

Holographic bound in covariant loop quantum gravity

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We investigate puncture statistics based on the covariant area spectrum in loop quantum gravity. First, we consider Maxwell-Boltzmann statistics with a Gibbs factor for punctures. We establish formulas which relate physical quantities such as horizon area to the parameter characterizing holographic degrees of freedom. We also perform numerical calculations and obtain consistency with these formulas. These results tell us that the holographic bound is satisfied in the large area limit and the correction term of the entropy-area law can be proportional to the logarithm of the horizon area. Second, we also consider Bose-Einstein statistics and show that the above formulas are also useful in this case. By applying the formulas, we can understand intrinsic features of Bose-Einstein condensate which corresponds to the case when the horizon area almost consists of punctures in the ground state. When this phenomena occurs, the area is approximately constant against the parameter characterizing the temperature. When this phenomena is broken, the area shows rapid increase which suggests the phase transition from quantum to classical area.

DOI: [10.1103/PhysRevD.94.024045](https://doi.org/10.1103/PhysRevD.94.024045)**I. INTRODUCTION**

Canonical quantization of general relativity has a long history beginning in the 1960s [1]. Basically, metric and its conjugate momentum had been used as canonical variables in these early days. In this case, the Hamiltonian constraint is nonpolynomial about these variables. So, it is almost impossible to solve its quantized counterpart. In [2], it was shown that if we use the complex Ashtekar connection and its conjugate, the Hamiltonian constraint can be written as polynomial about these variables. Surprisingly, it was found that Wilson loop for this connection is a solution of quantized Hamiltonian constraint [3]. Using Wilson loop, spin network, basic ingredients of the loop quantum gravity (LQG), has been constructed [4]. Discrete area spectrum is one of the main predictions in LQG [5,6].

However, it has been recognized that the reality conditions for the physical quantities to be real are difficult to be solved. Then, the SU(2) real connection has been introduced where imaginary number i in the complex Ashtekar connection was replaced by the real parameter γ called the Barbero-Immirzi (BI) parameter [7]. Although the Hamiltonian constraint becomes nonpolynomial, this complication can be relieved if we rewrite it using the technique developed in [8]. If we apply it in the symmetry-reduced model, we can discuss singularity avoidance which has been paid much attention [9].

The microscopic origin of black hole entropy in LQG had also been discussed in [10], where the number of degrees of freedom of the edge configuration for a fixed SU(2) area spectrum was counted. Then, Ashtekar, Baez, Corichi, and Krasnov refined this idea based on the isolated

horizon framework (so-called ABCK framework where the isolated horizon itself was described by U(1) connection [11,12]) and determined γ to satisfy the Bekenstein-Hawking entropy-area law $S = A/(4G)$ where S , A and G are black hole entropy, horizon area and the gravitational constant, respectively. Here, ambiguity of γ turned out to be the merit of using a real connection. Including the correction of error in original counting [13], or ambiguity in counting [14–16], relation with the quasinormal mode [17–19], various aspects have been discussed related to the ABCK framework [20–22].

The situation slightly changed when it was found that the isolated horizon can be written using the SU(2) connection [23]. This means that the horizon Hilbert space can be described by the SU(2) Chern-Simons state. Its dimension is written by the spin freedom j and the level of the Chern-Simons state k . Using a suitable analytic continuation of these variables to complex variables, it was obtained that the complex Ashtekar connection is desirable to reproduce $S = A/(4G)$ [24,25]. Furthermore, when we introduce the geometric temperature by demanding the horizon state be a Kubo-Martin-Schwinger state, we can also arrive at the complex connection [26].

Is there an essential reason why the complex Ashtekar connection is preferable? One of the reasons would be that the covariance is satisfied in this connection while it is violated in the real connection [27]. This should be taken seriously, and we should pay attention how to choose the Lorentz covariant connection which has been investigated in [28]. The connection obtained in [28] is called shifted connection which includes the BI parameter. Surprisingly, the Hamiltonian constraint can be written as a polynomial equation in this case again. Using a shifted connection, covariant LQG has been formulated, and a covariant area

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spectrum has been obtained [29,30]. Making the consistent relation between covariant LQG and the spin foam models became the important realm recently [31]. We should also notice that the covariant area spectrum does not include the BI parameter although the shifted connection itself does. Then, it is natural to ask whether or not we can obtain consistency with the entropy-area law if we consider counting microscopic freedom of black holes in covariant LQG. In [32], by assuming the horizon area consists of the minimum area eigenvalue, it was argued that the answer is in the affirmative.

Here, we consider the generality of the holographic bound and argue the correction term of the entropy-area law discussed in [33,34]. These are motivated by the quasilocal first law of black hole thermodynamics where the quasilocal energy is defined using the horizon area [35]. Then, regarding the puncture, which is an intersection of the edge at the horizon, as a particle, we can argue its statistical mechanics. One of the important points in [33,34] is that if we assume the degeneracy of matter fields close to the horizon as $\exp(\lambda A/G)$ where λ is a dimensionless constant, λ must approach $1/4$ in the large area limit when punctures are indistinguishable. The correction term of the entropy-area law is basically proportional to \sqrt{A} unless we assume the special form for the fugacity. Then, our concerns are whether these properties hold or not in the covariant area spectrum. The answer is in the affirmative for the holographic bound while the correction term depends on the ambiguity of the covariant area spectrum as we discuss later.

This paper is organized as follows. In Sec. II, we introduce tools necessary for constructing puncture statistics following [33,34]. In Sec. III, we consider the case when Maxwell-Boltzmann statistics with a Gibbs factor for punctures is assumed. We establish formulas which relate physical quantities such as horizon area to the parameter characterizing holographic degrees of freedom. We also perform numerical calculations and obtain consistency with these formulas. These results show that the holographic bound is saturated in the large area limit and that the correction term of the entropy-area law can be proportional to $\ln A$. In Sec. IV, we consider the case when Bose-Einstein statistics is assumed and argue that the above formulas are also useful in this case. By applying the formulas, we can understand the intrinsic features of Bose-Einstein condensate which corresponds to the case when black holes almost consist of punctures in the ground state. We show that when this phenomena occurs, the area is approximately constant against the parameter characterizing the temperature. When this phenomena is broken, the area shows a rapid increase which suggests the phase transition from quantum to classical area spectrum. Concluding remarks follow in Sec. V.

II. PREPARATION FOR PUNCTURE STATISTICS

Following [33,34], we introduce several notions necessary for arguing puncture statistics. First, we mention the

quasilocal law of black hole thermodynamics which holds for the stationary observer at proper distance l from the horizon [35],

$$E = \frac{A}{8\pi l}, \quad (2.1)$$

where E is quasilocal horizon energy of black hole. We rewrite (2.1) using the inverse Unruh temperature $\beta_U := \frac{2\pi l}{G}$ as

$$\beta_U E = \frac{A}{4G}. \quad (2.2)$$

Then, we can discuss the energy spectrum of the puncture by combining (2.2) with the area spectrum. In [33,34], the $SU(2)$ area spectrum written as

$$A = 8\pi\gamma G \sum_i \sqrt{j_i(j_i + 1)}, \quad (2.3)$$

has been used. Here, j_i is a half-integer associated with the puncture i . Here, we use the covariant area spectrum written as

$$A = 8\pi G \sum_i \sqrt{j_i(j_i + 1) - n_i^2 + \rho_i^2 + 1}, \quad (2.4)$$

where n_i is a half-integer with $j_i \geq n_i$ and ρ_i is a real number [29]. Notice that there is no ambiguity related to γ . In [30], it has been shown that it is enough for counting the degrees of freedom to consider the simple representation $n_i = 0$, which we assume here.

The important point is how to determine ρ_i . The relation (2.1) and the spectrum (2.4) show that the puncture i has quasilocal energy

$$E_i = \frac{G}{l} \sqrt{j_i(j_i + 1) + \rho_i^2 + 1}. \quad (2.5)$$

Thus, the simplest possibility is to choose $\rho_i = 0$, which we include considering below. The next simplest possibility would be to regard ρ_i as a dependent variable of j_i . In this case, $j_i = 1/2$ does not necessarily correspond to the ground state, which is important when we discuss Bose-Einstein condensate as shown in [34]. Although it is an interesting possibility, it is reasonable to assume that E_i is monotonic with j_i as a first extension of the previous case in [33,34]. Here, we choose ρ_i^2 as

$$\rho_i^2 = 0, \quad j_i^m (m > 1), \quad e^{2j_i}, \quad (2.6)$$

which correspond to the cases,

$$\frac{l}{G} E_i \rightarrow j_i, \quad j_i^m, \quad e^{j_i}, \quad (2.7)$$

in the limit $j_i \rightarrow \infty$. The reason why we choose a monomial or an exponential as (2.6) is supposed by the observation that only the qualitative behavior in the limit $j_i \rightarrow \infty$ determines the holographic property and the correction term of the entropy-area law in [33,34].

Let us consider puncture statistics. In general, we do not require that the inverse temperature is equal to β_U . We write the inverse temperature β using β_U as

$$\beta = \beta_U(1 + \delta_\beta), \quad (2.8)$$

where δ_β is a parameter. We only demand that δ_β vanishes in the semiclassical limit $A \rightarrow \infty$ to satisfy the relation (2.2).

We define n_j as the number of punctures carrying spin j and N as the total number of punctures. So, we have

$$N = \sum_j n_j. \quad (2.9)$$

We also define $D(\{n_j\})$ as the number of holographic degrees of freedom for a given configuration $\{n_j\}$. Here, we assume

$$D(\{n_j\}) = \exp((1 - \delta_h)\bar{A}), \quad (2.10)$$

where $\bar{A} := \frac{A}{4G}$ and δ_h is a free parameter. We suppose that the freedom $D(\{n_j\})$ comes from the matter fields close to the horizon motivated by the entanglement entropy hypothesis [36].

III. MAXWELL-BOLTZMANN STATISTICS

We include the Gibbs factor $N!$ in the Maxwell-Boltzmann statistics. The case without the Gibbs factor is discussed later in this section. Then the canonical partition function $Q(N, \beta)$ is given by

$$Q(N, \beta) = \frac{1}{N!} \sum_{n_j} D(\{n_j\}) \frac{N!}{\prod_j n_j!} \prod_j e^{-\beta n_j E_j}. \quad (3.1)$$

Here, we abbreviate the puncture index i and write the spin index j in the quasilocal energy as

$$E_j = \frac{G}{l} \sqrt{j(j+1) + \rho^2 + 1}. \quad (3.2)$$

Using (2.10), we can express the partition function as

$$Q = \frac{q^N}{N!}, \quad (3.3)$$

where

$$q = \sum_{j=1/2}^{\infty} \exp\left(-2\pi\delta\sqrt{j(j+1) + \rho^2 + 1}\right). \quad (3.4)$$

Here, we defined

$$\delta := \delta_\beta + \delta_h. \quad (3.5)$$

We introduce the fugacity $z = \exp(\beta\mu)$ where μ is a chemical potential. In this case, we can express the grand canonical partition function by

$$Z_{\text{MB}} = \sum_N \frac{(zq)^N}{N!} = \exp(zq). \quad (3.6)$$

The total number N and the mean energy E are

$$N = z \frac{\partial}{\partial z} (\ln Z_{\text{MB}}) = zq, \quad (3.7)$$

$$E = -\partial_\beta (\ln Z_{\text{MB}}) + N\mu = -z\partial_\beta q. \quad (3.8)$$

We can express the entropy as

$$\begin{aligned} S &= \beta E - N \ln z + \ln Z_{\text{MB}} \\ &= \bar{A}(1 + \delta_\beta) + N(1 - \ln z). \end{aligned} \quad (3.9)$$

As we said above, we required $\delta_\beta \rightarrow 0$ in the limit $A \rightarrow \infty$. So, if $z = e$, the correction term of the entropy-area law proportional to N disappears as pointed out in [34].

Since one of the purposes using the covariant area spectrum is to investigate the correction term, we consider the case $z \neq e$. In other treatments, it is often argued that the correction term proportional to $\ln \bar{A}$ appears [37–39]. From (3.7) and (3.8), we have

$$\bar{A} : N(1 - \ln z) = -\beta\partial_\beta q : q(1 - \ln z). \quad (3.10)$$

Thus, in discussing the ratio between \bar{A} and the correction term, it is enough if we investigate the ratio between $\partial_\beta q$ and q . Since z plays a minor role for this reason, we set $z = 1$ below, for simplicity.

How can we estimate the relation between q and A ? We should first notice that convergence of the sequence (3.4) highly depends on δ . So, our strategy is to analyze the dependencies of q and A as a function of δ for obtaining the relation between q and A .

Since it would be difficult in calculating (3.4) exactly, we suppose using numerical calculation. In this case, it is important to know j_{max} should sum up, which is a key to understand above the property. Concretely, we assume that we need to sum up from $j = 1/2$ to j_{max} in obtaining the value q_{fix} for enough precision toward the true value q , e.g., relative error $|q - q_{\text{fix}}|/q < 10^{-20}$. To accomplish the above task, we need to estimate the dependence of j_{max} on δ

as a first step, which is also a difficult task, in general. However, we can expect that $j_{\max} \rightarrow \infty$ in the limit $\delta \rightarrow 0$, and we can estimate (3.4) using the asymptotic form in the limit $j \rightarrow \infty$. For this reason, we assume $|\delta| \ll 1$.

Let us consider the cases (2.7). If we have $\bar{E}_j := \frac{1}{G} E_j \rightarrow j^n$ ($n = 1, m$) in $j \rightarrow \infty$, we can write

$$q_{\text{fix}} \simeq \sum_{j=1/2}^{j_{\max}} \exp(-2\pi\delta j^n) = \sum_{j=1/2}^{j_{\max}} \exp[-2\pi(\epsilon j)^n], \quad (3.11)$$

where $\delta =: \epsilon^n$. If we define $x := \epsilon j$ and $f(x) := \exp(-2\pi x^n)$, we can rewrite

$$q_{\text{fix}} \simeq \sum_{x=x_1}^{x_2} f(x), \quad (3.12)$$

where $x_1 = \epsilon/2$ and $x_2 = \epsilon j_{\max}$.

Using these notations, we comment on following important properties.

- (i) If we reduce $\epsilon_{\text{old}} \rightarrow \epsilon_{\text{new}} = \epsilon_{\text{old}}/10$,
 - (a) we should change $j_{\max, \text{old}} \rightarrow j_{\max, \text{new}} = 10j_{\max, \text{old}}$ in preserving the same precision.
 - (b) we obtain $q_{\text{fix, old}} \rightarrow q_{\text{fix, new}} = 10q_{\text{fix, old}}$ approximately.

To understand these properties, we should first notice that interval $\Delta x = \frac{\epsilon}{2}$ in the sum (3.12) becomes $1/10$ while x_2 does not change by (a). This means that there are $2j_{\max, \text{old}}$ terms we should sum up in the former case while $20j_{\max, \text{old}}$ terms in the latter case in (3.12). Thus, we obtain $q_{\text{fix, old}} \rightarrow q_{\text{fix, new}} = 10q_{\text{fix, old}}$ approximately. Since $|q - q_{\text{fix}}| = \sum_{x_2}^{\infty} f(x)$, we also have $|q_{\text{old}} - q_{\text{fix, old}}| \rightarrow 10|q_{\text{old}} - q_{\text{fix, old}}|$ approximately. Therefore, we have same relative error and the precision is preserved.

For this approximation to be valid, following conditions should hold.

Conditions

- (i) Changing $x_{1, \text{old}} \rightarrow x_{1, \text{new}}$ is negligible.
- (ii) $f(x)$ does not have the property,

$$|f(x + \Delta x)/f(x)| \ll 1, \text{ or } \gg 1.$$

The former assumption is implicitly used when we use the asymptotic form in the limit $j \rightarrow \infty$. The latter assumption holds when δ is small enough in the above case.

From these consideration, we obtain

$$q \propto \epsilon^{-1} = \delta^{-1/n}. \quad (3.13)$$

Since $A \propto -\partial_{\beta} q \propto -\partial_{\delta} q$, we also have

$$A \propto \delta^{-(n+1)/n}. \quad (3.14)$$

We mention that our results (3.13) and (3.14) are consistent with those in [34] where (2.3) was used which corresponds to the case $n = 1$.

Next, we consider the case $\bar{E}_j \rightarrow e^j$ in $j \rightarrow \infty$. In this case, we can write

$$q_{\text{fix}} \simeq \sum_{j=1/2}^{j_{\max}} \exp(-2\pi\delta e^j). \quad (3.15)$$

As in the previous case, if we want to obtain $q_{\text{fix}} \rightarrow Bq_{\text{fix}}$ ($B \gg 10$), we need to change the number of terms we should sum up from $2j_{\max}$ to $2Bj_{\max}$ ($B \gg 10$) for preserving the precision. This means that δ should change to satisfy

$$\delta_{\text{old}} e^{j_{\max}} = \delta_{\text{new}} e^{Bj_{\max}}. \quad (3.16)$$

So, we have $\delta_{\text{new}} = \delta_{\text{old}} e^{(-B+1)j_{\max}} \simeq \delta_{\text{old}} e^{-Bj_{\max}}$. This means $B \simeq -\frac{1}{j_{\max}} \ln\left(\frac{\delta_{\text{new}}}{\delta_{\text{old}}}\right)$. As a result, we have

$$q \propto -\frac{1}{j_{\max}} \ln\left(\frac{\delta}{C}\right), \quad (3.17)$$

where C is a constant. So, we have

$$A \propto -\frac{1}{j_{\max} \delta}. \quad (3.18)$$

The formulas (3.13), (3.14), (3.17), and (3.18) play quite important roles in this paper.

If we use the relations $(\Delta E)^2 = -\partial_{\beta} E$, (2.1), and (3.14), we obtain

$$\frac{\Delta E}{E} = \frac{\Delta A}{A} \propto \delta_{2n}^{\perp}. \quad (3.19)$$

The case of (3.18) is included in the limit $n \rightarrow \infty$. It is surprising that fluctuations of both energy and horizon area are summarized in this simple manner.

In the above estimate, we used the asymptotic form in the limit $j \rightarrow \infty$. Thus, it is desirable to check consistency using a numerical calculation. For this purpose, we choose

$$\rho^2 = 0, \quad j^4, \quad j^6, \quad e^{2j}, \quad (3.20)$$

which correspond to the cases,

$$\bar{E}_j \rightarrow j, \quad j^2, \quad j^3, \quad e^j, \quad (3.21)$$

in the limit $j \rightarrow \infty$. However, we stress that we use the exact expression (3.2) by substituting (3.20). We show $\delta - q$, \bar{A} relations in Fig. 1 which have complete consistency with (3.13), (3.14), (3.17), and (3.18). Especially, in all cases, $A \rightarrow \infty$ for $\delta \rightarrow 0$. So we confirmed that the holographic bound is saturated, i.e., $\delta_h \rightarrow 0$, in the semiclassical limit where the temperature should approach Unruh temperature $\beta \rightarrow \beta_U$. This is a generalization of the result in [34].

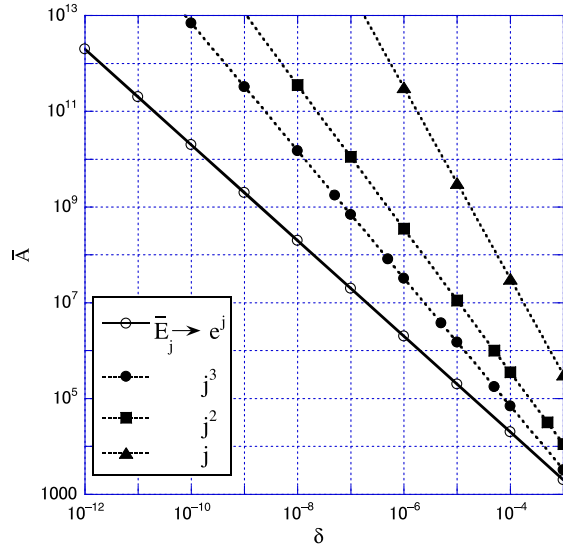
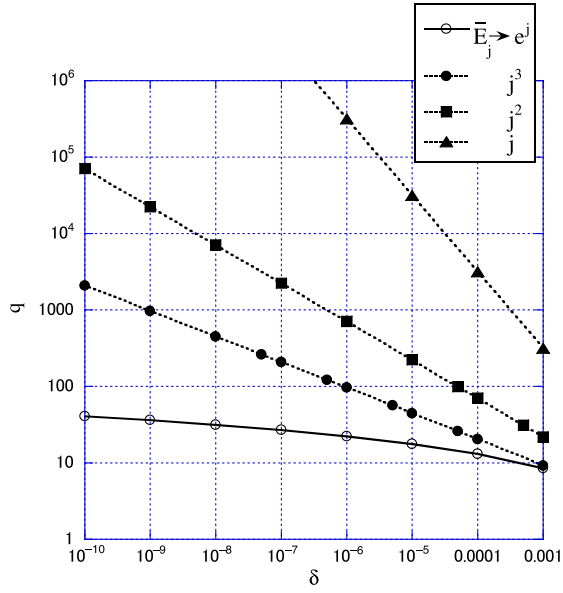


FIG. 1. The δ - q and \bar{A} relations showing the consistency with (3.13), (3.14), (3.17), and (3.18).

Then, we should also notice the results $q \propto \ln \bar{A}$ for $\rho^2 = e^{2j}$ derived by (3.17) and (3.18). To check its accuracy, we also show that $\exp(q/2)/\bar{A}$ is almost constant for $\rho^2 = e^{2j}$ in Fig. 2. The deviation from constant for large δ would be due to it from the asymptotic form. So, we obtain the log correction if we use the freedom ρ^2 . This is also our new results obtained by considering the covariant area spectrum.

Finally, we comment on the case without the Gibbs factor. In this case, we have

$$Z_{\text{MB}} = \sum q^N. \quad (3.22)$$

So, $q < 1$ is required. However, it is impossible in the small δ as we see from (3.13) and (3.17).

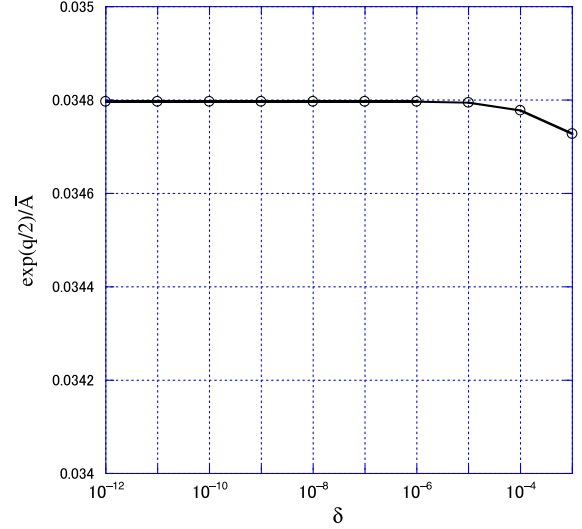


FIG. 2. The $\delta - \exp(q/2)/\bar{A}$ relation for $\rho^2 = e^{2j}$.

IV. BOSE-EINSTEIN STATISTICS

Here, we consider Bose-Einstein statistics as a candidate of the puncture statistics. First, we discuss the case $z = 1$ as an extension of the case in Maxwell-Boltzmann statistics. In this case, the grand canonical partition function can be written as

$$Z_{\text{BE}}(\beta) = \prod_j [1 - \exp(-\delta\beta_U E_j)]^{-1}. \quad (4.1)$$

So, we have

$$q := \ln Z_{\text{BE}}(\beta) = -\sum_j \ln [1 - \exp(-\delta\beta_U E_j)]. \quad (4.2)$$

We can perform an analogous discussion in the previous section. For example, if we have $\bar{E}_j \rightarrow j^n$ ($n = 1, m$) in $j \rightarrow \infty$, we replace $f(x)$ by $g(x) = \ln [1 - \exp(-2\pi x^n)]$ in (3.12). Then, the discussion below (3.12) holds, and we obtain (3.13) and (3.14). Similarly, for $\bar{E}_j \rightarrow e^j$, we obtain (3.17) and (3.18).

The conclusions are that we have a holographic bound in the large area limit, and the correction term of the entropy-area law behaves same as the case in Maxwell-Boltzmann statistics qualitatively. The result for $n = 1$ is consistent with [34] where the correction term is shown to be proportional to $\sqrt{\bar{A}}$ both in Maxwell-Boltzmann statistics and in Bose-Einstein statistics. We have shown that these can be understood in a unified way including the cases in covariant area spectrum.

Next, we discuss the case $z \neq 1$. The grand canonical partition function can be written as

$$Z_{\text{BE}}(\beta, \mu) = \prod_j [1 - \exp(\beta\mu - \delta\beta_U E_j)]^{-1}. \quad (4.3)$$

So, we should require

$$\delta\beta_{\text{U}}E_j - \beta\mu > 0. \quad (4.4)$$

Since we assumed that E_j is monotonic function with j , we obtain

$$\begin{aligned} \text{if } \beta > \beta_{\text{U}}(1 - \delta_h), \quad & \text{then } \mu < \frac{\beta_{\text{U}}\delta}{\beta}E_{1/2}, \\ \text{if } \beta < \beta_{\text{U}}(1 - \delta_h), \quad & \text{then } \mu = -\infty, \end{aligned}$$

as an extension of [34]. So, the high temperature region with $\beta < \beta_{\text{U}}(1 - \delta_h)$ should be described by a Maxwell-Boltzmann statistics. We concentrate on the case with $\beta > \beta_{\text{U}}(1 - \delta_h)$.

We consider whether or not above discussion can be extendible for the case $z \neq 1$. We define

$$q := \ln Z_{\text{BE}} = -\sum_j \ln [1 - \exp(\beta\mu - \delta\beta_{\text{U}}E_j)]. \quad (4.5)$$

The rhs of this equation includes two independent parameters δ_β and δ_h . To avoid complication, we set $\delta_h = 0$ below. Then, if we have $\bar{E}_j \rightarrow j^n$ in $j \rightarrow \infty$, we replace $g(x)$ by $g'(x) = \ln [1 - \exp(-2\pi x^n + \beta\mu)]$ to perform analogous discussion.

However, in this case, $g'(x)$ does not necessarily satisfy *the conditions* in the previous section. This depends on the ratio between $(2\pi x_1^n - \beta\mu)$ and ϵ . Concretely, if $(2\pi x_1^n - \beta\mu)$ is small enough, $g'(x_1 + \frac{\epsilon}{2})/g'(x_1)$ can be much smaller than 1. Of course, if we take $\epsilon \rightarrow 0$, we can obtain same conclusion as above. Below, we consider the case where *the conditions* are violated.

We can understand physical meaning of *the conditions* by using the number of punctures n_j for general \bar{E}_j . Here, n_j is represented by

$$n_j = [\exp(\delta\beta_{\text{U}}E_j - \beta\mu) - 1]^{-1}. \quad (4.6)$$

We define

$$\alpha := \delta\beta_{\text{U}}E_{1/2} - \beta\mu. \quad (4.7)$$

If $\alpha \ll 1$, the mean number of the grand state can be approximated as

$$n_{1/2} \simeq \frac{1}{\alpha}, \quad (4.8)$$

which is quite large. Thus, $n_1/n_{1/2} \ll 1$ is possible, which corresponds to the case $g'(x_1 + \frac{\epsilon}{2})/g'(x_1) \ll 1$.

Moreover, the total sum of the mean number $j > 1/2$, $n_{\text{ex}} := \sum_{j=1}^{\infty} n_j$ can be much smaller than $n_{1/2}$ which corresponds to the Bose-Einstein condensate state defined as a state where the horizon is almost dominated by spin 1/2 puncture. Since n_j ($j \geq 1$) can satisfy *the conditions*,

n_{ex} can be estimated by following the analogous discussion as above. That is, if we have $\bar{E}_j \rightarrow j^n$ or e^j in $j \rightarrow \infty$, we can estimate that $n_{\text{ex}} \propto \delta^{-1/n}$ or $-\ln \delta$, respectively. So the criteria for the Bose-Einstein condensate are

$$\text{if } \bar{E}_j \rightarrow j^n, \quad \text{then } \frac{1}{\delta^{1/n}} \ll \frac{1}{\alpha}, \quad (4.9)$$

$$\text{if } \bar{E}_j \rightarrow e^j, \quad \text{then } -\ln \delta \ll \frac{1}{\alpha}. \quad (4.10)$$

We show the relation between j and its number density corresponding to $\bar{E}_j \rightarrow j$ or e^j for $\delta = 10^{-4}$ and $\alpha = 10^{-8}$ in Fig. 3. Although both are the cases of the Bose-Einstein condensate, decays of n_j make a contrast in these cases.

We are interested in changes of physical quantities caused by the Bose-Einstein condensate. We show \bar{A} as a function of δ for the case $\alpha = 10^{-8}$ in Fig. 4. Surprisingly, a plateau appears for large δ while \bar{A} increases as $\delta \rightarrow 0$ following (3.14) or (3.18) for small δ . If we use the criteria (4.9) and (4.10), the Bose-Einstein condensate occurs for all δ in this diagram. Then, how can we understand this plateau?

We can discuss that \bar{A} in the plateau corresponds to the case where \bar{A} almost consists of the area spectrum $j = 1/2$, $\bar{A}_{1/2}$. The reason is as follows. To estimate the area \bar{A}_{ex} consisting of the area spectrum $j > 1/2$, we use

$$\frac{\bar{A}_{\text{ex}}}{\beta} = E_{\text{ex}} = n_{\text{ex}}\mu - \partial_\beta \ln Z_{\text{BE,ex}}. \quad (4.11)$$

If $\bar{E}_j \rightarrow j^n$ in $j \rightarrow \infty$, we have

$$n_{\text{ex}} \propto \delta^{-1/n}, \quad \partial_\beta \ln Z_{\text{BE,ex}} \propto \delta^{-(n+1)/n}, \quad (4.12)$$

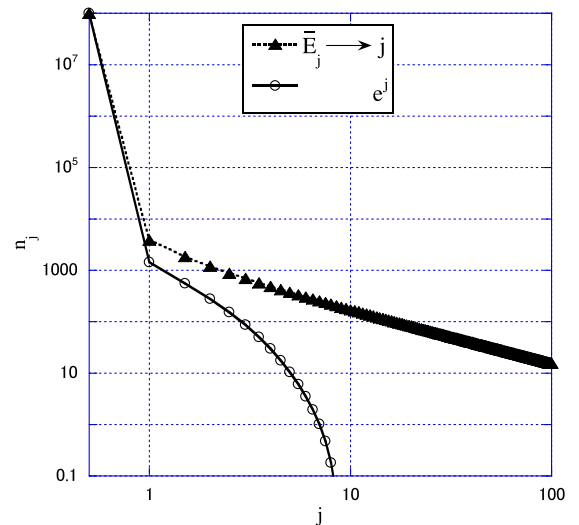


FIG. 3. The relation between j and its number density corresponding to $\bar{E}_j \rightarrow j$ and e^j for $\delta = 10^{-4}$, $\alpha = 10^{-8}$ and $\delta_h = 0$.

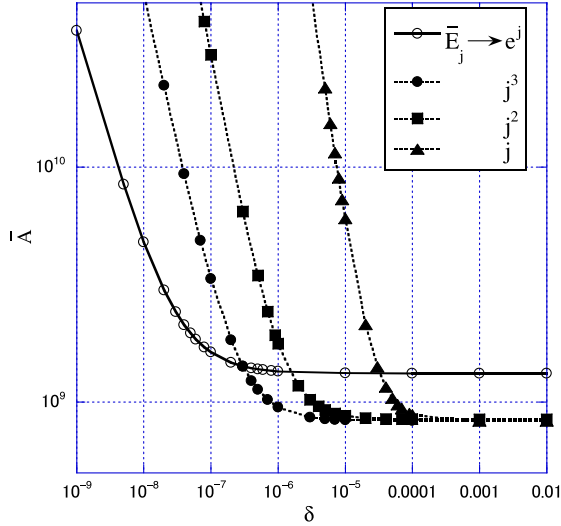


FIG. 4. We show \bar{A} as a function of δ for the cases $\alpha = 10^{-8}$ and $\delta_h = 0$.

where we used (3.13) and (3.14). So, if δ is small enough, first term of rhs in (4.11) can be negligible. Thus, we have

$$\bar{A}_{\text{ex}} \propto \delta^{-(n+1)/n}. \quad (4.13)$$

Similarly, we can consider the case $\bar{E}_j \rightarrow e^j$ in $j \rightarrow \infty$, and this case is included in the limit $n \rightarrow \infty$ in (4.13). So, the condition for $\bar{A}_{\text{ex}} \ll \bar{A}_{1/2} \propto \alpha^{-1}$ can be estimated as

$$\delta \gg \alpha^{n/(n+1)}. \quad (4.14)$$

We can find that this is consistent with the results in Fig. 4. This result is also newly revealed in this paper.

If we consider what observables in black hole physics are, we may adopt the criterion (4.14) as a condition for the Bose-Einstein condensate. When this condition is broken, \bar{A} shows rapid grow as $\delta \rightarrow 0$. If we can discuss this phenomena as a phase transition from the quantum black hole to the classical black hole, it is very interesting.

V. CONCLUSION AND DISCUSSION

We have investigated the puncture statistics based on the covariant area spectrum. First, we have considered Maxwell-Boltzmann statistics with a Gibbs factor for

punctures. If we assume the fugacity $z \neq 1$, we have reconfirmed the results in [34] that the correction term of the entropy-area law disappears for $z = e$. When we assume the fugacity $z = 1$, we have established formulas which relate physical quantities such as horizon area to the parameter characterizing holographic degrees of freedom using asymptotic form of the area spectrum in the large spin limit. We have also performed numerical calculations and obtained consistency with these formulas. From these results, we have obtained that the holographic bound is satisfied in the large area limit which is the extension of the previous research. We have found that the correction term of the entropy-area law can be proportional to the logarithm of the horizon area as it has been pointed out in other researches.

Second, we have also considered Bose-Einstein statistics and shown that above formulas are also useful in this case. By applying the formulas, we have understood intrinsic features of the Bose-Einstein condensate which correspond to the case when the horizon area almost consists of punctures in the ground state. We have shown that when this phenomena occurs, the area is approximately constant against the parameter δ characterizing the temperature. When this phenomena is broken, the area shows a rapid increase as $\delta \rightarrow 0$, which suggests the phase transition from quantum to classical area.

What should we consider as a next step? Although we have assumed that ρ is a dependent function of j , the validity should be checked by other method. For example, to reveal the property of ρ in the covariant area spectrum and the puncture statistics, it is important to investigate the Hawking radiation as in [40] which is one of our future work. It is also interesting to discuss possibility of the phase transition using covariant area spectrum as in [41]. In a long span, we should also investigate a covariant volume spectrum, which would lead us to the covariant loop quantum cosmology. This must be the interesting arena in the next decade.

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[1] For reviews, see, e.g. J. J. Halliwell, in *Quantum Cosmology and Baby Universes*, edited by S. Coleman, J. B. Hartle, T. Piran, and S. Weinberg (World Scientific, Singapore, 1991); C. Kiefer, *Quantum Gravity* (Clarendon Press, Oxford,

2004); D. H. Coule, *Classical Quantum Gravity* **22**, R125 (2005).

[2] A. Ashtekar, *Phys. Rev. Lett.* **57**, 2244 (1986); *Phys. Rev. D* **36**, 1587 (1987).

- [3] T. Jacobson and L. Smolin, *Nucl. Phys.* **B299**, 295 (1988).
- [4] C. Rovelli and L. Smolin, *Phys. Rev. D* **52**, 5743 (1995).
- [5] C. Rovelli and L. Smolin, *Nucl. Phys.* **B442**, 593 (1995); **456**, 753(E) (1995).
- [6] A. Ashtekar and J. Lewandowski, *Classical Quantum Gravity* **14**, A55 (1997).
- [7] J. F. Barbero G., *Phys. Rev. D* **51**, 5507 (1995); G. Immirzi, *Nucl. Phys. B, Proc. Suppl.* **57**, 65 (1997).
- [8] T. Thiemann, *Phys. Lett. B* **380**, 257 (1996).
- [9] M. Bojowald, *Living Rev. Relativ.* **8**, 11 (2005); A. Ashtekar, T. Pawłowski, and P. Singh, *Phys. Rev. D* **74**, 084003 (2006).
- [10] C. Rovelli, *Phys. Rev. Lett.* **77**, 3288 (1996).
- [11] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, *Phys. Rev. Lett.* **80**, 904 (1998); A. Ashtekar, J. Baez, and K. Krasnov, *Adv. Theor. Math. Phys.* **4**, 1 (2000).
- [12] A. Ashtekar, A. Corichi, and K. Krasnov, *Adv. Theor. Math. Phys.* **3**, 419 (1999).
- [13] M. Domagala and J. Lewandowski, *Classical Quantum Gravity* **21**, 5233 (2004); K. A. Meissner, *Classical Quantum Gravity* **21**, 5245 (2004).
- [14] A. Alekseev, A. P. Polychronakos, and M. Smedback, *Phys. Lett. B* **574**, 296 (2003); A. P. Polychronakos, *Phys. Rev. D* **69**, 044010 (2004).
- [15] A. Ghosh and P. Mitra, *Phys. Lett. B* **616**, 114 (2005); *Indian J. Phys.* **80**, 867 (2006); *Phys. Rev. D* **74**, 064026 (2006).
- [16] T. Tamaki and H. Nomura, *Phys. Rev. D* **72**, 107501 (2005); T. Tamaki, *Classical Quantum Gravity* **24**, 3837 (2007); T. Tanaka and T. Tamaki, *Eur. Phys. J. C* **73**, 2314 (2013).
- [17] For review, see, e.g. J. Natario and R. Schiappa, *Adv. Theor. Math. Phys.* **8**, 1001 (2004).
- [18] O. Dreyer, *Phys. Rev. Lett.* **90**, 081301 (2003); S. Hod, *Phys. Rev. Lett.* **81**, 4293 (1998).
- [19] T. Tamaki and H. Nomura, *Phys. Rev. D* **70**, 044041 (2004); H. Nomura and T. Tamaki, *Phys. Rev. D* **71**, 124033 (2005).
- [20] J. F. Barbero G. and E. J. S. Villaseñor, *Classical Quantum Gravity* **26**, 035017 (2009).
- [21] H. Sahlmann, *Phys. Rev. D* **76**, 104050 (2007); *Classical Quantum Gravity* **25**, 055004 (2008).
- [22] M. H. Ansari, *Nucl. Phys.* **B783**, 179 (2007); **795**, 635 (2008).
- [23] J. Engle, A. Perez, and K. Noui, *Phys. Rev. Lett.* **105**, 031302 (2010); J. Engle, K. Noui, A. Perez, and D. Pranzetti, *Phys. Rev. D* **82**, 044050 (2010); A. Perez and D. Pranzetti, *Entropy* **13**, 744 (2011).
- [24] E. Frodden, M. Geiller, K. Noui, and A. Perez, *J. High Energy Phys.* **05** (2013) 139.
- [25] E. Frodden, M. Geiller, K. Noui, and A. Perez, *Europhys. Lett.* **107**, 10005 (2014); B. A. Jibril, A. Mouchet, and K. Noui, *J. High Energy Phys.* **06** (2015) 145.
- [26] D. Pranzetti, *Phys. Rev. D* **89**, 104046 (2014).
- [27] J. Samuel, *Classical Quantum Gravity* **17**, L141 (2000); *Phys. Rev. D* **63**, 068501 (2001).
- [28] S. Alexandrov, *Phys. Rev. D* **65**, 024011 (2001).
- [29] S. Alexandrov and D. Vassilevich, *Phys. Rev. D* **64**, 044023 (2001).
- [30] S. Alexandrov, *Phys. Rev. D* **66**, 024028 (2002).
- [31] For review, see, e.g. S. Alexandrov, M. Geiller, and K. Noui, *SIGMA*, **8**, 055 (2012); S. Alexandrov and P. Roche, *Phys. Rep.* **506**, 41 (2011).
- [32] S. Alexandrov, arXiv:gr-qc/0408033.
- [33] A. Ghosh and A. Perez, *Phys. Rev. Lett.* **107**, 241301 (2011).
- [34] A. Ghosh, K. Noui, and A. Perez, *Phys. Rev. D* **89**, 084069 (2014); O. Asin, J. B. Achour, M. Geiller, K. Noui, and A. Perez, *Phys. Rev. D* **91**, 084005 (2015).
- [35] E. Frodden, A. Ghosh, and A. Perez, *Phys. Rev. D* **87**, 121503(R) (2013).
- [36] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, *Phys. Rev. D* **34**, 373 (1986).
- [37] S. Carlip, *Classical Quantum Gravity* **17**, 4175 (2000).
- [38] S. Das, P. Majumdar, and R. K. Bhaduri, *Classical Quantum Gravity* **19**, 2355 (2002).
- [39] A. Sen, *J. High Energy Phys.* **04** (2013) 156.
- [40] A. Barrau, X. Cao, K. Noui, and A. Perez, *Phys. Rev. D* **92**, 124046 (2015).
- [41] J. Mäkelä, *Phys. Rev. D* **93**, 084002 (2016).