-\frac{t_p}{g_p}}\right] f(i\mathrm{sgn}(k)t), \end{aligned} \tag{15}$$

where

$$\begin{aligned} f(\tau) &= \frac{\tau^{\frac{\dim(G)}{2}-1}}{2^n}\int_{\mathrm{sphere}} d\hat{x}\Delta(\hat{x})h(\tau,\hat{x}), \\ h(\tau,\hat{x}) &= \frac{Z_{\mathrm{VdM}}(\hat{x})Z_{\mathrm{1loop}}(\sigma)}{Z_{\mathrm{VdM}}(\sigma)}\bigg|_{\sigma_i^{(p)}=\sqrt{\tau_p}\hat{x}_i^{(p)}}, \\ \tau^{\frac{\dim(G)}{2}-1} &= \prod_{p=1}^n \tau_p^{\frac{\dim(G_p)}{2}-1}. \end{aligned} \tag{16}$$

We can always prove that $f(\tau)$ is related to the Borel transformation of the original perturbative series as

$$\left[\prod_{p=1}^{n} i\mathrm{sgn}(k_p)\right] f(\{\tau_p\}) = \mathcal{B}Z_{S^3}(\{-i\mathrm{sgn}(k_p)\tau_p\}), \qquad (17)$$

since the small-τ_p expansion of $h(\tau,\hat{x})$ is uniform convergent if Z_{S^3} is well defined. This immediately leads us to

$$Z_{S^3}(g) = \left[\prod_{p=1}^{n}\int_0^{-i\mathrm{sgn}(k_p)\infty} dt_p e^{-\frac{t_p}{g_p}}\right]\mathcal{B}Z_{S^3}(t), \qquad (18)$$

which is a generalization of (11).

A sufficient condition for Borel summability along $\mathbb{R}_+$ is again the absence of singularities along $\sigma_j \in (-e^{\frac{\pi i}{4}\mathrm{sgn}(k)}\infty, e^{\frac{\pi i}{4}\mathrm{sgn}(k)}\infty)$ in $Z_{1\mathrm{loop}}$. When the perturbative series is Borel summable along $\mathbb{R}_+$ and "level shift" is not so very large, we obtain

$$Z_{S^3}(g) = \left[\prod_{p=1}^{n}\left(\int_0^{\infty} dt_p + \oint_C dt_p\right) e^{-\frac{t_p}{g_p}}\right]\mathcal{B}Z_{S^3}(t). \qquad (19)$$

If the second term is zero, the Borel resummation along $\mathbb{R}_+$ is the same as the exact result. In the rest of this paper, we prove our main result for various observables:

$$\mathcal{O}(g) = \left[\prod_{p=1}^{n}\int_0^{-i\mathrm{sgn}(k_p)\infty} dt_p e^{-\frac{t_p}{g_p}}\right]\mathcal{B}\mathcal{O}(t). \qquad (20)$$

B. Other observables

1. Supersymmetric Wilson loop

We can generalize the above considerations to other observables. Let us begin with the Wilson loop,

$$W_{\mathbf{R}}(C) = \mathrm{tr}_{\mathbf{R}} P \exp\left[\oint_C ds(iA_\mu \dot{x}^\mu + \sigma|\dot{x}|)\right], \qquad (21)$$

with the adjoint scalar σ in the vector multiplet. The Wilson loop preserves two supercharges when the contour C is the great circle of S^3. Applying the localization method, the VEV of the Wilson loop is given by

$$\langle W_{\mathbf{R}}(\mathrm{Circle})\rangle = \langle \mathrm{tr}_{\mathbf{R}} e^{\sigma}\rangle_{\mathrm{M.M.}}, \qquad (22)$$

where $\langle\cdots\rangle_{\mathrm{M.M.}}$ denotes the VEV in the matrix model (3). This is just the linear combination of the exponential function of σ, and we can obviously write the Wilson loop as in (20).

2. Bremsstrahlung function in SCFT on $\mathbb{R}^3$

The bremsstrahlung function B determines an energy radiated by accelerating quarks in small velocities as $E = 2\pi B\int dt\dot{v}^2$. It was conjectured that B in three-dimensional $\mathcal{N}=2$ superconformal theory is given by [22]

$$B(g) = \frac{1}{4\pi^2}\frac{\partial}{\partial b}\log\langle \mathrm{tr} e^{b a}\rangle_{\mathrm{M.M.}}\Big|_{b=1}, \qquad (23)$$

which is technically a derivative of the Wilson loop in fundamental representation with winding number b. As in the Wilson loop, we can also rewrite $B(g)$ as in (20).

3. Two-point function of $U(1)$ flavor symmetry currents in SCFT on $\mathbb{R}^3$

Next we consider the two-point function of the $U(1)$ flavor symmetry current j_μ^a for the superconformal case. The three-dimensional conformal symmetry fixes the two-point function as

$$\langle j_a^\mu(x) j_b^\nu(0)\rangle = \frac{\tau_{ab}}{16\pi^2}(\delta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)\frac{1}{x^2} + \frac{i\kappa_{ab}}{2\pi}\epsilon^{\mu\nu\rho}\partial_\rho\delta^{(3)}(x), \qquad (24)$$

where $\tau_{ab}(g)$ and $\kappa_{ab}(g)$ are independent of x but nontrivially dependent on parameters. We can exactly compute $\tau_{ab}(g)$ and $\kappa_{ab}(g)$ by the localization [23]. This is generated by the S^3 partition function $Z_{S^3}(m,g)$ deformed by the real mass $\{m_a\}$ associated with the $U(1)$ symmetries:

$$\tau_{ab}(g) = -\frac{2}{\pi^2}\mathrm{Re}\left[\frac{1}{Z_{S^3}(0,g)}\frac{\partial^2 Z_{S^3}(m,g)}{\partial m_a \partial m_b}\right]_{\{m_a\}=0},$$
$$\kappa_{ab}(g) = \frac{1}{2\pi}\mathrm{Im}\left[\frac{1}{Z_{S^3}(0,g)}\frac{\partial^2 Z_{S^3}(m,g)}{\partial m_a \partial m_b}\right]_{\{m_a\}=0}. \qquad (25)$$

Repeating the argument on Z_{S^3}, we can show that $\tau_{ab}(g)$ and $\kappa_{ab}(g)$ satisfy (20).

4. Partition function and Wilson loop on squashed S^3

Let us consider the partition function on squashed sphere S_b^3 with the squashing parameter b [24]. This has a simple relation to the supersymmetric Renyi entropy [28]. The only difference from Z_{S^3} in the localization formula is the one-loop determinant [25],

$$Z_{1\mathrm{loop}}(\sigma) = \frac{\prod_{\alpha\in\mathrm{root}_+} 4\sinh(\pi b\alpha\cdot\sigma)\sinh(\pi b^{-1}\alpha\cdot\sigma)}{\prod_{m=1}^{N_f}\prod_{\rho_m\in R_m} s_b(\rho_m\cdot\sigma - \frac{iQ}{2}(1-\Delta_m))}, \qquad (26)$$

with $Q = b + b^{-1}$. Note that the partition function is ill defined when one of $m_1 b + m_2 b^{-1}$ ($m_{1,2}\in\mathbb{Z}$) is purely imaginary. Otherwise, we arrive at the same conclusion (20) by a similar argument. An important difference from the round sphere case is that the poles of the Borel transformation rotate as varying the argument of b and hit the integral contour of (20) when the partition function becomes ill defined.

One can also consider the SUSY Wilson loop on the ellipsoid constructed in [29]. This Wilson loop has a topology of the torus knot when b^2 is a rational number. As in (22), the localization formula of the Wilson loop is the VEV of $\mathrm{tr}_{\mathbf{R}} e^{\sigma}$ in the matrix model of the squashed sphere. Hence, the Wilson loop can be also written as in (20).

5. Two-point function of stress tensor in SCFT on $\mathbb{R}^3$

In three-dimensional CFT, the two-point function of the canonically normalized stress tensor at separate points takes the form [30]

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{c_T}{64}(P_{\mu\rho}P_{\nu\sigma}+P_{\nu\rho}P_{\mu\sigma}-P_{\mu\nu}P_{\rho\sigma})\frac{1}{16\pi^2x^2}, \tag{27}$$

where $P_{\mu\nu}=\delta_{\mu\nu}\partial^2-\partial_\mu\partial_\nu$ [31]. The coefficient $c_T(g)$ can be computed by $Z_{S_b^3}$ as [32]

$$c_T(g) = -\frac{32}{\pi^2}\mathrm{Re}\left[\frac{1}{Z_{S^3}(g)}\frac{\partial^2 Z_{S_b^3}(g)}{\partial b^2}\right]_{b=1}. \tag{28}$$

By a similar argument, (20) holds also for $c_T(g)$.

6. Partition function on squashed lens space

We suppose an orbifold of a biaxially squashed sphere: $S_b^3/\mathbb{Z}_n$ [33]. Gauge theory on the lens space has degenerate vacua specified by $m=\frac{n}{2\pi}\oint A$, where the integral contour is an element of $\pi_1(S_b^3/\mathbb{Z}_n)$. Therefore, the partition function on this space is decomposed as

$$Z_{S_b^3/\mathbb{Z}_n} = \sum_m Z^{(m)}_{S_b^3/\mathbb{Z}_n}. \tag{29}$$

The localization method tells us that $Z^{(m)}_{S_b^3/\mathbb{Z}_n}$ is expressed as in (3) with the different one-loop determinant [34],

$$Z^{(m)}_{\mathrm{1loop}} = \frac{\prod_{\alpha\in\mathrm{root}} s_{b,\alpha(m)}(\alpha\cdot\sigma - iQ/2)}{\prod_{f=1}^{N_f}\prod_{\rho_f\in\mathbf{R_f}} s_{b,\rho_f(m)}(\rho_f\cdot\sigma - iQ(1-\Delta_f)/2)}, \tag{30}$$

where

$$s_{b,h}(z) = \prod_{p=0}^{n-1} s_b\left(\frac{z}{n} + ib\langle p\rangle_n + ib^{-1}\langle p+h\rangle_n\right),$$
$$\langle m\rangle_n = \frac{1}{n}\left([m]_n+\frac{1}{2}\right)-\frac{1}{2}. \tag{31}$$

One can prove (20) for $Z^{(m)}_{S_b^3/\mathbb{Z}_n}$ by the same argument as the squashed S^3 partition function.

III. DISCUSSIONS

We have studied the perturbative series in general three-dimensional $\mathcal{N}=2$ SUSY CS matter theory. We have proven that the perturbative series is Borel summable along negative (positive) imaginary axis for positive (negative) CS levels and the Borel resummations along this direction are the same as the exact results for various observables. Thus we conclude that the Borel resummations of this direction are correct ways of resumming the perturbative series. We have found that this structure is already hidden in the localization formula.

We have found that the perturbative series are usually not Borel summable along $\mathbb{R}_+$ due to the singularities in the Borel transformations. It is interesting to find physical interpretations of the singularities. Technically the singularities come from poles in one-loop determinant of chiral multiplets. It is known in the context of factorization [35] that the poles for the squashed S^3 partition function correspond to Higgs branch solutions. Hence we expect that the singularities are related to such semiclassical solutions. It would be nice if one can make it clearer.

While the sufficient condition for Borel summability along $\mathbb{R}_+$ is the absence of singularities along $\sigma_j \in (-e^{\frac{\pi i}{4}\mathrm{sgn}(k)}\infty, e^{\frac{\pi i}{4}\mathrm{sgn}(k)}\infty)$ in Z_{1loop}, there should be many theories which do not satisfy this condition but are Borel summable along $\mathbb{R}_+$. One such example is the S^3 partition function of three-dimensional $\mathcal{N}=6$ superconformal theory (ABJM theory [36]) with $U(2)\times U(2)$ gauge group [5]. It is very important to find necessary or more sufficient conditions for Borel summability along $\mathbb{R}_+$. Since we have shown Borel summability along $\mathbb{R}_+$ for four-dimensional and five-dimensional theories with eight supercharges in [3], it might be natural to expect that the perturbative series in three-dimensional $\mathcal{N}=4$ CS matter theories is Borel summable along $\mathbb{R}_+$.

For theories describing M2-branes, the CS levels are not completely independent of each other and satisfy $\sum_{p=1}^n k_p = 0$. While our analysis includes such M2-brane theories as special cases, we could directly discuss these cases. One of the subtleties here is that if we take $\sum_{p=1}^n k_p = 0$ at first in our argument, then integral domain of $\hat{x}$ in (14) becomes noncompact. It is very nice if one can overcome the subtleties.

In the planar limit, we expect that the perturbative series becomes convergent [37] and, hence, Borel summable along the positive real axis. To be consistent with this, the second term in (19) should be suppressed in the $1/N$ expansion. It is illuminating if one can explicitly prove this statement. This would also be related to a simple connection between the planar limit and M-theory limit discussed in [38].

Recently it was discussed that some SUSY CS matter theories exhibit phase transitions as varying real masses or FI parameters [39]. Since real masses shift poles of Z_{1loop}, these also shift poles in the Borel plane. In general, this effect may change the direction of the Borel summability and be related to the phase transitions.

Finally, although we know the localization formula for the vortex loop [40], we have not discussed the perturbative series of the vortex loop. Technically, the localization formula for the vortex loop is like the S^3 partition function with a different integral contour, and we probably need to think of it more carefully.

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APPENDIX PROOF OF (10)

Here we explicitly prove (10) as in [3]. For this purpose, first we prove the uniform convergence of the small-τ expansion of $h(\tau,\hat{x})$. Let us rewrite $h(\tau,\hat{x})$ in a convenient form for the small-τ expansion. By using

$$\frac{\sinh \pi x}{\pi x} = \prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right),$$

$$s_1(z) = \prod_{n=1}^{\infty}\left(\frac{n-iz}{n+iz}\right)^n,$$

we find that the small-τ expansion is generated by

$$2^{N^2-N}Z_{\rm VdM}(\hat{x})\exp\Bigg[-2\sum_{i<j}\sum_{\ell=1}^{\infty}\frac{(-\tau)^\ell\zeta(2\ell)}{\ell}(\hat{x}_i-\hat{x}_j)^{2\ell} - N_a\sum_{i,j}\ln\tilde{s}_1(\sqrt{\tau}(\hat{x}_i-\hat{x}_j)-i(1-\Delta_a)) - N_f\sum_j \ln\tilde{s}_1(\sqrt{\tau}\hat{x}_j - i(1-\Delta_f)) - \bar{N}_f\sum_j\ln\tilde{s}_1(-\sqrt{\tau}\hat{x}_j-i(1-\bar{\Delta}_f))\Bigg], \tag{A1}$$

where $\tilde{s}_1(x)$ is a generating function of the small-x expansion of $s_1(x)$:

$$\tilde{s}_1(x) = \exp\left[-2ix\sum_{\ell=0}^{\infty}\frac{\zeta(2\ell)}{2\ell+1}(-x^2)^\ell\right]. \tag{A2}$$

To show the uniform convergence of the small-τ expansion, we apply Weierstrass's M-test, which asks if one can find a sequence $\{M_\ell\}$ satisfying $|h_\ell(\hat{x})| < M_\ell$ and $\sum_{\ell=0}^{\infty} M_\ell < \infty$ for fixed τ. Indeed, we can easily construct such a series. For instance, since $\zeta(\ell\geq 2) < 2$ and $\hat{x}\leq 1$, a generating function $\bar{h}(\tau)$ of M_ℓ can be obtained by the replacement in (A1):

$$(-1)^{\ell+1}\zeta(2\ell)(\hat{x}_i-\hat{x}_j)^{2\ell} \to 2,$$

$$-\ln\tilde{s}_1(\sqrt{\tau}\hat{x}-i(1-\Delta)) \to 4\sum_{\ell=0}^{\infty}\frac{(\sqrt{\tau}+|1-\Delta|)^{2\ell+1}}{2\ell+1} = 2\log\frac{1+\sqrt{\tau}+|1-\Delta|}{1-\sqrt{\tau}-|1-\Delta|},$$

which leads us to

$$\bar{h}(\tau) = \frac{2^{N^2-N}Z_{\rm VdM}(\hat{x})}{(1-\tau)^{2(N^2-N)}}\left(\frac{1+\sqrt{\tau}+|1-\Delta_a|}{1-\sqrt{\tau}-|1-\Delta_a|}\right)^{2N^2N_a} \times\left(\frac{1+\sqrt{\tau}+|1-\Delta_f|}{1-\sqrt{\tau}-|1-\Delta_f|}\right)^{2NN_f}\left(\frac{1+\sqrt{\tau}+|1-\bar{\Delta}_f|}{1-\sqrt{\tau}-|1-\bar{\Delta}_f|}\right)^{2N\bar{N}_f}.$$

Thus, the small-τ expansion of $h(\tau,\hat{x})$ is uniform convergent. This implies that $h(\tau,\hat{x})$ is the analytic continuation of the convergent series, and we can exchange the power series expansion of $h(\tau,\hat{x})$ and the integration over $\hat{x}$. Therefore, $f(\tau)$ is also identical to an analytic continuation of the convergent series. Finally, the integral transformation (8) gives (10).

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[18] Strictly speaking, we consider first ${\rm Im}(g) = \epsilon {\rm sgn}(k)$ and take $\epsilon \to +0$. This prescription is usually adopted in perturbative computation of CS-type matrix models by using the Gaussian matrix model.

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