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Higher derivative massive spin-3 models in $D = 2+1$

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We find new higher derivative models describing a parity doublet of massive spin-3 modes in $D = 2+1$ dimensions. One of them is of fourth order in derivatives while the other one is of sixth order. They are complete, in the sense that they contain the auxiliary scalar field required to remove spurious degrees of freedom. Both of them are obtained through the master action technique starting with the usual (second-order) spin-3 Singh-Hagen model, which guarantees that they are ghost free. The fourth- and sixth-order terms are both invariant under (transverse) Weyl transformations, quite similarly to the fourth-order K-term of the "new massive gravity." The sixth-order term slightly differs from the product of the Schouten by the Einstein tensor, both of third order in derivatives. It is also possible to write down the fourth-order term as a product of a Schouten-like by an Einstein-like tensor (both of second order in derivatives) in close analogy with the K-term.

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I. INTRODUCTION

The coupling of higher spin particles to themselves and to matter fields is a long-standing problem. In the spin-2 case, for both massless and modern massive gravity theories, the geometrical approach plays a crucial role in the coupling problem. Regarding massive spin-2 theories, the case of $D = 2+1$ dimensions is specially interesting from the geometrical perspective. It is possible to describe self-interacting massive spin-2 particles in $D = 2+1$ in a purely geometrical approach via the so called "new massive gravity" (NMG) [1] which is a fourth-order theory in derivatives but still ghost free. It would be certainly interesting to generalize it to higher spins, and we can start with the spin-3 case. The geometry of spin-3 particles has been already discussed in $D = 2+1$ in [2] and more recently in [3]. A spin-3 analogue of the NMG has been considered in [4]; however, it does contain ghosts. Another approach to such problem is to try to generalize to spin-3 the works [5,6]where the helicity $+2$ and the helicity -2 eingenstates represented by the linearized topologically massive gravity of [7] have been joined together (soldered) into a parity doublet which is exactly the linearized version of the NMG. Since a spin-3 analogue of the topologically massive gravity has been suggested in [2], one might try to solder two of such models into a parity doublet with both helicities $+3$ and -3. It turns out that such a procedure is not straightforward for spin-3, and we have not yet been able to implement it. It is still under investigation.

Here, we follow instead basically the same approach of [1] where the NMG has been deduced from the Fierz-Pauli [8] massive spin-2 theory via a master action [9]. We replace the Fierz-Pauli theory by the massive spin-3 Singh-Hagen [10] model. A key ingredient in the master action technique is to identify a trivial (no particle content) term in the starting theory which might be used to mix the old and new (dual) fields. In this sense we start by briefly reviewing the Singh-Hagen theory stressing the triviality of its massless limit just like the massless limit of the Fierz-Pauli theory (linearized Einstein-Hilbert) which has no particle content in $D = 2+1$. By using the massless limit of the Singh-Hagen model as a mixing term in the master action approach, we obtain a fourth-order spin-3 model $S^{(4)}$. It reminds us of the NMG, since the spin-3 fourth-order term can be written as a product of a Schouten-like ($\mathbb{S}_{\mu\nu\lambda}$) by an Einstein-like ($\mathbb{G}_{\mu\nu\lambda}$) tensor (Fronsdal tensor), both of second order in derivatives, just like their spin-2 counterparts [11]. The spin-3 Schouten-like tensor that we have used emerges automatically during the procedure, but differs from the usual (third order) definitions present for example in the recent work [3].

We finish our work introducing a sixth-order model $S^{(6)}$, which also reminds us of the linearized NMG, since the sixth-order term can be written almost completely in terms of the product of the usual Schouten tensor ($S_{\mu\nu\lambda}$) by the usual Einstein tensor ($G_{\mu\nu\lambda}$), both of third order in derivatives. Both fourth- and sixth-order models are invariant under spin-3 reparametrizations $\delta\phi_{\mu\nu\lambda} = \partial_{(\mu}\tilde{\xi}_{\nu\lambda)}$, with $\eta^{\nu\lambda}\tilde{\xi}_{\nu\lambda} = 0$. The sixth-order model is also invariant under a linearized transverse Weyl transformation $\delta\phi_{\mu\nu\rho} = \eta_{(\mu\nu}\zeta^T_{\rho)}$. They describe a parity doublet of helicities ± 3 and are complete in the sense that they inherit the scalar auxiliary field, required to get rid of ghosts, from the Singh-Hagen model.

In Sec. II we obtain the model $S^{(4)}$ from the Singh-Hagen model. In Sec. III we obtain the sixth-order model $S^{(6)}$

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from $S^{(4)}$. We draw our conclusions and perspectives in Sec. IV.

II. FROM THE SINGH-HAGEN THEORY TO $S^{(4)}$

In $D = 2 + 1$ a doublet of helicities $+3$ and -3 particles can be described by the Singh-Hagen model [10]:

$$S_{SH} = \int d^3x \left[\frac{1}{2}\phi_{\mu\nu\lambda}\mathbb{G}_{\mu\nu\lambda}(\phi) - \frac{m^2}{2}(\phi_{\mu\nu\lambda}\phi^{\mu\nu\lambda} - 3\alpha\phi_\mu\phi^\mu) - 3ma\phi_\mu\partial^\mu W + \frac{bm^2}{2}W^2 + \frac{c}{2}W\Box W\right]. \quad (1)$$

The spin-3 field $\phi_{\mu\nu\lambda}$, with trace $\phi_\lambda = \eta^{\mu\nu}\phi_{\mu\nu\lambda}$, is totally symmetric. We have introduced the auxiliary scalar field W through the constants a, b and c, so far arbitrary, in order to remove spurious degrees of freedom. Here, we have used the second-order Fronsdal tensor (Einstein-like) $\mathbb{G}_{\mu\nu\lambda}$ introduced in [2], which is given by

$$\mathbb{G}_{\mu\nu\lambda} \equiv \mathbb{R}_{\mu\nu\lambda} - \frac{1}{2}\eta_{(\mu\nu}\mathbb{R}_{\lambda)}. \quad (2)$$

The "Ricci" tensor is given in terms of $\phi_{\mu\nu\lambda}$ as follows:

$$\mathbb{R}_{\mu\nu\lambda} = \Box\phi_{\mu\nu\lambda} - \partial^\alpha\partial_{(\mu}\phi_{\alpha\nu\lambda)} + \partial_{(\mu}\partial_\nu\phi_{\lambda)}, \quad (3)$$

while its trace is $\mathbb{R}_\lambda = \eta^{\mu\nu}\mathbb{R}_{\mu\nu\lambda} = 2\Box\phi_\lambda - 2\partial^\mu\partial^\nu\phi_{\mu\nu\lambda} + \partial_\lambda(\partial\cdot\phi)$. Along this work, we use unnormalized symmetrization, meaning the minimal sum of terms to achieve symmetry, for instance, $\partial_{(\alpha}\partial_\beta\phi_{\gamma)} = \partial_\alpha\partial_\beta\phi_\gamma + \partial_\alpha\partial_\gamma\phi_\beta + \partial_\beta\partial_\gamma\phi_\alpha$.

As demonstrated in [12], in order to get rid of the spin-1 modes given by the transverse vectors ϕ_μ^T and $(\partial^\mu\partial^\nu\phi_{\mu\nu\lambda})^T$, one needs to write the equations of motion as a system of homogeneous equations:

$$\mathbb{M}\begin{pmatrix}\phi_\mu^T \\ (\partial^\mu\partial^\nu\phi_{\mu\nu\lambda})^T\end{pmatrix} \equiv \begin{pmatrix}\alpha\Box & -1 \\ \Box + \frac{m^2}{3}(5\alpha - 1) & -1\end{pmatrix}\begin{pmatrix}\phi_\mu^T \\ (\partial^\mu\partial^\nu\phi_{\mu\nu\lambda})^T\end{pmatrix} = 0, \quad (4)$$

where only trivial solutions are reached by imposing that $det\mathbb{M} \neq 0$. With this requirement, one has $\alpha = 1$, and in consequence, $\phi_\mu^T = 0 = (\partial^\mu\partial^\nu\phi_{\mu\nu\lambda})^T$. In order to guarantee that the remaining longitudinal terms (spin-0 modes) given by $\partial^\mu\phi_\mu, \partial^\mu\partial^\nu\partial^\lambda\phi_{\mu\nu\lambda}$ vanish, one needs to look for the scalar part of the equations of motion and write them again as a system of equations. Taking $\alpha = 1$, only trivial solutions $\partial^\mu\phi_\mu = 0 = \partial^\mu\partial^\nu\partial^\lambda\phi_{\mu\nu\lambda}$ will be obtained by setting the arbitrary coefficients to $a = 1/3$, $b = 18$ and $c = -8/3$.[1]

The spin-3 second-order term in (1), as in the spin-2 case, leads to the equations of motion $\mathbb{G}_{\mu\nu\lambda}(\phi) = 0$, which in $D = 2 + 1$ implies a pure gauge solution

$$\phi_{\mu\nu\lambda} = \partial_{(\mu}\tilde{\Lambda}_{\nu\lambda)}, \quad (5)$$

with traceless parameter $\tilde{\Lambda} = \eta^{\mu\nu}\tilde{\Lambda}_{\mu\nu} = 0$. This proves the trivial (pure gauge) nature of such term in $D = 2 + 1$ dimensions, and we can use it as a "mixing term" between dual fields in order to construct a master action. From the Singh-Hagen model, given in formula (1) with $(\alpha, a, b, c) = (1, 1/3, 18, -8/3)$, we build up the following master action:

$$S_M^{(2)} = \int d^3x \left[\frac{1}{2}\phi_{\mu\nu\lambda}\mathbb{G}_{\mu\nu\lambda}(\phi) - \frac{m^2}{2}(\phi_{\mu\nu\lambda}\phi^{\mu\nu\lambda} - 3\phi_\mu\phi^\mu) - m\phi_\mu\partial^\mu W - \frac{1}{2}(\phi - \psi)_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(\phi - \psi)\right] + S_1[W], \quad (6)$$

where we have introduced a dual field $\psi_{\mu\nu\alpha}$ and added a mixing second-order term. The auxiliary action is

$$S_1[W] = \int d^3x \left(9m^2W^2 - \frac{4}{3}W\Box W\right). \quad (7)$$

In order to interpolate among the dual models, let us introduce a source term $j_{\mu\nu\lambda}$ coupled to the totally symmetric field $\phi_{\mu\nu\lambda}$ and define the generating functional for the master action (6):

$$W_M[j] = \int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}W \exp i\left(S_M^{(2)} + \int d^3x\, j_{\mu\nu\lambda}\phi^{\mu\nu\lambda}\right). \quad (8)$$

By making the shift $\psi \to \psi + \phi$, the master action becomes the Singh-Hagen theory, $S_M \Rightarrow S_{SH}$, since $\psi_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}$ has no particle content. This shows that the master action $S_M^{(2)}$ describes one doublet of helicities ± 3. On the other hand, rearranging the action without any shift we have

$$S_M = \int d^3x \left[-\frac{1}{2}\psi_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(\psi) - \frac{m^2}{2}(\phi_{\mu\nu\lambda}\phi^{\mu\nu\lambda} - 3\phi_\mu\phi^\mu) + \phi_{\mu\nu\lambda}M^{\mu\nu\lambda}\right] + S_1[W], \quad (9)$$

where $M^{\mu\nu\lambda}$ is given by

[1] The values for b and c are actually given in terms of a. Here, in order to compare with [12], we have set $a = 1/3$.

$$M^{\mu\nu\lambda} = \mathbb{G}^{\mu\nu\lambda}(\psi) - \frac{m}{3}\eta^{(\mu\nu}\partial^{\lambda)}W + j^{\mu\nu\lambda}. \qquad (10)$$

Gaussian integrating over $\phi_{\mu\nu\lambda}$, we have

$$S_M^{(2)} = \int d^3x\left[-\frac{1}{2}\psi_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(\psi) + \frac{1}{2m^2}\left(M_{\mu\nu\lambda}M^{\mu\nu\lambda} - \frac{3}{4}M_\lambda M^\lambda\right)\right] + S_1[W], \qquad (11)$$

and after substituting back $M^{\mu\nu\lambda}$ from (10) in (11), we obtain the fourth-order theory

$$S^{(4)} = \int d^3x\left[-\frac{1}{2}\psi_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(\psi) + \frac{1}{2m^2}\mathbb{S}_{\mu\nu\lambda}(\psi)\mathbb{G}^{\mu\nu\lambda}(\psi) + \frac{1}{12m}\psi_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(\eta\partial W) + j_{\mu\nu\lambda}F^{\mu\nu\lambda} + \mathcal{O}(j^2)\right] + S_2[W], \qquad (12)$$

where $\eta\partial W$ stands for the fully symmetric tensor $\eta_{(\mu\nu}\partial_{\rho)}W = \eta_{\mu\nu}\partial_\rho W + \eta_{\nu\rho}\partial_\mu W + \eta_{\rho\mu}\partial_\nu W$, while

$$F^{\mu\nu\lambda} = \frac{1}{m^2}S^{\mu\nu\lambda}(\psi) + \frac{1}{12m}\eta^{(\mu\nu}\partial^{\lambda)}W, \qquad (13)$$

$$S_2[W] = \int d^3x\left(9m^2W^2 - \frac{9}{8}W\Box W\right). \qquad (14)$$

The fourth-order term has been defined with the help of the spin-3 Schouten-like tensor[2] $\mathbb{S}_{\mu\nu\lambda}$ given in terms of the Fronsdal (Einstein-like) tensor (2):

$$\mathbb{S}_{\mu\nu\lambda}(\psi) = \mathbb{G}_{\mu\nu\lambda}(\psi) - \frac{1}{4}\eta_{(\mu\nu}\mathbb{G}_{\lambda)}(\psi) = \mathbb{R}_{\mu\nu\lambda}(\psi) - \frac{1}{8}\eta_{(\mu\nu}\mathbb{R}_{\lambda)}(\psi). \qquad (15)$$

In this sense one could interpret this fourth-order term as the analogue of the K-term of [1] since as observed by the authors of [11] the K-term can be written in terms of the spin-2 Schouten tensor as $K = S_{\mu\nu}G^{\mu\nu}$ where in that case $S_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/4$ with $G_{\mu\nu}$ the Einstein tensor. Thus, in a certain point of view, the fourth model (12) corresponds to a spin-3 version of the linearized new massive gravity, differently from the analogy proposed in [4] where the authors have found a fifth-order massive spin-3 model which describes two excitations, but one of them is a ghost. The fact that the action (12) is equivalent to $S_M^{(2)}$, which on its turn is equivalent to the Singh-Hagen model (1), guarantees that $S^{(4)}$ is ghost free and describes the two helicities ± 3.

In an alternative way one could write the $\psi\psi$ piece of the action (12) as its spin-2 analogue [1]:

$$S_{\psi\psi}^{(4)} = \int d^3x\left[-\frac{1}{2}\psi_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(\psi) + \frac{1}{2m^2}\left(\mathbb{R}_{\mu\nu\lambda}\mathbb{R}^{\mu\nu\lambda} - \frac{15}{16}\mathbb{R}_\mu\mathbb{R}^\mu\right)_{\psi\psi}\right]. \qquad (16)$$

As in that case, the second-order term, which plays the role of the Einstein-Hilbert term also appears with the "wrong" sign in the action.

The dual field $F^{\mu\nu\lambda}$ in (12) allows us to obtain the dual map given by

$$\phi^{\mu\nu\lambda} \leftrightarrow F^{\mu\nu\lambda}. \qquad (17)$$

Taking derivatives with respect to the source in (8) and (12), one has the equivalence of the following correlation functions:

$$\langle\phi_{\mu_1\nu_1\lambda_1}\ldots\phi_{\mu_N\nu_N\lambda_N}\rangle_{S_M^{(2)}} = \langle F_{\mu_1\nu_1\lambda_1}\ldots F_{\mu_N\nu_N\lambda_N}\rangle_{S^{(4)}} + C.T., \qquad (18)$$

which guarantees the quantum equivalence between the Singh-Hagen model and the fourth-order model. The $C.T.$ terms in (18) are contact terms due to quadratic terms on the sources in (12). The same dual map at the level of the equations of motion becomes evident in the classical equivalence. In order to see this we can first consider the symmetric sector in both theories. The equations of motion with respect to $\phi_{\mu\nu\lambda}$ in the Singh-Hagen model give

$$\mathbb{G}^{\mu\nu\lambda}(\phi) - m^2(\phi^{\mu\nu\lambda} - \eta^{(\mu\nu}\phi^{\lambda)}) - \frac{m}{3}\eta^{(\mu\nu}\partial^{\lambda)}W = 0. \qquad (19)$$

On the other hand, the equations of motion with respect to $\psi_{\mu\nu\lambda}$ in (12) are

$$-\mathbb{G}^{\mu\nu\lambda}(\psi) + \frac{1}{m^2}\mathbb{G}^{\mu\nu\lambda}[\mathbb{S}(\psi)] + \frac{1}{12m}\mathbb{G}^{\mu\nu\lambda}(\eta\partial W) = 0, \qquad (20)$$

where we have used the self-adjoint property of the operator $\mathbb{G}_{\mu\nu\lambda}$ and the commutativity between the operators $\mathbb{S}_{\mu\nu\lambda}$ and $\mathbb{G}_{\mu\nu\lambda}$ in the sense that when integrated $\mathbb{S}_{\mu\nu\lambda}(\psi)\mathbb{G}^{\mu\nu\lambda}(\phi) = \mathbb{S}_{\mu\nu\lambda}(\phi)\mathbb{G}^{\mu\nu\lambda}(\psi)$. In (20) one can notice that it is possible to rewrite the last two terms in terms of the dual field $F_{\mu\nu\lambda}$ given by (17), giving us a unique term $\mathbb{G}_{\mu\nu\lambda}(F)$ which is the equivalent of the first term in (19). The

[2] Originally, the spin-3 Schouten tensor $S_{\mu\nu\rho}$ is of third order in derivatives, see [3] and also (37) where it is defined in terms of the Einstein tensor $G_{\mu\nu\rho}$ which is also of third order, see (36). In $D = 2+1$ dimensions $G_{\mu\nu\rho}$ is dual to a rank-6 spin-3 Riemann tensor $R_{[\mu\nu][\alpha\beta][\rho\gamma]}$. On the other hand, the second-order Fronsdal tensor that we have used here was once interpreted as an Einstein tensor in [2], our definition of $\mathbb{S}_{\mu\nu\lambda}$ as a Schouten-like tensor mimics (37).

rest of the equivalence can be achieved by observing that using (13) we have

$$\mathbb{G}_{\mu\nu\lambda}(\psi) = m^2(F^{\mu\nu\lambda} - \eta^{(\mu\nu}F^{\lambda)}) + \frac{m}{3}\eta^{(\mu\nu}\partial^{\lambda)}W. \quad (21)$$

Thus, (20) becomes

$$\mathbb{G}^{\mu\nu\lambda}(F) - m^2(F^{\mu\nu\lambda} - \eta^{(\mu\nu}F^{\lambda)}) - \frac{m}{3}\eta^{(\mu\nu}\partial^{\lambda)}W = 0, \quad (22)$$

showing us that the equations of motion derived from the fourth-order model can be written in the same form of the Singh-Hagen equations of motion (19). For a complete proof of equivalence, we also need to compare the equations of motion of the scalar field W in both formulations. Regarding that point, we notice from (14)) that the auxiliary action has been automatically corrected. This kind of correction has been already observed in the case of the maps among the spin-3 self-dual models [13,14]. This corrects the description from one formulation to another one, preventing ghosts. Besides the auxiliary field action, the linking term between the W and the rank-3 tensor has also been modified, in comparison with $mW\partial_\mu\phi^\mu$ in (6); namely, we have now $-W\partial_\mu\mathbb{G}^\mu(\psi)/(4m)$ in (12), see (A5) in the Appendix. The equation of motion of the W field from $S^{(4)}$, neglecting sources, is given by

$$18m^2W - \frac{9}{4}\Box W - \frac{1}{4m}\partial_\mu\mathbb{G}^\mu = 0, \quad (23)$$

while from the Singh-Hagen model (1) we have

$$18m^2W - \frac{8}{3}\Box W + m\partial_\mu\phi^\mu = 0. \quad (24)$$

The reader can check that (23) is equivalent to

$$18m^2W - \frac{8}{3}\Box W + m\partial_\mu F^\mu = 0, \quad (25)$$

which is of the same form of (24) with the dual map (17). This completes the proof of equivalence of the equations of motion of the fourth- and second-order theories $S^{(4)}$ and S_{SH}, respectively.

III. FROM $S^{(4)}$ TO $S^{(6)}$

An important difference between the spin-3 fourth-order term $\mathbb{S}_{\mu\nu\rho}\mathbb{G}^{\mu\nu\rho}$ of (12) and the spin-2 K-term $S_{\mu\nu}G^{\mu\nu}$ however is that the spin-2 K-term contains one massless mode in the spectrum, whereas we are going to show here that its spin-3 analogue has no particle content. In order to analyze the particle content of the fourth-order term in (12), let us consider the lower-order action:

$$S[\psi,\phi] = \frac{9}{m^2}\int d^3x\left[\phi_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(\psi) - \frac{1}{2}(\phi_{\mu\nu\lambda}\phi^{\mu\nu\lambda} - 3\alpha\phi_\mu\phi^\mu)\right]. \quad (26)$$

Notice that, by Gaussian integrating over $\phi_{\mu\nu\lambda}$ in (26), we have a fourth-order term for $\psi_{\mu\nu\alpha}$. On the other hand, if we first integrate over $\psi_{\mu\nu\lambda}$ in (26), we have $\mathbb{G}_{\mu\nu\lambda}(\phi) = 0$, which in turn implies the pure gauge solution $\phi_{\mu\nu\lambda} = \partial_{(\mu}\tilde{\Lambda}_{\nu\lambda)}$. Substituting back this result in the nonderivative term of (26), we have the rank-2 traceless theory below

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}(\phi_{\mu\nu\lambda}\phi^{\mu\nu\lambda} - 3\alpha\phi_\mu\phi^\mu)|_{\phi_{\mu\nu\lambda}=\partial_{(\mu}\tilde{\Lambda}_{\nu\lambda)}}\\ &= \frac{3}{2}[\tilde{\Lambda}_{\mu\nu}\Box\tilde{\Lambda}^{\mu\nu} + a(\partial^\mu\tilde{\Lambda}_{\mu\nu})^2], \quad (27)\end{aligned}$$

where $a = 4\alpha - 2$, and we have redefined the tensors in order to get rid of the overall factor $9/m^2$. In the case of the usual spin-3 mass term $\alpha = 1$, ($a = 2$), the traceless model (27) becomes exactly the WTDIFF model in $D = 3$, see [15], which has no particle content in $D = 3$. On the other hand, the integral over $\phi_{\mu\nu\lambda}$ in (26) with $\alpha = 1$ gives precisely the fourth-order term obtained in (12), i.e., the spin-3 K-term. Therefore, the fourth-order term $\mathbb{S}_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}$ has no particle content, and it can be used as a mixing term in a new master action as follows.

Back in the fourth-order model (12), but now with an extra mixing term, of fourth order in derivatives, we have a new master action:

$$\begin{aligned}S_M^{(2)} = \int d^3x\Big[&-\frac{1}{2}\psi_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(\psi) + \frac{1}{2m^2}\mathbb{S}_{\mu\nu\lambda}(\psi)\mathbb{G}^{\mu\nu\lambda}(\psi)\\ &+ \frac{1}{12m}\psi_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(\eta\partial W) + j_{\mu\nu\lambda}F^{\mu\nu\lambda}\\ &- \frac{1}{2m^2}\mathbb{S}_{\mu\nu\lambda}(\psi - \Phi)\mathbb{G}^{\mu\nu\lambda}(\psi - \Phi)\Big] + S_2[W]. \quad (28)\end{aligned}$$

Making the shift $\Phi \to \Phi + \psi$, we have essentially $S_M^{(4)} \Rightarrow S^{(4)}$ due to the lack of content of $\mathbb{S}_{\mu\nu\rho}[\Phi]\mathbb{G}^{\mu\nu\rho}[\Phi]$. On the other hand, without any shift the action can be written in the following way:

$$\begin{aligned}S_M^{(4)} = \int d^3x\Big[&-\frac{1}{2}(\psi - N)_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(\psi - N)\\ &- \frac{1}{2m^2}\mathbb{S}_{\mu\nu\lambda}(\Phi)\mathbb{G}^{\mu\nu\lambda}(\Phi) + \frac{1}{2}N_{\mu\nu\lambda}\mathbb{G}^{\mu\nu\lambda}(N)\Big]\\ &+ S_2[W, j], \quad (29)\end{aligned}$$

where we have introduced the convenient variable

$$N^{\mu\nu\lambda} = \frac{1}{m^2} S^{\mu\nu\lambda}(\Phi) + \frac{1}{12m}\eta^{(\mu\nu}\partial^{\lambda)}W + \frac{1}{m^2}\tilde{j}^{\mu\nu\lambda}, \tag{30}$$

with $\tilde{j}_{\mu\nu\lambda} = j_{\mu\nu\lambda} - \eta_{(\mu\nu}j_{\lambda)}/4$. Shifting $\psi_{\mu\nu\lambda} \to \psi_{\mu\nu\lambda} + N_{\mu\nu\lambda}$, we decouple a trivial Fronsdal term $\psi\mathbb{G}(\psi)$, and one can functionally integrate over $\psi_{\mu\nu\lambda}$. After that, we end up with a sixth-order theory in terms of $\Phi_{\mu\nu\lambda}$:

$$\begin{aligned} S^{(6)} = \int d^3x \Big[&-\frac{1}{2m^2}\mathbb{S}_{\mu\nu\lambda}(\Phi)\mathbb{G}^{\mu\nu\lambda}(\Phi) \\ &+ \frac{1}{2m^4}\mathbb{S}_{\mu\nu\lambda}(\Phi)\mathbb{G}^{\mu\nu\lambda}[\mathbb{S}(\Phi)] \\ &+ \frac{1}{12m^3}\mathbb{S}_{\mu\nu\lambda}(\Phi)\mathbb{G}^{\mu\nu\lambda}(\eta\partial W) \\ &+ j_{\mu\nu\lambda}H^{\mu\nu\lambda} + \mathcal{O}(j^2)\Big] + S_3[W]. \end{aligned} \tag{31}$$

In terms of spin-3 second-order curvature-like tensors, see Appendix, we can write

$$S^{(6)} = \int d^3x \left\{ \mathbb{R}_{\mu\nu\alpha}\frac{(\Box - m^2)}{2m^4}\left[\mathbb{R}^{\mu\nu\alpha} - \frac{5}{16}\eta^{(\mu\nu}\mathbb{R}^{\alpha)}\right] + \frac{9}{256m^4}(\partial_\mu\mathbb{R}^\mu)^2 + j_{\mu\nu\rho}H^{\mu\nu\rho}\right\} + S_3[W]. \tag{32}$$

Through the dual field $H^{\mu\nu\lambda}$, one has a dual map between the second-order Singh-Hagen theory and the sixth-order model (31):

$$\phi_{\mu\nu\lambda} \leftrightarrow H_{\mu\nu\lambda} \equiv \frac{1}{m^2}\mathbb{S}_{\mu\nu\lambda}\left[\frac{1}{m^2}\mathbb{S}(\Phi) + \frac{1}{12m}\eta\partial W\right] + \frac{1}{12m}\eta_{(\mu\nu}\partial_{\lambda)}W. \tag{33}$$

As we have done before, this dual map guarantees the quantum equivalence between the theories. By taking derivatives with respect to the source term in (28) and (31) and using (18), we have the equivalence between the correlation functions in the Singh-Hagen model and in $S^{(6)}$:

$$\langle\phi_{\mu_1\nu_1\lambda_1}\ldots\phi_{\mu_N\nu_N\lambda_N}\rangle_{S_{SH}} = \langle H_{\mu_1\nu_1\lambda_1}\ldots H_{\mu_N\nu_N\lambda_N}\rangle_{S^{(6)}} + \text{C.T.} \tag{34}$$

By using the self-adjoint property of the operator $\mathbb{G}_{\mu\nu\lambda}$ and the commutativity between the tensors $\mathbb{S}_{\mu\nu\lambda}$ and $\mathbb{G}_{\mu\nu\lambda}$, one could, as before, obtain the classical equivalence at the level of the equations of motion. Finally, the linking term between W and the rank-3 field has also been changed, see (A6). The auxiliary action $S_3[W]$ has now an extra higher derivative term:

$$S_3[W] = \int d^3x\left(9m^2W^2 - \frac{9}{4}W\Box W + \frac{9}{64m^2}W\Box^2W\right). \tag{35}$$

In the Appendix we give the explicit expressions for the sixth-order term and the linking term between W and $\Phi_{\mu\nu\rho}$ as a function of $\Phi_{\mu\nu\rho}$.

Since the usual spin-3 Einstein tensor $G^{\mu\nu\lambda}$ and the usual spin-3 Schouten tensor $S_{\mu\nu\lambda}$, see [3,11], are defined in terms of third-order differential operators[3]:

$$G_{\mu\nu\lambda} = \frac{1}{18}E_{(\mu}{}^{\alpha}E_{\nu}{}^{\beta}E_{\lambda)}{}^{\gamma}\Phi_{\alpha\beta\gamma} \tag{36}$$

$$S_{\mu\nu\lambda} = G_{\mu\nu\lambda} - \frac{1}{4}\eta_{(\mu\nu}G_{\lambda)} \tag{37}$$

with $E_{\mu\nu} = \epsilon_{\mu\nu\alpha}\partial^\alpha$, we may think[4] that the sixth-order term of $S^{(6)}$ might be a product of the type $S_{\mu\nu\lambda}G^{\mu\nu\lambda}$ just like the NMG fourth-order term is a product of the two second-order tensors $S_{\mu\nu}G^{\mu\nu}$. It turns out that this is almost true:

$$\begin{aligned} S^{(6)} = \int d^3x \Big[&-\frac{1}{2m^2}\mathbb{S}_{\mu\nu\lambda}(\Phi)\mathbb{G}^{\mu\nu\lambda}(\Phi) + \frac{1}{2m^4}S_{\mu\nu\lambda}(\Phi)G^{\mu\nu\lambda}(\Phi) \\ &+ \frac{1}{256m^4}(\partial_\mu\mathbb{R}^\mu)^2 + \frac{1}{12m^3}\mathbb{S}_{\mu\nu\lambda}(\Phi)\mathbb{G}^{\mu\nu\lambda}(\eta\partial W) \\ &+ j_{\mu\nu\lambda}H^{\mu\nu\lambda} + \mathcal{O}(j^2)\Big] + S_3[W]. \end{aligned} \tag{38}$$

The last term $(\partial_\mu\mathbb{R}^\mu)^2$ frustrates our expectations. We remark that such a term cannot be canceled by any field redefinition $\Phi_{\mu\nu\rho} \to \Phi_{\mu\nu\rho} + c\eta_{(\mu\nu}\Phi_{\rho)}$ where c is a constant.

Regarding local symmetries, a comment is in order. One can verify that both the fourth- and sixth-order terms in $S^{(6)}$ are invariant under the transformation

$$\delta\Phi^{\mu\nu\lambda} = \partial^{(\mu}\tilde{\xi}^{\nu\lambda)}, \tag{39}$$

with $\tilde{\xi} = \eta^{\mu\nu}\tilde{\xi}_{\mu\nu} = 0$. Since (39) is not a symmetry of the starting model (1), due to the mass term, it seems that we might be able to obtain $S^{(4)}$ from the Singh-Hagen model (1) alternatively via Noether gauge embedding (NGE) of (39). The same procedure when applied to the Fierz-Pauli theory (nongauge theory) in $D = 2+1$ leads to the linearized "new massive gravity" of [1] (a gauge theory) as we have shown in [16].[5] We believe that $S^{(6)}$ can also be derived from $S^{(4)}$ via gauge embedding of a local

[3] Our expressions for $G_{\mu\nu\lambda}$ and $S_{\mu\nu\lambda}$ are the usual ones, see, for instance [3]; however, as in the rest of the present paper, we have unnormalized symmetrization.

[4] We thank an anonymous referee for asking this question.

[5] We have also used such a method in [14] in order to obtain higher order (gauge invariant) spin-3 self-dual models from the first-order (nongauge theory) spin-3 self-dual model of [17].

symmetry. The model $S^{(6)}$ is invariant under transverse Weyl transformations:

$$\delta\Phi_{\mu\nu\lambda} = \eta_{(\mu\nu}\zeta^T_{\lambda)}, \tag{40}$$

with $\partial^\mu \zeta^T_\mu = 0$, while the second-order term (the Fronsdal term) of $S^{(4)}$ is not invariant under (40). Therefore, local symmetries are improved as we jump from $S^{(4)}$ to $S^{(6)}$ just like the jump from S_{SH} to $S^{(4)}$. A complete discussion on the symmetry behind the sequence of doublet models and the use of the Noether embedding procedure to derive $S^{(4)}$ and $S^{(6)}$ is in progress, [18].

As a last remark before the conclusions, we go back to the action (26) and notice that in the specific case of $\alpha = 7/8$ ($a = 3/2$), after integrating over $\phi_{\mu\nu\lambda}$, we have a fourth-order theory for the field $\psi_{\mu\nu\rho}$ which is exactly what appears in the last model of a chain of self-dual models obtained in [14]. In this case one can demonstrate that the resulting dual theory for the tensor $\tilde{\Lambda}_{\mu\nu}$ on the right-hand side of (27) contains a ghost. This is the reason why, see [13], the chain of spin-3 self-dual models stops at the fourth-order case contrary to some expectations, see [4], of a possible $2s$ rule for spin-s, which would lead us to a top sixth-order model for $s = 3$.

IV. CONCLUSIONS

We have used the master action technique to obtain higher-order (in derivatives) massive spin-3 gauge invariant models starting with the usual (second-order) Singh-Hagen model. The models include the auxiliary field action needed to guarantee that only spin-3 modes propagate and lower spin ghosts are excluded.

First, we have obtained the fourth-order model $S^{(4)}$, given in (12). In some sense, it can be interpreted as a spin-3 analogue of the linearized "new massive gravity" (NMG) [1]. Its fourth-order term has a structure similar to the spin-2 K-term of [1]; i.e., it is the product of a spin-3 Schouten-like $\mathbb{S}_{\mu\nu\rho}$ by an Einstein-like tensor $\mathbb{G}^{\mu\nu\rho}$. In fact, what we have called Einstein-like tensor is the second-order Fronsdal tensor. The Schouten-like tensor is defined from the Einstein-like tensor in analogy with the definition of the usual (third-order) Schouten tensor from the usual (third-order) Einstein tensor, see (15) and (37). Moreover, just like the linearized K-term is invariant under Weyl transformations $\delta h_{\mu\nu} = \eta_{\mu\nu}\Lambda$ unlike the second-order term (Einstein-Hilbert) of the NMG model, the fourth-order term of $S^{(4)}$ is invariant under a transverse Weyl transformation $\delta\Phi_{\mu\nu\lambda} = \eta_{(\mu\nu}\zeta^T_{\lambda)}$ contrary to the second-order piece of $S^{(4)}$, i.e., the Fronsdal action.

We have explicitly shown the equivalence of the equations of motion of $S^{(4)}$ and of the Singh-Hagen model via the dual map (13). A key difference between NMG and $S^{(4)}$ is the following: The spin-3 product $\mathbb{S}_{\mu\nu\rho}\mathbb{G}^{\mu\nu\rho}$ has no particle content, as we have shown here, while its spin-2 counter part $S_{\mu\nu}G^{\mu\nu}$ contains a massless mode in the spectrum. In the master action approach a dual (higher-order) model can be obtained by using a "mixing term" between dual fields. The spectrum equivalence between the dual models requires the absence of propagating modes in the mixing term [19] which is not possible in the spin-2 case, so we are not able to go beyond the fourth order in the spin-2 case without introducing ghosts. Thanks to the trivial fourth-order term $\mathbb{S}_{\mu\nu\rho}\mathbb{G}^{\mu\nu\rho}$, starting with $S^{(4)}$, we have been able to go even higher and obtain a sixth-order spin-3 ghost free model $S^{(6)}$ via master action, see (32). The sixth-order term of $S^{(6)}$ can be written almost as a product of the usual spin-3 Schouten by the usual Einstein tensors, both of third order, see (38).

The model $S^{(6)}$ describes a parity doublet with both helicities ± 3 while the fourth-order NMG theory describes ± 2 helicities. Since the sixth-order model is apparently the top (highest order in derivatives) spin-3 model, it can also be interpreted as the spin-3 analogue of the linearized NMG theory (highest-order spin-2 model). Recalling that for $s = 1$, the corresponding highest-order massive model is the Maxwell-Proca theory (second order) which describes both helicities ± 1; there seems to be a $2s$ rule for the highest-order spin-s, see comment in [4]. The same $2s$ rule seems to work for the parity singlet models which only describe one helicity mode, either $+s$ or $-s$. This has been confirmed in the spin-1 and spin-2 cases [16]. However, regarding the spin-3 singlets (self-dual models), in previous works [13,14] we have not been able to go beyond the fourth order without introducing ghosts. The model $S^{(6)}$ brings some hope of overcoming this barrier.

Regarding the local symmetries, we recall that dual models obtained via master actions in the spin-1 [9] and spin-2 [20] cases can be alternatively obtained via embedding of local symmetries, see [21] and [16], respectively. Likewise, we believe that is possible to obtain $S^{(4)}$ from S_{SH} via Noether gauge embedding of the traceless reparametrization $\delta\Phi^{\mu\nu\lambda} = \partial^{(\mu}\tilde{\xi}^{\nu\lambda)}$ and $S^{(6)}$ from $S^{(4)}$ via embedding of the transverse Weyl transformation $\delta\Phi_{\mu\nu\lambda} = \eta_{(\mu\nu}\zeta^T_{\lambda)}$; this is currently under investigation as well as the possibility of going beyond the sixth-order barrier.

Last, we know that is possible to systematically join together (solder) the helicity eigenstates (self-dual models) even of different masses into local field theories with both helicities. The soldering procedure works in the spin-1 [22,23] and spin-2 cases [5,6,24], but it has been only partially successfully in the spin-3 case [25]. We believe that the action $S^{(6)}$ may be useful also in order to extend the soldering program from the spin-1 and spin-2 cases to spin-3.

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APPENDIX: EXPLICIT EXPRESSIONS

Here, we give explicit expressions for some of the terms we have found previously. The fourth-order term given in the expression (12) can be written in terms of the fields $\phi_{\mu\nu\lambda}$ as follows:

$$\frac{1}{2m^2}\mathbb{S}_{\mu\nu\lambda}(\phi)\mathbb{G}^{\mu\nu\lambda}(\phi) = \frac{1}{2m^2}\mathbb{R}_{\mu\nu\lambda}\mathbb{R}^{\mu\nu\lambda} - \frac{15}{32m^2}\mathbb{R}_{\mu}\mathbb{R}^{\mu} \tag{A1}$$

$$= \frac{1}{2m^2}\phi_{\mu\nu\lambda}\Box^2\phi^{\mu\nu\lambda} - \frac{3}{2m^2}\phi_{\mu\nu\lambda}\Box\partial^{\mu}\partial_{\alpha}\phi^{\alpha\nu\lambda} + \frac{3}{4m^2}\phi_{\mu}\Box\partial_{\nu}\partial_{\lambda}\phi^{\mu\nu\lambda} + \frac{9}{8m^2}\phi_{\mu\nu\lambda}\partial^{\nu}\partial^{\lambda}\partial_{\alpha}\partial_{\beta}\phi^{\mu\alpha\beta}$$
$$- \frac{9}{8m^2}\phi_{\mu\nu\lambda}\partial^{\mu}\partial^{\nu}\partial^{\lambda}\partial_{\alpha}\phi^{\alpha} + \frac{21}{32m^2}\phi_{\mu}\Box\partial^{\mu}\partial_{\alpha}\phi^{\alpha} - \frac{3}{8m^2}\phi_{\mu}\Box^2\phi^{\mu}, \tag{A2}$$

which is gauge invariant under (39) and (40). In fact, using $\delta_{\xi}\mathbb{R}_{\mu\nu\lambda} = 0$, the gauge invariance of (A1) becomes obvious.

Let us give now an explicit expression for the sixth-order term presented in (31). Inside space-time integrals, we can write

$$\frac{1}{2m^4}\mathbb{S}_{\mu\nu\lambda}(\phi)\mathbb{G}^{\mu\nu\lambda}[\mathbb{S}(\phi)] = \frac{1}{2m^4}\mathbb{R}_{\mu\nu\lambda}\Box\mathbb{R}^{\mu\nu\lambda} - \frac{15}{32m^4}\mathbb{R}_{\mu}\Box\mathbb{R}^{\mu} + \frac{9}{256m^4}(\partial_{\alpha}\mathbb{R}^{\alpha})^2 \tag{A3}$$

$$= \frac{1}{2m^4}\mathbb{S}_{\mu\nu\rho}(\phi)\Box\mathbb{G}^{\mu\nu\rho}(\phi) + \frac{9}{256m^4}(3\Box\partial_{\mu}\phi^{\mu} - 2\partial_{\mu}\partial_{\nu}\partial_{\rho}\phi^{\mu\nu\rho})^2. \tag{A4}$$

About the gauge invariance, it is possible to check that also the sixth-order term is invariant under (39) and (40). Using the self-adjoint property of the Einstein tensor in the linking term of (12), we have (inside integrals)

$$\frac{1}{12m}\eta_{(\mu\nu}\partial_{\lambda)}W\mathbb{G}^{\mu\nu\lambda}(\psi) = \frac{1}{4m}\partial_{\mu}W\mathbb{G}^{\mu}(\psi) = \frac{3}{8m}W\partial_{\mu}\mathbb{R}^{\mu}(\psi)$$
$$= \frac{9}{8m}W\left[\Box\partial_{\mu}\psi^{\mu} - \frac{2}{3}\partial_{\mu}\partial_{\nu}\partial_{\lambda}\psi^{\mu\nu\lambda}\right], \tag{A5}$$

which by its turn guarantees gauge invariance under (39) and (40). Finally, in (31) we have another linking term which can be rewritten with the help of the self-adjoint property of the Einstein tensor as follows:

$$\frac{1}{12m^3}\mathbb{S}_{\mu\nu\lambda}(\Phi)\mathbb{G}^{\mu\nu\lambda}(\eta\partial W) = \frac{1}{12m^3}\eta_{(\mu\nu}\partial_{\lambda)}W\mathbb{G}^{\mu\nu\lambda}[\mathbb{S}(\Phi)] = -\frac{3}{64m^3}W\Box\partial_{\mu}\mathbb{R}^{\mu}(\Phi)$$
$$= -\frac{9}{64m^3}W\Box\left[\Box\partial_{\mu}\Phi^{\mu} - \frac{2}{3}\partial_{\mu}\partial_{\nu}\partial_{\lambda}\Phi^{\mu\nu\lambda}\right]. \tag{A6}$$

Up to a d'Alembertian, the structure of the fifth-order linking term is the same one of the third-order linking term, which makes evident its gauge invariance under (39) and (40).

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