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One-loop chiral perturbation theory with two fermion representations

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We develop chiral perturbation theory for chirally broken theories with fermions in two different representations of the gauge group. Any such theory has a nonanomalous singlet $U(1)_A$ symmetry, yielding an additional Nambu-Goldstone boson when spontaneously broken. We calculate the next-to-leading order corrections for the pseudoscalar masses and decay constants, which include the singlet Nambu-Goldstone boson, as well as for the two condensates. The results can be generalized to more than two representations.

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I. INTRODUCTION

Within the Standard Model, all of the quarks transform in the fundamental representation of the QCD group $SU(3)_c$. However, exotic fermions in higher irreducible representations (irreps) of $SU(3)_c$ have long been considered an intriguing possibility for physics beyond the Standard Model, with a potentially rich phenomenology [1–19]. More generally, fermions in multiple representations of a strongly coupled gauge group can appear in other extensions of the Standard Model, including composite Higgs [20–22] and composite dark matter [23] models. In particular, composite Higgs models of "partial compositeness" [24], in which the elementary top quark mixes with a composite top partner, tend to require the presence of fermions in two different representations of the new strongly coupled gauge group [25–29].

In any of the extensions of the Standard Model noted above, the presence of strong gauge interactions impedes the use of perturbation theory for most quantities of interest. If the strong sector exhibits spontaneous chiral symmetry breaking, then the dynamics of the resulting Nambu-Goldstone bosons (NGBs) can be described by a low-energy effective theory known as chiral perturbation theory (ChPT) [30–33]. ChPT is an invaluable tool for understanding the associated phenomenology, and has been used with great success in the context of QCD. Looking beyond QCD, ChPT is also well understood for the case of an arbitrary number of fermions in a single representation, including complex [34] as well as real or pseudoreal representations [35–38]. However, ChPT has not been systematically explored in the case of a strong sector containing two or more fermion representations.

With fermions in two different representations, the chiral symmetry breaking pattern remains mostly unchanged: if each fermion species r in isolation has an associated global chiral symmetry G_r which is spontaneously broken to H_r, then when multiple species are present the global symmetry contains the product group $G_1 \times G_2 \times \cdots \times G_n$, and the residual unbroken symmetry group is $H_1 \times H_2 \times \cdots \times H_n$. However, this is not the whole story; with two or more fermion representations, additional Abelian axial symmetries appear as linear combinations of the individually anomalous flavor-singlet axial rotations of each fermion species. These additional symmetries are then spontaneously broken, giving rise to singlet NGBs [11,39].

Any additional singlet NGB which appears in a theory with multiple fermion representations is a particularly interesting object. It can play the role of a composite axion [40–44], offering a potential solution to the strong CP problem. In various extensions of the Standard Model, the singlet may provide a candidate to explain the 750 GeV diphoton excess observed by ATLAS and CMS [45,46]; within composite Higgs models it appears quite naturally as a relatively isolated light state with anomaly induced couplings to pairs of Standard Model vector bosons [29,47].

In this paper, we study ChPT through next-to-leading order (NLO) for a theory with two fermion species charged under distinct representations of a confining gauge group; generalization to more than two species is straightforward. All fermion masses for a particular representation are taken to be degenerate for simplicity. We derive formulas for the pseudoscalar masses and decay constants of all states, including the singlet NGB. We also give formulas for the two chiral condensates.

The outline of the paper is as follows: In Sec. II we describe the symmetries and patterns of breaking for the three types of irreps. While being well established, we found it useful to include this discussion, to make this paper more self-contained. In Sec. III we write down the chiral Lagrangian through order p^4 for a theory with two representations of fermions. One-loop results for the pseudoscalar masses, decay constants, and condensates are presented in Sec. IV. We conclude in Sec. V. The three appendixes deal with technicalities.

II. SYMMETRIES AND PATTERNS OF BREAKING

There are three types of irreps: complex, real, and pseudoreal. We consider a vectorlike field content, which implies that fermions in a complex or pseudoreal irrep fit into N Dirac fermions. For a real irrep, we will allow any number N_w of Weyl (or, equivalently, Majorana) fermions. In the chiral limit, where all masses are zero, the familiar symmetry breaking patterns are [48]

$$\begin{aligned} \text{complex:}\ & SU(N)_L \times SU(N)_R \to SU(N)_V, \\ \text{pseudoreal:}\ & SU(2N) \to Sp(2N), \\ \text{real:}\ & SU(N_w) \to SO(N_w). \end{aligned} \tag{2.1}$$

As a natural generalization of the familiar terminology of QCD, for all types of irreps we will refer to the spontaneously broken symmetries as axial symmetries, and to the unbroken ones as vector symmetries. For simplicity, we will consider only mass matrices that do not break explicitly any of the vector symmetries, so that all pions made out of a single fermion species will have the same mass.

A. Symmetry breaking patterns

In this subsection we describe in some detail the symmetry breaking pattern for each type of irrep, and how it is reflected in the field content of the effective chiral theory.

1. Complex representations

We consider N Dirac fermions ψ_i, $\overline{\psi}_i$, where the flavor index is $i = 1, \ldots, N$. We suppress color and Dirac indices. The global symmetry of the massless theory is $SU(N)_L \times SU(N)_R$, which is spontaneously broken to the diagonal subgroup $SU(N)_V$. The effective field Σ takes values in the coset $SU(N)_L \times SU(N)_R/SU(N)_V \cong SU(N)$. It describes the long-distance fluctuations of the bilinears

$$\begin{aligned} \Sigma_{ij} &\leftrightarrow \mathrm{tr}(P_L \psi_i \overline{\psi}_j) = \mathrm{tr}(\psi_{L,i} \overline{\psi}_{R,j}), \\ \Sigma^*_{ij} &\leftrightarrow \mathrm{tr}(P_R \psi_j \overline{\psi}_i) = \mathrm{tr}(\psi_{R,j} \overline{\psi}_{L,i}), \end{aligned} \tag{2.2}$$

where the traces on the right-hand side are over color and Dirac indices, $P_{R,L} = (1 \pm \gamma_5)/2$, and $\psi_{L,R} = P_{L,R}\psi$, $\overline{\psi}_{L,R} = \overline{\psi} P_{R,L}$. The chiral spurion $\chi_{ij}(x)$ is introduced by adding to the Lagrangian of the massless theory the following source term:

$$\mathcal{L}_{src} = \overline{\psi}_L \chi \psi_R + \overline{\psi}_R \chi^\dagger \psi_L. \tag{2.3}$$

The symmetry transformations act as

$$\psi_{L,R} \to g_{L,R} \psi_{L,R}, \qquad \overline{\psi}_{L,R} \to \overline{\psi}_{L,R} g^\dagger_{L,R}, \tag{2.4a}$$

$$\Sigma \to g_L \Sigma g_R^\dagger, \qquad \chi \to g_L \chi g_R^\dagger, \tag{2.4b}$$

where $g_{L,R} \in SU(N)_{L,R}$.

The mass matrix is given by the "expectation value" of the chiral spurion, $M_{ij} = \langle \chi_{ij} \rangle$. By applying an $SU(N)_L \times SU(N)_R$ transformation the mass term can be brought to a diagonal form, $M_{ij} = m_i \delta_{ij}$, where in general m_i are complex numbers. In this paper, we will consider only the equal-mass limit, $m_i = m$, and we take m to be real and positive. The fermion condensate will therefore be oriented in the direction of the identity matrix, $\langle \overline{\psi}_i \psi_j \rangle \propto \delta_{ij}$. Correspondingly, for the effective field we will have $\langle \Sigma_{ij} \rangle = \delta_{ij}$.

2. Real and pseudoreal representations

For any real or pseudoreal irrep there exists a matrix S with the invariance property

$$g^T S g = S, \tag{2.5}$$

for any element g of the gauge group. Here S is a real orthogonal matrix. Equivalently, the Hermitian generators of the Lie algebra T_a satisfy

$$T_a^T S = T_a^* S = -S T_a. \tag{2.6}$$

For a real representation S is symmetric, whereas for a pseudoreal representation it is antisymmetric.

We start by considering again N Dirac fermions, and begin by studying their properties under charge conjugation. The massless action for any number of Dirac fermions in a complex irrep is invariant under charge conjugation, which acts on the fermion and gauge fields as

$$\begin{aligned} \psi &\to C \overline{\psi}^T, \\ \overline{\psi} &\to \psi^T C, \\ A_\mu &\to -A_\mu^*, \end{aligned} \tag{2.7}$$

where the charge-conjugation matrix C satisfies $C\gamma_\mu = -\gamma_\mu^T C$, and $C^{-1} = C^\dagger = C^T = -C$.

For Dirac fermions that belong to a real or a pseudoreal irrep, the massless fermion action is invariant under an additional, similar-looking discrete symmetry that leaves the gauge field invariant, and acts nontrivially on the fermion fields only, according to[1]

$$\begin{aligned} \psi &\to S C \overline{\psi}^T, \\ \overline{\psi} &\to \psi^T C S^T. \end{aligned} \tag{2.8}$$

[1]This is referred to as "anti-unitary" symmetry in Ref. [49].

Because the gauge field is invariant, the transformation (2.8) can be applied to each Dirac fermion individually.

Motivated by this symmetry, we express the microscopic theory in terms of purely left-handed Weyl fermions, $\xi_I \equiv P_L\xi_I \equiv \xi_{L,I}$, $\bar{\xi}_I \equiv \bar{\xi}_I P_R \equiv \bar{\xi}_{L,I}$, where $I = 1, \ldots, 2N$, which are related to the left- and right-handed components of the Dirac fermions via[2]

$$\begin{aligned} \xi_{L,i} &= \psi_{L,i}, \\ \xi_{L,N+i} &= SC\bar{\psi}^T_{R,i}, \\ \bar{\xi}_{L,i} &= \bar{\psi}_{L,i}, \\ \bar{\xi}_{L,N+i} &= \psi^T_{R,i} S^T C. \end{aligned} \tag{2.9}$$

In terms of the Weyl fields, the Lagrangian takes the form

$$\mathcal{L} = \sum_{I=1}^{2N} \bar{\xi}_{L,I} \not{D} \xi_{L,I}, \tag{2.10}$$

which is invariant under the $SU(2N)$ flavor transformation

$$\xi_L \to g\xi_L, \qquad \bar{\xi}_L \to \bar{\xi}_L g^\dagger. \tag{2.11}$$

Because of the Grassmann nature of the field, we have

$$\begin{aligned} \xi^T_{L,I} CS \xi_{L,J} &= \xi^T_{L,J} CS^T \xi_{L,I}, \\ \bar{\xi}_{L,I} CS \bar{\xi}^T_{L,J} &= \bar{\xi}_{L,J} CS^T \bar{\xi}^T_{L,I}. \end{aligned} \tag{2.12}$$

It follows that these bilinears are (anti)symmetric in $I \leftrightarrow J$ when S is (anti)symmetric.

The chiral field Σ now lives in $SU(2N)$, with the correspondence

$$\mathrm{tr}(\xi_{L,I}\xi^T_{L,J} CS) \leftrightarrow \Sigma_{IJ} = s\Sigma_{JI}, \tag{2.13a}$$

$$\mathrm{tr}(\bar{\xi}^T_{L,I}\bar{\xi}_{L,J} CS^T) \leftrightarrow \Sigma^*_{IJ} = s\Sigma^*_{JI}, \tag{2.13b}$$

where it follows from Eq. (2.12) that $s = 1$ ($s = -1$) for a real (pseudoreal) irrep. In both cases we have the transformation rules

$$\Sigma \to g\Sigma g^T, \qquad \chi \to g\chi g^T, \tag{2.14}$$

and the source term in the Lagrangian is now

$$\mathcal{L}_{src} = \bar{\xi}_L \chi CS^T \bar{\xi}^T_L + \xi^T_L CS \chi^\dagger \xi_L. \tag{2.15}$$

For a real irrep, we will allow the number of Weyl fields N_w to be either even or odd. In the latter case, one can then use a Weyl basis or a Majorana basis (see Appendix A), but not a Dirac basis. For all values of N_w, we have that Σ is an element of the coset generated by the broken generators, and thus an element of $SU(N_w)$.

As usual, the symmetry-breaking order parameter is a fermion bilinear, now given by the expectation value of Eq. (2.12). We will assume that the mass matrix orients the fermion condensate such that

$$\langle \xi^T_{L,J} CS \xi_{L,I} \rangle \propto J_{IJ}, \tag{2.16}$$

where J is a real orthogonal matrix, and where J is symmetric (antisymmetric) for a real (pseudoreal) irrep. While we will be making specific choices for the explicit form of the matrix J, our discussion of the chiral effective theory applies assuming only that $\langle\Sigma\rangle = J$, where J has the properties listed above, and, in addition, $\det J = 1$.

For a pseudoreal irrep, we will again assume that the Dirac mass matrix is given by $M_{ij} = m\delta_{ij}$, with $m \geq 0$. When translated to the Weyl basis, the mass term takes the form

$$m\bar{\psi}\psi \to \frac{1}{2} m(\xi^T_L CSJ_A \xi_L + \bar{\xi}_L CSJ_A \bar{\xi}^T_L), \tag{2.17}$$

with

$$J_A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.18}$$

The fermion condensate is oriented in the direction of the $2N \times 2N$ matrix J_A, and the symmetry breaking pattern is $SU(2N) \to Sp(2N)$. Note that $\det J_A = +1$, independent of N. It follows that the ground state is represented in the effective theory as $\langle\Sigma_{IJ}\rangle = (J_A)_{IJ}$, consistent with the fact that $\Sigma \in SU(2N)$.

In the case of a real irrep one can conceive of two simple choices for the mass matrix. First, for any number N_w of Weyl (or Majorana) fermions, we may consider the Majorana mass $M_{IJ} = m\delta_{IJ}$, where again $m \geq 0$. The fermion condensate is then $\propto \delta_{IJ}$, and the ground state of the effective theory is $\langle\Sigma_{IJ}\rangle = \delta_{IJ}$. By taking the chiral limit $m \to 0$, we see that the symmetry breaking pattern is indeed $SU(N_w) \to SO(N_w)$.

In the case of an even number of Majorana fermions, we may regroup the fields into $N = N_w/2$ Dirac fermions. Let us endow these Dirac fermions with a common mass, $M_{ij} = m\delta_{ij}$. Upon translating back to the Weyl or Majorana basis, the mass matrix takes the same form as in Eq. (2.17), except the $2N \times 2N$ matrix J_A gets replaced by J_S, with

$$J_S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.19}$$

Note, however, that

[2]Technically, we define the Weyl fermions as 4-component fields whose right-handed components vanish identically.

$$\det J_S = \begin{cases} +1, & N \text{ even}, \\ -1, & N \text{ odd}. \end{cases} \tag{2.20}$$

Only in the case that the number of Dirac fermions is even (equivalently, the number of Majorana fermions is a multiple of 4) is J_S an element of $SU(2N)$, so that we may assume that $\langle \Sigma_{IJ} \rangle = (J_S)_{IJ}$. In the case of an odd number of Dirac fermions, it is in general not possible to have $\langle \Sigma \rangle = J_S$. This elementary fact is sometimes overlooked in the literature.

B. Parametrization of the coset field: single irrep

In the case of a complex irrep we have $\langle \Sigma_{ij} \rangle = \delta_{ij}$. The expansion around this classical vacuum is facilitated by writing $\Sigma(x) = \hat{U}(x) \in SU(N)$, with

$$\hat{U}(x) = \exp\left(\frac{i\sqrt{2}\pi(x)}{F}\right) = \exp\left(\frac{i\sqrt{2}\pi_a(x)T_a}{F}\right), \tag{2.21}$$

with π_a the Nambu-Goldstone bosons associated with the spontaneous symmetry breaking, and where T_a, $a = 1, \ldots, N^2 - 1$, are the generators of $SU(N)$, normalized as[3]

$$\mathrm{tr}(T_a T_b) = \delta_{ab}, \tag{2.22}$$

and F is the pion decay constant in the chiral limit. Following Ref. [36] we adopt the convention

$$\langle 0|A_{\mu a}(x)|\pi_b\rangle = i p_\mu \sqrt{2} F \delta_{ab} e^{ipx}, \tag{2.23}$$

where A^a_μ is the axial current. Introducing the (external) vector gauge field $v_\mu = v_{\mu a}(x)T_a$ and the axial gauge field $a_\mu(x) = a_{\mu a}(x)T_a$, the covariant derivative takes the form

$$D_\mu \Sigma = D_\mu \hat{U} = \partial_\mu \hat{U} + i[v_\mu, \hat{U}] + i\{a_\mu, \hat{U}\}. \tag{2.24}$$

Moving on to real and pseudoreal irreps, we first split the generators of the global symmetry group $SU(N_w)$ into broken generators $X_{\hat{a}}$ and unbroken generators $Q_{\tilde{a}}$, which satisfy

$$J Q_{\tilde{a}} = -Q^T_{\tilde{a}} J, \tag{2.25}$$

$$J X_{\hat{a}} = +X^T_{\hat{a}} J, \tag{2.26}$$

where the matrix J was introduced in Eq. (2.16). In both cases, the expansion of the nonlinear field can be written as

$$\Sigma(x) = \hat{U}(x) J, \tag{2.27}$$

where

$$\hat{U}(x) = \exp\left(\frac{i\sqrt{2}\pi(x)}{F}\right) = \exp\left(\frac{i\sqrt{2}\pi_{\hat{a}}(x)X_{\hat{a}}}{F}\right). \tag{2.28}$$

It can be verified that $\Sigma(x)$ is symmetric (antisymmetric) for a real (pseudoreal) irrep, as it should be. As before, the vector gauge field is constructed from the unbroken generators, while the axial one is constructed from the broken ones, i.e.,

$$v_\mu = v_{\mu\tilde{a}} Q_{\tilde{a}}, \qquad a_\mu = a_{\mu\hat{a}} X_{\hat{a}}. \tag{2.29}$$

By using the infinitesimal form of the transformation (2.14), the covariant derivative is

$$\begin{aligned} D_\mu \Sigma &= \partial_\mu \hat{U} J + i(v_\mu + a_\mu)\hat{U} J + i\hat{U} J (v_\mu + a_\mu)^T \\ &= (\partial_\mu \hat{U} + i[v_\mu, \hat{U}] + i\{a_\mu, \hat{U}\}) J \\ &\equiv (D_\mu \hat{U}) J. \end{aligned} \tag{2.30}$$

In writing down the chiral Lagrangian it will be convenient to use notation which is as uniform as possible for all three cases. To this end, we generalize Eq. (2.27) to the case of a complex irrep by simply taking J to be the $N \times N$ identity matrix in this case. In all three cases, complex, real, and pseudoreal, the covariant derivative is then given by Eq. (2.30).

While we have discussed convenient choices for the matrix J for the three types of irreps, our results are valid more generally. In particular, for the real and pseudoreal cases, the derivation is valid for any matrix J which satisfies the properties discussed in the previous subsection. For the convenience of the reader we summarize them: J must be an $N_w \times N_w$ real orthogonal matrix with $\det J = 1$, and it should be symmetric (antisymmetric) for the real (pseudoreal) case.

C. Singlet axial symmetries

In addition to the non-Abelian flavor symmetry group, we may apply to the fermions of each irrep a flavor-singlet axial transformation. For Dirac fermions, this transformation is given by

$$\psi_i \to e^{-i\theta\gamma_5}\psi_i, \qquad \overline{\psi}_i \to \overline{\psi}_i e^{-i\theta\gamma_5}. \tag{2.31}$$

with a similar transformation for Majorana fermions. The corresponding $U(1)_A$ current is

$$A_\mu = \begin{cases} \sum_{i=1}^N \overline{\psi}_i \gamma_\mu \gamma_5 \psi_i, & \text{Dirac fermions}, \\ \sum_{I=1}^{N_w} \overline{\Psi}_I \gamma_\mu \gamma_5 \Psi_I, & \text{Majorana fermions}. \end{cases} \tag{2.32}$$

[3]The same normalization is used for the real and pseudoreal cases.

The Dirac version may be used for complex and pseudoreal irreps, whereas the Majorana version is used for real irreps.[4]

The individual $U(1)_A$ currents are anomalous

$$\partial_\mu A_\mu = \frac{g^2}{32\pi^2} N_w T F_{a\mu\nu} \tilde{F}_{a\mu\nu}, \qquad (2.33)$$

where the group-invariant T is defined by

$$\mathrm{tr}(T_a T_b) = T\delta_{ab}, \qquad (2.34)$$

where the T_a are the generators of the gauge group in the given irrep, with $T = \frac{1}{2}$ for the fundamental irrep. As usual, $N_w = 2N$ in the case of Dirac fermions.

Consider an asymptotically free theory with fermions in n different irreps. We will assume that if a given irrep, r, is real, the fermions are arranged as $N_{w,r}$ Majorana fields. If r is complex or pseudoreal, we assume that the fermions may be assembled into $N_r = N_{w,r}/2$ Dirac fermions.[5] In any such theory, only the overall $U(1)_A$ transformation is anomalous, whereas $n-1$ linearly independent combinations of the individual $U(1)_A$ currents are anomaly free.

D. Parametrization of the coset fields: two irreps

From now on, we specialize to theories with fermions in two different irreps. The irreps can be of the same type, e.g., both complex; or they can be of different types, e.g., one complex irrep and one real irrep, as in the model of Ref. [27]. Out of the two flavor-singlet axial currents, we can make one linear combination which is anomaly free. Using indices $r, s, \ldots = 1, 2$ to label the two irreps, the nonanomalous current is[6]

$$A_\mu = \sum_r q_r A_{r,\mu}, \qquad (2.35)$$

where, adopting a convenient normalization, the axial charges of the two irreps are

$$q_1 = \frac{N_{w,2} T_2}{\sqrt{N_{w,1}^2 T_1^2 + N_{w,2}^2 T_2^2}},$$
$$q_2 = -\frac{N_{w,1} T_1}{\sqrt{N_{w,1}^2 T_1^2 + N_{w,2}^2 T_2^2}}. \qquad (2.36)$$

[4]See Appendix A for the definition of the Majorana fermion Ψ.

[5]If r is a complex irrep, we count both r and its complex conjugate as the same irrep, for the obvious reason that a Dirac fermion in a complex irrep corresponds to two same-handedness Weyl fermions in the two complex conjugate irreps.

[6]For Eq. (2.35) to be true to all orders, $A_{r,\mu}$ on the right-hand side should be the renormalized singlet axial current of the rth irrep.

The requirement that the current A_μ be anomaly free only fixes the ratio q_1/q_2. As is usually the case for an Abelian symmetry, the overall normalization of the current is arbitrary. Obviously, physics should not depend on this choice. In Appendix C we discuss the choice of normalization in a little more detail, showing that this is indeed the case.

For each irrep, the fermion condensate carries twice the axial charge of a single field. It follows that the nonanomalous $U(1)_A$ is spontaneously broken, too. To account for the corresponding NGB, we introduce a new effective field,

$$\Phi(x) = \exp\left(\frac{i\zeta(x)}{\sqrt{2}F_\zeta}\right) \in U(1), \qquad (2.37)$$

with unit charge under $U(1)_A$. The covariant derivative of this field is

$$D_\mu \Phi = \partial_\mu \Phi + i\alpha_\mu \Phi = i\Phi\left(\frac{\partial_\mu \zeta}{\sqrt{2}F_\zeta} + \alpha_\mu\right), \qquad (2.38)$$

where α_μ is the (external) $U(1)_A$ gauge field.

In order to match all quantum numbers of the order parameters, including their $U(1)_A$ charges, Eq. (2.2) gets replaced by

$$\mathrm{tr}(\psi_{L,i}\bar{\psi}_{R,j}) \leftrightarrow \Phi^{2q}\Sigma_{ij},$$
$$\mathrm{tr}(\psi_{R,j}\bar{\psi}_{L,i}) \leftrightarrow \Phi^{-2q}\Sigma_{ij}^*, \qquad (2.39)$$

for the complex case, while (2.13) gets replaced by

$$\mathrm{tr}(\xi_{L,I}\xi_{L,J}^T CS) \leftrightarrow \Phi^{2q}\Sigma_{IJ},$$
$$\mathrm{tr}(\bar{\xi}_{L,I}^T \bar{\xi}_{L,J} CS^T) \leftrightarrow \Phi^{-2q}\Sigma_{IJ}^*, \qquad (2.40)$$

for the real and pseudoreal cases. In all cases, the chiral source χ carries charge $+2q$.

III. CHIRAL LAGRANGIAN

We are now ready to write down the chiral Lagrangian for two different irreps, labeled by indices $r, s, \ldots = 1, 2$. As before, when r is a complex irrep the flavor indices are $i, j, \ldots = 1, \ldots, N_r$, where N_r is the number of Dirac fermions. For real and pseudoreal irreps, the flavor indices are $I, J, \ldots = 1, \ldots, N_{w,r}$, where $N_{w,r}$ is the number of Weyl fermions.[7] To allow for more uniformity of our notation, we also introduce n_r, which will be equal N_r for a complex irrep, and to $N_{w,r}$ for real and pseudoreal irreps.

A. Leading order

The leading-order (LO) Lagrangian consists of kinetic terms and mass terms,

[7]Recall that N_w is even for a pseudoreal irrep.

$$\mathcal{L}_2 = \mathcal{L}_k + \mathcal{L}_m. \tag{3.1}$$

There is a separate kinetic term for each coset field,

$$\mathcal{L}_k = F_\zeta^2 (D_\mu \Phi)^\dagger D_\mu \Phi + \sum_r \frac{F_r^2}{4} \langle (D_\mu \Sigma_r)^\dagger D_\mu \Sigma_r \rangle, \tag{3.2}$$

where, from now on, we will use the notation $\langle \cdots \rangle$ to indicate tracing over the flavor indices.

The mass terms take the form

$$\mathcal{L}_m = -\sum_r \frac{F_r^2}{4} \langle \chi_r^\dagger U_r + U_r^\dagger \chi_r \rangle, \tag{3.3}$$

where we have introduced the product fields

$$U_r(x) = \Phi(x)^{2q_r} \Sigma_r(x) = \Phi(x)^{2q_r} \hat{U}_r(x) J_r. \tag{3.4}$$

The presence of Φ^{2q_r} is forced upon us because χ_r carries charge $2q_r$. Using the results of Sec. II D it can be checked that $\mathcal{L}_m$ is invariant under all the flavor symmetries, including the nonanomalous $U(1)_A$. With the external gauge fields turned on, the entire Lagrangian $\mathcal{L}_2$ is thus invariant under local flavor transformations.

The LO Lagrangian is also invariant under an "intrinsic" parity symmetry that acts simultaneously on all fields. For a complex irrep, the intrinsic parity is

$$\begin{aligned} \Sigma_r &\to \Sigma_r^\dagger, & \chi_r &\to \chi_r^\dagger, \\ v_{r\mu} &\to v_{r\mu}, & a_{r\mu} &\to -a_{r\mu}, \end{aligned} \tag{3.5}$$

whereas for the other two cases it is

$$\begin{aligned} \Sigma_r &\to s_r \Sigma_r^\dagger, & \chi_r &\to s_r \chi_r^\dagger, \\ v_{r\mu} &\to -v_{r\mu}^T, & a_{r\mu} &\to -a_{r\mu}^T, \end{aligned} \tag{3.6}$$

where, as in Eq. (2.13), $s_r = 1$ ($s_r = -1$) for a real (pseudoreal) irrep. The transformation of the pion fields is $\pi_r \to -\pi_r$ for a complex irrep, and $\pi_r \to -\pi_r^T$ for real and pseudoreal irreps.[8] Finally, for the singlet sector, the intrinsic parity is

$$\Phi \to \Phi^*, \qquad \alpha_\mu \to -\alpha_\mu. \tag{3.7}$$

In order to develop the perturbative expansion we let the chiral sources assume their "expectation values," i.e., we set

$$\chi_r = 2 m_r B_r J_r, \tag{3.8}$$

where $m_r \geq 0$, and the allowed choices for J_r are summarized in Sec. II B. Using Eqs. (2.30), (3.4) and (3.8) it can be checked that the J_r matrices completely drop out when the LO Lagrangian is expressed in terms of the fields Φ and $\hat{U}_r$. We next use Eqs. (2.21), (2.28) and (2.37) to extract the quadratic part of the LO Lagrangian, obtaining

$$\mathcal{L}_2^{\rm quad} = \frac{1}{2}(\partial_\mu \zeta \partial_\mu \zeta + M_\zeta^2 \zeta^2) + \frac{1}{2} \sum_r \langle \partial_\mu \pi_r \partial_\mu \pi_r + M_r^2 \pi_r^2 \rangle, \tag{3.9}$$

where we have now turned off the external gauge fields. The tree-level masses are

$$M_r^2 = 2 m_r B_r, \tag{3.10}$$

for the pions, and

$$M_\zeta^2 = 2 \sum_r \frac{F_r^2}{F_\zeta^2} q_r^2 m_r B_r \langle \mathbf{1}_r \rangle = \sum_r \frac{F_r^2}{F_\zeta^2} q_r^2 M_r^2 n_r, \tag{3.11}$$

for the flavor singlet pseudoscalar ζ, where n_r is defined at the beginning of Sec. III. Note that M_ζ^2 vanishes only when the fermion masses of both irreps vanish. The tree-level "quark flow" propagators are obtained using closure relations that we have collected in Appendix B. For a complex irrep (dropping the irrep's index r) the propagator is[9]

$$\langle \pi_{ij}(x) \pi_{k\ell}(y) \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 + M^2} \left(\delta_{i\ell} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{k\ell} \right). \tag{3.12}$$

For a real or pseudoreal irrep, it is

$$\langle \pi_{IJ}(x) \pi_{KL}(y) \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 + M^2} \times \left(\frac{1}{2} (\delta_{IL} \delta_{JK} + J_{IK} J_{JL}) - \frac{1}{N_w} \delta_{IJ} \delta_{KL} \right). \tag{3.13}$$

This structure follows from the fact that the pion matrix obeys the relation $\pi = \frac{1}{2}(\pi + J \pi^T J^T)$ and that it is traceless.

An advantage of the quark-flow Feynman rules is that the vertices can be read off mechanically, and the coset structure is reflected only in the above expressions for the propagators. In particular, this is the only place where one encounters the J_r matrices once the Lagrangian has been expressed in terms of the Φ and $\hat{U}_r$ fields.

[8]It follows from Eq. (2.26) that if X_a is a coset generator, so is X_a^T.

[9]In Eqs. (3.12) and (3.13) the notation $\langle \cdots \rangle$ stands for an expectation value, not a flavor-index trace.

The expansion of the kinetic terms in the pion fields is standard. In the mass terms, on the other hand, we encounter terms that depend on both the pion and flavor singlet fields. For example, the quartic part of $\mathcal{L}_m$ is

$$\mathcal{L}_m^{\rm quart} = -\sum_r M_r^2 \left(\frac{n_r q_r^4 F_r^2}{12F_\zeta^4}\zeta^4 + \frac{q_r^2}{2F_\zeta^2}\zeta^2\langle \pi_r^2 \rangle + \frac{q_r}{3F_\zeta F_r}\zeta \langle \pi_r^3\rangle + \frac{1}{12F_r^2}\langle \pi_r^4 \rangle \right). \quad (3.14)$$

The $\zeta\langle\pi_r^3\rangle$ term appearing in this Lagrangian is a novel feature, as, by itself, $\langle\pi_r^3\rangle$ violates intrinsic parity. This interaction allows the decay $\zeta \to 3\pi$ to proceed at tree level even when all fermion masses for a single irrep are degenerate (if that mass is small enough), unlike the similar decay $\eta \to 3\pi$ in QCD which requires isospin violation to occur.

B. Next-to-leading order

The next-to-leading order Lagrangian

$$\mathcal{L}_4 = \mathcal{L}_s + \mathcal{L}_d + \mathcal{L}_\zeta, \quad (3.15)$$

consists of several kinds of terms. Following closely the classification of the QCD case [31,32], we start with the single-trace terms

$$\mathcal{L}_s = \sum_r (L_{0r}P_{0r} - L_{3r}P_{3r} + L_{5r}P_{5r} - L_{8r}P_{8r} - H_{2r}X_{2r}), \quad (3.16)$$

where[10]

$$P_{0r} = \langle (D_\mu \hat{U}_r)^\dagger D_\nu \hat{U}_r (D_\mu \hat{U}_r)^\dagger D_\nu \hat{U}_r \rangle, \quad (3.17a)$$

$$P_{3r} = \langle (D_\mu \hat{U}_r)^\dagger D_\mu \hat{U}_r (D_\nu \hat{U}_r)^\dagger D_\nu \hat{U}_r \rangle, \quad (3.17b)$$

$$P_{5r} = \langle (D_\mu \hat{U}_r)^\dagger D_\mu \hat{U}_r (\chi_r^\dagger U_r + U_r^\dagger \chi_r) \rangle, \quad (3.17c)$$

$$P_{8r} = \langle \chi_r^\dagger U_r \chi_r^\dagger U_r + U_r^\dagger \chi_r U_r^\dagger \chi_r \rangle, \quad (3.17d)$$

$$X_{2r} = \langle \chi_r^\dagger \chi \rangle. \quad (3.17e)$$

Here H_{2r} is a "high-energy" constant, multiplying a contact term. The minus signs in Eq. (3.16) and following, relative to Refs. [31,32], are present because we work in Euclidean metric while their metric is Minkowski, and we want our results for observables to agree with theirs in the single-representation case. Note that, through U_r, some of these operators depend on the singlet field Φ. Next, there are double-trace terms

[10] P_{0r} is redundant for $N \le 3$ for complex irreps, or $N_w \le 3$ for real and pseudoreal irreps [33].

$$\mathcal{L}_d = \sum_{rs} (-L_{1rs}P_{1rs} - L_{2rs}P_{2rs} + L_{4rs}P_{4rs} - L_{6rs}P_{6rs} - L_{7rs}P_{7rs}), \quad (3.18)$$

where all low-energy constants (LECs) except L_{4rs} are symmetric under $r \leftrightarrow s$, and

$$P_{1rs} = \langle (D_\mu \hat{U}_r)^\dagger D_\mu \hat{U}_r \rangle \langle (D_\nu \hat{U}_s)^\dagger D_\nu \hat{U}_s \rangle, \quad (3.19a)$$

$$P_{2rs} = \langle (D_\mu \hat{U}_r)^\dagger D_\nu \hat{U}_r \rangle \langle (D_\mu \hat{U}_s)^\dagger D_\nu \hat{U}_s \rangle, \quad (3.19b)$$

$$P_{4rs} = \langle (D_\mu \hat{U}_r)^\dagger D_\mu \hat{U}_r \rangle \langle \chi_s^\dagger U_s + U_s^\dagger \chi_s \rangle, \quad (3.19c)$$

$$P_{6rs} = \langle \chi_r^\dagger U_r + U_r^\dagger \chi_r \rangle \langle \chi_s^\dagger U_s + U_s^\dagger \chi_s \rangle, \quad (3.19d)$$

$$P_{7rs} = \langle \chi_r^\dagger U_r - U_r^\dagger \chi_r \rangle \langle \chi_s^\dagger U_s - U_s^\dagger \chi_s \rangle. \quad (3.19e)$$

Finally, there are additional terms that involve the singlet field's two-derivative operator

$$\mathcal{L}_\zeta = L_0' P_0' - \sum_r (L_{1r}' P_{1r}' + L_{2r}' P_{2r}' + L_{3r}' P_{3r}'), \quad (3.20)$$

where

$$P_0' = ((D_\mu \Phi)^\dagger D_\mu \Phi)^2, \quad (3.21a)$$

$$P_{1r}' = \langle (D_\mu \hat{U}_r)^\dagger D_\mu \hat{U}_r \rangle (D_\mu \Phi)^\dagger D_\mu \Phi, \quad (3.21b)$$

$$P_{2r}' = \langle (D_\mu \hat{U}_r)^\dagger D_\nu \hat{U}_r \rangle (D_\mu \Phi)^\dagger D_\nu \Phi, \quad (3.21c)$$

$$P_{3r}' = \langle \chi_r^\dagger U_r + U_r^\dagger \chi_r \rangle (D_\mu \Phi)^\dagger D_\mu \Phi. \quad (3.21d)$$

IV. NEXT-TO-LEADING ORDER RESULTS

In this section we will present the NLO corrections for the masses and the decay constants of the (pseudo) NGBs, and for the condensates. Since the calculations leading to these results are straightforward, we will not give any details. We have cross-checked all applicable single-representation results in these formulas (i.e., analytic terms and chiral logarithms which do not involve ζ) against the corresponding NLO results in the literature [36].

A. Pseudoscalar masses

The inverse propagator takes the general form

$$p^2 + M^2 + \Gamma(p^2), \quad (4.1)$$

where M^2 is the tree-level mass, and $\Gamma(p^2)$ is the NLO self-energy. The physical mass-squared $M_{\rm phys}^2 = M^2 + \delta M^2$ is equal to the value of $-p^2$ for which this vanishes. At NLO, we may set $p^2 = -M_{\rm phys}^2 \to -M^2$ in $\Gamma(p^2)$, and we obtain

$$\delta M^2 = \Gamma(-M^2). \tag{4.2}$$

We first consider the pions. The NLO correction can be expressed as

$$\delta M_r^2 = \delta M_{r,an}^2 + \delta M_{r,\pi}^2 + \delta M_{r,\zeta}^2. \tag{4.3}$$

The origin of the various terms is the following. $\delta M_{r,an}^2$ is the analytic contribution from the NLO Lagrangian, given by

$$\delta M_{r,an}^2 = \frac{8M_r^2}{F_r^2}\left((2L_{8r} - L_{5r})M_r^2 + \sum_s (2L_{6rs} - L_{4rs})M_s^2 n_s\right). \tag{4.4}$$

$\delta M_{r,\pi}^2$ is the usual nonanalytic contribution from a pion tadpole, which arises from a single quartic vertex of the LO Lagrangian. It is given by [36]

$$\delta M_{r,\pi}^2 = M_r^2 C_r \Delta_r, \tag{4.5}$$

where $C_r = 1/n_r$ for complex, $-1/2 + 1/n_r$ for real, and $1/2 + 1/n_r$ for pseudoreal representations, and with

$$\Delta_r = \frac{M_r^2}{16\pi^2 F_r^2} \log \frac{M_r^2}{\mu^2}. \tag{4.6}$$

We are using the standard ChPT subtraction scheme in which the tadpole is given entirely by the logarithm and its constant terms are absorbed into the renormalized L_i's [32].

Finally $\delta M_{r,\zeta}^2$ is a similar nonanalytic contribution involving a ζ tadpole, which arises from the second term on the right-hand side of Eq. (3.14). Explicitly

$$\delta M_{r,\zeta}^2 = -q_r^2 M_r^2 \Delta_\zeta, \tag{4.7}$$

$$\Delta_\zeta = \frac{M_\zeta^2}{16\pi^2 F_\zeta^2} \log \frac{M_\zeta^2}{\mu^2}. \tag{4.8}$$

For the one-loop correction to the mass of the pseudoscalar singlet we similarly find

$$\delta M_\zeta^2 = \delta M_{\zeta,an}^2 + \delta M_{\zeta,\zeta}^2 + \delta M_{\zeta,\pi}^2. \tag{4.9}$$

The analytic contribution is

$$\begin{aligned} \delta M_{\zeta,an}^2 = {} & \frac{1}{F_\zeta^2} \sum_r M_r^2 n_r (16 L_{8r} M_r^2 q_r^2 + 2L'_{3r} M_\zeta^2) \\ & + \frac{1}{F_\zeta^2} \sum_{rs} M_r^2 M_s^2 n_r n_s (8 L_{6rs} (q_r^2 + q_s^2) \\ & + 16 L_{7rs} q_r q_s). \end{aligned} \tag{4.10}$$

The nonanalytic contribution from a ζ tadpole is

$$\delta M_{\zeta,\zeta}^2 = -\sum_r M_r^2 n_r q_r^4 \frac{F_r^2}{F_\zeta^2} \Delta_\zeta, \tag{4.11}$$

and the nonanalytic contribution from pion tadpoles is

$$\delta M_{\zeta,\pi}^2 = -\sum_r D_r M_r^2 q_r^2 \frac{F_r^2}{F_\zeta^2} \Delta_r, \tag{4.12}$$

where the dimensionality of the coset, D_r, is equal to $n_r^2 - 1$ for a complex representation, $\frac{1}{2} n_r(n_r + 1) - 1$ for a real representation, and $\frac{1}{2} n_r(n_r - 1) - 1$ for a pseudoreal representation.

B. Decay constants

As in the case of the pion mass, we write

$$\delta F_r = \delta F_{r,an} + \delta F_{r,\pi} + \delta F_{r,\zeta}. \tag{4.13}$$

We find that

$$\begin{aligned} \delta F_{r,an} &= 4F_r \left(L_{5r} \frac{M_r^2}{F_r^2} + \sum_s L_{4rs} n_s \frac{M_s^2}{F_r^2} \right), \\ \delta F_{r,\pi} &= -\frac{1}{2} F_r n_r \Delta_r, \\ \delta F_{r,\zeta} &= 0. \end{aligned} \tag{4.14}$$

There are no loop contributions to F_ζ, and we find a purely analytic result

$$\delta F_\zeta = -F_\zeta \sum_r L'_{3r} n_r \frac{M_r^2}{F_\zeta^2}. \tag{4.15}$$

C. Condensates

The condensate $\Sigma_r \equiv \langle \overline{\psi}_r \psi_r \rangle$ per flavor of irrep r is defined by

$$\Sigma_r = -\frac{1}{n_r} \frac{\partial \log Z}{\partial m_r}. \tag{4.16}$$

To leading order this yields $\Sigma_r^0 = -F_r^2 B_r$, using Eq. (3.3). At NLO, we again define

$$\delta\Sigma_r = \delta\Sigma_{r,an} + \delta\Sigma_{r,\pi} + \delta\Sigma_{r,\zeta}, \tag{4.17}$$

for the analytic, pion-loop and ζ-loop contributions. A straightforward calculation finds

$$\delta\Sigma_{r,an} = 4\Sigma_r^0\left((2L_{8r} + H_{2r})\frac{M_r^2}{F_r^2} + 4\sum_s L_{6rs} n_s \frac{M_s^2}{F_r^2}\right),$$
$$\delta\Sigma_{r,\pi} = -\Sigma_r^0 \frac{D_r}{n_r}\Delta_r,$$
$$\delta\Sigma_{r,\zeta} = -\Sigma_r^0 q_r^2 \Delta_\zeta, \tag{4.18}$$

where D_r was defined below Eq. (4.12).

V. CONCLUSION

In this paper, we developed chiral perturbation theory for a vectorlike gauge theory with fermions transforming in two different irreducible representations of the gauge group. We considered fermions in any type of representation of the gauge group, complex, real or pseudoreal. We assumed that bilinear condensates develop for each of these fermions, breaking the flavor symmetry of each fermion species spontaneously.

The low-energy effective field theory contains Nambu-Goldstone bosons associated with the vacuum manifolds for each of the two condensates. In addition, it contains one more singlet Nambu-Goldstone boson, because a linear combination of the two axial $U(1)$ symmetries remains nonanomalous. The two fermion condensates both break this singlet axial $U(1)$, and the associated axial current thus creates a singlet Nambu-Goldstone boson from the vacuum.

We allowed for degenerate masses for each fermion species; of course, they are not degenerate between different irreducible representations. This turns the Nambu-Goldstone bosons into massive pseudo Nambu-Goldstone bosons. The (tree-level) mass-squared of the singlet Nambu-Goldstone boson is a linear combination of the masses of the two fermion species, and it is thus not possible to give this Nambu-Goldstone boson a mass without giving at least one of the nonsinglet Nambu-Goldstone boson multiplets a mass. We presented next-to-leading order results for all meson masses, decay constants, and the two condensates. It should be straightforward to generalize the framework of this paper to a theory with more than two different types of fermions.

We can imagine two potential uses for these results. The first, as mentioned in the Introduction, is that theories as considered here have applications in models for physics beyond the standard model. We have not seen a systematic construction of the chiral Lagrangian for theories with more than one representation of fermions presented to date, so our results might be a resource for model builders. From this perspective, we think that the most interesting aspect of these systems is the appearance of the additional $U(1)$ Nambu-Goldstone boson. The nonsinglet Nambu-Goldstone bosons come in degenerate-mass multiplets, if degenerate masses are given to the fermions in the underlying theory. On the other hand, the singlet appears as a somewhat isolated state, particularly if the masses of the two nonsinglet multiplets are somewhat separated. This is a distinctive feature in the context of e.g. composite Higgs models, where new resonances tend to appear with large multiplicity and similar masses. Unusually for a Nambu-Goldstone boson, the singlet can decay at tree level as $\zeta \to 3\pi$ even when all fermion masses for each representation are degenerate, which may have interesting phenomenological consequences in some theories.

The second potentially useful application of these results is as a theoretical benchmark for the interpretation of lattice simulations relevant for various extensions of the Standard Model [50]. Three of us are involved in such an effort [51]. The interesting physics issues are whether such a system is confining and chirally broken, and if so, how the dimensionful parameters (decay constants, masses, condensates) for the different representations are related to each other. Is it possible that there are ranges of bare parameters in which the fermions in one representation condense, while those in the others do not? Speculations about such behavior are longstanding (see e.g. Ref. [52]). Of course, the results of this paper apply only in the case that the fermions in both representations condense.

Seeing the additional $U(1)$ Nambu-Goldstone boson in a lattice calculation might be difficult. One would have to measure "quark-disconnected" diagrams like those used in the measurement of the η' mass in QCD. An elaborate multichannel analysis along the lines of Ref. [53] might be needed to observe them. The ordinary pions will be easier to study. There, the interesting physics is the dependence of the squared mass M_r^2 of a pseudoscalar, or of its decay constant F_r, on the mass of a fermion in a representation $s \neq r$ when the mass of a fermion in representation r is fixed.

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APPENDIX A: MAJORANA FERMIONS

When the Dirac fermions belong to a real irrep we may alternatively introduce Majorana fermions Ψ_I, $I = 1, \ldots, 2N$, where

$$\begin{aligned}\Psi_{L,i} &= \psi_{L,i},\\ \Psi_{L,N+i} &= SC\overline{\psi}^T_{R,i},\\ \Psi_{R,i} &= SC\overline{\psi}^T_{L,i},\\ \Psi_{R,N+i} &= \psi_{R,i},\end{aligned} \tag{A1}$$

and where $i = 1, \ldots, N$ as before. Defining

$$\overline{\Psi} \equiv \Psi^T CS, \tag{A2}$$

the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\sum_{I=1}^{2N} \overline{\Psi}_I \not{D} \Psi_I. \tag{A3}$$

For a real irrep, the number of Majorana (or Weyl) fermions N_w is also allowed to be odd, in which case we can use a Majorana (or Weyl) basis, but not a Dirac basis. Correspondingly, allowing the range of summation in Eq. (A3) to be an arbitrary positive integer N_w, the flavor symmetry acts as

$$\Psi \to (P_L g + P_R g^*)\Psi, \tag{A4}$$

where $g \in SU(N_w)$. For $N_w = 2N$, it can be checked that Eq. (A4) agrees with Eq. (2.11).

In terms of the Majorana fields, we have [compare Eq. (2.12), and recall $S^T = S$)]

$$\Psi_I^T CS\Psi_J = \Psi_J^T CS\Psi_I. \tag{A5}$$

Moreover, since $C\gamma_5 = \gamma_5^T C$, this remains true if the same chiral projector is inserted on both sides of the equation. All these bilinears are therefore symmetric on their flavor indices, as expected.

APPENDIX B: PROJECTORS

In writing down the expressions for the tree-level propagators we use that, with the normalization (2.22), the projector on the traceless Hermitian generators of $SU(N)$ is

$$P_{IJKL} \equiv T_{aIJ}T_{aKL} = \delta_{IL}\delta_{JK} - \frac{1}{N}\delta_{IJ}\delta_{KL}. \tag{B1}$$

Splitting it into a projector on the space spanned by the Q's of Eq. (2.25) and the X's of Eq. (2.26), we have $P = P^Q + P^X$ where

$$P^Q_{IJKL} \equiv Q_{\tilde{a}IJ}Q_{\tilde{a}KL} = \frac{1}{2}(\delta_{IL}\delta_{JK} - J_{IK}J_{JL}), \tag{B2}$$

$$P^X_{IJKL} \equiv X_{\hat{a}IJ}X_{\hat{a}KL} = \frac{1}{2}(\delta_{IL}\delta_{JK} + J_{IK}J_{JL}) - \frac{1}{N}\delta_{IJ}\delta_{KL}. \tag{B3}$$

These results can be proved by rewriting Eqs. (2.25) and (2.26) as

$$Q_{\tilde{a}} = \frac{1}{2}(Q_{\tilde{a}} - JQ^T_{\tilde{a}}J^T), \tag{B4}$$

$$X_{\hat{a}} = \frac{1}{2}(X_{\hat{a}} + JX^T_{\hat{a}}J^T), \tag{B5}$$

which are valid for any real orthogonal matrix J.

APPENDIX C: NORMALIZATION OF THE SINGLET AXIAL CURRENT

In Eq. (2.36) we chose a particular normalization of the charges q_1 and q_2, and thus a particular normalization of the nonanomalous singlet axial current defined in Eq. (2.35). Furthermore, the singlet's decay constant F_ζ is defined by[11]

$$\langle 0|A_\mu(x)|\zeta\rangle = ip_\mu\sqrt{2}F_\zeta e^{ipx}, \tag{C1}$$

in analogy with Eq. (2.23). F_ζ will show up in, for example, the ζ decay rate, and it is therefore instructive to check that ζ physics is not affected by the choice of normalization.

If we change the normalization of A_μ by a factor λ, it follows from Eq. (C1) that the decay constant is rescaled as $F_\zeta \to \lambda F_\zeta$, and from Eq. (2.35) that $q_r \to \lambda q_r$. If we now turn off the external gauge fields, and reexpress the LO Lagrangian in terms of the ζ field, the result will depend only on the ratios q_r/F_ζ, which are invariant. This is true, in particular, for the factor of Φ^{2q_r} that occurs in Eq. (3.3). Other concrete examples are provided by the LO singlet mass (3.11), and the interaction vertices (3.14).

Proceeding to the NLO results we have calculated, corrections to masses, to the pion decay constants, and to the condensates, should be invariant under the rescaling, whereas corrections to the singlet decay constant should scale in the same way as F_ζ itself.[12] One can then read off from our NLO results how the NLO LECs should scale. The unprimed NLO LECs are invariant, while the primed ones rescale as $L'_{ir} \to \lambda^2 L'_{ir}$, $i = 1, 2, 3$, and $L'_0 \to \lambda^4 L'_0$. Alternatively, these scaling rules can be inferred from the contribution of these NLO terms to the singlet axial current, in comparison with the LO term following from Eq. (3.2). Indeed, in our explicit NLO results, L'_{3r} always appears in the combination L'_{3r}/F^2_ζ, which is independent of λ.

We conclude with one more example. In the context of composite Higgs models, when the couplings to Standard Model gauge fields are turned on, the A_μ current becomes anomalous, and ζ develops anomaly-induced couplings to

[11] F_ζ is the decay constant in the full chiral limit, where the masses of all fermions in the underlying theory vanish.

[12] Note that in order to probe the singlet decay constant we need to turn back on the singlet axial gauge field.

pairs of Standard Model vector bosons. If, for example, we turn on electromagnetism, this anomaly takes the form

$$\partial_\mu A_\mu = e^2 F_{\mu\nu}\tilde{F}_{\mu\nu}\sum_r q_r c_r, \tag{C2}$$

where $F_{\mu\nu}$ is the electromagnetic field strength, and where c_r is a weighted sum over the squared electric charges of the fermions that belong to the rth irrep. Using that

$$A_\mu = \sqrt{2}F_\zeta \partial_\mu \zeta + \text{higher orders}, \tag{C3}$$

we find what is essentially the ζ equation of motion to this order, i.e.,

$$\Box\zeta = \frac{e^2}{\sqrt{2}} F_{\mu\nu}\tilde{F}_{\mu\nu}\sum_r \frac{q_r}{F_\zeta} c_r. \tag{C4}$$

Again, only the ratio q_r/F_ζ appears, implying that the decay rate is independent of the arbitrary choice of normalization of the singlet axial current A_μ.

[1] E. Ma, Quarks with charm and color: A new model of their strong and weak interactions, Phys. Lett. B **58**, 442 (1975).
[2] G. Karl, The classification of new particles, Phys. Rev. D **14**, 2374 (1976).
[3] F. Wilczek and A. Zee, A new zoology of quarks and hadrons, Phys. Rev. D **16**, 860 (1977).
[4] Y. J. Ng and S. H. H. Tye, Is the Upsilon a Bound State of Exotic Quarks?, Phys. Rev. Lett. **41**, 6 (1978).
[5] H. Georgi and S. L. Glashow, A quixotic interpretation of the Υ particle, Nucl. Phys. **B159**, 29 (1979).
[6] C. B. Dover, T. K. Gaisser, and G. Steigman, Cosmological Constraints On New Stable Hadrons, Phys. Rev. Lett. **42**, 1117 (1979).
[7] W. J. Marciano, Exotic new quarks and dynamical symmetry breaking, Phys. Rev. D **21**, 2425 (1980).
[8] B. Holdom and M. E. Peskin, Raising the axion mass, Nucl. Phys. **B208**, 397 (1982).
[9] K. Konishi and R. Tripiccione, Is weak interaction symmetry broken by ordinary color forces?, Phys. Lett. B **121**, 403 (1983).
[10] D. Lust, E. Papantonopoulos, K. H. Streng, and G. Zoupanos, Phenomenology of high color fermions, Nucl. Phys. **B268**, 49 (1986).
[11] T. E. Clark, C. N. Leung, S. T. Love, and J. L. Rosner, Sextet quarks and light pseudoscalars, Phys. Lett. B **177**, 413 (1986).
[12] E. Braaten, A. R. White, and C. R. Willcox, Infrared anomalous interactions as a signal for electroweak symmetry breaking by color sextet quarks, Int. J. Mod. Phys. A **01**, 693 (1986).
[13] G. V. Borisov, Y. F. Pirogov, and K. R. Rudakov, Pair production of exotic particles at anti-p p colliding beams, Z. Phys. C **36**, 217 (1987).
[14] A. R. White, Dynamical symmetry breaking, axion dynamics and instanton induced second class τ decays, Mod. Phys. Lett. A **02**, 945 (1987).
[15] K. Fukazawa, T. Muta, J. Saito, I. Watanabe, M. Yonezawa, and M. Inoue, Dynamical breaking of electroweak symmetry by color sextet quark condensates, Prog. Theor. Phys. **85**, 111 (1991).
[16] R. S. Chivukula, M. Golden, and E. H. Simmons, Six jet signals of highly colored fermions, Phys. Lett. B **257**, 403 (1991).
[17] E. H. Simmons, Creeping up on condensed quixes: A new test of color mediated electroweak symmetry breaking, Phys. Lett. B **259**, 125 (1991).
[18] R. S. Chivukula, M. Golden, and E. H. Simmons, Multi-jet physics at hadron colliders, Nucl. Phys. **B363**, 83 (1991).
[19] T. Kahara, M. Ruggieri, and K. Tuominen, Deconfinement vs chiral symmetry and higher representation matter, Phys. Rev. D **85**, 094020 (2012).
[20] R. Contino, The Higgs as a composite Nambu-Goldstone boson, arXiv:1005.4269.
[21] B. Bellazzini, C. Csáki, and J. Serra, Composite Higgses, Eur. Phys. J. C **74**, 2766 (2014).
[22] G. Panico and A. Wulzer, The composite Nambu-Goldstone Higgs, Lect. Notes Phys. **913**, 1 (2016).
[23] G. D. Kribs and E. T. Neil, Review of strongly coupled composite dark matter models and lattice simulations, arXiv:1604.04627.
[24] D. B. Kaplan, Flavor at SSC energies: A new mechanism for dynamically generated fermion masses, Nucl. Phys. **B365**, 259 (1991).
[25] G. Ferretti and D. Karateev, Fermionic UV completions of composite Higgs models, J. High Energy Phys. 03 (2014) 077.
[26] J. Barnard, T. Gherghetta, and T. S. Ray, UV descriptions of composite Higgs models without elementary scalars, J. High Energy Phys. 02 (2014) 002.
[27] G. Ferretti, UV completions of partial compositeness: The case for a SU(4) gauge group, J. High Energy Phys. 06 (2014) 142.
[28] M. Golterman and Y. Shamir, Top quark induced effective potential in a composite Higgs model, Phys. Rev. D **91**, 094506 (2015).
[29] G. Ferretti, Gauge theories of partial compositeness: Scenarios for Run-II of the LHC, J. High Energy Phys. 06 (2016) 107.
[30] S. Weinberg, Phenomenological Lagrangians, Physica (Amsterdam) **96A**, 327 (1979).

[31] J. Gasser and H. Leutwyler, Chiral perturbation theory to one loop, Ann. Phys. (N.Y.) **158**, 142 (1984).
[32] J. Gasser and H. Leutwyler, Chiral perturbation theory: Expansions in the mass of the strange quark, Nucl. Phys. **B250**, 465 (1985).
[33] M. Golterman, Applications of chiral perturbation theory to lattice QCD, arXiv:0912.4042.
[34] J. Gasser and H. Leutwyler, Light quarks at low temperatures, Phys. Lett. B **184**, 83 (1987).
[35] K. Splittorff, D. Toublan, and J. J. M. Verbaarschot, Diquark condensate in QCD with two colors at next-to-leading order, Nucl. Phys. **B620**, 290 (2002).
[36] J. Bijnens and J. Lu, Technicolor and other QCD-like theories at next-to-next-to-leading order, J. High Energy Phys. 11 (2009) 116.
[37] J. Bijnens and J. Lu, Meson-meson scattering in QCD-like theories, J. High Energy Phys. 03 (2011) 028.
[38] J. Bijnens and J. Lu, Two-point functions and S-parameter in QCD-like theories, J. High Energy Phys. 01 (2012) 081.
[39] A. R. White, The Mmss of the heavy axion η_6, arXiv: hep-ph/9307287.
[40] J. E. Kim, A Composite Invisible Axion, Phys. Rev. D **31**, 1733 (1985).
[41] L. Randall, Composite axion models and Planck scale physics, Phys. Lett. B **284**, 77 (1992).
[42] V. A. Rubakov, Grand unification and heavy axion, JETP Lett. **65**, 621 (1997).
[43] A. Kobakhidze, Solving the strong *CP* problem with high-colour quarks and composite axion, arXiv:1602.06363.
[44] N. D. Barrie, A. Kobakhidze, M. Talia, and L. Wu, 750 GeV composite axion as the LHC diphoton resonance, Phys. Lett. B **755**, 343 (2016).
[45] The ATLAS Collaboration, Report No. ATLAS-CONF-2015-081.
[46] The CMS Collaboration, Report No. CMS-PAS-EXO-15-004.
[47] A. Belyaev, G. Cacciapaglia, H. Cai, T. Flacke, A. Parolini, and H. Serôdio, Singlets in composite Higgs models in light of the LHC di-photon searches, Phys. Rev. D **94**, 015004 (2016).
[48] M. E. Peskin, The alignment of the vacuum in theories of technicolor, Nucl. Phys. **B175**, 197 (1980).
[49] J. B. Kogut, M. A. Stephanov, D. Toublan, J. J. M. Verbaarschot, and A. Zhitnitsky, QCD-like theories at finite baryon density, Nucl. Phys. **B582**, 477 (2000).
[50] T. DeGrand, Lattice tests of beyond standard model dynamics, Rev. Mod. Phys. **88**, 015001 (2016).
[51] T. DeGrand, D. Hackett, W. Jay, E. Neil, B. Svetitsky, Y. Shamir, and N. Tamir (unpublished).
[52] S. Raby, S. Dimopoulos, and L. Susskind, Tumbling gauge theories, Nucl. Phys. **B169**, 373 (1980).
[53] J. J. Dudek, R. G. Edwards, B. Joo, M. J. Peardon, D. G. Richards, and C. E. Thomas, Isoscalar meson spectroscopy from lattice QCD, Phys. Rev. D **83**, 111502 (2011).