BPS pion domain walls in the supersymmetric chiral Lagrangian

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We construct exact solutions of BPS pion domain walls in the four-dimensional $\mathcal{N}=1$ supersymmetric $SU(N)$ chiral Lagrangian with pion masses introduced via linear and quadratic superpotentials. The model admits N discrete vacua in the center of $SU(N)$ for the linear superpotential. In addition to the latter, new vacua appear for the quadratic superpotential. We find that the domain wall solutions of pions (Nambu-Goldstone bosons) that interpolate between a pair of (pion) vacua preserve half of supersymmetry. Contrary to our expectations, we have not been able to find domain walls involving the quasi-Nambu-Goldstone bosons present in the theory, which in turn has the consequence that not all vacua of the theory are connected by a BPS domain wall solution.

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I. INTRODUCTION

Domain walls that separate two vacua are topological defects appearing in various subjects of physics from condensed matter physics to field theory, high energy physics [1], QCD [2], and cosmology [3]. In supersymmetric theories, Bogomol'nyi-Prasad-Sommerfield (BPS) domain walls are the most stable configurations, studied extensively in the literature, such as supergravity [4] and $\mathcal{N}=1$ supersymmetric QCD [5]. They preserve half of supersymmetry (therefore called $1/2$ BPS states) and their tension is given by the central charge in $1+1$ dimensions [6]. In $3+1$ dimensions the tension of the $1/2$ BPS domain walls coincides instead with a tensorial charge present only in theories with broken translational invariance [5]. Domain walls were also studied in theories with extended supersymmetry such as $\mathcal{N}=2$ supersymmetric hyper-Kähler sigma models [7] and $\mathcal{N}=2$ supersymmetric Abelian [8] and non-Abelian [9] gauge theories. If multiple domain walls with different angles join at a junction, the total configuration is a $1/4$ BPS state preserving a quarter of supersymmetry both in $\mathcal{N}=1$ [10,11] and $\mathcal{N}=2$ [12] supersymmetric gauge theories. See Refs. [13–15] for reviews.

In this paper, we study BPS pion domain walls in the $\mathcal{N}=1$ supersymmetric chiral Lagrangian with pion mass terms. The model appears as the low-energy effective theory of supersymmetric QCD in supersymmetric vacua with broken chiral symmetry. The $SU(N)$ chiral Lagrangian with the simplest pion mass term admits N symmetric discrete vacua in the center elements of $SU(N)$. We construct exact solutions of BPS $SU(2K)$ pion domain walls interpolating between the pion vacua present in the theory and find that these domain walls carry $SU(2K)/[SU(K)\times SU(K)\times U(1)]$ orientational moduli as well as translational moduli. These domain walls are special solutions interpolating only 2 of $2K$ vacua. We have not been able to find any domain wall solutions connecting any of the other $2K-2$ vacua. We construct the low-energy effective field theory on the domain wall for the $N=2$ case and obtain the $\mathbb{C}P^1$ model. This case is similar to the moduli space found for vortices in $U(2)$ gauge theories [16]. The $SU(2)$ case ($N=2$) reduces to a domain wall in the $O(4)$ model admitting two discrete vacua [17,18], in which the $\mathbb{C}P^1$ moduli of the domain wall were already found. In contrast to pion domain walls in nonsupersymmetric theories [2,19] that are topologically and dynamically unstable, pion domain walls found in this paper saturate the BPS bound and are therefore stable classically and quantum mechanically (even non-perturbatively).

In supersymmetric theories, a global symmetry G is extended to its complex extension $G^{\mathbb{C}}$ since the potential is constructed from a superpotential which is holomorphic in the chiral superfields. Consequently, spontaneously broken global symmetry in supersymmetric vacua results in additional massless bosons, called quasi-Nambu-Goldstone (NG) bosons [20,21] in addition to the usual NG bosons. These massless bosons together with their fermionic superpartners, called quasi-NG fermions [22], constitute chiral multiplets. The NG and quasi-NG bosons must parametrize a Kähler manifold as required from supersymmetric nonlinear sigma models [23]. The general framework to construct low-energy effective theories was provided in Refs. [24]. In the case of chiral symmetry breaking $SU(N)_{\rm L}\times SU(N)_{\rm R}\to SU(N)_{\rm L+R}$, there must appear the same number of quasi-NG bosons as the number of NG bosons (pions) and the target space is $SU(N)^{\mathbb{C}}\simeq SL(N,\mathbb{C})$

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[25]. The most general Kähler potential is an arbitrary function of G-invariants, corresponding to the deformation of directions of quasi-NG bosons, which cannot be fixed by G [25–28]. Manifestly supersymmetric higher-derivative corrections have recently been constructed, including the example of chiral symmetry breaking [29]. A supersymmetric Skyrme term has been constructed recently [30] but the usual kinetic term canceled out as in the case of baby (lower dimensional) Skyrmions [31]. In this paper, we study—for chiral symmetry breaking—supersymmetric pion mass terms preserving the vector symmetry $H = SU(N)_{\mathrm{L+R}}$. In the case of the simplest superpotential, the potential admits N symmetric discrete vacua for the $SU(N)$ case. We construct BPS pion domain walls interpolating between the pion vacua of the theory. These vacua for which we are able to find domain wall solutions are antipodal points on the target space. However, as we mentioned, not all the supersymmetric vacua are connected by domain walls; vacua with an imaginary part require quasi-NG bosons to be turned on. For this type of domain wall—although we have found the BPS equations—we have not been able to find a domain wall solution, neither analytically nor numerically. Using an appropriate Ansatz, we have reduced the BPS matrix equation to a complex scalar equation which describes one NG mode and one quasi-NG mode, for which we can show that no solutions exist. Although we do not yet have a solid proof of absence of the remaining domain wall, our results provide some evidence.

As a similar model, the (nonsupersymmetric) $U(N)$ chiral Lagrangian with the pion mass term admits a non-Abelian sine-Gordon soliton that carries $\mathbb{C}P^{N-1}$ moduli [32]. The low-energy effective theory on said domain wall is given by the $\mathbb{C}P^{N-1}$ model [33]. Such a $U(N)$ chiral Lagrangian appears e.g. in the Josephson junction of two non-Abelian superconductors, in which a non-Abelian sine-Gordon soliton describes a non-Abelian vortex (color-magnetic flux tube) from the bulk point of view [34], that is a non-Abelian extension of Josephson vortices in field theory [35]. For the non-Abelian sine-Gordon soliton in the $U(N)$ chiral Lagrangian, one has to consider the group $U(N)$ instead of $SU(N)$. We do not need a $U(1)$ part and consider instead the simple group $SU(N)$. Consequently, our configurations separate into two *different* vacua so they are domain walls, but two spatial infinities of a sine-Gordon soliton are in the *same* vacuum. We also show that there is no BPS domain wall interpolating the same vacuum in our model. Only two physically distinct vacua can be connected by a BPS pion domain wall. As a consequence, we find no domain wall solutions for the $SU(2K+1)$ case.

This paper is organized as follows. In Sec. II, we give a brief review of the supersymmetric nonlinear sigma model and chiral symmetry breaking in supersymmetric theories, and discuss supersymmetric pion mass terms. In Sec. III, we construct non-Abelian BPS domain walls. In Sec. IV, we construct the effective theory on the domain wall which is the $\mathbb{C}P^1$ model. Section V is devoted to a summary as well as discussions. We use the notation of the textbook of Wess and Bagger [36].

II. SUPERSYMMETRIC CHIRAL LAGRANGIAN

Subsections II A and II B are devoted to a review of supersymmetric nonlinear sigma models and chiral symmetry breaking in supersymmetric theories, respectively, while the supersymmetric mass term in Subsection II C has not been discussed in the literature.

A. Supersymmetric nonlinear sigma models

In four-dimensional $\mathcal{N} = 1$ supersymmetric theories, we have N chiral superfields Φ^i, $(i = 1, \ldots, N)$ whose component expansion in the chiral basis, $y^m = x^m + i\theta\sigma^m\bar{\theta}$, is given by

$$\Phi^i(y,\theta) = \varphi^i(y) + \theta\psi^i(y) + \theta^2 F^i(y), \tag{2.1}$$

where φ^i is a complex scalar field, ψ^i is a Weyl fermion and F^i is a complex auxiliary field. The supersymmetric Lagrangian is described by a Kähler potential $K(\Phi,\Phi^\dagger)$ as well as a superpotential $W(\Phi)$, where the first is a function of the superfields, Φ^i, and the latter is a holomorphic function

$$\begin{aligned}\mathcal{L} &= \int d^4\theta K(\Phi,\Phi^\dagger) + \left(\int d^2\theta W(\Phi) + \text{c.c.}\right)\\ &= -g_{i\bar{j}}(\varphi,\bar{\varphi})\partial_m\varphi^i\partial^m\bar{\varphi}^{\bar{j}} + g_{i\bar{j}}(\varphi,\bar{\varphi})F^iF^{*\bar{j}}\\ &\quad + F^i\frac{\partial W}{\partial\varphi^i} + F^{*\bar{j}}\frac{\partial W^*}{\partial\bar{\varphi}^{\bar{j}}} + (\text{fermion terms}),\end{aligned} \tag{2.2}$$

where $g_{i\bar{j}} \equiv \frac{\partial}{\partial\varphi^i}\frac{\partial}{\partial\bar{\varphi}^{\bar{j}}}K(\varphi,\bar{\varphi})$ is the Kähler metric. The potential V can be written in terms of the superpotential as

$$V = g_{i\bar{j}}F^iF^{*\bar{j}} = g^{i\bar{j}}\frac{\partial W}{\partial\varphi^i}\frac{\partial W^*}{\partial\bar{\varphi}^{\bar{j}}}, \tag{2.3}$$

while the auxiliary field is solved by

$$F^i = -g^{i\bar{j}}\frac{\partial W^*}{\partial\bar{\varphi}^{\bar{j}}}. \tag{2.4}$$

Here $g^{i\bar{j}}$ is the inverse of the Kähler metric $g_{i\bar{j}}$. The G-invariance of the Kähler potential implies that the following transformation

$$K(\Phi,\Phi^\dagger)\xrightarrow{g}K(\Phi',\Phi'^\dagger) = K(\Phi,\Phi^\dagger) + F(\Phi,g) + F^*(\Phi^\dagger,g), \tag{2.5}$$

is preserved; i.e. the transformation with F (F^*) being a (n anti-)holomorphic function of Φ ($\Phi^\dagger$) which are determined by a group element $g \in G$. This transformation is called a Kähler transformation and the latter two terms in the above equation vanish under the superspace integral $\int d^4\theta$.

B. Supersymmetric chiral Lagrangian

Let us now consider chiral symmetry breaking of the form

$$G = SU(N)_{\rm L} \times SU(N)_{\rm R} \to H = SU(N)_{\rm L+R}. \qquad (2.6)$$

The NG modes corresponding to the above symmetry breaking span the following coset space

$$G/H = \frac{SU(N)_{\rm L} \times SU(N)_{\rm R}}{SU(N)_{\rm L+R}} \simeq SU(N). \qquad (2.7)$$

We denote the generators of the coset by $T_A \in \mathfrak{su}(N)$, which take value in the $SU(N)$ algebra. It was shown in Ref. [21] that when the vacuum expectation value (VEV) giving rise to the symmetry breaking belongs to a real representation of $SU(N)$, then the number of quasi-NG boson is exactly the same as the number of NG bosons; this is also called a maximal realization.

Chiral symmetry breaking belongs to said class and the total target space is given by

$$G^{\mathbb{C}}/H^{\mathbb{C}} \simeq SU(N)^{\mathbb{C}} \simeq SL(N,\mathbb{C}) \simeq T^*SU(N). \qquad (2.8)$$

The NG supermultiplet is expressed as the following coset representative

$$M = \exp(i\Phi^i T_A \delta_i^A) \in G^{\mathbb{C}}/H^{\mathbb{C}}, \qquad (2.9)$$

where the NG superfields take the form

$$\Phi^i(y,\theta) = \pi^i(y) + i\sigma^i(y) + \theta\psi^i(y) + \theta\theta F^i(y), \qquad (2.10)$$

with π^i being NG bosons, σ^i quasi-NG bosons—both of which are real fields—and finally ψ^i quasi-NG fermions. The NG supermultiplets obey the following nonlinear transformation law

$$M \to M' = g_{\rm L} M g_{\rm R}^\dagger, \qquad (g_{\rm L}, g_{\rm R}) \in SU(N)_{\rm L} \times SU(N)_{\rm R}. \qquad (2.11)$$

In the vacuum $M = \mathbf{1}_N$, the unbroken symmetry $H = SU(N)_{\rm L+R}$ defined by $g_L = g_R$ remains.[1] From the following transformation

$$MM^\dagger \to g_{\rm L} MM^\dagger g_{\rm L}^\dagger, \qquad (2.12)$$

the simplest Kähler potential, that is invariant under the $SU(N)_{\rm L} \times SU(N)_{\rm R}$ symmetry, is just

$$K_0 = f_\pi^2 {\rm tr}(MM^\dagger), \qquad (2.13)$$

where f_π is a constant. The bosonic part of the Lagrangian—corresponding to the above Kähler potential—to leading order in the derivative expansion is

$$\mathcal{L}_0 = -f_\pi^2 {\rm tr}(\partial_m M \partial^m M^\dagger), \qquad (2.14)$$

where M is the lowest component of the NG superfield given in Eq. (2.9).

From the left-invariant Maurer-Cartan one-form $iM^{-1}\frac{\partial M}{\partial\varphi^i}$ we define the holomorphic vielbein $E_i^A(\varphi)$ and their conjugates as

$$iM^{-1}\frac{\partial M}{\partial\varphi^i} = E_i^A(\varphi)T_A, \qquad -i\frac{\partial M^\dagger}{\partial\overline{\varphi}^{\bar{i}}}M^{-1\dagger} = E_{\bar{i}}^{*\overline{A}}(\overline{\varphi})T_{\overline{A}}. \qquad (2.15)$$

Their pull-backs to space-time give

$$iM^{-1}\partial_m M = E_i^A(\varphi)T_A\partial_m\varphi^i,$$
$$-i(\partial_m M^\dagger)M^{-1\dagger} = E_{\bar{i}}^{*\overline{A}}(\overline{\varphi})T_{\overline{A}}\partial_m\overline{\varphi}^{\bar{i}}. \qquad (2.16)$$

By using the vielbein, the Lagrangian for the bosonic fields can be rewritten as

$$\begin{aligned}\mathcal{L}_0 &= -f_\pi^2{\rm tr}(MT_AT_{\overline{B}}M^\dagger)E_i^A(\varphi)E_{\bar{j}}^{*\overline{B}}(\overline{\varphi})\partial_m\varphi^i\partial^m\overline{\varphi}^{\bar{j}} \\ &= -G_{A\overline{B}}E_i^A(\varphi)E_j^{*\overline{B}}(\overline{\varphi})\partial_m\varphi^i\partial^m\overline{\varphi}^{\bar{j}} \\ &= -g_{i\bar{j}}(\varphi,\overline{\varphi})\partial_m\varphi^i\partial^m\overline{\varphi}^{\bar{j}}, \end{aligned} \qquad (2.17)$$

with the Kähler metric $g_{i\bar{j}}$ and the metric $G_{A\overline{B}}$ on the tangent space, defined by

$$\begin{aligned} g_{i\bar{j}}(\varphi,\overline{\varphi}) &= f_\pi^2{\rm tr}(MT_AT_{\overline{B}}M^\dagger)E_i^A(\varphi)E_{\bar{j}}^{*\overline{B}}(\overline{\varphi}) \\ &= G_{A\overline{B}}E_i^A(\varphi)E_{\bar{j}}^{*\overline{B}}(\overline{\varphi}), \end{aligned} \qquad (2.18)$$

$$G_{A\overline{B}} = f_\pi^2{\rm tr}(MT_AT_{\overline{B}}M^\dagger), \qquad (2.19)$$

respectively.

The Kähler potential in Eq. (2.13) is the simplest one, while the most general Kähler potential can be written as [25,28]

$$K = f({\rm tr}(MM^\dagger), {\rm tr}[(MM^\dagger)^2], \ldots, {\rm tr}[(MM^\dagger)^{N-1}]), \qquad (2.20)$$

with an *arbitrary* function of $N-1$ arguments.

[1] For chiral symmetry breaking in supersymmetric vacua, the unbroken group $H = SU(N)_{\rm L+R}$ is not unique, and is further broken to a subgroup when some quasi-NG bosons get VEVs [25], where some of the quasi-NG bosons change to NG bosons [25,28].

If we set all quasi-NG bosons to zero [25,26]

$$U = M|_{\sigma^i=0} \in SU(N), \tag{2.21}$$

we get the $SU(N)$ chiral Lagrangian

$$\mathcal{L} = -f_\pi^2 \mathrm{tr}(\partial_m U \partial^m U^\dagger) = f_\pi^2 \mathrm{tr}(U^\dagger \partial_m U)^2, \tag{2.22}$$

where the decay constant f_π is determined from the function f. Here, we have used that

$$G_{A\bar{B}}|_{\sigma=0} = f_\pi^2 \delta_{A\bar{B}}, \qquad E_i^A|_{\sigma^j=0} = e_i^A(\pi). \tag{2.23}$$

with the normalization of generators $\mathrm{tr}[T_A T_{\bar{B}}] = \delta_{A\bar{B}}$ and the vielbein $e_i^A(\pi)$ for $SU(N)$.

C. Supersymmetric mass term

The pion mass term in the chiral Lagrangian breaks the $G = SU(N)_\mathrm{L} \times SU(N)_\mathrm{R}$ symmetry explicitly. It is often considered that explicit breaking terms do not break the vector symmetry $SU(N)_{\mathrm{L+R}}$. Here we consider such mass terms preserving the vector symmetry $SU(N)_{\mathrm{L+R}}$. The superpotential invariant under $SU(N)_{\mathrm{L+R}}$ is

$$W = g(\mathrm{tr}(M), \mathrm{tr}(M^2), \ldots, \mathrm{tr}(M^{N-1})), \tag{2.24}$$

with an arbitrary function g of $N-1$ arguments.

In this paper, we consider only functions of the trace M, for simplicity:

$$W = w(\mathrm{tr}M), \tag{2.25}$$

with an arbitrary function w. The auxiliary fields are solved as

$$F^i = -g^{i\bar{j}} \frac{\partial W^*}{\partial \bar{\varphi}^{\bar{j}}} = -i\overline{w}'(\mathrm{tr}M^\dagger)\mathrm{tr}(M^\dagger T_{\bar{A}}) g^{i\bar{j}} E_{\bar{j}}^{*\bar{A}}(\bar{\varphi}), \tag{2.26}$$

where the prime denotes differentiation with respect to the argument, and so the potential term can be written as

$$V = g_{i\bar{j}} F^i F^{*\bar{j}} = |w'(\mathrm{tr}M)|^2 \mathrm{tr}(M^\dagger T_{\bar{B}}) \mathrm{tr}(M T_A) G^{A\bar{B}}. \tag{2.27}$$

Here $G^{A\bar{B}}$ is the inverse of the metric $G_{A\bar{B}}$ on the tangent space. The supersymmetric vacua are given by

$$F = 0 \Leftrightarrow w'(\mathrm{tr}M)\mathrm{tr}(MT_A) = 0. \tag{2.28}$$

In the next two subsections we will consider the two simplest possibilities for a chiral symmetry breaking mass term, conserving the vector symmetry $SU(N)_{\mathrm{L+R}}$.

1. Linear superpotential

The simplest superpotential

$$W = w(\mathrm{tr}M) = \frac{m}{N}\mathrm{tr}M, \tag{2.29}$$

with mass $m \in \mathbb{R}$, admits N symmetric supersymmetric vacua, given by[2]

$$M = \omega^k \mathbf{1}_N, \qquad k = 0, 1, 2, \ldots, N-1,$$
$$\omega \equiv \exp(2\pi i/N), \tag{2.30}$$

namely the center elements of $SL(N, \mathbb{C})$.

Let us point out a crucial fact about the restriction to the NG subspace: $M = M|_{\sigma^i=0} = U \in SU(N)$. In this subspace, we can write

$$\mathrm{tr}[U] = \mathrm{tr}[\exp\{i\theta^A T^A\}] \in \mathbb{R}, \tag{2.31}$$

if and only if $\theta^A \in \mathbb{R}$ are real parameters. Since for the NG restriction θ^A are indeed real parameters, the above expression holds.[3] Therefore, in this subspace only the vacua

$$U = \mathbf{1}_N, \qquad U = -\mathbf{1}_N, \tag{2.32}$$

can be reached for even N and only the vacuum

$$U = \mathbf{1}_N, \tag{2.33}$$

is possible for odd N. In order to reach the general $\omega^k \neq \pm \mathbf{1}_N$ vacua, we need to turn on the quasi-NG directions.

We note that for the $N = 2$ case, the NG boson part of the Lagrangian with the superpotential (2.29) reduces to the well-known $O(4)$ model:

$$\mathcal{L} = -f_\pi^2 \partial_m \mathbf{m} \cdot \partial^m \mathbf{m}, \tag{2.34}$$

with $\mathbf{m} = (m_1, \ldots, m_4)$ with the constraint $\mathbf{m}^2 = 1$ and the potential [17,18],

$$V = \frac{m^2}{2f_\pi^2}(m_1^2 + m_2^2 + m_3^2) = \frac{m^2}{2f_\pi^2}(1 - m_4^2), \tag{2.35}$$

admitting two vacua $m_4 = \pm 1$.

[2] A phase for the mass will just rotate all the supersymmetric vacua, so we can set the phase to zero without loss of generality.

[3] To realize that the expression holds, it is enough to realize that an i can only come from the product of an odd number of generators which is traceless and therefore does not contribute to the trace. All even powers of the generators have no i and thus the trace is a real quantity.

2. *Quadratic superpotential*

We will also consider the next-simplest potential, i.e. a quadratic potential of the form

$$W = w(\mathrm{tr}M) = \frac{m}{2N^2}(\mathrm{tr}M)^2, \tag{2.36}$$

such that the vacuum equation now reads

$$\mathrm{tr}(M)\mathrm{tr}(MT_A) = 0, \tag{2.37}$$

which has both the old type of vacua

$$M = \omega^k \mathbf{1}_N, \qquad \omega = \exp\frac{2\pi i}{N}, \tag{2.38}$$

as well as new vacua

$$\mathrm{tr}M = 0. \tag{2.39}$$

These new vacua are sections of $SL(N,\mathbb{C})$ and probably connected spaces, but not connected to the old type of vacua.

The $SU(2)$ case of $N = 2$, i.e. the NG subspace of the model, now reduces to the $O(4)$ model with the following potential

$$V = \frac{m^2}{2f_\pi^2} m_4^2(1 - m_4^2), \tag{2.40}$$

admitting three vacua: $m_4 = \pm 1$ and $m_4 = 0$. Notice that the vacua $m_4 = \pm 1$ are pointlike on the space of vacua, whereas the vacuum $m_4 = 0$ is the manifold S^2: $m_1^2 + m_2^2 + m_3^2 = 1$. The latter can be interpreted as vacuum moduli.

The $m_4 = 0$ vacuum breaks the global $SU(2)$ symmetry to $U(1)$.

III. BPS PION DOMAIN WALLS

A. BPS equation and Bogomol'nyi bound for domain walls

BPS equations are obtained by the condition that the supersymmetry transformation of fermions vanish. The transformation law of the fermions in the chiral multiplet is given by

$$\delta\psi^i = i\sqrt{2}\sigma^m\bar{\xi}\partial_m\varphi^i + \sqrt{2}\xi F^i, \tag{3.1}$$

where ξ and $\bar{\xi}$ are transformation parameters. Assuming that the fields φ^i depend only on the x^1-direction and imposing the half-BPS condition $i\sigma^1\bar{\xi} = \xi$, we obtain the following BPS equation for domain walls:

$$\partial_1\varphi^i + F^i = 0. \tag{3.2}$$

From Eq. (2.26), the above equation reads

$$\partial_1\varphi^i = i\bar{w}'(\mathrm{tr}M^\dagger)\mathrm{tr}(M^\dagger T_{\bar{A}})g^{i\bar{j}}E_{\bar{j}}^{*\bar{A}}. \tag{3.3}$$

By multiplying by $E_i^B T_B$ on the both sides, we obtain the invariant form of the BPS equation

$$iM^{-1}\partial_1 M = i\bar{w}'(\mathrm{tr}M^\dagger)\mathrm{tr}(M^\dagger T_{\bar{B}})T_A G^{A\bar{B}}. \tag{3.4}$$

If we restrict to the NG-boson subspace, $M = M|_{\sigma^i=0} = U$, we get

$$iU^\dagger\partial_1 U = \frac{i}{f_\pi^2}\bar{w}'(\mathrm{tr}U^\dagger)\mathrm{tr}(U^\dagger T_A)T_A. \tag{3.5}$$

The BPS equation (3.4) can also be obtained from the Bogomol'nyi bound. The Lagrangian can be written as

$$\mathcal{L} = -G_{A\bar{B}}E_i^A(\varphi)E_{\bar{j}}^{*\bar{B}}(\bar{\varphi})\partial_1\varphi^i\partial^1\bar{\varphi}^{\bar{j}} - G^{A\bar{B}}|w'(\mathrm{tr}M)|^2\mathrm{tr}(MT_A)\mathrm{tr}(M^\dagger T_{\bar{B}}), \tag{3.6}$$

yielding the energy for domain walls

$$\begin{aligned} E &= \int dx^1(G_{A\bar{B}}E_i^A(\varphi)E_{\bar{j}}^{*\bar{B}}(\bar{\varphi})\partial_1\varphi^i\partial^1\bar{\varphi}^{\bar{j}} \\ &\quad + G^{A\bar{B}}|w'(\mathrm{tr}M)|^2\mathrm{tr}(MT_A)\mathrm{tr}(M^\dagger T_{\bar{B}})) \\ &= \int dx^1 G_{A\bar{B}}[E_i^A(\varphi)\partial_1\varphi^i - iG^{A\bar{C}}\bar{w}'(\mathrm{tr}M^\dagger)\mathrm{tr}(M^\dagger T_{\bar{C}})] \\ &\quad \times [E_{\bar{j}}^{*\bar{B}}(\bar{\varphi})\partial_1\bar{\varphi}^{\bar{j}} + iG^{D\bar{B}}w'(\mathrm{tr}M)\mathrm{tr}(MT_D)] + T, \end{aligned} \tag{3.7}$$

where the domain wall topological charge is defined by

$$\begin{aligned} T &\equiv \int dx^1(-iE_i^A(\varphi)\partial_1\varphi^i w'(\mathrm{tr}M)\mathrm{tr}(MT_A) \\ &\quad + iE_{\bar{j}}^{*\bar{B}}(\bar{\varphi})\partial_1\bar{\varphi}^{\bar{j}}\bar{w}'(\mathrm{tr}M^\dagger)\mathrm{tr}(M^\dagger T_{\bar{B}})) \\ &= \int dx^1(w'(\mathrm{tr}M)\mathrm{tr}(\partial_1 M) + \bar{w}'(\mathrm{tr}M^\dagger)\mathrm{tr}(\partial_1 M^\dagger)) \\ &= |[2\Re(W)]_{x=-\infty}^{x=+\infty}|. \end{aligned} \tag{3.8}$$

If we now consider the restriction to the NG subspace (i.e. setting $\sigma^i = 0$), then we get the energy for the NG domain walls

$$\begin{aligned} E &= \int dx^1[f_\pi^2\mathrm{tr}(iU^\dagger\partial_1 U)^2 + f_\pi^{-2}|w'(\mathrm{tr}U)|^2 \\ &\quad \times \mathrm{tr}(UT_A)\mathrm{tr}(U^\dagger T_A)] \\ &= \int dx^1\mathrm{tr}[(f_\pi U^\dagger\partial_1 U - f_\pi^{-1}w'(\mathrm{tr}U)\mathrm{tr}(UT_A)T_A) \\ &\quad \times (f_\pi\partial_1 U^\dagger U - f_\pi^{-1}\bar{w}'(\mathrm{tr}U^\dagger)\mathrm{tr}(U^\dagger T_B)T_B)] + T, \end{aligned} \tag{3.9}$$

in turn reproducing the BPS equation for the NG subspace (3.5) and the domain wall topological charge T is now given by

$$T = \int dx^1 (w'(\mathrm{tr}U)\mathrm{tr}(\partial_1 U) + \bar{w}'(\mathrm{tr}U^\dagger)\mathrm{tr}(\partial_1 U^\dagger)) = |[2\Re(W)]_{x=-\infty}^{x=+\infty}|. \quad (3.10)$$

The energy E is most severely bounded from below by $|T|$. The bound is saturated when the quantity in the parentheses in Eq. (3.9) vanishes. This condition is nothing but the BPS equation (3.5).

B. Linear superpotential

In this subsection we consider the simplest superpotential, namely the linear one of Eq. (2.29). In this case, the BPS equation reads

$$iM^{-1}\partial_1 M = \frac{im}{N}\mathrm{tr}(M^\dagger T_{\bar{B}})T_A G^{A\bar{B}}. \quad (3.11)$$

Restricting to the NG-boson subspace, $M = M|_{\sigma^i=0} = U$, we get

$$iU^\dagger\partial_1 U = \frac{im}{Nf_\pi^2}\mathrm{tr}(U^\dagger T_A)T_A, \quad (3.12)$$

where we have used the expression of the inverse metric on the tangent space: $G^{A\bar{B}} = f_\pi^{-2}\delta^{A\bar{B}}$.

With this superpotential, we can calculate the tension of the domain wall using Eq. (3.8), which for the vacua (2.30) gives

$$T_k = 2m|\Re(\omega^k) - 1| = 4m\sin^2\frac{\pi k}{N}, \quad k \in \mathbb{Z}, \quad (3.13)$$

where we have assumed that the domain wall starts from the vacuum $M = \mathbf{1}_N$ and goes to the vacuum $M = \omega^k\mathbf{1}_N$. The fundamental domain wall, i.e. interpolating between two nearest vacua, thus has the tension

$$T_1 = 2m|\Re(\omega) - 1| = 4m\sin^2\frac{\pi}{N}. \quad (3.14)$$

A domain wall with the maximum tension is given by

$$\frac{k}{N} = \frac{1}{2} \quad \text{for even } N, \qquad \frac{k}{N\pm 1} = \frac{1}{2} \quad \text{for odd } N. \quad (3.15)$$

If we now restrict to the NG subspace, $M = M|_{\sigma^i=0} = U$, then only real vacua exists and thus the single domain wall exists only for even N and interpolates between $U = \mathbf{1}_N$ and $U = -\mathbf{1}_N$, giving the domain wall tension

$$T = 4m\sin^2\frac{\pi}{2} = 4m. \quad (3.16)$$

A double domain wall for even N or a single domain wall for odd N would wind 2π and thus have a vanishing tension. Since the superpotential is not double valued, these solutions do not exist. Alternatively, we can think of two domain walls in the NG subspace for even N as a domain wall and an anti-domain wall, which thus have zero overall topological charge. They may exist locally if well separated, but they are likely to decay to the vacuum, i.e. to the trivial topological sector.

1. SU(2) solution

We will begin with the simplest possible solution, which is in the NG subspace and for $N = 2$; namely the $SU(2)$ case. The linear superpotential (2.29) gives rise to two discrete vacua $U = \pm\mathbf{1}_2$. The general element of $SU(2)$ can be written as

$$U = \exp\left(i\frac{\theta}{2}\mathbf{n}\cdot\sigma\right) = \cos\frac{\theta}{2}\mathbf{1}_2 + i\mathbf{n}\cdot\sigma\sin\frac{\theta}{2}, \quad (3.17)$$

with a unit vector $\mathbf{n} = (n_1, n_2, n_3)$, ($\mathbf{n}^2 = 1$) and the Pauli matrices σ_A. We construct a domain wall interpolating between $U = \mathbf{1}_2$, ($\theta = 0$) at $x \to +\infty$ and $U = -\mathbf{1}_2$, ($\theta = 2\pi$) at $x \to -\infty$. By using an $SU(2)$ transformation, Eq. (3.17) can be diagonalized without loss of generality to $\mathbf{n} = (0, 0, 1)$, yielding:

$$U_0 = \mathrm{diag}(e^{i\theta/2}, e^{-i\theta/2}). \quad (3.18)$$

Then, the BPS equation (3.12) reduces to

$$\partial_1\theta = -\frac{m}{f_\pi^2}\sin\frac{\theta}{2}, \quad (3.19)$$

which is the BPS equation for the sine-Gordon soliton. A single soliton solution is

$$\theta(x^1) = 4\arctan\exp\left[-\frac{m}{2f_\pi^2}(x^1 - X)\right], \quad (3.20)$$

with the constant $X \in \mathbb{R}$ corresponding to the position of the soliton. We thus find that the most general single soliton solution is Eq. (3.17) with Eq. (3.20). The general solution therefore has the moduli

$$S^2 \simeq \frac{SO(3)}{SO(2)} \simeq \mathbb{C}P^1 \simeq \frac{SU(2)}{U(1)}, \quad (3.21)$$

characterized by $\mathbf{n}$. The tension for this domain wall is $T = 4m$.

2. SU(2K) solutions

In this section we consider the NG subspace for even $N = 2K$, with $K \in \mathbb{Z}$. We now choose an Ansatz for the element U for a single domain wall as

$$U_0 = \mathrm{diag}\left(\exp\left(\frac{i\theta}{2}\right),\ldots,\exp\left(\frac{i\theta}{2}\right),\right.$$
$$\left.\exp\left(-\frac{i\theta}{2}\right),\ldots,\exp\left(-\frac{i\theta}{2}\right)\right) = \exp(i\theta T_0), \qquad (3.22)$$

$$T_0 \equiv \mathrm{diag}\left(\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2},\ldots,-\frac{1}{2}\right). \qquad (3.23)$$

The boundary conditions of θ for the domain wall are: $\theta=0$, $(U=\mathbf{1}_{2K})$ at $x\to+\infty$ and $\theta=2\pi$, $(U=-\mathbf{1}_{2K})$ at $x\to-\infty$.

The BPS equation (3.12) can now readily be calculated as

$$\begin{aligned} i\partial_1\theta T_0 &= \frac{m}{2Kf_\pi^2}\mathrm{tr}[U_0^\dagger T_A]T_A \\ &= \frac{m}{2Kf_\pi^2}\sum_{k=K+1}^{2K}\frac{1}{k(k-1)} \\ &\quad\times[Ke^{-\frac{i\theta}{2}}+(k-1-K)e^{\frac{i\theta}{2}}-(k-1)e^{\frac{i\theta}{2}}] \\ &\quad\times\mathrm{diag}\Big(\underbrace{1,\ldots,1}_{k-1},1-k,\underbrace{0,\ldots,0}_{2K-k}\Big) \\ &= -\frac{im}{f_\pi^2}\sin\frac{\theta}{2}\sum_{k=K+1}^{2K}\frac{1}{k(k-1)}\mathrm{diag}\Big(\underbrace{1,\ldots,1}_{k-1},1-k,\underbrace{0,\ldots,0}_{2K-k}\Big) \\ &= -\frac{im}{Kf_\pi^2}T_0\sin\frac{\theta}{2}. \end{aligned} \qquad (3.24)$$

The solutions are thus given by Eq. (3.20) with $m\to m/K$.

Since there are only two real vacua, this is the general single domain wall in the restricted NG subspace. The tension is again $4m$.

The solution has the moduli

$$\frac{SU(2K)}{SU(K)\times SU(K)\times U(1)}, \qquad (3.25)$$

in addition to the translational modulus.

3. SU(2) double domain wall case

In this section we consider the NG subspace for the $N=2$ case, with a double domain wall, interpolating from $\mathbf{1}_2$ back to $\mathbf{1}_2$. We now choose an Ansatz for the element U for a single domain wall as

$$U_0 = \mathrm{diag}(\exp(i\theta),\exp(-i\theta)) = \exp(i\theta T_0), \qquad (3.26)$$

$$T_0 \equiv \mathrm{diag}(1,-1). \qquad (3.27)$$

The boundary conditions of θ for the domain wall are: $\theta=0$, $(U=\mathbf{1}_2)$ at $x\to+\infty$ and $\theta=2\pi$, $(U=\mathbf{1}_2)$ at $x\to-\infty$.

The BPS equation (3.12) now reads

$$\partial_1\theta = -\frac{m}{2f_\pi^2}\sin\theta. \qquad (3.28)$$

θ can interpolate from π to 0, which is the normal domain wall solution of Sec. III B 1 or from π to 2π, which is simply the anti-domain wall solution (mod 2π). Due to the fact that the right-hand side of Eq. (3.28) is negative (positive) semidefinite for θ in the range $[0,\pi]$ ($[\pi,2\pi]$), no BPS pion domain wall solution (i.e. NG boson domain wall) can interpolate between 2π and 0.

This result extends trivially to $SU(2K)$ and since $SU(2K+1)$ only has the single vacuum of the double domain wall, also no BPS solutions exist for odd $N=2K+1$.

4. SL(3,ℂ) case

We now attempt to relax the restriction to the NG subspace, which is a necessity if we are to consider the domain wall between the general vacua $M=\mathbf{1}_N$ and $M=\omega\mathbf{1}_N$. We will start by considering $SL(3,\mathbb{C})$. We first diagonalize an $SL(3,\mathbb{C})$ element M as

$$M_0 = \mathrm{diag}\left(\exp\left(\frac{i\theta}{3}\right),\exp\left(\frac{i\theta}{3}\right),\right.$$
$$\left.\exp\left(-\frac{i2\theta}{3}\right)\right) = \exp(i\theta T_0), \qquad (3.29)$$

$$T_0 \equiv \mathrm{diag}\left(\frac{1}{3},\frac{1}{3},-\frac{2}{3}\right). \qquad (3.30)$$

We consider the boundary conditions of θ for a domain wall: $\theta=0$, $(M=\mathbf{1}_3)$ at $x\to+\infty$ and $\theta=2\pi$, $(M=\omega\mathbf{1}_3)$ at $x\to-\infty$.

Now the situation is a little more complicated because when we are not restricting to the NG subspace, we need also to take into account the metric on the tangent space $G_{A\bar{B}}$. The BPS equation (3.4) now reads

$$i\partial_1\theta = \frac{me^{\frac{i2\theta}{3}}(1-e^{i\bar{\theta}})}{f_\pi^2(e^{i\theta}+2e^{i\bar{\theta}})}, \qquad (3.31)$$

where we have used the inverse metric on the tangent space

$$G^{1\bar{1}} = G^{2\bar{2}} = G^{3\bar{3}} = \frac{1}{f_\pi^2}e^{2\Im(\theta)/3}, \qquad (3.32)$$

$$G^{4\bar{4}} = G^{5\bar{5}} = G^{6\bar{6}} = G^{7\bar{7}} = \frac{1}{2f_\pi^2}e^{-4\Im(\theta)/3} + \frac{1}{2f_\pi^2}e^{2\Im(\theta)/3}, \qquad (3.33)$$

$$\begin{aligned} G^{4\bar{5}} &= -G^{5\bar{4}} = G^{6\bar{7}} = -G^{7\bar{6}} \\ &= -i\frac{1}{2f_\pi^2}e^{-4\Im(\theta)/3} + i\frac{1}{2f_\pi^2}e^{2\Im(\theta)/3}, \end{aligned} \qquad (3.34)$$

$$G^{8\bar{8}} = \frac{3e^{i2\theta/3+i\bar{\theta}/3}}{f_\pi^2(e^{i\theta}+2e^{i\bar{\theta}})}, \tag{3.35}$$

and the generators are $T_A = \frac{1}{\sqrt{2}}\lambda_A$, where λ_A are the Gell-Mann matrices.

Let us decompose Eq. (3.31) into real and imaginary parts

$$\partial_1 a = -\frac{me^{b/3}(\sin\frac{a}{3} + e^b \sin\frac{2a}{3})}{f_\pi^2(1+2e^{2b})}, \tag{3.36}$$

$$\partial_1 b = -\frac{me^{b/3}(\cos\frac{a}{3} - e^b \cos\frac{2a}{3})}{f_\pi^2(1+2e^{2b})}, \tag{3.37}$$

where we have defined the complex function $\theta = a + ib$, in terms of two real-valued functions. Notice that the only fixed points (vacua) of this system is $a = 2\pi n$ and $b = 0$ with $n \in \mathbb{Z}$. If we consider the imaginary function, b, then around the vacuum $a = 2\pi$, the asymptotic behavior of b when large and negative is

$$b \sim -3\log\frac{mx}{f_\pi^2} + \text{const}, \tag{3.38}$$

whereas if b is large and positive, it goes as

$$b \sim \frac{3}{2}\log\frac{mx}{f_\pi^2} + \text{const} \tag{3.39}$$

Neither of these behaviors allow for b to return to the vacuum $b = 0$. This means that the system exhibits an instability such that when $|b|$ is larger than some critical value, it cannot return to the vacuum even if $a \simeq 2\pi$. This, however, does not prove the absence of solutions to the Eq. (3.31). We will leave this task to future studies. We have nevertheless been seeking for numerical solutions without finding any.

5. $SL(N,\mathbb{C})$ case

Here we generalize the previous section to $SL(N,\mathbb{C})$. We first diagonalize an $SL(N,\mathbb{C})$ element M as

$$M_0 = \text{diag}\left(\exp\left(i\frac{\theta}{N}\right), \ldots, \exp\left(i\frac{\theta}{N}\right), \exp\left(-i\theta\frac{N-1}{N}\right)\right) = \exp(i\theta T_0), \tag{3.40}$$

$$T_0 \equiv \text{diag}\left(\frac{1}{N}, \ldots, \frac{1}{N}, -\frac{N-1}{N}\right). \tag{3.41}$$

We consider the boundary conditions of θ for a domain wall: $\theta = 0$, $(M = \mathbf{1}_N)$ at $x \to -\infty$ and $\theta = 2\pi$, $(M = \omega\mathbf{1}_N)$ at $x \to +\infty$.

Substituting this form into the BPS equation (3.4), we get

$$i\partial_1\theta = \frac{me^{\frac{i(N-1)\theta}{N}}(1-e^{i\bar{\theta}})}{f_\pi^2(e^{i\theta}+(N-1)e^{i\bar{\theta}})}. \tag{3.42}$$

Since, as we have seen in the previous section, it is difficult at best to find solutions in the generic case where the quasi-NG bosons are turned on, we will first attempt a simplification. We want to take the large N limit of the above equation. Let us define

$$\tilde{m} \equiv \frac{m}{Nf_\pi^2}. \tag{3.43}$$

In the large N limit, Eq. (3.42) reduces to

$$i\partial_1\theta = \tilde{m}e^{i\theta}(e^{-i\bar{\theta}} - 1). \tag{3.44}$$

The vacua are clearly $\theta = 2\pi n$, with $n \in \mathbb{Z}$. A domain wall solution would thus go from 0 to 2π. Let us again decompose the equation into real functions

$$\partial_1 a = -\tilde{m}e^{-b}\sin a, \tag{3.45}$$

$$\partial_1 b = \tilde{m}e^{-2b}(e^b\cos a - 1), \tag{3.46}$$

where $\theta = a + ib$. Expanding Eq. (3.46) in small a yields

$$\partial_1 b = \tilde{m}e^{-2b}\left[e^b\left(1 - \frac{1}{2}a^2 + \mathcal{O}(a^4)\right) - 1\right], \tag{3.47}$$

which for $b = 0$ and a small but positive will drive b negative. It is easy to see from the right-hand side of (3.46) that once b is negative, it will always decrease and hence become more and more negative. Since all the vacua has $b = 0$, no solution exists to this equation.

Since we have used a particular—albeit well motivated—Ansatz for the domain wall field M and we have taken the large N limit, this is not a general proof of nonexistence.

Finally, let us consider the finite N case. Decomposing Eq. (3.42) into real functions, we get

$$\partial_1 a = -\frac{m}{f_\pi^2}\frac{e^{\frac{b}{N}}(\sin\frac{a}{N} + e^b\sin\frac{(N-1)a}{N})}{1+(N-1)e^{2b}}, \tag{3.48}$$

$$\partial_1 b = -\frac{m}{f_\pi^2}\frac{e^{\frac{b}{N}}(\cos\frac{a}{N} - e^b\cos\frac{(N-1)a}{N})}{1+(N-1)e^{2b}}, \tag{3.49}$$

where $\theta = a + ib$. Expanding Eq. (3.49) in small a yields

$$\partial_1 b = -\frac{m}{f_\pi^2} e^{\frac{b}{N}} \left(1 - \frac{a^2}{2N^2} - e^b \left(1 - \frac{(N-1)^2 a^2}{2N^2} + \mathcal{O}(a^4) \right) + \mathcal{O}(a^4) \right). \tag{3.50}$$

Since the $SU(2)$ case is already solved, we will consider only $N > 2$, in which case the second cosine dominates and hence for $b = 0$ and small a again drives b negative. If b attains a negative value and it has to return to zero for when a goes to 2π, then a positive value of the right-hand side of Eq. (3.49) is a necessity. The larger negative values b takes on, the harder it is for the function to be positive; therefore we will consider $b = 0$ as the most conservative choice for a negative value of b. If the function cannot attain positive values for $b = 0$, then even less so for $b < 0$. It is thus enough to realize that

$$-\cos\frac{a}{N} + \cos\frac{(N-1)a}{N} \leq 0, \quad \text{for } N \geq 4. \tag{3.51}$$

Hence, no solution exists for $N \geq 4$. This is of course consistent with the large N limit considered above. The only possibility is $N = 3$ for which we do not have a proof at present. Numerically, however, we have not been able to find a solution to the BPS equation.

Of course the proof of nonexistence is limited to the use of our Ansatz. We leave a general proof for future work.

C. Quadratic potential

In this section we turn to the case of the quadratic potential (2.36), hence the BPS equation reads

$$iM^{-1}\partial_1 M = \frac{im}{N^2} \mathrm{tr}(M^\dagger) \mathrm{tr}(M^\dagger T_{\overline{B}}) T_A G^{A\overline{B}}. \tag{3.52}$$

Restricting again to the NG-boson subspace, $M = M|_{\sigma^i=0} = U$, we have

$$iU^\dagger \partial_1 U = \frac{im}{N^2 f_\pi^2} \mathrm{tr}(U^\dagger) \mathrm{tr}(U^\dagger T_A) T_A. \tag{3.53}$$

With this superpotential, we can also calculate the domain wall tension using Eq. (3.8), which for a domain wall between $M = \mathbf{1}_N$ and the new vacuum yields

$$T = \frac{m}{N^2} |\Re((\mathrm{tr}\mathbf{1}_N)^2)| = m, \tag{3.54}$$

while for a domain wall between the vacuum $M = \omega^k \mathbf{1}_N$ and the new vacuum, we have

$$T = \frac{m}{N^2} |\Re((\omega^k \mathrm{tr}\mathbf{1}_N)^2)| = m \cos\frac{4\pi k}{N}. \tag{3.55}$$

If we restrict to the NG subspace, $M = M|_{\sigma^i=0} = U$, only real vacua exist and so the tension is always given by Eq. (3.54).

1. SU(2) solution

Let us consider $N = 2$ as a warm up. The old vacua have $\omega = e^{i\pi}$ and so are given by

$$M_1 = \mathbf{1}_2, \qquad M_2 = -\mathbf{1}_2, \tag{3.56}$$

whereas the new vacuum is given by

$$\mathrm{tr} M = 0, \tag{3.57}$$

which we can flesh out as

$$M_3 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{3.58}$$

whose determinant is

$$-a^2 - bc = 1, \tag{3.59}$$

yielding

$$M_3 = \begin{pmatrix} a & -\frac{1+a^2}{c} \\ c & -a \end{pmatrix}. \tag{3.60}$$

The simplest possibility is $a = 0$ and $c = 1$, i.e.,

$$M_3 = -i\tau^2. \tag{3.61}$$

The complication of the $N = 2$ case is that there is no new diagonal vacuum. We will now consider an Ansatz that will interpolate between one of the old vacua and the new vacuum, namely from M_1 to M_3:

$$U = \mathbf{1}_2 \cos\theta - i\tau^2 \sin\theta. \tag{3.62}$$

Since both vacua are in the subspace spanned by the NG bosons, it is consistent to restrict to the NG submanifold, if a solution exists. The boundary conditions are $\theta = 0$ ($U = \mathbf{1}_2$) at $x \to +\infty$ and $\theta = \pi/2$ ($U = -i\tau^2$) at $x \to -\infty$. Notice that due to the two vacua not being proportional to the identity matrix ($\mathbf{1}_2$), the global $SU(2)$ symmetry is broken to $U(1)$ by the vacuum. Plugging the above Ansatz into Eq. (3.5) we get

$$\tau^2 \partial_1 \theta = -\tau^2 \frac{m}{4 f_\pi^2} \sin 2\theta, \tag{3.63}$$

which has the solution

$$\theta(x^1) = \arctan \exp\left[-\frac{m}{2 f_\pi^2} (x^1 - X) \right], \tag{3.64}$$

where X is again a position modulus.

The $U(1)$ symmetry possessed by the vacuum is unbroken by the domain wall solution. Consequently, the domain wall has no orientational moduli.

2. $SU(2K)$ solutions

We can straightforwardly extend the $SU(2)$ solution to $SU(2K)$, by embedding K blocks of the $SU(2)$ Ansatz (3.62) in U_0 as

$$U_0 = \begin{pmatrix} \mathbf{1}_2 \cos\theta_1 - i\tau^2 \sin\theta_1 & & \\ & \ddots & \\ & & \mathbf{1}_2 \cos\theta_K - i\tau^2 \sin\theta_K \end{pmatrix}, \tag{3.65}$$

which interpolates between the vacuum $U = \mathbf{1}_{2K}$ and

$$U = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}. \tag{3.66}$$

It is straightforward to show that the BPS equation is exactly the same as (3.63) in each K block along the diagonal. The solution is therefore Eq. (3.64),

$$\theta_i(x^1) = \arctan \exp\left[-\frac{m}{2f_\pi^2}(x^1 - X_i)\right], \tag{3.67}$$

with $i = 1, \ldots, K$ and the moduli space is now given by

$$\frac{SU(K)}{U(1)^{K-1}}, \tag{3.68}$$

for generic position moduli $X_1 \neq X_2 \neq \cdots \neq X_K$. If however $X_1 = X_2 = \cdots = X_K$ then no orientational (NG) moduli exist for this solution.

IV. LOW-ENERGY EFFECTIVE THEORY ON THE DOMAIN WALL

In this section, we construct the low-energy effective theory on the $SU(2)$ domain wall for the linear superpotential (2.29) by using the moduli (or Manton's) approximation [37]. The most general solution is obtained by performing the $SU(2)_{\rm L+R}$ transformation in Eq. (3.18) and is given by

$$U = VU_0V^\dagger = \exp(i\theta VT_0V^\dagger), \quad V \in SU(2). \tag{4.1}$$

Now we define the complex 2-vector ϕ by the following relation

$$VT_0V^\dagger = \phi\phi^\dagger - \frac{1}{2}\mathbf{1}_2. \tag{4.2}$$

The vector ϕ satisfies the constraint $\phi^\dagger\phi = 1$. Using this vector, the general solution is rewritten as

$$\begin{aligned} U &= \exp\left[i\theta\left(\phi\phi^\dagger - \frac{1}{2}\mathbf{1}_2\right)\right] \\ &= [\mathbf{1}_2 + (e^{i\theta} - 1)\phi\phi^\dagger]\exp(-i\theta/2). \end{aligned} \tag{4.3}$$

The vector ϕ parametrizes $\mathbb{C}P^1$ and the moduli of the solution are given by X and ϕ.

We first promote the moduli X and ϕ in the solution to fields $X(x^\alpha)$ and $\phi(x^\alpha)$ depending on the coordinates x^α of the domain wall world-volume, substitute it into the original Lagrangian, and then perform an integration over the codimension.

The differentiation of U with respect to the world-volume coordinates x^α can be calculated as

$$\begin{aligned} \partial_\alpha U = \Big[&-\frac{i}{2}(\mathbf{1}_2 + (e^{i\theta} - 1)\phi\phi^\dagger)\partial_\alpha\theta + i\partial_\alpha\theta e^{i\theta}\phi\phi^\dagger \\ &+ (e^{i\theta} - 1)(\partial_\alpha\phi\phi^\dagger + \phi\partial_\alpha\phi^\dagger)\Big]\exp(-i\theta/2). \end{aligned} \tag{4.4}$$

By using the relations

$$\partial_\alpha e^{i\theta(x^1;X(x^\alpha))} = i\partial_\alpha X\frac{\partial\theta}{\partial X}e^{i\theta} = -i\partial_\alpha X\partial_1\theta e^{i\theta}, \tag{4.5}$$

$$2|1 - e^{i\theta}|^2 = 2(2 - e^{i\theta} - e^{-i\theta}) = 8\sin^2\frac{\theta}{2}, \tag{4.6}$$

we obtain

$$\begin{aligned} \mathrm{tr}(\partial_\alpha U\partial^\alpha U^\dagger) = &\frac{1}{2}(\partial_1\theta)^2(\partial_\alpha X)^2 \\ &+ 8\sin^2\frac{\theta}{2}[\partial^\alpha\phi^\dagger\partial_\alpha\phi + (\phi^\dagger\partial_\alpha\phi)^2]. \end{aligned} \tag{4.7}$$

By noting the formulas

$$\begin{aligned} &\int dx^1 \sin^2\frac{\theta}{2} = \frac{2}{m}, \\ &\int dx^1(\partial_1\theta)^2 = \int dx^1\left(\frac{m}{f_\pi^2}\sin\frac{\theta}{2}\right)^2 = \frac{2m}{f_\pi^4}, \end{aligned} \tag{4.8}$$

where we have used the BPS equation (3.19) in the second relation, the integration of Eq. (4.7) over the codimensional coordinate x yields the final form of the effective Lagrangian on the wall:

$$\mathcal{L}_{\rm eff} = -\frac{m}{f_\pi^2}\partial_\alpha X \partial^\alpha X - \frac{16 f_\pi^2}{m}[\partial_\alpha \phi^\dagger \partial^\alpha \phi + (\phi^\dagger \partial_\alpha \phi)(\phi^\dagger \partial^\alpha \phi)] - \frac{m}{f_\pi^2}, \tag{4.9}$$

where the last term is the tension of the domain wall. The first term describes the translational zero modes while the second term represents the orientational zero modes, which is described by the $\mathbb{C}P^1$ model.

V. SUMMARY AND DISCUSSION

We have studied the BPS domain walls in the $\mathcal{N}=1$ supersymmetric chiral Lagrangian with $SU(N)_{\rm L+R}$ invariant pion mass terms. The bosonic components of the model consist of both NG and quasi-NG bosons. We have constructed exact solutions of BPS pion domain walls for the case of a linear and a quadratic superpotential. In all cases we have considered the simplest Kähler potential; the difference with the most general Kähler potential of the chiral invariant amounts simply to a change in the Kähler modulus (pion decay constant). All the domain wall solutions are topologically stable. We have, however, found not all vacua are connected by domain walls of BPS type. In particular, we have only been able to find BPS domain wall solutions connecting pion vacua, i.e. vacua with no imaginary part. These domain wall solutions, in turn, are described only by the NG-boson subspace and not by the quasi-NG bosons, which are left turned off in the solutions. For a well-motivated Ansatz, we have found the complex BPS equation not restricted to the NG submanifold. We have, however, proved that this BPS equation has no solutions for $N \geq 4$. The $N=2$ case has only real vacua and the analytic domain wall solution is simply the sine-Gordon solution. We have not been able to find analytical or numerical solutions to the BPS equation for $N=3$, although we do not at present have a proof of nonexistence. The understanding of the absence of domain walls between all the vacua with an imaginary part still needs some progress. This may in turn teach us about the dynamics of the quasi-NG bosons in nonperturbative solutions. We leave this interesting open issue for future studies.

The BPS bound gives a tension which is the absolute value of the real part of the difference between the superpotential evaluated at two given vacua. The fact that the tension is the real part of this difference, means that if the vacua are purely imaginary (for instance $M=i$ and $M=-i$), then the BPS bound gives a vanishing tension. Since, physically, no domain wall can interpolate two such vacua with vanishing tension, they are necessarily not saturating the bound and thus are non-BPS. Whether non-BPS solutions exist or not is beyond the scope of this paper, although it is an interesting problem which we leave for future work.

Non-Abelian vortices in $U(2)$ gauge theories also carry $\mathbb{C}P^1$ moduli [16] and the $U(N)$ gauge group was generalized to an arbitrary gauge group [38] such as $SO(N)$ and $USp(2N)$ [39]. Our model itself could straightforwardly be extended to a chiral Lagrangian of an arbitrary group G, but one nontrivial question is which coset space G/H is realized on the domain wall. We have already observed a more complicated structure of the domain walls in our model than simply the $\mathbb{C}P^{N-1}$ model; further cosets appear already in the $SU(N)$ case.

Our model should admit a domain wall junction as a $1/4$ BPS state [10]. In particular, the simplest superpotential with N vacua is expected to admit a $\mathbb{Z}_N$ symmetric domain wall junction as in Ref. [11]. However, since we already have observed that not all the vacua are connected in our model, the domain wall junctions may be either absent or modified compared to the usual case.

The effective theory of a non-Abelian vortex in $U(N)$ gauge theory is the $\mathbb{C}P^{N-1}$ model, and lump solutions on it correspond to Yang-Mills instantons in the bulk [40]. The total configuration of lumps inside a vortex is a $1/4$ BPS state. In the same way, lump solutions in our domain wall, which will correspond to Skyrmions in the bulk as the case of a non-Abelian sine-Gordon soliton [33], may be $1/4$ BPS states.

Recently, a supersymmetric Skyrme term has been constructed in Ref. [30] in which it has been found that the usual kinetic term cancels out. In this case, the introduction of a superpotential can be done only perturbatively [31]. Construction of such a model and its BPS domain wall solutions—that may be of compacton type—remain as a future problem.

The chiral Lagrangian can be realized on a non-Abelian domain wall in $\mathcal{N}=2$ supersymmetric $U(N)$ gauge theory with two $N\times N$ complex scalar fields (hypermultiplets) [41,42]. If we find a suitable mass deformation preserving (part of) the supersymmetry in the original bulk action that induces the superpotential $W=\frac{m}{N}{\rm tr}(M)$ on the wall, then our solution may describe a wall inside a wall as a $1/4$ BPS state. The non-Abelian domain wall in Refs. [41,42] describes a non-Abelian Josephson junction in the presence of a Josephson term in the bulk that breaks supersymmetry, and a sine-Gordon soliton on the wall that describes a non-Abelian vortex absorbed into the junction [34]. A supersymmetry-preserving mass deformation, if it exists, would describe a supersymmetric Josephson junction and would give a BPS non-Abelian vortex absorbed into the junction, that is, a BPS non-Abelian Josephson vortex.

Finally, in (non-supersymmetric) QCD, topological solitons in chiral symmetry breaking were studied, see, e.g. Refs. [2,43]. Our BPS configurations may have implications for these more realistic cases as well.

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