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Composite gauge-bosons made of fermions

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We construct a class of Abelian and non-Abelian local gauge theories that consist only of matter fields of fermions. The Lagrangian is local and does not contain an auxiliary vector field nor a subsidiary condition on the matter fields. It does not involve an extra dimension nor supersymmetry. This Lagrangian can be extended to non-Abelian gauge symmetry only in the case of SU(2) doublet matter fields. We carry out an explicit diagrammatic computation in the leading $1/N$ order to show that massless spin-one bound states appear with the correct gauge coupling. Our diagram calculation exposes the dynamical features that cannot be seen in the formal auxiliary vector-field method. For instance, it shows that the s-wave fermion-antifermion interaction in the $^3S_1$ channel ($\bar{\psi}\gamma_\mu\psi$) alone cannot form the bound gauge bosons; the fermion-antifermion pairs must couple to the d-wave state too. One feature common to our class of Lagrangian is that the Noether current does not exist. Therefore it evades possible conflict with the no-go theorem of Weinberg and Witten on the formation of the non-Abelian gauge bosons.

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I. INTRODUCTION

The U(1) gauge theory normally consists of a gauge field and matter fields. The Lagrangian is invariant under the simultaneous gauge transformation of the gauge field and the matter fields. After this was generalized to the non-Abelian group [1], we learned that the non-Abelian extension underlies the dynamics of the fundamental particles.

Let us take a side step and ask out of curiosity the following question: is it possible to construct a gauge-invariant Lagrangian with matter fields alone? For instance, can we construct a *local* field theory with the electron-positron field alone such that it is invariant under the space-time-dependent rotation $\psi(x) \to e^{i\alpha(x)}\psi(x)$ even in the absence of an auxiliary gauge field? If the particles are bosons, the CP^N/CP^{N-1} model [2] would probably be the best known example of this type. Its supersymmetric extension was also discussed [3]. In the case that the matter fields are fermions alone, the history actually goes much further back to the work by Bjorken [4], but the work along this line has not been fruitful.[1]

The method of the auxiliary vector fields was often used in the past to proceed in this kind of argument. It introduces nonpropagating gauge fields at the start and their kinetic energy terms are added later by the loop contribution, ending up with the Lagrangian of matter and propagating gauge fields. Many argued that the nonpropagating gauge field implanted as an auxiliary field in the Lagrangian should be interpreted as turning into a bound state once it has acquired its kinetic energy from the loop contributions. But it is an inevitable consequence of the gauge invariance of the Lagrangian that such an auxiliary field, elementary or otherwise, ought to acquire a gauge-invariant kinetic energy term $-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}$ after loops are included. Would it not be more illuminating if the composition of the massless vector state can be seen explicitly in terms of the constituent matter fields? Such a diagrammatic computation was indeed made by Haber, Hinchliffe and Ravinovici [5] for the CP^{N-1} model many years ago. Unfortunately, this demonstration cannot be repeated when the constituents are fermions, since a simple *local* gauge-invariant Lagrangian corresponding to that of the CP^N model is not known in the case of fermion constituents.

More recently, an attempt has been made to introduce composite gauge bosons through the fifth dimension of the Randall-Sundrum model [6]. The gauge bosons live in the branes and can be interpreted as wholly or partially composite. This is a new class or concept of composite gauge bosons. Models were built and phenomenology was discussed for possible extensions of the standard model along this line [7,8].

In this paper we would like to focus on the dynamics of the formation of composite gauge bosons at an elementary level of particle physics. Many of us have the underlying conviction or speculation that when a Lagrangian is locally gauge invariant, gauge bosons must emerge as composite states even if they are not placed as elementary particles. We would like to see it with our model Lagrangians in an explicit diagrammatic way. In order to separate the issue from the argument based on the auxiliary vector field trick, we study the Lagrangians consisting of fermion fields alone. Furthermore, since our Lagrangian consists only of fermions, supersymmetry is not relevant to our argument, barring the nonlinear realization [9]. We stay in the flat space-time of dimension four all the time. We have no need of an extra

[1] A review of some of the early history can be found at the beginning of Ref. [3], including references.

dimension explicitly or implicitly. Given our Lagrangian, we can carry out the diagram calculation at the leading $1/N$ order with no further approximation or assumption. In this way we can observe how the composite gauge bosons are made of their constituents dynamically. Our reasoning for the construction of the Lagrangian is simple and resorts to no sophisticated mathematical argument or technique.

The primary purpose of this paper is to give model Lagrangians that advocate the inevitability of gauge bosons in gauge-symmetric theories. Although the application of our class of model Lagrangians to the real world is not our primary concern at this moment, short comments are made at the end on issues in electroweak phenomena. At the end, looking back at the history of "compositeness" including findings in some supersymmetric theories, we wonder if it is really a meaningful concept at a fundamental level.

At present, we do not have in mind an immediate application of our model Lagrangian to particle phenomenology. The gauge bosons have been generally accepted as the "elementary" particles and, experimentally, there is no compelling evidence of compositeness. Therefore we shall not pursue the experimental relevance of our models seriously in this paper. Our emphasis at present is primarily on their theoretical implications in composite gauge bosons in general. When Yang and Mills introduced the non-Abelian gauge field theory [1], it had no immediate application. Even the ρ meson was not known at that time although the concept of the weak intermediate bosons was entertained by theorists. The Yang-Mills theory became a subject of intense phenomenological interest only after the Higgs mechanism [10], Weinberg's "A Model of Leptons" [11], and quantum chromodynamics were unexpectedly developed one after another. If we recall this history, we may have the chance to see some feature of our models develop into a subject of experimental interest as the Large Hadron Collider upgrades its luminosity and energy.

We organize the paper as follows. In Sec. II, following in the footsteps of the CP^N model, we introduce the U(1) gauge model of charged Dirac fields alone. We emphasize that, in contrast to the CP^N model, one cannot write a *local* Lagrangian of fermion fields alone with the so-called auxiliary field trick. In Sec. III we show that the Noether current is inevitably absent in the gauge theories that consist of matter fields alone. In Sec. IV, we show the dynamics of the U(1) gauge-boson formation first in the bosonic matter model and then in the fermionic matter model. We introduce, as usual, the N families of matter fields and take the large-N limit in order to solve the models explicitly in a compact form. We find that a massless bound state appears in the $^3S_1$ channel of elastic fermion-antifermion scattering, but that the fermion-antifermion pair must interact in the $^3D_1$ channel as well in order to form the massless bound state of spin one. In Sec. V we extend our models to the non-Abelian gauge symmetry. Choosing the matter fields in the SU(2) doublet, we can build a non-Abelian model with Dirac fields. Computing the elastic scattering amplitude, we find the non-Abelian gauge bosons in the SU(2)-triplet channel as bound states with the correct self-couplings as required by the non-Abelian gauge invariance. In our class of models, the SU(2)-doublet matter plays a special role; it is impossible to extend the model to matter fields of general SU(2) multiplets or general Lie groups. The special role of the SU(2) doublet is discussed in the text and also with two examples in one of the Appendixes. In the final Sec. VI, we discuss the relevance of the missing Noether currents to the no-go theorem of Weinberg and Witten [12]. We conclude with comments on possible relevance to the electroweak phenomenology and on the historical mutation of the concept of compositeness.

II. U(1) MODELS

We proceed by following an elementary line of argument. The first step is to construct a local Lagrangian $L(\psi, \overline{\psi})$ such that

$$L(e^{i\alpha(x)}\psi(x), e^{-i\alpha(x)}\overline{\psi}(x)) = L(\psi(x), \overline{\psi}(x)), \qquad (1)$$

where $L(\psi(x), \overline{\psi}(x))$ depends on space-time coordinates x_μ only through the unconstrained fields $\psi(x)/\overline{\psi}(x)$. We cannot construct such a Lagrangian backward from the QED Lagrangian by integrating out the gauge field $A_\mu(x)$: we would need a gauge fixing to integrate over $A_\mu(x)$, but fixing a gauge breaks manifest gauge invariance. We make our search here with the CP^N model as a guide.

Quantum electrodynamics cannot be modified or extended in our way if both renormalizability and locality are required in the space-time of $(3+1)$ dimensions. We do not consider here genuinely or intrinsically nonlocal field theories in which the fundamental fields and/or interaction contains nonlocality.[2] In contrast to nonlocality, nonrenormalizability can be controlled formally by dimensional regularization or by a covariant cutoff in phenomenology. Therefore, here we abandon renormalizability in $(3+1)$ dimensions for the moment and move to a world of $(3+1)$ dimensions or consider a covariant cutoff theory in $(3+1)$ dimensions.

A. Boson matter

In order to construct a local Lagrangian with fermion matter fields alone, we first reexamine the gauge invariance of the bosonic matter model—namely, the CP^N model—from a slightly different viewpoint.

In the CP^N model the gauge noninvariance of the free Lagrangian L_0 due to $\partial^\mu\phi$ under $\phi \to e^{i\alpha(x)}\phi$ must be counterbalanced with that of the interaction $L_{\rm int}$. Therefore, $L_{\rm int}$ must have at least the same number of derivatives as L_0. Since L_0 and $L_{\rm int}$ have the same space-time dimension,

[2]For example, the field theories once considered by Yukawa [13] and his followers.

we must introduce an inverse of $(\phi^*\phi)$ in $L_{\rm int}$ to make up for the dimension due to ∂^μ in the numerator of $L_{\rm int}$. Keeping the number of ∂^μ in $L_{\rm int}$ the smallest, we reach almost uniquely the simplest form of the gauge-invariant Lagrangian made of the matter fields alone as

$$L_{\rm tot} = L_0 + L_{\rm int}, \tag{2}$$

where L_0 is the standard free Lagrangian,

$$L_0 = \sum_{i=1}^{N} \partial^\mu \phi_i^* \partial_\mu \phi_i - \sum_{i=1}^{N} m^2 \phi_i^* \phi_i, \tag{3}$$

and the interaction Lagrangian $L_{\rm int}$ is given by

$$L_{\rm int} = \lambda \frac{\sum_{i=1}^{N}(\phi_i^* \overleftrightarrow{\partial}^\mu \phi_i) \sum_{j=1}^{N}(\phi_j^* \overleftrightarrow{\partial}_\mu \phi_j)}{4\sum_{k=1}^{N}(\phi_k^*\phi_k)}, \quad (\lambda \to 1). \tag{4}$$

The indices (i, j, k) run from 1 to N so that the model is solvable in the leading order of $1/N$. They are referred to as the *copy* indices hereafter. From time to time, however, the summation over the copy indices will be suppressed unless we need to recall it.

Under the local U(1) gauge transformation, the fields transform with a space-time-dependent phase $\alpha(x)$ common to all copy indices i as

$$\phi_i \to e^{i\alpha(x)}\phi_i, \quad \text{and} \quad \phi_i^* \to e^{-i\alpha(x)}\phi_i^*. \tag{5}$$

For the total Lagrangian, both L_0 and $L_{\rm int}$ vary nontrivially under the gauge transformation (5), but the variations δL_0 and $\delta L_{\rm int}$ are so made as to be proportional to each other:

$$\delta L_0 = -i\left(\sum_i \phi_i^* \overleftrightarrow{\partial}_\mu \phi_i\right)\partial^\mu\alpha + \left(\sum_i \phi_i^*\phi_i\right)\partial^\mu\alpha\partial_\mu\alpha,$$
$$\delta L_{\rm int} = -\lambda \delta L_0. \tag{6}$$

These gauge variations cancel each other between L_0 and $L_{\rm int}$ for

$$\lambda = 1 \quad \text{(gauge limit)}. \tag{7}$$

If we remove the mass term and impose the constraint $\sum_i \phi_i^*\phi_i = N/2f$ in Eq. (4), we recognize this Lagrangian (with $\lambda = 1$) as that of the CP^{N-1} model [2]. However, we have introduced N copies solely to simplify the computation of the leading $1/N$ expansion. Our interest is not in the SU(N) symmetry among the different copies.

As far as U(1) gauge invariance is concerned, we may add to Eq. (2) the terms that are gauge invariant by themselves, for instance, nonderivative ϕ^4 couplings such as

$$L'_{\rm int} = -\sum_{i,j=1}^{N} \lambda_{ij}(\phi_i^*\phi_i)(\phi_j^*\phi_j), \tag{8}$$

where λ_{ij} are arbitrary real constants. However, in the leading $1/N$ order the interactions such as $L'_{\rm int}$ do not affect bound-state formation.[3] Therefore, we leave out such interactions hereafter. It is reassuring to see later that the vector bound state becomes massless with the correct gauge coupling irrespective of the additional gauge-invariant interactions such as $L'_{\rm int}$.

B. Fermionic model

Following the reasoning outlined above, we can obtain—with a little stretch of the imagination—a fermionic extension of the bosonic model Lagrangian (2). Since the free Lagrangian L_0 contains only one the first derivative of ψ, the interaction $L_{\rm int}$ can counterbalance the gauge variation of L_0 with only one first derivative of the field. Just as in the bosonic case, we need to introduce the inverse of the scalar density $\overline{\psi}\psi$ in $L_{\rm int}$ in order to match the dimension. Following the same reasoning as in the bosonic model, we reach the Lagrangian $L_0 + L_{\rm int}$,

$$L_0 = \sum_i \overline{\psi}_i(i\partial\!\!\!/ - m)\psi_i,$$
$$L_{\rm int} = -i\lambda \frac{\sum_i(\overline{\psi}_i\gamma_\mu\psi_i)\sum_j(\overline{\psi}_j\overleftrightarrow{\partial}^\mu\psi_j)}{2\sum_k\overline{\psi}_k\psi_k}, \quad (\lambda \to 1), \tag{9}$$

where the gauge invariance is realized at $\lambda = 1$. Under the gauge transformation

$$\psi \to e^{i\alpha(x)}\psi,$$
$$\overline{\psi} \to \overline{\psi}e^{-i\alpha(x)}, \tag{10}$$

the Lagrangian of Eq. (9) is invariant due to the cancellation between the gauge variations of L_0 and $L_{\rm int}$ at $\lambda = 1$:

$$\delta L_0 = -\overline{\psi}(\partial\!\!\!/\alpha)\psi,$$
$$\delta L_{\rm int} = \lambda\overline{\psi}(\partial\!\!\!/\alpha)\psi. \tag{11}$$

We may add to $L_{\rm int}$ the self-gauge-invariant terms such as

$$L'_{\rm int} = -\frac{fm}{4}(\overline{\psi}\gamma_\mu\psi)\frac{1}{(\overline{\psi}\psi)}(\overline{\psi}\gamma^\mu\psi), \tag{12}$$

where the insertion of the fermion mass m is just to make the constant f dimensionless. The constant f is unconstrained by gauge invariance. After we compute the massless bound state with $L_{\rm int}$ of Eq. (9) alone, we shall examine how the

[3] Because we compute the bound state of spin one, not of spin zero.

interactions like $L'_{\rm int}$ affect its mass and coupling. Since they will turn out to be irrelevant to the determination of the mass and coupling of the massless bound state, we shall not include them in our diagram calculation. Before calculating the diagram, some may suspect that the fermion-antifermion interaction through $\propto (\overline{\psi}\gamma_\mu\psi)(\overline{\psi}\gamma^\mu\psi)$ might be responsible for or relevant to binding a gauge boson. This is wrong. Such an interaction does not exist in our $L_{\rm int}$. Even if one includes it in $L_{\rm int}$, it does not participate in the formation of the massless gauge boson nor in the determination of the gauge coupling, as we shall see later.

Our fermionic Lagrangian (9) is obviously nonrenormalizable in four space-time dimensions just like that of the CP^N model. As we know, the only renormalizable U(1) gauge field theory with a charged fermion is quantum electrodynamics: the propagating gauge field A_μ is needed explicitly in the Lagrangian.

C. Auxiliary vector-field trick

Our bosonic Lagrangian (2) with $\lambda = 1$ takes the same form as what we could obtain by starting with the gauge-invariant Lagrangian of a nonpropagating auxiliary gauge field A_μ,

$$L_{\rm aux} = \sum_i (\partial_\mu - ieA_\mu)\phi_i^*(\partial^\mu + ieA^\mu)\phi_i - m^2\phi_i^*\phi_i. \quad (13)$$

Either by integrating Eq. (13) over A^μ or by substituting the equation of motion for A^μ,

$$eA_\mu = \frac{i}{2}\left(\sum_i \phi_i^* \overleftrightarrow{\partial}_\mu \phi_i\right) \Big/ \left(\sum_j \phi_j^*\phi_j\right), \quad (14)$$

we obtain for $m^2 \to 0$ the CP^N Lagrangian (before imposing the constraint and turning it into CP^{N-1}) [14].

When we compute the dimension-four operator of A^μ for the effective action using the loop correction, we obtain the "kinetic energy term" $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. One cannot obtain anything other than the gauge-invariant FF term ("the Maxwell term") since the Lagrangian (13) is gauge invariant by construction. Whether this appearance of the FF term is to be interpreted as the "generation of a bound state" or not should be subject to debate. If we accepted such an interpretation, a massless spin-one state would emerge irrespectively of the strength of the interaction e^2 which is implanted in Eq. (13) at the beginning. After a rescaling of the A_μ field, the physical coupling of A_μ to ϕ/ϕ^* is fixed to some number, which is independent of e at one loop and logarithmically divergent in four dimensions. The field A_μ is guaranteed to turn into a massless boson once the field is introduced as an auxiliary field. In contrast, in our model the strength of the interaction $L_{\rm int}$ must be tuned to the optimum value ($\lambda = 1$) in order to make the bound state massless. In this way we see that the masslessness of the vector bound state is a dynamical consequence of gauge invariance rather than a kinematical outcome.

The substitution of the equation of motion (14) also needs scrutiny: if one computes $\partial_\mu F^{\mu\nu}$ with this A^μ, one would obtain $\partial_\mu F^{\mu\nu} = 0$ instead of $\partial_\mu F^{\mu\nu} = J^\nu$. Therefore, the field A^μ of Eq. (14) is not acceptable as the composite gauge field. One would need contributions from loops to write a dynamical gauge field that obeys the correct equation of motion. We do not know how to write such an object in a local composite field.

What would happen if one attempted to introduce the auxiliary field A_μ in the fermionic model? For the fermionic matter, the Lagrangian with a nonpropagating auxiliary field is simply equal to

$$L_{\rm aux} = \sum_i \overline{\psi}_i(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi_i. \quad (15)$$

The equation of motion with respect to A_μ is trivially equal to $\sum_i \overline{\psi}_i\gamma_\mu\psi_i = 0$ and provides us with nothing. As for the functional integration over the auxiliary field A_μ, one cannot carry it out at the tree level since the auxiliary Lagrangian (15) is not quadratic in A_μ, unlike that of the bosonic model. When the two-point loop diagrams of $A_\mu A_\nu$ are computed, the local limit of the two-point functions ought to be proportional to $F_{\mu\nu}F^{\mu\nu}$ by the underlying gauge invariance. But we cannot obtain a compact local Lagrangian of the matter fields alone such as ours out of the auxiliary Lagrangian of Eq. (15).

The auxiliary vector-field trick bypasses the important part of the dynamics of the matter fields. In contrast, our explicit Lagrangian models provide dynamical details of binding which are either missing in the auxiliary field trick or are very different from it.

III. NOETHER CURRENT

When we attempt to write a conserved current in our models, we encounter one peculiar problem: we are unable to construct a conserved current with the prescription of the Noether theorem. In fact, such a current simply does not exist.

According to the general prescription, the Noether current J_μ^N is obtained when the Lagrangian is invariant under a set of space-time-independent phase transformations of fields. In the bosonic model, it would be generated by the transformation

$$\phi_i \to (1 + i\alpha)\phi_i \quad \text{and} \quad \phi_i^* \to (1 - i\alpha)\phi_i^*, \quad (16)$$

where α is infinitesimal and *independent* of the space-time. The variation $\delta L_{\rm tot}$ of $O(\alpha)$ under this transformation leads to the divergence of the Noether current through the identification

$$\partial^\mu J^N_\mu = -\delta L_{\rm tot}/\delta\alpha. \tag{17}$$

Using the equation of motion on the right-hand side, one ought to obtain the Noether current J^N_μ as

$$J^N_\mu = -i\sum_i\left(\frac{\partial L_{\rm tot}}{\partial(\partial^\mu\phi_i)}\phi_i - \phi_i^*\frac{\partial L_{\rm tot}}{\partial(\partial^\mu\phi_i^*)}\right). \tag{18}$$

When we follow this standard procedure in our models, we find that the right-hand side of Eq. (18) is identically zero in the gauge symmetry limit due to the cancellation between the contributions from L_0 and $L_{\rm int}$:

$$J^N_\mu = i(1-\lambda)\sum_i(\phi_i^*\overleftrightarrow{\partial}_\mu\phi_i), \tag{19}$$

where the term proportional to λ comes from $L_{\rm int}$ and the gauge symmetry holds at $\lambda = 1$. One may be puzzled when one thinks of the perturbative calculation: since ϕ and ϕ^* always appear pairwise in a product in the Lagrangian, one may assign the conserved U(1) charge ±1 to ϕ and ϕ^*. Then this charge ought to be conserved in all diagrams of physical processes (such as scattering and decay) even in the gauge symmetry limit where the Noether current disappears.

The same happens in the fermionic model too. Just as in the bosonic model, the conserved Noether current disappears in the gauge symmetry limit:

$$J^N_\mu = (1-\lambda)\sum_i\overline{\psi}_i\gamma_\mu\psi_i. \tag{20}$$

The current $\sum\overline{\psi}_i\gamma_\mu\psi_i$ is not the Noether current. It is a general property of the gauge theories with no gauge field that the Noether current is identically zero; $J^N_\mu \equiv 0$. It is easy to trace the root cause of this absence of the Noether current to local gauge invariance itself. An almost trivial proof is given in Appendix A. The proof can be easily extended to the non-Abelian models. It has an important implication in the non-Abelian case: if the Noether current existed, the generation of the massless gauge bosons would face a potential conflict with the no-go theorem of Weinberg and Witten [12].

Unlike the Noether current, the conserved energy-momentum tensor exists in the Abelian and non-Abelian gauge theories of matter fields alone. For the fermionic U (1) model with the Lagrangian of Eq. (9), the conserved energy-momentum tensor is given by

$$T^{\mu\nu} = i\sum_i\overline{\psi}_i\gamma_\mu\partial_\nu\psi_i - \frac{i\lambda(\sum_i\overline{\psi}_i\gamma^\mu\psi_i)(\sum_j\overline{\psi}_j\overleftrightarrow{\partial}^\nu\psi_j)}{2\sum_k(\overline{\psi}_k\psi_k)} - g^{\mu\nu}L_{\rm tot}. \tag{21}$$

It is manifestly gauge invariant with the matter fields alone.

IV. COMPOSITE U(1) GAUGE BOSON

It is natural to wonder if our U(1) models contain a gauge boson as a composite state even though we have not inserted it by hand. In order to answer to this question, we carry out a diagram calculation in this section in order to exhibit the dynamical mechanism of formation of the composite gauge boson. We compute our models perturbatively in the $1/N$ expansion: we sum an infinite series of the leading-$1/N$-order terms and show explicitly that a massless vector boson indeed appears as a pole in scattering amplitudes with the properties required by gauge symmetry both in the bosonic and the fermionic model. In the case of the CP^{N-1} model in which $\phi^*\phi$ is subject to a constraint, this diagram computation was done by Haber *et al.* [5]. Our primary interest is in the fermionic model, which is technically complex since channel coupling occurs between the $^3S_1$ and $^3D_1$ channels. Unlike the formal argument based on the auxiliary vector-field trick [15], the diagrammatic computation allows us to see explicitly how a massless bound state is formed dynamically with the matter particles. For instance, when we examine elastic fermion-antifermion scattering of $J^{PC} = 1^{--}$, we find that the massless bound state appears in the $^3S_1$ channel, not in the $^3D_1$ channel. That is, the bound state couples with the fermions through the vertex $\overline{\psi}\gamma_\mu\psi$, not through $\overline{\psi}\overleftrightarrow{\partial}_\mu\psi$. Nonetheless, the interactions of both types are needed to form a massless bound state.

A. Gauge boson in the bosonic model

We start with our U(1) bosonic model to study a composite gauge boson before our study of the fermionic model since the computation is simpler for the bosonic model, yet it demonstrates the essential aspects of the diagram calculation.

We consider the two-body $\phi^+\phi^-$ p-wave scattering ($J^{PC} = 1^{--}$), treating all N copies of the fields ($i = 1, \cdots N$) as independent. We show that a pole of a massless bound state appears in this channel. Then we proceed to make sure that the pattern and magnitude of the coupling of this bound state indeed obey what we expect for the U(1) gauge boson.

We study the p-wave amplitude for the two-body scattering,

$$\phi_i^+(p_1) + \phi_i^-(p_2) \to \phi_j^+(p_3) + \phi_j^-(p_4). \tag{22}$$

We compute the amplitude in the leading $1/N$ order since a compact explicit solution can be obtained only at this order. In the scattering (22), the copy indices are chosen to be the same for the initial particles and also for the final particles. In the diagram calculation, $L_{\rm int}$ is separated from $L_{\rm tot}$ in Eq. (4) and treated as *the interaction*. While this statement sounds trivial, we point out one subtlety. That is, when we carry out the perturbative calculation by splitting the Lagrangian into L_0 and L_{int}, we have fixed once and for all the gauge ambiguity of our Lagrangian (2). That is, when we write the propagator of ϕ/ϕ^* in the momentum space as $1/(p^2-m^2)$, we do not need more gauge fixing since there is no A^μ field in the Lagrangian. With this separation, the fields obey the equation of motion of L_0 that violates gauge symmetry. Consequently, the Noether current of L_0 is the conserved current in the diagrams. For the purpose of visualizing how the gauge-invariance limit is reached, we float λ in L_{int} as a free parameter until we set it to unity at the end of the calculation.

In the diagram calculation of the leading $1/N$ order, we normal-order the operator $\phi^*\phi$ in the denominator of L_{int} and expand it around its vacuum value as

$$1/\sum\phi^*\phi = 1/\left(\sum\langle 0|\phi^*\phi|0\rangle + \sum :\phi^*\phi:\right) = \frac{1}{\sum\langle 0|\phi^*\phi|0\rangle}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sum :\phi^*\phi:}{\sum\langle 0|\phi^*\phi|0\rangle}\right)^n, \tag{23}$$

where the summation $\sum$ with no index attached means the summation over the copy index $i(=1,\cdots N)$. This separation of the vacuum value is important for handling the denominator of L_{int} in a systematic $1/N$ expansion. [5] The vacuum expectation value $\langle 0|\phi^*\phi|0\rangle$ is infinite in the $(3+1)$ space-time, so it is regularized dimensionally as

$$\sum\langle 0|\phi^*\phi|0\rangle = \lim_{x\to 0}\sum\langle 0|T(\phi^*(x)\phi(0))|0\rangle, = \frac{N\Gamma(1-D/2)}{(4\pi)^{D/2}(m^2)^{1-D/2}}, \tag{24}$$

where N copies of bosons contribute to the vacuum value of the scalar density. The space-time dimension D is eventually set to four. Hereafter, we denote this vacuum expectation value by I_0^b,

$$I_0^b \equiv \sum\langle 0|\phi^*\phi|0\rangle. \tag{25}$$

Now we are ready to compute the two-body scattering of Eq. (22). The great simplification of the leading $1/N$ order is that for elastic scattering we only have to sum the chain of the bubble diagrams, as shown in Fig. 1, in which the copy index i runs within a loop of each bubble.

Let us define with the S matrix the two-body scattering amplitude $T(p_3,p_4;p_1,p_2)$ as

$$\langle p_3,p_4|S-1|p_1,p_2\rangle = i(2\pi)^4\delta^4(p_3+p_4-p_1-p_2) \times T(p_3,p_4;p_1,p_2). \tag{26}$$

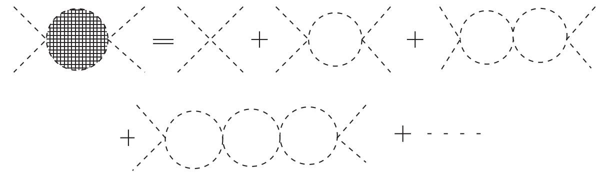

FIG. 1. The chain of the bubble diagrams for the elastic boson scattering.

The amplitude T has the Lorentz structure of the form

$$T(p_4,p_3;p_1,p_2) = (p_3-p_4)^\mu(p_1-p_2)^\nu T(q)_{\mu\nu}, \tag{27}$$

where $q=p_1+p_2=p_3+p_4$ and the one-particle states are normalized as $\langle p_i|p_j\rangle = 2E_i\delta(\mathbf{p}_i-\mathbf{p}_j)$ so that the amplitude $T(p_3,p_4;p_2,p_1)$ is a Lorentz scalar. For the elastic scattering in the leading $1/N$ order, it is sufficient to keep only the first term of the expansion (23) in the denominator of L_{int}. The normal-ordered product $(\sum :\phi^*\phi:)$ starts contributing to the next-to-leading order of $1/N$ in the elastic scattering.

The amplitude $T(q)_{\mu\nu}$ starts with a contact interaction term with no bubble, the first term on the right-hand side of Fig. 1, which is equal to

$$T^0_{\mu\nu} = \frac{\lambda}{2I_0^b}g_{\mu\nu}, \tag{28}$$

where the superscript “zero” of $T^0_{\mu\nu}$ indicates the zero-loop contribution of $O(\lambda)$. The bubble summation can be carried out by solving the algebraic equation (Fig. 2)

$$T(q)_{\mu\nu} = T^0_{\mu\nu} + K(q)_{\mu\kappa}T^\kappa_\nu(q). \tag{29}$$

where the kernel $K(q)_{\mu\kappa}$ is given by the single bubble diagram in which the copy index flows around the loop. Equation (29) will become powerful later when we sum the corresponding series in the fermionic model in which two eigenchannels contribute and entangle in the formation of a bound state.

A straightforward computation gives us the kernel as

$$K_{\mu\kappa}(q) = \frac{\lambda N\Gamma(1-D/2)}{(4\pi)^{D/2}(m^2)^{1-D/2}I_0^b} \times\left(g_{\mu\kappa} + \frac{1-D/2}{6m^2}(g_{\mu\kappa}q^2 - q_\mu q_\kappa)\right) + O(q^4). \tag{30}$$

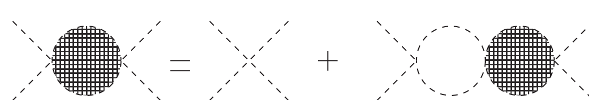

FIG. 2. The iteration equation of bubbles into a chain.

Since we want to extract the pole and residue of a massless bound state at $q^2 = 0$, we need $K_{\mu\kappa}(q)$ only to orders no higher than $O(q^2)$. The factor outside the large bracket in Eq. (30) is simply equal to λ when Eq. (24) is substituted for I_0^b, so that

$$K_{\mu\kappa}(q) = \lambda\left(g_{\mu\kappa} + \frac{1-D/2}{6m^2}(g_{\mu\kappa}q^2 - q_\mu q_\kappa)\right) + O(q^4). \tag{31}$$

Note here that $K_{\mu\kappa}(q)$ does not satisfy transversality, $q^\mu K_{\mu\kappa} \neq 0$. This is not a violation of gauge invariance. In the standard Lagrangian where the elementary A_μ field is present, one would need the $A_\mu A^\mu \phi^* \phi$ term to realize transversality of the photon self-energy, $q^\mu \Pi(q)_{\mu\kappa} = 0$, namely, gauge invariance. The term needed for transversality does exist in our model, but it is tucked away elsewhere at this stage. As we shall see in a moment, it is this nontransversality of $K_{\mu\kappa}(q)$ that makes the composite boson massless.[4]

Let us substitute Eq. (31) into the iteration equation (29) and move the term $\lambda g_{\mu\kappa}$ of the kernel $K_{\mu\kappa}(q)$ to the left-hand side. We may drop the term proportional to $q_\mu q_\mu$ by using $q \cdot (p_1 - p_2) = 0 = q \cdot (p_3 - p_4)$ on the external boson lines. Then Eq. (29) turns into

$$(1-\lambda)T(q)_{\mu\nu} = T^0_{\mu\nu} + \frac{\lambda(1-D/2)q^2}{6m^2}T(q)_{\mu\nu} + O(q^4). \tag{32}$$

Now we go to the gauge limit $\lambda \to 1$. Since $T^0_{\mu\nu}$ is independent of q, Eq. (32) tells us that in this limit there is a pole at $q^2 = 0$ in the amplitude $T(q)_{\mu\nu}$ as

$$T(q)_{\mu\nu} = -\frac{6m^2}{(1-D/2)q^2}T^0_{\mu\nu} + O(q^2), \quad (\lambda = 1). \tag{33}$$

When the parameter λ is off the gauge limit ($\lambda \neq 1$), the pole is located away from zero at $q^2 = [6(1-\lambda)/\lambda(1-D/2)]m^2$ so that the bound state would be either a massive vector boson or a tachyon. We extract the residue of the pole at $q^2 = 0$ for $\lambda = 1$ and compare this residue with what we would obtain from the Feynman diagram of the standard U(1) gauge Lagrangian of the charged scalar fields,

$$L_{\text{tot}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial^\mu\phi^* - ieA^\mu\phi^*)(\partial_\mu\phi + ieA_\mu\phi) - m^2\phi^*\phi. \tag{34}$$

By equating our residue with that of the Feynman diagram, we obtain the coupling e^2 of our model as

$$e^2 = \frac{3(4\pi)^{D/2}(m^2)^{2-D/2}}{N\Gamma(2-D/2)}. \tag{35}$$

When we approach the space-time dimension of $D = 4$, this coupling can be expressed in terms of the logarithmic cutoff of divergence as

$$e^2 = \frac{48\pi^2}{N\ln(\overline{\Lambda}^2/m^2)}, \tag{36}$$

where $\ln\overline{\Lambda}^2 = (2-D/2)^{-1} + \ln 4\pi - \gamma_E$ (γ_E = Euler constant). The sign of e^2 comes out to be positive. It is amusing to observe that the factor $(1-D/2)$ in the denominator of Eq. (33) is combined with $\Gamma(1-D/2)$ in $1/I_b^0$ of $T^0_{\mu\nu}$ to turn into $\Gamma(2-D/2)$, which is the logarithmic divergence in the space-time dimension of $D = 4$. That is, a quadratic divergence $\Gamma(1-D/2)$ metamorphoses into a logarithmic divergence, which can happen in dimensional regularization.

If we had started with the auxiliary A_μ field and generated the $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ to the leading $1/N$ order, we would have obtained a coupling constant identical to Eq. (36) after rescaling A_μ by wave-function renormalization. [3] This equality is not unexpected since the one-loop self-energy diagram of the auxiliary A_μ field leading to Eq. (36) is identical to the bubble diagram of the p-wave $\phi^\dagger\phi$ scattering in the leading $1/N$ order. There is no guarantee that this equality holds beyond the leading $1/N$ order since noncontact interactions enter the scattering amplitude while the self-energy diagram remains the two-point function.

In order to claim that the massless bound state discovered above is indeed the U(1) gauge boson, we must show that other couplings of this state obey the pattern required for the gauge boson. One may bypass this part by resorting to the gauge invariance that has been embedded in the Lagrangian of our model. But we show here explicitly how the U(1) gauge invariance arises diagrammatically for the coupling of the massless bound state.

The absence of the coupling $eA_\mu\partial^\mu(\phi^*\phi)$ is obvious since the form of our L_{int} requires the bound state to couple with ϕ^*/ϕ through $(\phi^*\overleftrightarrow{\partial}^\mu\phi)$, not through $\partial^\mu(\phi^*\phi)$. This is also required by C invariance of our Lagrangian. However, there must exist a coupling $e^2\phi^*\phi A_\mu A^\mu$, where A_μ is the effective gauge field and e^2 is given by Eq. (35). Aside from this coupling, there should be no coupling of dimension four such as a nonderivative quartic coupling of A_μ.

The coupling of $\phi^*\phi A_\mu A^\mu$ requires a little computation. Here the first nontrivial term of the expansion of $1/(\phi^*\phi)$ enters the computation,

$$-\frac{\lambda}{4I_0^2}(\phi^*\overleftrightarrow{\partial}^\mu\phi)(\phi^*\overleftrightarrow{\partial}_\mu\phi)(:\phi^*\phi:). \tag{37}$$

[4]This is the case in the CP^{N-1} model analyzed in Ref. [5] too.

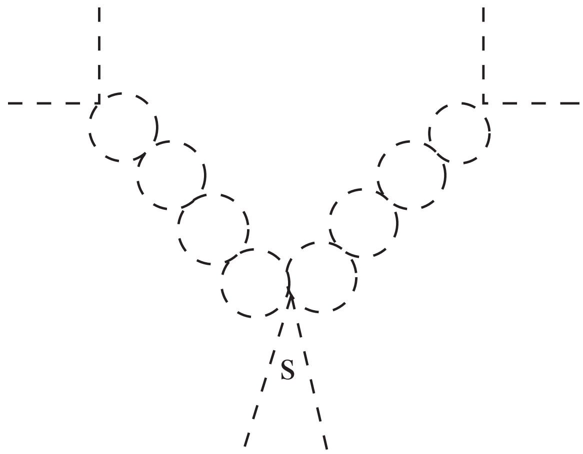

FIG. 3. The diagram for the formation of the $\phi^*\phi A_\mu A^\mu$ coupling. The $\phi^*\phi$ pair arises from the six-body interaction of Eq. (37) at the center. The letter S denotes that the external $\phi^*\phi$ pair at the center is in the scalar state $\phi^*\phi$, not in the vector state $\phi^*\overleftrightarrow{\partial}_\mu\phi$.

At the leading $1/N$ order, we attach a chain of the bubble diagrams to $(\phi^*\overleftrightarrow{\partial}^\mu\phi)$ and another chain to $(\phi^*\overleftrightarrow{\partial}_\mu\phi)$ to form the composite A^μ and A_μ bosons, respectively (see Fig. 3). Then we equate this diagram at the poles of the A^μ and A_μ bosons to the diagram of Fig. 4 which is obtained with the interaction $e^2\phi^*\phi A_\mu A^\mu$ of the standard U(1) gauge Lagrangian (34).

This calculation gives us the relation

$$e^4 = \left(\frac{3(4\pi)^{D/2}(m^2)^{2-D/2}}{N\Gamma(2-D/2)}\right)^2. \tag{38}$$

Two powers e^2 out of e^4 in Eq. (38) are to be attributed to the couplings of the $\phi^*\phi$ pairs with A_μ and with A_ν at the outer ends of two bubble chains in Fig. 3. The remaining e^2 is to be assigned to the four-body $A_\mu A^\mu\phi^*\phi$ coupling at the center. Therefore, the coupling e^4 of Eq. (38) is precisely what we want to see.

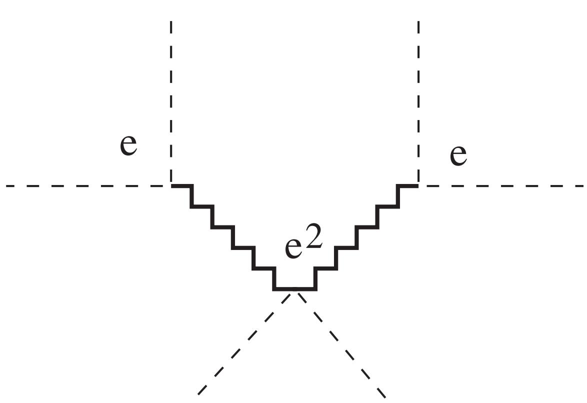

FIG. 4. The corresponding Feynman diagram for $e^2\phi^*\phi A_\mu A^\mu$.

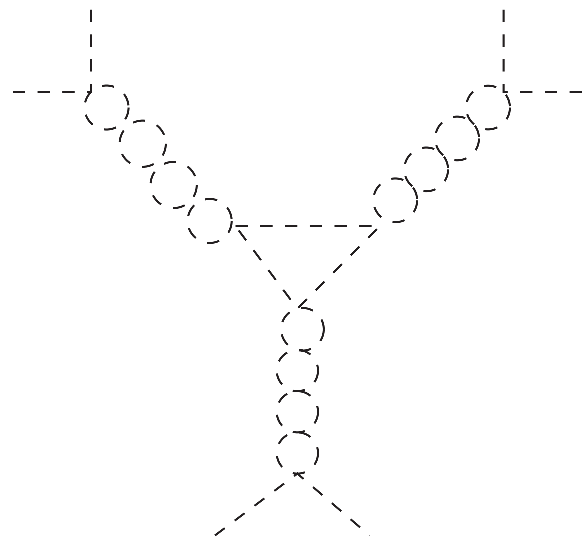

FIG. 5. The triple self-coupling of the composite A_μ, which can appear potentially from the center of the diagrams containing three chains of ϕ^*/ϕ bubbles.

The absence of the triple self-coupling of A_μ is a consequence of C invariance. Diagrammatically, this is assured in the U(1) model by the cancellation between a pair of diagrams where the two chains are interchanged. Since they do not cancel in the non-Abelian models and there is some subtlety, we add a few comments here in anticipation of the non-Abelian cases. The relevant diagram is depicted in Fig. 5.

If we indeed compute this coupling with individual diagrams, we must be careful about the surface-term ambiguity. The triangular loop at the center is linearly divergent in four space-time dimensions and therefore its constant term is ambiguous by the surface term of the loop integral. The value depends on how the loop momentum is routed, just as in the chiral anomaly or the finite part of the electron self-energy in QED. To fix this finite ambiguity, one must impose invariance and/or symmetry that must be preserved in theory. In this case the C invariance of $L_{\rm tot}$ and/or the Bose statistics of the composite A_μ fixes the ambiguity. With the right choice of the routing momentum, a pair of triangular loop diagrams cancel each other and change the net triple self-coupling to zero in the U(1) model.

In comparison, we need an explicit computation of diagrams to show that the net quartic self-coupling vanishes, although there is no subtlety with regards to the surface-term ambiguity. In the presence of the six-body coupling of Eq. (37), three classes of loop diagrams can potentially contribute to the quartic self-coupling of the composite gauge boson in the leading $1/N$ order (Fig. 6).

The square box diagrams [Fig. 6(a)] alone do not cancel among themselves. When we add all three classes of the diagrams, however, they sum up to zero at the zero external momentum limit where the on-shell quartic coupling

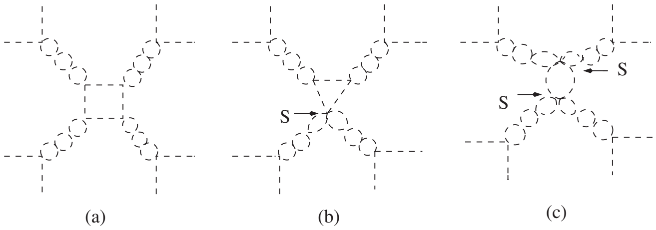

FIG. 6. Three classes of diagrams can contribute to the quartic self-coupling of composite A_μ. The letter S for the six-body ϕ^*/ϕ interaction point in the loop at the center denotes that the $\phi^*\phi$ pair is in the scalar state. (a) No six-body coupling, (b) one six-body coupling, and (c) two six-body couplings.

constant is defined. Up to an overall constant, the cancellation occurs among the three types of diagrams in Fig. 6 as

$$\propto \left(\frac{1}{4} - \frac{1}{2} + \frac{1}{4}\right)\frac{1}{\Gamma(2 - D/2)}, \tag{39}$$

where the first, second, and third terms in the bracket are from the three types of diagrams in Figs. 6(a), 6(b), and 6(c), respectively. Of course, this cancellation is not an accident. Its origin is traced back to the U(1) gauge invariance incorporated into the Lagrangian.[5]

Our fundamental Lagrangian L_{tot} is invariant under the gauge transformation $\phi(x) \to e^{i\alpha(x)}\phi(x)$ and the conjugate. Once a massless vector bound state emerges with the effective coupling $ie(\phi^* \overleftrightarrow{\partial}_\mu \phi)A^\mu$, the only way for it to be compatible with the gauge invariance is that the additional interaction $e^2\phi^*\phi A_\mu A^\mu$ exists for this effective A_μ field and that A_μ transforms as $eA_\mu \to eA_\mu + i\partial_\mu\alpha$ under $\phi(x) \to e^{i\alpha(x)}\phi(x)$. As far as the interactions of dimension four are concerned, there is no other way known to us that satisfies the U(1) gauge invariance incorporated in L_{tot}. As for the self-couplings of A_μ, we would have to satisfy U(1) gauge invariance with the A_μ fields alone without derivatives. That is, there is no room to accommodate a nonderivative self-interaction of A_μ in four dimensions. When we argue in this way, the gauge invariance of the composite A_μ coupling is an inevitable and trivial consequence of the gauge symmetry of L_{tot}, once a massless spin-one bound state emerges with the coupling $ie(\phi^* \overleftrightarrow{\partial}_\mu \phi)A^\mu$. When we take this viewpoint, the crucial step is whether or not a massless bound state of spin one is indeed formed out of the interactions among the matter fields themselves. The rest may be interpreted as logical inevitability.

[5]We freely switch between $\phi^*\phi$ and $:\phi^*\phi:$ in this calculation since our computation of the couplings involves only those diagrams in which a ϕ/ϕ^* particle emitted from one L_{int} annihilates at another L_{int} in the center of the diagram; see Figs. 6(b) and 6(c). The normal ordering makes no difference in Figs. 6(b) and 6(c) for this reason.

Before closing this subsection, we comment on the interactions of dimension higher than four (in the world of space-time dimension four or $3+1$). The interaction $(\phi^*\phi)^2 A_\mu A^\mu$ has dimension six. It can arise from the third term ($n = 2$) of the expansion of the denominator $1/(\phi^*\phi)$ in Eq. (4), that is,

$$L_{\mathrm{int}} = \frac{1}{4(I_0^b)^3}(:\phi^*\phi:)^2(\phi^* \overleftrightarrow{\partial}_\mu \phi)(\phi^* \overleftrightarrow{\partial}^\mu \phi). \tag{40}$$

By attaching the chains of the ϕ bubbles to $(\phi^* \overleftrightarrow{\partial}_\mu \phi)$ and $(\phi^* \overleftrightarrow{\partial}^\mu \phi)$, then going to the gauge-boson mass shells on the chains, we can extract the effective interaction of dimension six for the composite gauge boson,

$$L_{\mathrm{int}} = \frac{e^2}{I_0^b}(\phi^*\phi)^2 A_\mu A^\mu, \tag{41}$$

where the coupling e^2 is given by Eq. (35). This coupling is not gauge invariant by itself. However, there is another effective coupling of dimension six, which contains only a single A_μ. We can compute it with the interaction of Eq. (40) and put it in the form of an effective interaction,

$$L_{\mathrm{int}} = \frac{ie}{I_0^b}(\phi^*\phi)(\phi^* \overleftrightarrow{\partial}_\mu \phi)A^\mu. \tag{42}$$

When the two interactions (41) and (42) of dimension six are combined and added to the first term of the expansion of $1/(\phi^*\phi)$,

$$\frac{1}{4I_0^b}(\phi^* \overleftrightarrow{\partial}_\mu \phi)(\phi^* \overleftrightarrow{\partial}^\mu \phi), \tag{43}$$

the sum total is gauge invariant. That is, when all the couplings of $O(1/I_0^b)$ [Eqs. (41), (42), and (43)] are combined, the interaction of dimension six for the effective field A_μ is gauge invariant. The combined effective interaction can be cast into the form

$$L_{\mathrm{int}}^{\mathrm{eff}} = \frac{1}{4I_0^b}(\phi^* \overleftrightarrow{D}^\mu \phi)(\phi^* \overleftrightarrow{D}_\mu \phi), \tag{44}$$

where $D_\mu = \partial_\mu + ieA_\mu$ and $(\phi^* \overleftrightarrow{D}_\mu \phi) \equiv \phi^* D^\mu \phi - (D_\mu \phi)^* \phi$. The interaction of Eq. (44) illustrates what happens for the effective interactions of higher dimension in general. It is obvious for dimensional reasons that $L_{\mathrm{int}}^{\mathrm{eff}}$ must be inversely proportional to powers of I_0^b. Although I_0^b is formally proportional to m^2 in the dimensional regularization, it is quadratically divergent in the cutoff ($\sim N\overline{\Lambda}^2$) in the world of $D = 4$. If we give a physical meaning to the cutoff, therefore, the interactions of dimension six are suppressed by $O(p^2/N\overline{\Lambda}^2)$ in the region of energy scale $O(p^2)$ relative to those of dimension four. Meanwhile, the divergences of $O(N \ln \overline{\Lambda}^2)$ are absorbed into the gauge coupling e^2 as we have seen in Eq. (35). Therefore, if our model should turn out to be phenomenologically relevant in one way or another, its cutoff $\overline{\Lambda}$ would place these higher-dimensional interactions under control. Whether or not these interactions can generate anything phenomenologically interesting is a separate question.

We can cast the amplitudes of higher-dimension processes in the standard U(1) gauge theory with the elementary gauge boson into the form of effective interactions. However, such effective interactions are generally not identical to the higher-dimensional interactions that have been obtained above from our Lagrangian (2). The loop-diagram amplitudes produced by the standard U(1) gauge theory do exist equally in our model since the gauge boson exists as a composite. Our model contains the additional terms that are generated by matter fields and suppressed by the large cutoff scale of I_b. At these orders the physics is generally different from the standard gauge theory of the elementary gauge boson. If our model were identical to the standard U(1) gauge theory, it would be perfectly renormalizable in our world of four dimensions. But that is not the case: our model contains the higher-dimensional local interactions that are additional to the standard gauge theory and suppressed by powers of $1/I_b = O(\overline{\Lambda}^2)$.

B. Gauge boson in the fermionic model

The computation of the massless bound state is technically a little complex in the fermionic model since there exist two channels of $J^{PC} = 1^{--}$. We compute the elastic fermion-antifermion scattering

$$f^+(p_1, s_1) + f^-(p_2, s_2) \to f^+(p_3, s_3) + f^-(p_4, s_4) \quad (45)$$

in the leading $1/N$ order with the Lagrangian (9). The copy indices are chosen to be the same for the initial f^+f^- and the final f^+f^-. We shall suppress spin indices $s_i (i = 1, \cdots 4)$ in the following since they are obvious in most places. We leave out the *self-gauge-invariant* interactions such as Eq. (12). Although those interactions certainly contribute to the fermion-fermion scattering in general, we show later that the omission of such interactions does not affect the properties of the massless bound state.

We follow our path taken for the bosonic model: we separate $\overline{\psi}\psi$ in the denominator of L_{int} into a sum of the vacuum expectation values and the normal-ordered products $:\overline{\psi}\psi:$ and then expand it in the power series of $\sum :\overline{\psi}\psi:/\sum\langle 0|\overline{\psi}\psi|0\rangle$. The vacuum expectation value $\langle 0|\overline{\psi}\psi|0\rangle$ is divergent and dimensionally regularized as

$$\begin{aligned}\sum\langle 0|\overline{\psi}\psi|0\rangle &= -\lim_{x\to 0}\text{tr}\langle 0|T(\psi(x)\overline{\psi}(0)|0\rangle \\ &= -\frac{4Nm\Gamma(1-D/2)}{(4\pi)^{D/2}(m^2)^{1-D/2}},\end{aligned} \quad (46)$$

where the trace (tr) in the first line of the right-hand side refers to the spinor indices of ψ and $\overline{\psi}$. We shall denote the right-hand side of Eq. (46) by I_0^f hereafter,

$$I_0^f \equiv \langle 0|\overline{\psi}\psi|0\rangle = -4mI_0^b. \quad (47)$$

I_0^f is opposite in sign to I_0^b of the boson (25) and its dimension is three instead of two.

Now we proceed to compute the two-body scattering amplitude of $J^{PC} = 1^{--}$. There exist two eigenchannels in the fermion scattering. The fermion-antifermion pair is in the configuration of $\overline{v}_{-\mathbf{p}}\boldsymbol{\gamma}u_{\mathbf{p}}$ in one channel and in $2\mathbf{p}\overline{v}_{-p}u_{\mathbf{p}}$ in the other in the center-of-momentum frame. The spins of $\overline{v}_{-\mathbf{p}}$ and $u_{\mathbf{p}}$ are combined into a triplet in both cases so that they make the ${}^3S_1$ and ${}^3D_1$ states of f^+f^-, respectively. With our choice of L_{int} in Eq. (9), the fermion-antifermion pair turns from $\overline{\psi}\gamma_\mu\psi$ on one side to $(\overline{\psi}\overleftrightarrow{\partial}^\mu\psi)$ on the other, or conversely from $(\overline{\psi}\overleftrightarrow{\partial}_\mu\psi)$ to $\overline{\psi}\gamma^\mu\psi$ at every interaction point in the chain of bubbles.

Let us define the Lorentz scalar amplitude $T(p_1, p_2; p_3, p_4)$ with the S matrix as

$$\begin{aligned}&\langle p_3, p_4|S-1|p_1, p_2\rangle \\ &\quad = i(2\pi)^4\delta^4(p_3 + p_4 - p_1 - p_2) \\ &\quad\quad \times T(p_3, p_4; p_1, p_2),\end{aligned} \quad (48)$$

where the one-fermion states are so normalized that the amplitude $T(p_3, p_4; p_1, p_2)$ is a Lorentz scalar and its Lorentz structure is given in the (2×2) matrix form by

$$\begin{aligned}T = (\overline{u}_{p_3}\gamma_\mu v_{p_4}, \overline{u}_{p_3}(p_3 - p_4)_\mu v_{p_4}/m)\begin{pmatrix} T_{11}^{\mu\nu}(q) & T_{12}^{\mu\nu}(q) \\ T_{21}^{\mu\nu}(q) & T_{22}^{\mu\nu}(q)\end{pmatrix} \\ \times\begin{pmatrix}\overline{v}_{p_2}\gamma_\nu u_{p_1} \\ \overline{v}_{p_2}(p_1 - p_2)_\nu u_{p_1}/m\end{pmatrix},\end{aligned} \quad (49)$$

where $q = (p_1 + p_2) = (p_3 + q_4)$. The perturbation series for $T(q)^{\mu\nu}$ starts with the tree diagram, which gives $-(\lambda/2I_0^f)g_{\mu\nu}$ to the off-diagonal elements of $T_{\mu\nu}^0$:

$$T_{\mu\nu}^0 = -\frac{1}{2I_0^f}\begin{pmatrix} 0 & \lambda \\ \lambda & 0\end{pmatrix}g_{\mu\nu}. \quad (50)$$

The summation of the bubble chains can be carried out by solving the matrix equation,

$$T(q)_{\mu\nu} = T_{\mu\nu}^0 + K(q)_{\mu\kappa}T_\nu^\kappa(q), \quad (51)$$

where the kernel $K(q)_{\mu\kappa}$ is the 2×2 matrix of the four single-bubble diagrams that connect between the γ_μ-type vertex (${}^3S_1$) and the $\overleftrightarrow{\partial}_\mu$-type vertex (${}^3D_1$) (see Fig. 7),

$$K^{\mu\kappa}(q) = \begin{pmatrix} K(q)_{11} & K(q)_{12} \\ K(q)_{21} & K(q)_{22}\end{pmatrix}^{\mu\kappa}. \quad (52)$$

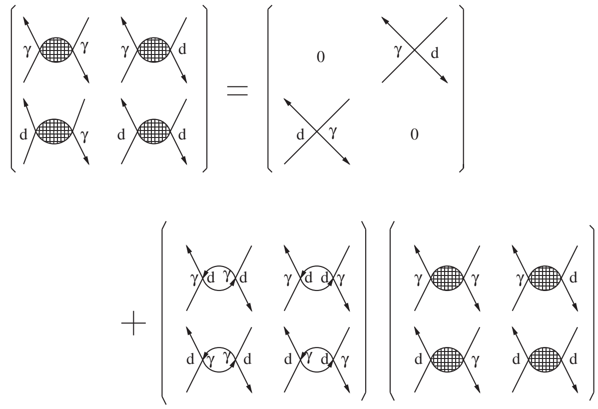

FIG. 7. Iteration of bubble diagrams for fermion scattering. The letters γ and d denote that the fermion pair at the interaction point is $\bar{\psi}\gamma_\mu\psi$ and $\bar{\psi}\overleftrightarrow{\partial}_\mu\psi$, respectively.

In order to extract the mass and coupling of the composite boson from $T(q)_{\mu\nu}$, we need $(I-K(q))_{\mu\kappa}$ near $q^2=0$ in Eq. (51). To be more specific, we need the terms of $g_{\mu\kappa}$ and $(q^2g_{\mu\kappa}-q_\mu q_\kappa)$ for K_{ij}. In fact, for the off-diagonal elements K_{12} and K_{21}, all we need are the leading terms that give $K_{12}K_{21}=O(q^2)$. By straightforward diagram computation, we find the relevant terms of $K^{\mu\kappa}(q)$ near $q^2=0$ as

$$\begin{aligned}K^{\mu\kappa}(q)_{11}&=\lambda\left(g^{\mu\kappa}+\frac{\Gamma(2-D/2)}{6m^2\Gamma(1-D/2)}(g^{\mu\kappa}q^2-q^\mu q^\kappa)\right)\\&=K^{\mu\kappa}(q)_{22},\\K^{\mu\kappa}(q)_{12}&=-\lambda\left(\frac{\Gamma(-D/2)}{\Gamma(1-D/2)}-2\right)g^{\mu\kappa},\\K^{\mu\kappa}(q)_{21}&=-\lambda\left(\frac{\Gamma(2-D/2)}{6m^2\Gamma(1-D/2)}\right)(g^{\mu\kappa}q^2-q^\mu q^\kappa).\end{aligned}\tag{53}$$

We have kept Γ functions above since they are partially canceled with $\Gamma(1-D/2)$ coming from $1/I_0^f$ of T^0 when $(I-K)^{-1}$ is operated on T^0 later. The terms in Eq. (53) that turn out to determine the pole and residue of the massless

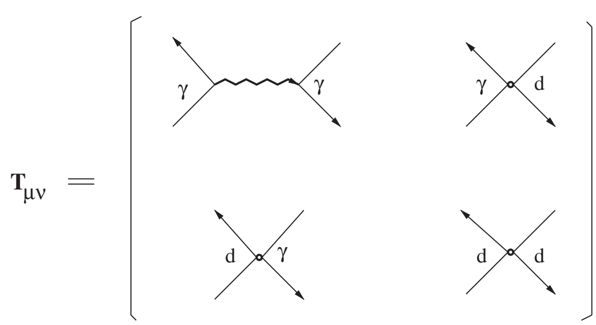

FIG. 8. The massless bound state appears only in the upper left corner, which is the $^3S_1$ channel.

bound state are the first term $\lambda g^{\mu\kappa}$ of the diagonal element $K(q)_{11}^{\mu\kappa}(=K(q)_{22}^{\mu\kappa})$ and the off-diagonal element $K(q)_{12}^{\mu\kappa}\neq 0$ at $q^2=0$.

Let us examine the pole and residue of the matrix amplitude $T_{\mu\nu}$ at $q^2=0$ by solving Eq. (51) as

$$T_{\mu\nu}=\left(\frac{1}{I-K}\right)_\mu^{\ \kappa}T^0_{\kappa\nu}.\tag{54}$$

Since the external fermion lines are on mass shell, the terms proportional to $q_\mu q_\kappa$ in $K_{\mu\kappa}$ have been removed by use of the Dirac equation and the mass shell condition on the external lines. We then approach the gauge symmetry limit $\lambda=1$ of $T=(I-K)^{-1}T^0$. The result is

$$T(q)_{\mu\nu}=\frac{(4\pi)^{D/2}(m^2)^{2-D/2}}{\Gamma(2-D/2)}\begin{pmatrix}\frac{3}{4q^2}&\frac{C}{m^2}\\\frac{C}{m^2}&\frac{C}{m^2}\end{pmatrix}g_{\mu\nu},\tag{55}$$

where

$$C=\frac{D(D-2)}{32(D+1)}.\tag{56}$$

A pole appears only in the (11)-matrix element at the upper left corner in Eq. (55) and the other entries are regular at $q^2=0$. This is depicted in Fig. 8.

This means that the bound state appears in the channel of $\overline{\psi}\gamma_\mu\psi\to\overline{\psi}\gamma_\mu\psi$, that is, in the $^3S_1$ channel, not in the $^3D_1$ channel.[6] If either end of the chain is $\overline{\psi}\overleftrightarrow{\partial}_\mu\psi$, no massless pole appears in such a chain.

By comparing the matrix element $T_{11}^{\mu\nu}$ with the one-photon pole diagram of the standard U(1) gauge interaction $-e\overline{\psi}\gamma_\mu\psi A^\mu$, we can identify the gauge coupling e^2 with the residue at the pole to obtain

$$e^2=\frac{3(4\pi)^{D/2}(m^2)^{(2-D/2)}}{4N\Gamma(2-D/2)},\tag{57}$$

or, in terms, of the covariant ultraviolet cutoff in the space-time of $D=4$,

$$e^2=\frac{12\pi^2}{N\ln(\overline{\Lambda}^2/m^2)}.\tag{58}$$

This is the parallel of Eq. (35) in the bosonic model. While the quartic divergence $[\propto\Gamma(-D/2)\sim\Lambda^4]$ and quadratic divergence $(\sim\Lambda^2)$ are present in $T(q)_{\mu\nu}$, they do not enter the residue of the pole at $q^2=0$. Therefore, the coupling e^2

[6]This has nothing to do with the d-wave threshold behavior $\sim|\mathbf{p}|^l(l=2)$. The threshold behaviors reside in the spinorial factors in Eq. (49) and have been separated out in defining $T(q)_{ij}^{\mu\nu}$.

involves only the logarithmic divergence ($\sim 1/N \ln \Lambda^2$) as it does for the bosonic model.

As we have pointed out, we may add to our fermionic model the interaction $L'_{\rm int}$ of Eq. (12) which is gauge invariant by itself. Let us denote the shifts of the matrices $K(q)$ and T^0 due to $L'_{\rm int}$ as $K(q) \to K(q) + \Delta K(q)$ and $T^0 \to T^0 + \Delta T^0$. Near $q^2 = 0$, these shifts are given by

$$\Delta T^0_{\mu\nu} = \frac{1}{2I^f_0} g_{\mu\nu} \begin{pmatrix} fm & 0 \\ 0 & 0 \end{pmatrix} \tag{59}$$

and

$$\Delta K^{\mu\kappa} = \frac{1 - D/2}{6m^2} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} (g^{\mu\kappa} q^2 - q^\mu q^\kappa). \tag{60}$$

It is not difficult to see that these modifications, Eqs. (59) and (60), do not alter either the location of the pole at $q^2 = 0$ or its residue. In terms of diagrams, we can visualize the effect of Eqs. (59) and (60) as follows. We should first notice the fact that the newly added bubble consisting of γ_μ on one end and γ_κ on the other end vanishes like $(g_{\mu\kappa} q^2 - q_\mu q_\kappa)$ at $q = 0$. Let us say that this bubble is of the type $\gamma_\mu \otimes \gamma_\kappa$. When the $\gamma_\mu \otimes \gamma_\kappa$ bubble enters the middle of the eigenchannel that produces the bound state, the chain would thus acquire a factor of $O(q^2)$ from this bubble. Therefore it cancels the pole and becomes irrelevant to the formation of the massless bound state. The pole at $q^2 = 0$ is produced only by the $g_{\mu\kappa}$ term of $K(q)_{\mu\kappa}$ in the chain of bubbles of the types $\gamma_\mu \otimes \overleftrightarrow{\partial}_\nu$ and $\overleftrightarrow{\partial}_\mu \otimes \gamma_\nu$ alone. With the addition of $L'_{\rm int}$, therefore, the massless pole is undisturbed and its residue is unaffected.

Let us move on to the self-coupling of the gauge field. Charge-conjugation invariance forbids the triple self-coupling, but the quartic self-coupling is not forbidden by any discrete symmetry. Since the massless bound state couples only to the $^3S_1$ vertex, namely, to $\overline{\psi}\gamma_\mu\psi$, the relevant diagrams have a square box at the center with six permutations of the four γ vertices, that is, the diagram of Fig. 6(a) in which the boson lines are replaced by the fermion lines and the γ matrices sit at the four corners of the box. However, the sum of these box diagrams vanishes in the zero-energy-momentum limit of the bound-state bosons—not just the leading divergent term ($\sim \ln \overline{\Lambda}^2$) but rather all finite terms in this limit. This fact is well known as the gauge-invariance requirement $\sim e^4 F^\nu_\mu F^\kappa_\nu F^\lambda_\kappa F^\mu_\lambda$ on the photon-photon scattering amplitude in quantum electrodynamics.

For the diagrams corresponding to Figs. 6(b) and 6(c) with the boson lines replaced by fermions, the two chains of bubbles are attached to the six-body fermion interaction. However, since the six-body fermion interaction is of the form $(\overline{\psi}\gamma_\mu\psi)(\overline{\psi}\overleftrightarrow{\partial}^\mu\psi)(\overline{\psi}\psi)$, one of the vector vertices starts with the γ vertex but the other starts with the $\overleftrightarrow{\partial}$ vertex. As we have already observed, the massless bound-state pole cannot appear in the latter chain. Therefore, the massless vector bound state can be formed only in one of the two chains attached to the six-body interaction point, not in both. That is, only three massless bound states can be formed in Fig. 6(b) and two in Fig. 6(c). Combining this observation with that for Fig. 6(a) above, we conclude that there is no nonderivative quartic self-coupling of the massless U(1) bound state in the fermion model either, just as gauge invariance requires.

The lowest possible coupling of higher dimension with fermion fields is the Pauli term $i\overline{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu}$. This coupling is gauge invariant by itself. With our interaction $L_{\rm int}$, however, our composite boson does not have this coupling. To see this, recall the decomposition of the photon-fermion vertex for the fermion on mass shell, $i\overline{u}\sigma_{\mu\nu}q^\nu v' = \overline{u}(p + p')_\mu v' - 2m\overline{u}\gamma_\mu v'$. This relation tells us that if the massless bound state had the Pauli-term interaction, we would have its pole in the channels of both $\overline{\psi} \overleftrightarrow{\partial}_\mu \psi$ and $\overline{\psi}\gamma_\mu\psi$. In our preceding study, however, we have found a massless pole only in $\overline{\psi}\gamma_\mu\psi$. This means that there is no Pauli term.

The effective interaction $\overline{\psi}\psi A_\mu A^\mu$ is also of dimension five and not gauge invariant by itself. As in the bosonic model, if an interaction of A_μ appears with a dimension higher than four, it ought to appear in a gauge-invariant combination since the underlying Lagrangian is gauge invariant. As for this specific interaction, the accompanying gauge-covariant partners are $\partial^\mu\overline{\psi}\partial_\mu\psi$ and $ie(\overline{\psi}\overleftrightarrow{\partial}_\mu\psi)A^\mu$. But we have already found that the coupling $(\overline{\psi}\overleftrightarrow{\partial}_\mu\psi)A^\mu$ does not exist in our model, and $\partial_\mu\psi\partial^\mu\psi$ does not exist in $L_{\rm tot}$. Therefore the coupling $(\overline{\psi}\psi)A_\mu A^\mu$ can be generated as an effective interaction in our model.

One of the merits of our fermionic model is that it reveals the dynamical details explicitly in regard to how the self-interaction of the constituent fermions conspires to generate the composite gauge boson. Specifically, the composite gauge boson is formed with fermions in the presence of the process of the transition between the $^3S_1$ and the $^3D_1$ channel. No massless bound state can be formed with the $^3S_1$ channel alone. There is no place to see this dynamics in the auxiliary field trick on fermions in which the auxiliary vector field only has the $^3S_1$ interaction.

V. NON-ABELIAN EXTENSIONS

It is possible to extend our U(1) models to non-Abelian models. The non-Abelian extension turns out to be quite easy if we choose matter fields in the SU(2) doublet. In this section we present the SU(2)-doublet model for both bosons and fermions and compute the composite gauge bosons again in the leading $1/N$ order. The extension of our U(1) models to a general Lie group or even to a SU(2) representation other than the doublet encounters difficulty. This is not a simple technical difficulty; rather, it involves some problem at a fundamental level in our class of models. We explain this difficulty in the text, then go a little further with a few examples of the bosonic models in Appendix B.

Those who approach the problem with the auxiliary field trick would trivially extend the U(1) model to general groups and representations by simply replacing the 2×2 matrices $\frac{1}{2}\tau_a$ of SU(2) with the $n \times n$ generator matrices T_a of a general Lie group. In our case, however, such a simple substitution does not extend our models to those of general groups or representations.[7] This is another indication of the fact that our models are physically different at some fundamental level from what the auxiliary field trick gives.

A. Non-Abelian bosonic model

Let us introduce N families of scalar boson fields in the SU(2) doublet,

$$\Phi^i = \begin{pmatrix} \phi_1^i \\ \phi_2^i \end{pmatrix}, \qquad (i = 1, \cdots N), \tag{61}$$

and their conjugates $\Phi^{i\dagger}$, which we write in a row. The subscripts (1,2) are those of SU(2). We shall suppress the copy index and/or the SU(2) index wherever there is no confusion. Our bosonic Lagrangian is given simply by

$$L_0 = \sum_i \partial^\mu \Phi^{i\dagger} \partial_\mu \Phi^i - \sum_i m^2 \Phi^{i\dagger}\Phi^i,$$
$$L_{\rm int} = \lambda \frac{\sum_i (\Phi^{i\dagger}\boldsymbol{\tau}\overleftrightarrow{\partial}_\mu \Phi^i) \cdot \sum_j (\Phi^{j\dagger}\boldsymbol{\tau}\overleftrightarrow{\partial}^\mu \Phi^j)}{4\sum_k (\Phi^{k\dagger}\Phi^k)} \quad (\lambda \to 1), \tag{62}$$

where i, j, and k are copy indices and $\boldsymbol{\tau}$ denotes the Pauli matrices $\tau_a (a = 1, 2, 3)$.[8] For the SU(2) gauge invariance of $L_0 + L_{\rm int}$, we give the proof here for the infinitesimal rotation,

$$\Phi \to \left(1 + \frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\right)\Phi,$$
$$\Phi^\dagger \to \Phi^\dagger\left(1 - \frac{i}{2}\boldsymbol{\tau}\cdot\boldsymbol{\alpha}\right), \tag{63}$$

where $\boldsymbol{\alpha}$ is a space-time-dependent vector function. Let us compute the variations $L_0 \to L_0 + \delta L_0$ and $L_{\rm int} \to L_{\rm int} + \delta L_{\rm int}$ separately and confirm tthe cancellation to $O(\alpha)$ between the two variations. For L_0, it is easy to obtain

$$\delta L_0 = -\frac{i}{2}(\Phi^\dagger \boldsymbol{\tau}\overleftrightarrow{\partial}_\mu \Phi)\cdot\partial^\mu\alpha + O(\alpha^2). \tag{64}$$

The computation of $\delta L_{\rm int}$ requires a little care. To the order $O(\alpha)$, it is not difficult to obtain the transformation

$$(\Phi^\dagger\boldsymbol{\tau}\overleftrightarrow{\partial}^\mu\Phi) \to \Phi^\dagger U^\dagger \boldsymbol{\tau} U \partial^\mu \Phi$$
$$- (\partial^\mu\Phi^\dagger)U^\dagger\boldsymbol{\tau}U\Phi + 2i(\Phi^\dagger\Phi)\partial^\mu\alpha + O(\alpha^2), \tag{65}$$

where $U = 1 + i\boldsymbol{\tau}\cdot\boldsymbol{\alpha}/2$. The third term proportional to $\partial^\mu\alpha$ on the right-hand side has been obtained by use of the relation

$$\boldsymbol{\tau}(\boldsymbol{\tau}\cdot\partial^\mu\alpha) + (\boldsymbol{\tau}\cdot\partial^\mu\alpha)\boldsymbol{\tau} = 2\partial^\mu\alpha. \tag{66}$$

Since an isoscalar product remains unchanged under global SU(2) rotations, it holds for arbitrary SU(2)-doublet functions A, B, C, and D that

$$((UA)^\dagger\boldsymbol{\tau}UB)\cdot((UC)^\dagger\boldsymbol{\tau}UD) = (A^\dagger\boldsymbol{\tau}B)\cdot(C^\dagger\boldsymbol{\tau}D). \tag{67}$$

Thanks to this relation, when we take the product of Eq. (65) with itself in $L_{\rm int}$, four products made of the first two terms are invariant by themselves as

$$(\Phi^\dagger U^\dagger\boldsymbol{\tau}U\partial_\mu\Phi)\cdot(\Phi^\dagger U^\dagger\boldsymbol{\tau}U\partial^\mu\Phi) = (\Phi^\dagger\boldsymbol{\tau}\partial_\mu\Phi)\cdot(\Phi^\dagger\boldsymbol{\tau}\partial^\mu\Phi), \tag{68}$$

and so forth. The product of the third term with itself is $O(\alpha^2)$. In the cross products of the first two terms with the third term $2i(\Phi^\dagger\Phi)\partial^\mu\alpha$, we may set $U = 1$ since we are computing to $O(\alpha)$. Dividing these terms of $O(\alpha)$ in the numerator of $\delta L_{\rm int}$ by $4(\Phi^\dagger\Phi)$, we obtain that the variation of $L_{\rm int}$ is equal to

$$+\frac{i}{2}\lambda(\Phi^\dagger\boldsymbol{\tau}\overleftrightarrow{\partial}_\mu\Phi)\cdot\partial^\mu\alpha + O(\alpha^2), \tag{69}$$

which cancels δL_0 for $\lambda = 1$.

The proof to all orders of $\boldsymbol{\alpha}$ is not difficult, although it is a bit tedious. We can carry it out with brute force using the local rotation matrix U for the SU(2) doublet matter fields,

$$U = \cos\alpha + i(\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\tau})\sin\alpha, \tag{70}$$

where $\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}/\alpha$. Alternatively, in the case of bosons, we could introduce the auxiliary fields and integrate over them to reach the Lagrangian (62). Operationally, this turns out to be a much simpler avenue. While its physical meaning is subject to debate and some people may feel it is questionable, we can use the auxiliary field method as a mathematical tool of manipulation without a problem. If one wants to proceed along this line, one starts with

[7] One well-known example of the special role of SU(2) may come to mind, i.e., the instanton. The instanton is special to SU(2), as it is not extendable to SU(N) ($N \geq 3$) or other general groups because of its topological property. In our case, however, topology is not an issue. What is important is the *self-duality* of the group and the representation.

[8] This bosonic Lagrangian as well as its Abelian version appear in the earlier paper [3].

$$L_{\rm tot} = (\partial^\mu + i\mathbf{A}^\mu\Phi)^\dagger \cdot (\partial_\mu + i\mathbf{A}_\mu)\Phi - m^2\Phi^\dagger\Phi + \frac{1}{2}\mu^2\mathbf{A}^\mu\mathbf{A}_\mu, \tag{71}$$

where $\mathbf{A}^\mu = \frac{1}{2}\tau_a A_a^\mu$. Although we do not really need it here, we have added the mass term μ^2 to $\mathbf{A}^\mu$ for gauge fixing, which is to be removed after the functional interaction is completed.

Having seen the Lagrangian of Eq. (62), it is tempting to speculate that if the isospin $\frac{1}{2}\tau_a$ is replaced by the $n \times n$ matrices of the generator T_a of some other group G, we could obtain the non-Abelian extension to the case where the matter fields form the n-dimensional multiplets of the group G; namely,

$$L_{\rm tot} = \sum_i \partial^\mu\Phi^{i\dagger}\partial_\mu\Phi^i - m^2\Phi^{i\dagger}\Phi^i + \lambda\frac{\sum_i(\Phi^{i\dagger}T_a\overleftrightarrow{\partial}_\mu\Phi^i)\cdot\sum_j(\Phi^{j\dagger}T_a\overleftrightarrow{\partial}^\mu\Phi^j)}{\sum_k(\Phi^{k\dagger}\Phi^k)} \quad (\lambda \to 1), \tag{72}$$

where $T_a \neq \frac{1}{2}\tau_a$. Unfortunately, this does not work. The Lagrangian of Eq. (72) is not gauge invariant. We can pinpoint the step where the proof fails in this attempt: the relation of Eq. (66) is crucial in achieving non-Abelian gauge invariance in the Lagrangian (62). This relation holds only for the SU(2) doublet.

Some may yet wonder why one cannot resort to the auxiliary field trick starting with

$$L_{\rm tot} = (\partial^\mu + i\mathbf{A}^\mu\Phi)^\dagger \cdot (\partial_\mu + i\mathbf{A}_\mu)\Phi - m^2\Phi^\dagger\Phi, \tag{73}$$

where $\mathrm{A}_\mu = T_a A_\mu^a$. The equation of motion for the auxiliary field A_μ^a is to be obtained by solving

$$-i(\Phi^\dagger T_a\overleftrightarrow{\partial}_\mu\Phi) + \Phi^\dagger\{T_a, T_b\}\Phi A_\mu^b = 0. \tag{74}$$

The $n \times n$ matrix $\{T_a, T_b\}$ is not proportional to a unit matrix except in the case of $T_a = \frac{1}{2}\tau_a$. In fact, its determinant is zero in most cases. Consequently, the set of the algebraic equations (74) is generally unsolvable. This same problem derails an attempt to integrate over the field A_μ^a to get an effective action in terms of Φ and $\Phi^\dagger$ alone. We have illustrated this difficulty using two examples in Appendix B.

When one attempts the diagram calculation with the wrong Lagrangian of Eq. (72), one could tune the location of a pole in the chain of the bubble diagrams to zero by setting λ off unity. However, when one proceeds to calculate the coupling of $\Phi^\dagger\Phi A_\mu A^\mu$ (see Fig. 3), the Lagrangian of Eq. (72) would generate the form

$$\Phi^\dagger\Phi\mathbf{A}_\mu\cdot\mathbf{A}^\mu, \tag{75}$$

where the structure $\mathbf{A}_\mu\cdot\mathbf{A}^\mu$ arises from the denominator of $L_{\rm int}$ and enters the center of the triangular loop in Fig. 3. However, the correct non-Abelian structure for these couplings ought to be

$$\Phi^\dagger\{T_a, T_b\}\Phi A_\mu^a A^{\mu b}. \tag{76}$$

This conflict is another manifestation of the fact that the Lagrangian of Eq. (72) is not gauge invariant.

These arguments are more than what we really need, but they hopefully clarify the special role of the SU(2) doublet matter fields when we attempt to write a *local* non-Abelian gauge-invariant Lagrangian with matter fields alone. We have not succeeded in finding such a Lagrangian in a reasonably simple form except for the SU(2) doublet matter.

B. Non-Abelian fermionic model

The non-Abelian extension is possible for the fermionic model if one follows the bosonic model given above. For the SU(2) gauge group where the Dirac fields form SU(2) doublets with N copies,

$$\Psi^i = \begin{pmatrix}\psi_1^i \\ \psi_2^i\end{pmatrix}, \quad \overline{\Psi}^i = (\overline{\psi}_1^i, \overline{\psi}_2^i) \quad (i = 1, 2\cdots N), \tag{77}$$

the gauge-invariant Lagrangian is given by

$$L_0 = \sum_{i=1}\overline{\Psi}^i(i\partial\!\!\!/ - m)\Psi^i,$$
$$L_{\rm int} = -i\lambda\frac{\sum_i(\overline{\Psi}^i\boldsymbol{\tau}\gamma_\mu\Psi^i)\cdot\sum_j(\overline{\Psi}^j\boldsymbol{\tau}\overleftrightarrow{\partial}^\mu\Psi^j)}{2\sum_k(\overline{\Psi}^k\Psi^k)} \quad (\lambda \to 1). \tag{78}$$

Gauge invariance can be proved in a similar way as in the bosonic model, although the auxiliary field method never leads us to this Lagrangian. To the first order in $\boldsymbol{\alpha}(x)$ under the space-time-dependent rotation $\Psi \to \exp(i\boldsymbol{\tau}\cdot\boldsymbol{\alpha}(\mathrm{x})/2)\Psi$ and its conjugate, the gauge variations are given by

$$\delta L_0 = -\frac{1}{2}(\overline{\Psi}\gamma_\mu\boldsymbol{\tau}\Psi)\cdot\partial^\mu\alpha + O(\alpha^2),$$
$$\delta L_{\rm int} = -\lambda\delta L_0(\lambda \to 1). \tag{79}$$

We can prove the gauge invariance to all orders of $\boldsymbol{\alpha}(x)$ using Eq. (70). In fact, a brute-force proof to all orders of α is mathematically less cumbersome for the fermionic model than for the bosonic model.

Just as in the case of bosonic matter, this simple form of the non-Abelian model is possible only for the doublet matter fields in SU(2) gauge symmetry. It should be emphasized that our non-Abelian fermionic model cannot be obtained from the Lagrangian of nonpropagating auxiliary vector fields.

C. Noether current

As it happens in the Abelian models, the Noether current does not exist in our bosonic and fermionic non-Abelian models. The reason is the same as in the Abelian case: for the Lagrangians with the matter fields alone, the contributions to the Noether current from L_0 and $L_{\rm int}$ cancel each other as a consequence of gauge invariance. The proof in Appendix A can be trivially extended to the non-Abelian models. Even without such a general proof, the Noether currents off the gauge symmetry limit (which are given below) clearly show their absence in the gauge symmetry limit.

The Noether current exists off the gauge symmetry limit. Following the standard prescription, we obtain the Noether currents from our Lagrangians of Eqs. (62) and (78) in the form

$$\mathbf{J}^N_\mu = i(1-\lambda)\Phi^\dagger \frac{\boldsymbol{\tau}}{2}\overleftrightarrow{\partial}_\mu \Phi \quad \text{(bosonic)},$$
$$\mathbf{J}^N_\mu = (1-\lambda)\overline{\Psi}\frac{\boldsymbol{\tau}}{2}\gamma_\mu \Psi \quad \text{(fermionic)}. \tag{80}$$

As for the energy-momentum tensor, the conserved tensor operator exists for any value of λ just as in the U(1) models.

D. Composite gauge bosons

In the case of the SU(2)-doublet matter fields, the non-Abelian diagram calculation is almost identical to the Abelian one. The only difference is in the insertion of the $\boldsymbol{\tau}$ matrix at every point of the vectorial interactions in Figs. 1 and 7. The massless composite bosons emerge in the $J^{PC} = 1^{--}$ channels of the adjoint representation of SU(2). In the case of fermion matter the composite massless bosons appear in the $^3S_1$ eigenchannel, that is, they couple only through $\overline{\Psi}\boldsymbol{\tau}\gamma_\mu\Psi$. The correct properties of the massless bound states are confirmed just as in the Abelian cases.

We summarize the differences between the SU(2)-doublet models and the Abelian models:

(A) For the non-Abelian models of SU(2)-doublet matter fields, the vacuum expectation values $I^b_0 = \langle 0|\Phi^\dagger\Phi|0\rangle$ and $I^f_0 = \langle 0|\overline{\Psi}\Psi|0\rangle$ are twice as large as their Abelian values, respectively, since both the upper and lower components of the doublet matter contribute.

(B) The bubble diagrams entering the kernel K of the iteration equation are scaled upward by the same factor of 2 since a trace is taken within the bubble loop: $\mathrm{tr}(\tau_a\cdot\tau_b) = 2\delta_{ab}$.

(C) Since the multiplication of the factor of 2 in (A) and (B) occurs in both the numerator and the denominator of the kernel K in Eqs. (31) and (53), it keeps the kernel K unchanged from the Abelian value. Meanwhile, the lowest-order T matrix T^0 is scaled down by a factor of 2 since it is inversely proportional to I^b_0 (I^f_0), as is the amplitude $T = (I-K)^{-1}T_0$.

Since the kernel $K^{\mu\nu}$ remains unchanged, $(I-K)$ is still transverse and starts with a term proportional to $g^{\mu\nu}q^2 - q^\mu q^\nu$ with the same nonvanishing coefficient. Consequently, the solution for the iterated amplitude T takes the same form as in the corresponding Abelian models, but the residue at $q^2 = 0$ is half as large, reflecting the fact that the lowest-order term T^0 is scaled down by a factor of 2.

Summing up this argument, the location of the pole at $q^2 = 0$ remains the same and its residue is scaled down by a factor of 2, relative to the Abelian models, for both the bosonic and the fermionic model. We describe below some more details specific to each of the non-Abelian models.

1. The bosonic model

We compute the chain of bubble diagrams as shown in Fig. 1 where the $\boldsymbol{\tau}$ matrices are inserted at every point of interaction. The residue at the massless pole is compared with that of the corresponding Feynman diagram computed with the standard Lagrangian of the SU(2) gauge symmetry,

$$L_{\rm int} = ig_2(\Phi^\dagger \mathbf{A}^\mu\partial_\mu\Phi - \partial_\mu\Phi^\dagger\mathbf{A}^\mu\Phi) + g_2^2\Phi^\dagger(\mathbf{A}^\mu\cdot\mathbf{A}_\mu)\Phi, \tag{81}$$

where $\mathbf{A}^\mu = \frac{1}{2}\tau_a A^\mu_a$. We obtain the gauge coupling of the composite SU(2) gauge bosons $\mathbf{A}^\mu$ to the matter fields,

$$\frac{g_2^2}{4\pi} = \frac{96\pi^2}{N\ln(\overline{\Lambda}^2/m^2)}, \tag{82}$$

when it is expressed with the cutoff $\overline{\Lambda}$ in four space-time dimensions.[9] Recall that the standard definition of g_2 accompanies the generators $\frac{1}{2}\boldsymbol{\tau}$ instead of just $\boldsymbol{\tau}$. [See the definition of $\mathbf{A}^\mu$ following Eq. (81).] In the leading $1/N$ order, the magnitude of the coupling (82) coincides with what one would obtain in the auxiliary field trick since it comes from the same single bubble diagram with $\boldsymbol{\tau}$ on both ends.

The four-body interaction $\Phi^\dagger\Phi A_\mu A^\mu$ can be computed with the second term of the expansion for $1/(\Phi^\dagger\Phi)$ around its vacuum value in $L_{\rm int}$, namely,

$$-\frac{1}{4(I^b_0)^2}(\Phi^\dagger\boldsymbol{\tau}\overleftrightarrow{\partial}_\mu\Phi)\cdot(\Phi^\dagger\boldsymbol{\tau}\overleftrightarrow{\partial}^\mu\Phi)(:\Phi^\dagger\Phi:). \tag{83}$$

[9] For $\overline{\Lambda}$, see Eq. (36) and the line following it.

Attaching chains of bubbles to $(\Phi^\dagger \boldsymbol{\tau} \overleftrightarrow{\partial}^\mu \Phi)$ and $(\Phi^\dagger \boldsymbol{\tau} \overleftrightarrow{\partial}_\mu \Phi)$ of this interaction and approaching the zero-momentum limit, we obtain g_2^4, of which g_2^2 is assigned to the gauge couplings of two composite gauge bosons with the external $\Phi^\dagger \boldsymbol{\tau} \Phi$ at the outer ends of the chains and the remaining g_2^2 is assigned to the $\Phi^\dagger \Phi A_\mu A^\mu$ coupling. This step is a repeat of what we have done for the Abelian model depicted in Figs. 3 and 4. Going through this computation, we find that the resulting g_2^2 for $\Phi^\dagger (\boldsymbol{A}^\mu \cdot \boldsymbol{A}_\mu)\Phi$ is equal to the value given in Eq. (82), as we expected.

For the non-Abelian gauge bosons, there must be the triple self-coupling and the quartic self-coupling. They are computed with the diagrams of Figs. 5 and 6 after inserting the τ matrices appropriately. The triple self-coupling diagrams, of course, do not cancel among themselves in the non-Abelian case. Charge-conjugation invariance allows for the triple self-coupling since the non-Abelian charge flowing in the opposite directions in a pair of triangular diagrams survives with $\tau_a \tau_b - \tau_b \tau_a = 2i\epsilon_{abc}\tau_c \neq 0$. Paying attention to the subtlety of the linear divergence that has been cautioned earlier, we find that the value obtained for the triple self-coupling agrees with what the SU(2) gauge symmetry requires by $-\frac{1}{4}\mathbf{G}_{\mu\nu} \cdot \mathbf{G}^{\mu\nu}$. The quartic self-coupling arises from the diagrams with four-corner, three-corner, and two-corner loops at the center [i.e., Figs. 6(a), 6(b), and 6(c)] and survives in the limit of zero external momenta. They have the correct magnitude and group structure as required by the SU(2) gauge symmetry.

All this should not be surprising after we have found a triplet of spin-one massless bound states out of the manifestly gauge-invariant Lagrangian. Once we have found that the effective fields of these bound states couple with the matter fields in the form

$$L_{\text{int}} = ig_2(\Phi^\dagger \mathbf{A}^\mu \partial_\mu \Phi - \partial_\mu \Phi^\dagger \mathbf{A}^\mu \Phi), \tag{84}$$

with $\mathbf{A}^\mu = \frac{1}{2}\tau_a A_a^\mu$, all other couplings of $\mathbf{A}_\mu$ necessary to satisfy the SU(2) gauge invariance ought to be generated by loop and chain diagrams in the same $1/N$ order. Otherwise, the models would violate the SU(2) gauge invariance that was embedded in the Lagrangian at the beginning. We know of no other way for this to be compatible with the SU(2) gauge symmetry once the interaction of Eq. (84) emerges.

2. *The fermionic model*

Let us turn to the fermionic model. While the presence of two $J^{PC} = 1^{--}$ channels requires a 2×2 matrix calculation, the diagram computation of the bound-state generation is identical to that of the Abelian case except for the insertion of the $\boldsymbol{\tau}$ matrices into the 2×2 matrix equation of Fig. 7 after replacing the boson lines with the fermion lines. Massless bound states appear in the ${}^3S_1$ channel here again, and the squared SU(2) gauge coupling expressed in g_2^2 is larger than that of the U(1) fermionic model by a factor of 2 just as in the bosonic case:

$$\frac{g_2^2}{4\pi} = \frac{24\pi^2}{N \ln(\bar{\Lambda}^2/m^2)}, \tag{85}$$

where the coupling g_2 is defined by

$$L_{\text{int}} = -g_2 \bar{\Psi} \gamma_\mu \mathbf{A}^\mu \Psi. \tag{86}$$

When we work on the other couplings of dimension four, we do not encounter any complication new to the non-Abelian symmetry. The reason is that the massless bound states couple to the matter fields only through the vertex of $(\bar{\Psi}\gamma_\mu \boldsymbol{\tau} \Psi)$, not through $(\bar{\Psi}\boldsymbol{\tau}\overleftrightarrow{\partial}_\mu \Psi)$. Therefore the computation of the triple and quartic self-couplings can be carried out in the same way as in the U(1) model. The relevant diagrams are those of Figs. 5 and 6 where the boson lines are replaced by the fermion lines. Since the composite bound states generated in the chains of bubbles couple with the fermions only through $(\bar{\Psi}\boldsymbol{\tau}\gamma_\mu \Psi)$ and not through $(\bar{\Psi}\boldsymbol{\tau}\overleftrightarrow{\partial}^\mu \Psi)$, the vertices of the triangle (Fig. 5) and the box [Fig. 6(a)] at the center of the diagram are only those of γ_μ, not of $\overleftrightarrow{\partial}_\mu$. The diagrams of Figs. 6(b) and 6(c) do not contribute since the six-body interaction $(\bar{\Psi}\boldsymbol{\tau}\gamma_\mu \Psi)(\bar{\Psi}\boldsymbol{\tau}\overleftrightarrow{\partial}^\mu \Psi)(\bar{\Psi}\Psi)$ is incapable of producing two composite bosons. (Recall the argument in the Abelian fermionic model.) As for the fermionic triangular and box diagrams corresponding to Figs. 5 and 6(a), the same large-N computation was actually carried out 20 years ago in a similar model [16] that contains an explicit gauge-symmetry breaking but only through the gauge boson mass. We do not repeat the calculation of the self-couplings for the non-Abelian fermionic model here. The bottom line is that the same coupling g_2 as the matter-gauge-boson coupling of Eq. (85) appears in the self-interaction of the gauge bosons, as we expected.

All these beautiful outcomes conforming to non-Abelian gauge symmetry are manifestations of the gauge invariance that was embedded in the Lagrangian at the beginning. Hoping that we are not overly repetitious, we emphasize that once the massless bound states of spin one appearing and their effective fields $\mathbf{A}^\mu$ couple with the matter fields like $g_2\bar{\Psi}\gamma_\mu \mathbf{A}^\mu \Psi$, the bound states must be gauge bosons and the associated gauge self-couplings of $\mathbf{A}^\mu$ in $-\frac{1}{4}\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}$ must be generated in order to satisfy the SU(2) gauge invariance of L_{tot}. We know no other way to realize the non-Abelian gauge invariance.

VI. DISCUSSION

We start this final section with an obvious observation common to all of our models. In our models we cannot introduce an elementary gauge field using the substitution rule $\partial_\mu \to \partial_\mu + ieA_\mu$ in our Lagrangian. The reason is obvious from the structure of the models: this substitution operation is nothing other than one special gauge transformation. Take for example the fermion fields ψ in our

U(1) Lagrangian. The substitution $\partial_\mu \psi \to (\partial_\mu + ieA_\mu)\psi$ is realized by the rotation

$$\psi(x) \to \exp\left(ie \int^x A_\mu(y) dy^\mu\right)\psi(x). \tag{87}$$

Since Eq. (87) is one of the gauge transformations with

$$\alpha(x) = e \int^x A_\mu(y) dy^\mu, \tag{88}$$

the function $\alpha(x)$ is canceled out between L_0 and $L_{\rm int}$ by gauge invariance and disappears from the Lagrangian entirely. Therefore the elementary A_μ field cannot be introduced into our Lagrangians in this way. The inability to introduce the elementary A_μ field in our Lagrangians using the so-called substitution rule closely parallels the vanishing of the Noether current.

The next observation concerns the no-go theorem of Weinberg and Witten. The theorem was stated in the following way [12].

Theorem: A theory that allows the construction of a Lorentz-covariant conserved four-vector current J^μ cannot contain massless particles of spin $j > 1/2$ with nonvanishing values of the conserved charge $\int J^0 d^3x$.

The proof is simple. First fix the Lorentz scalar value of the matrix element $\langle p'|J_\mu|p\rangle$ for the massless spin-one particle in the forward limit $p' \to p$. Then make a Lorentz transformation and examine its rotational property around the momentum $\mathbf{p}$ in the brick-wall frame ($\mathbf{p}' = -\mathbf{p}$). We need the conserved current J_μ that provides the Lorentz scalar charge $\int J_0 d^3x$.

The theorem holds whether the massless boson is elementary or composite. As was emphasized by the authors [12], however, the theorem does not apply to the standard non-Abelian gauge bosons (without spontaneous symmetry breaking). The catch is in the word "Lorentz-covariant." The state of zero helicity does not exist for massless gauge bosons. In order to make the theory manifestly Lorentz covariant and gauge invariant at the same time, one has to fix a gauge by introducing an unphysical ghost state in the Lagrangian. Otherwise, one cannot carry out the diagram calculation. Fixing a gauge by a subsidiary condition either violates manifest gauge invariance or introduces a state that does not exist as a physical particle state. Therefore, Lorentz scalar charges that meet the conditions of the Theorem do not exist in the standard non-Abelian gauge theory.[10]

What should we do with this theorem for our non-Abelian models? If we could write the non-Abelian Noether currents with the matter fields alone, we would potentially interfere with this theorem. However, the Lorentz-covariant conserved currents do not exist in our models. They exist only off the gauge symmetry limit ($\lambda \neq 1$) and disappear as we go to the gauge symmetry limit of $\lambda = 1$, and it is only at this point that the vector bound states become massless. We thus circumvent the theorem. Is this really the answer to the potential conflict of the composite non-Abelian gauge bosons with the Weinberg-Witten theorem? To be frank, we are not totally comfortable with this answer. But it appears in our models that the generation of the massless non-Abelian composite bosons evades the conflict with the Weinberg-Witten theorem.[11]

It is explicitly visible in our models that gauge invariance requires that the force in the 1^{--} channel be attractive ($\lambda > 0$) and that the bound state in this channel be massless ($\lambda \to 1$). Repulsive forces ($\lambda < 0$) cannot be gauge invariant. We are tempted to speculate that even if gauge fields are not introduced explicitly, gauge bosons must appear as composite states if a theory is gauge invariant. While it sounds like a trivial proposition, it is desirable to elevate it to a rigorous theorem of field theory.

One obvious question is whether our models have anything to do with the real world. At an early stage of the electroweak theory, people discussed the possibility of composite W and Z bosons. [17,18] A quarter century ago we also proposed a nonrenormalizable phenomenological model of composite W and Z bosons where an explicit symmetry breaking enters only through the W/Z masses [16,19]. This occurred right after the experimental confirmation of the W and Z bosons at accelerators [20,21]. At that time very little was known experimentally about the properties of W and Z bosons. One sensitive theoretical test was to study how much deviation from the gauge symmetry could be accommodated for the self-couplings of dimension four through their loop contributions [22]. A more general test irrespective of sources was proposed [23] and is still being used for experimental tests of the minimal standard model. Now that the Higgs boson has been discovered with its properties roughly in agreement with theoretical expectations, the next step is to raise the precision in the interaction of W and Z bosons by direct measurement. The early indication of the two-photon anomaly at 750 GeV is one example that may open up a new window. However, since the invariant mass of 750 GeV is near the upper end of the two-photon phase space in the current data and "the anomaly" is still no more than a three-standard-deviation effect even with the ATLAS and CMS data combined, we need to wait some time before

[10] If one takes the purist viewpoint that the initial and final states of the matrix element $\langle p'|J_\mu|p\rangle$ must be asymptotic states, the theorem does not apply to the non-Abelian gauge theory like QCD, which is singular in the infrared limit so that one-gluon states are not asymptotic states. Our non-Abelian models contain $N(\to \infty)$ doublets of matter particles so that the infrared limit is nonsingular, i.e., not confining.

[11] The W and Z bosons in the extra-dimension model [7] are the lowest-lying Kaluza-Klein modes with mass so that they do not conflict with the theorem.

a consensus is reached among experimentalists on this anomaly. Both experimentalists and phenomenologists are working toward to this goal [24,25].

When our model is expressed as a composite gauge theory with the effective fields $\mathbf{A}_\mu$, the difference from the minimal standard model would appear in the interactions of dimension higher than four which are suppressed by powers of $p^2/\overline{\Lambda}^2$ at $|p^2| < \overline{\Lambda}^2$. When experiments explore the region of energies comparable or higher than $\overline{\Lambda}$, we shall be able to directly discriminate our model Lagrangian from the standard model of W and Z bosons. But we currently have no theoretical basis to speculate on the magnitude of $\overline{\Lambda}$.

We conclude with one disturbing question to which we give no good answer. Is it really possible to tell experimentally or even theoretically whether a given particle is elementary or composite? This is a nagging question that confronted theorists [26] at the height of nuclear democracy in the early 1960s. Theorists proposed various criteria of compositeness, but no consensus emerged. Although we have started with the matter fields alone and constructed the massless gauge bosons explicitly as their bound states, can we exactly describe the same physics with some other Lagrangian in which all particles are elementary? Can we really answer the question of elementarity vs compositeness once for all?

The following theorem was given by Kamefuchi, O'Rafeartaigh, and Salam [27] in 1961. If a composite local operator carries all quantum numbers of a given particle in regard to space-time (J^{PC}) and other properties (charge, isospin, etc.), it gives the same S-matrix amplitudes on the particle mass shell up to overall normalizations. The difference shows up only off the mass shell. But the "off-shell amplitudes" are not really scattering amplitudes of the particle, but include continuum contributions. According to this theorem, therefore, the definition of particle fields is infinitely ambiguous with respect to their continua. When a different particle field is used, its interaction Lagrangian takes a different form. To avoid this ambiguity and the issue of renormalizability, we were tempted to replace the field theory with the S-matrix theory in the 1960s so as to deal only with the on-shell amplitudes and the observables. As we know, it led us to the dual resonance model and then back to the Lagrangian theory of strings with the Nambu-Goto action.

Meanwhile, our attention has been drawn to one interesting observation in supersymmetric theory. Along the line of the Olive-Montonen conjecture, Seiberg and Witten [28] showed in the $N = 2$ supersymmetric theory that the strong- and weak-coupling limits are dual to each other. To be more specific, the roles of a particle and a soliton of the same spin parity are interchanged between the strong and weak limits of coupling. Since solitons are composite in everyone's picture, in such theories elementarity vs compositeness loses its absolute meaning. It depends on the strength of coupling. A similar duality was shown earlier for an $N = 4$ model as well [29]. The proof of this duality relies on the simple holomorphicity special to supersymmetry. If something similar holds in nonsupersymmetric theory as well, the meaning of elementarity and compositeness of particles would finally disappear and the naming would become just a matter of convenience; if a Lagrangian takes the simplest form with a certain choice for a set of particle fields, one would call such particles *elementary* for convenience.

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APPENDIX A: NONEXISTENCE OF THE NOETHER CURRENT

The Noether current does not exist in the theories that satisfy local gauge invariance with matter fields alone. The proof is almost trivial. We give it here only for the U(1) bosonic model since the extension to fermions and non-Abelian theories is straightforward.

Under the U(1) gauge transformation, the Lagrangian satisfies the local invariance

$$L(e^{-i\alpha(x)}\phi^*, e^{i\alpha(x)}\phi) = L(\phi^*, \phi), \tag{A1}$$

where $\alpha(x)$ is an arbitrary function of space-time that satisfies mild conditions such as differentiability. The copy index i $(= 1, \cdots N)$ has been suppressed in Eq. (A1). For the infinitesimal $\alpha(x)$, gauge invariance requires

$$\begin{aligned} &-i\left(\phi^*\frac{\partial L}{\partial\phi^*} + \partial_\mu\phi^*\frac{\partial L}{\partial(\partial_\mu\phi^*)}\right)\alpha \\ &+ i\left(\frac{\partial L}{\partial\phi}\phi + \frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\mu\phi\right)\alpha \\ &+ i\left(-\phi^*\frac{\partial L}{\partial(\partial_\mu\phi^*)} + \frac{\partial L}{\partial(\partial_\mu\phi)}\phi\right)\partial_\mu\alpha = 0. \end{aligned} \tag{A2}$$

Since $\alpha(x)$ and $\partial_\mu\alpha(x)$ are two independent functions when $\alpha(x)$ is an arbitrary function of x_μ, the condition of Eq. (A2) requires that the terms proportional to $\alpha(x)$ and to $\partial_\mu\alpha(x)$ must be separately equal to zero. After using the equations of motion, the coefficient of $\alpha(x)$ equal to zero gives

$$-\partial_\mu\left(\phi^*\frac{\partial L}{\partial\phi^*_\mu}\right)+\partial_\mu\left(\phi\frac{\partial L}{\partial\phi_\mu}\right)=0. \tag{A3}$$

Normally this would be the statement of conservation of the Noether current, $\partial^\mu J^N_\mu=0$. However, the third term proportional to $\partial_\mu\alpha(x)$ in Eq. (A2) gives

$$-\frac{\partial L}{\partial(\partial_\mu\phi^*)}\phi^*+\frac{\partial L}{\partial(\partial_\mu\phi)}\phi=0. \tag{A4}$$

This is nothing other than the statement of

$$J^N_\mu\equiv 0 \tag{A5}$$

at all space-time locations. In the case that the elementary gauge field A_μ exists in the Lagrangian, the gauge transformation $A_\mu\to A_\mu+i\partial_\mu\alpha$ generates an additional term proportional to $\partial_\mu\alpha(x)$ and adds to the third term in Eq. (A2) to exactly cancel the variation due to ϕ/ϕ^*. This cancellation is nothing other than gauge invariance itself. Consequently, Eq. (A5) does not follow in the conventional gauge theory. An extension of this proof to the fermion models and the non-Abelian models is just as simple and easy.

Despite this general proof of $J^N_\mu\equiv 0$, some may wonder if it is possible to define a conserved current in the gauge symmetry limit by factoring out $(1-\lambda)$ from the current J_μ defined by Eq. (19) off the gauge limit ($\lambda\neq 1$) and then going to the limit of $\lambda=1$. If physics is somehow "continuous" in this respect in the neighborhood of $\lambda=1$, this might allow us to circumvent the difficulty. That is, we choose as a conserved current simply the current

$$J'_\mu=i\sum_i(\phi^*_i\overleftrightarrow{\partial}_\mu\phi_i), \tag{A6}$$

so that the charge is $Q\equiv\int J'_0 d^3x$. This charge is not gauge invariant, but let us leave it aside for a moment. If one computes by brute force the divergence of this current J'_μ with the equation of motion, one would not be led to $\partial^\mu J'_\mu=0$. Instead, one would end up with the trivial circular identity as follows: since $\partial^\mu J'_\mu=i\sum_i(\phi^*_i\Box\phi_i-\Box\phi^*_i\phi_i)$, one multiplies the equation of motion for ϕ_i with the field ϕ^*_i and subtracts the corresponding bilinear object with $\phi_i\leftrightarrow\phi^*_i$. Then the result is a trivial identity: $i\sum_i(\phi^*_i\Box\phi_i-\phi^*_i\Box\phi_i)=i\sum_i(\phi^*_i\Box\phi_i-\Box\phi^*_i\phi_i)$. Therefore the conclusion from this exercise is as follows: only when one violates gauge invariance by staying away from the symmetry limit ($\lambda\neq 1$) can the Noether theorem define a conserved current in the familiar form with strength reduced by $(1-\lambda)$.

The same happens for our fermion model. Just as in the bosonic model, the current $\sum_i\overline{\psi}_i\gamma_\mu\psi_i$ is not the conserved Noether current in the gauge symmetry limit.[12] The equation of motion of $L_{\rm tot}$ does not allow us to compute $\partial^\mu(\overline{\psi}\gamma_\mu\psi)$ in the gauge symmetry limit: such a computation drives us around a circular loop just as in the case of bosons.

In the perturbative diagram calculation which is performed in the interaction picture, however, the fields obey the equation of *free* motion. Therefore $\phi^*\overleftrightarrow{\partial}_\mu\phi$ and $\overline{\psi}\gamma_\mu\psi$ are both divergence free, that is, they are conserved currents.

APPENDIX B: DIFFICULTY IN GENERAL NON-ABELIAN MODELS

The local Lagrangian of matter fields alone has been easily obtained by the auxiliary gauge field method for the SU(2) model with the doublet matter. But we cannot extend it to other groups and representations. We show it here using two explicit examples.

Let us start with the Lagrangian of the nonpropagating auxiliary gauge fields,

$$\begin{aligned}L&=\Phi^\dagger(\overleftarrow{\partial}^\mu-i\mathbf{A}^\mu)(\partial_\mu+i\mathbf{A}_\mu)\Phi\\&\quad-m^2\Phi^\dagger\Phi+\frac{1}{2}\mu^2A_{a,\mu}A^\mu_a,\ (\mu^2\to 0),\end{aligned} \tag{B1}$$

where Φ and $\Phi^\dagger$ are the column and row fields belonging to the n-dimensional representation of group G. We have absorbed the coupling e into $\mathbf{A}_\mu$. Let the group G be induced by the generators T_a $(a=1,\cdots k)$, which are $n\times n$ matrices. We represent the nonpropagating gauge fields $A^\mu_a(a=1\cdots k)$ in the $n\times n$ matrices,

$$\mathbf{A}^\mu=\sum_{a=1}^k T_aA^\mu_a. \tag{B2}$$

The Lagrangian (B1) is invariant under the local gauge transformation

$$\begin{aligned}&\Phi\to U\Phi,\\&A^\mu\to UA^\mu U^\dagger-i(\partial^\mu U)U^\dagger,\end{aligned} \tag{B3}$$

where $U=\exp(iT_a\alpha_a)$. In order to integrate the exponentiated action of L over A^μ_a, we combine the terms bilinear and linear in A^μ_a into a quadrature and "shift the origin." In the case of the SU(2)-doublet matter fields, we see with $\{\tau_a,\tau_b\}=2\delta_{ab}$ that the coefficients of the bilinear terms of A^μ_a are simply $\delta_{ab}\Phi^\dagger\Phi$ so that no diagonalization is needed for the symmetrized product of the generators $\{T_a,T_b\}=\frac{1}{4}\{\tau_a,\tau_b\}=\frac{1}{2}\delta_{ab}$. Upon integration over A^μ_a, the

[12]Unlike the corresponding object in the bosonic case, this current is gauge invariant.

denominator of $L_{\rm int}(\Phi^\dagger,\Phi)$ comes out to be the singlet $\Phi^\dagger\Phi$, as given in Eq. (62). Upon integration, an additional term

$$-2{\rm tr}\ln(\Phi^\dagger\Phi) \tag{B4}$$

appears in the effective action. But we may remove this term since it is gauge invariant by itself. We retain the remainder as the gauge-invariant Lagrangian in terms of $\Phi^\dagger/\Phi$.

However, this procedure does not work in the cases other than the SU(2) doublet. When $\{T_a,T_b\}\not\propto\delta_{ab}I$, it happens that the integral over $\mathbf{A}_\mu$ is generally impossible. Even if it were possible, the *trace-log* term would not be invariant by itself under rotations of the group G, not even under global rotations. While the whole action is gauge invariant, it is not separately so for the effective Lagrangian and the trace-log term. Unfortunately, this is what happens in the cases other than the SU(2) doublet. We show two simple examples below.

Let us first examine the case of the real triplets of SU(2). In this case the coefficient of the bilinear terms of A_a^μ ($a=1,\ 2,\ 3$) is written in terms of the 3×3 matrices $(T_a)_{bc}=-i\varepsilon_{abc}$ and the matter fields $\Phi=(\phi_1,\phi_2,\phi_3)^t$ and $\Phi^\dagger=\Phi^t$. The bilinear terms of A_a^μ are given by

$$(\Phi^tT_aT_b\Phi)A_a^\mu A_{b,\mu}. \tag{B5}$$

It can be diagonalized by the orthogonal transformation $A'_\mu=\mathbf{O}A_\mu$ into

$$(A_1'^\mu,A_2'^\mu,A_3'^\mu)\begin{pmatrix}\Phi^t\Phi & 0 & 0\\ 0 & \Phi^t\Phi & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}A'_{1\mu}\\ A'_{2\mu}\\ A'_{3\mu}\end{pmatrix}. \tag{B6}$$

When this is placed in the action and exponentiated, we cannot integrate it over the third component of $\mathbf{A}'_\mu$ since the action is flat along that direction (at $\mu\to0$). The action blows up as $\mu\to0$ and there is no way to keep it well defined.

How about the SU(3)-triplet matter fields as the next-to-simplest example? For the triplet matter fields, the bilinear terms in $A_a^\mu(a=1,\cdots8)$ can be written as

$$A_\mu^aM_{ab}A^{b,\mu}, \tag{B7}$$

where $M_{ab}=\frac{1}{8}\Phi^\dagger\{\lambda_a,\lambda_b\}_+\Phi$ is a symmetric matrix under $a\leftrightarrow b$. The matrix M_{ab} can be diagonalized into $\mathbf{D}$ by some orthogonal rotation $\mathbf{O}$ as

$$(A'_\mu)^t\mathbf{O}^tM\mathbf{O}A'^\mu=A'_{a,\mu}D_{aa}A_a'^\mu. \tag{B8}$$

Can the diagonal matrix $\mathbf{D}$ be proportional to the unit matrix? If so, the functional integral over A_μ^a would produce a denominator common to all a in $L_{\rm int}$ just as in the case of SU(2). But that is obviously not the case: if $\mathbf{D}\propto\mathbf{I}$, then $M_{ab}=(\mathbf{ODO}^t)_{ab}$ would also have to be proportional to δ_{ab} even before the rotation. We can easily see by simple inspection (using the representation $T_a=\frac{1}{2}\lambda_a$ familiar to physicists) that M_{ab} is not proportional to an 8×8 unit matrix. Consequently, the resulting Lagrangian in terms of matter fields alone would not take a form as compact as in the SU(2)-doublet case, if one could write it at all.[13]

These two examples show that the auxiliary field method can lead to a simple *local* field theory only for the U(1) and the SU(2)-doublet models of bosonic matter fields.

[13] This does not conflict with what Rabinovici and Smolkin [15] did for general Lie groups: they integrated over the matter fields at one loop for a general group and representation to show that the $-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}$ is indeed generated. Their purpose was to see whether or not this *Maxwell term* can be generated by loops upon integrating over matter fields in the auxiliary vector-field Lagrangian. They did not address finding a *local* non-Abelian gauge-invariant Lagrangian written in matter fields alone.

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