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Towards the fundamental spectrum of the quantum Yang-Mills theory

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In this work we focus on the quantum Einstein-Yang-Mills sector quantized by the methods of loop quantum gravity. We point out the improved UV behavior of the coupled system as compared to pure quantum Yang-Mills theory on a fixed, classical background spacetime as was considered in a seminal work by Kogut and Susskind. Furthermore, we develop a calculational scheme by which the fundamental spectrum of the quantum Yang-Mills Hamiltonian can be computed in principle and by which one can make contact with the Wilsonian renormalization group, possibly purely within the Hamiltonian framework. Finally, we comment on the relationship of the fundamental spectrum to that of pure Yang-Mills theory on a (flat) classical spacetime.

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I. INTRODUCTION

The Hamiltonian approach to pure quantum Yang-Mills theory on Minkowski space was much developed by Kogut and Susskind [1]. These authors regularized the classical expression for the Yang-Mills Hamiltonian on a regular spatial lattice of cubic topology embedded in $\mathbb{R}^3$, which comes with a lattice length parameter ϵ as measured by the spatial Euclidean background metric induced by the Minkowski metric on spatial hypersurfaces of Minkowski space. The quantum Hamiltonian was written in terms of non-Abelian fluxes through the faces of the cubic cell complex dual to the lattice for the electric degrees of freedom and in terms of non-Abelian holonomies along the plaquette loops of the lattice. Furthermore, those authors assumed a representation of holonomies and fluxes on a Hilbert space of square integrable functions of the magnetic loop functions just introduced, where the natural Haar measure on the compact gauge group is used in order to define the Hilbert space measure.

While well defined at finite ϵ, the necessary continuum limit $\epsilon \to 0$ is problematic in this approach: Namely, the regularized Hamiltonian involves an inverse power of ϵ and thus blows up at fixed Yang-Mills coupling. This leads to the conclusion that the Yang-Mills coupling entering the Hamiltonian is to be considered a bare coupling that must be renormalized suitably in the continuum limit. Since the renormalization is, arguably, easier to study in the path integral formulation, the Hamiltonian approach to quantum Yang-Mills theory was basically dropped and research focused on the functional integral approach, whose underlying mathematical framework is the constructive Euclidean program [2–7]. Starting from the Euclidean action, not the Hamiltonian, involves an additional integral and thus in four spacetime dimensions does not involve ϵ explicitly. The well-established and very active research field of lattice quantum chromodynamics (LQCD) is the practical implementation of that program and has produced many spectacular results, see, e.g., [8,9], yet the existence of pure quantum Yang-Mills theory has not been proven. In fact, the Clay Mathematical Institute[1] has devoted one of its millennium prizes to this research topic. To circumvent these problems this paper does not deal with the Euclidean formulation at all. Furthermore, we leave the realm of Quantum Field Theory (QFT) on curved spacetime [10–13] completely and pass to quantum gravity, because we wish to examine here the old idea that quantum gravity itself resolves the UV divergences of QFT. We do this in the Hamiltonian approach to quantum gravity, one incarnation of which is loop quantum gravity (LQG) [14–16]. This approach is ideally suited to Yang-Mills theory, because the gravitational field, in its canonical formulation, can be viewed as a Yang-Mills theory for the gauge group $SU(2)$ with a very complicated interaction. Thus the quantization methods developed for Yang-Mills fields, in fact pioneered by Kogut and Susskind, can also be applied to the gravitational degrees of freedom, as has been done in [17].

Indeed, a rigorous Hilbert space representation can be found for the so-called holonomy flux algebra, in fact for any compact gauge group and any spacetime dimension, which consists of holonomies along one-dimensional paths and non-Abelian fluxes through two-dimensional surfaces (in $3+1$ spacetime dimensions). This is in fact very similar to the Kogut-Susskind program, but the difference is that in LQG there is no fixed lattice and dual cell complex; there is also no lattice regulator ϵ at all. Rather, one considers *all paths and all surfaces* in one big Hilbert space; that is to say, one considers all graphs and dual cell complexes. LQG

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[1] http://www.claymath.org/millenium-problems/yang-mills-and-mass-gap.

is therefore a continuum theory without a lattice cutoff. We see that in the corresponding quantum operator the factor $1/\epsilon$ of the Kogut-Susskind Hamiltonian is replaced by the $1/\ell_P$ where ℓ_P is the Planck length. At that level, therefore, there is no problem in taking the continuum limit. However, renormalization group ideas are still important as we see later on.

Just in order to avoid possible confusion from the outset, we mention here that LQG comes in two versions. In the first version one solves the constraints of the theory, which arise due to the spacetime diffeomorphism invariance of Einstein's theory, in the quantum theory [18,19]. In the second version one solves those constraints classically by gauge fixing the freedom to choose coordinates in terms of scalar matter fields (see, e.g., [20–23]). These two approaches are technically and conceptually very different, because in the first version the primary task is to solve the quantum constraints and to supply a Hilbert space structure on the resulting space of (distributional) solutions and it is a nontrivial task to find appropriate gauge invariant observables acting on it. There is no Hamiltonian in this first approach, because time translations are regarded as gauge transformations. In the second approach these tasks are already implemented classically. Furthermore, the classical construction automatically supplies a Hamiltonian that generates time evolution. In this paper we therefore follow the second route, specifically the choice of scalar matter considered in [24,25] as this brings us maximally close to the situation of pure Yang-Mills theory on Minkowski space.

The LQG Hilbert space, which was originally designed for the first approach, is necessarily nonseparable. This comes about because one considers the huge algebra of *all* fluxes and *all* holonomies, which in turn are needed if one wishes to implement the (spatial) diffeomorphism invariance of the theory in a (cyclic) representation of the holonomy—flux algebra [26,27]. On the other hand, classically, far fewer functions on the phase space suffice in order to separate all of its points; that is to say, much fewer paths and surfaces suffice. In [25,28–30] the observation was made that—since in the second approach one has fixed the (spatial) diffeomorphism invariance of the theory—one may indeed restrict to a much smaller algebra. For instance, if the topology of spacetime is that of $\mathbb{R}^4$ then it suffices to consider rectangular paths and surfaces along the coordinate axes and planes, respectively. A further reduction of the number of degrees of freedom is obtained by passing to an abstract infinite graph and dual cell complex, respectively, which have no information about their embedding into $\mathbb{R}^3$. The quantum theory is then formulated in terms of these abstract elementary holonomy and flux operators. The embedding scale reappears in the semiclassical limit in terms of coherent states [31] for the gravitational degrees of freedom and can be chosen as small as one wishes.

In this paper we therefore consider the approach of [25] to Einstein-Yang-Mills theory on the differential manifold $\mathbb{R}^4$ in the gauge fixed version of LQG[2] with scalar matter content and focus on the Yang-Mills contribution to the Hamiltonian, which then in the classical theory simply reads

$$H = \frac{1}{2Q^2}\int_{\mathbb{R}^3} d^3x \frac{q_{ab}}{\sqrt{\det(q)}}[\mathrm{Tr}(\underline{E}^a\underline{E}^b) + \mathrm{Tr}(\underline{B}^a\underline{B}^b)]. \tag{1.1}$$

Here $\underline{E}, \underline{B}$ denote the electric and magnetic Yang-Mills field, Q is the Yang-Mills coupling constant, and q_{ab} is the induced spatial metric on the Cauchy surface $\mathbb{R}^3$. The spatial indices are $a, b, c, .. = 1, 2, 3$ and the traces are taken in the adjoint representation of the Lie algebra $\mathfrak{g}$ of the Yang-Mills gauge group G, e.g., $su(N)$ for $G = SU(N)$.

The architecture of this paper is as follows:

In Sec. II we briefly review the quantization of (1.1); more details can be found in [17,25]. We also review the essentials of [1] and compare these two theories.

Section III reviews useful facts about the representation theory of $SU(3)$ (QCD gauge group) needed in Secs. IV and V, while analogous knowledge for $SU(2)$ (gravitational gauge group) is shifted to the appendix.

In Sec. IV we compute basic building blocks necessary in order to compute the background spectrum of (1.1) with fixed Minkowski background metric, that is, $q_{ab} = \delta_{ab}$, on a lattice of size ϵ, i.e., we treat the Kogut and Susskind situation.

In Sec. V we do the same, but with q_{ab} being a quantum operator on the LQG Hilbert space. The calculational steps performed here are the preparation for computing the fundamental spectrum of H on the tensor product Hilbert space corresponding to both geometry and matter degrees of freedom.

In Sec. VI we summarize our findings and elucidate the necessary steps for our future research.

II. REVIEW OF EINSTEIN-YANG-MILLS THEORY

In this chapter we recap elements of the classical and quantum Einstein-Yang-Mills theories. In the first section we review the classical canonical formulation and in the second we formulate the quantum theory using the techniques of LQG. We also review the derivation of the Kogut-Susskind lattice Hamiltonian on Minkowski space. Notice that our quantization makes use of the presence of additional scalar matter fields that do not explicitly appear in the Hamiltonian since they serve to fix the general coordinate freedom and therefore are "Higgsed away." See [20] for all the details.

[2]That is, the coordinate freedom is fixed but not the Yang-Mills-like gauge freedom.

A. Classical Einstein-Yang-Mills theory

The Yang-Mills action for a unitary gauge group G in general relativity is

$$S_{\mathrm{YM}} = -\frac{1}{4Q^2}\int_M d^4x\sqrt{|\det(g)|}g^{\mu\nu}g^{\rho\sigma}\underline{F}^I_{\mu\rho}\underline{F}^I_{\nu\sigma}, \tag{2.1}$$

where $\underline{F}$ is the curvature of the G connection, $\underline{A}$ and Q are the coupling constant, and $g^{\mu\nu}$ is the metric on the manifold M. The aim of this paper is to cast this action into canonical form. This is done using the Arnowitt-Deser-Misner formalism, the details of which can be found in [32]. The idea is to assume that M may be splitted as $M = \mathbb{R}\times S$. This foliation into spacelike hypersurfaces allows the replacement of the ten components of the spacetime metric by the six components of the induced Riemann metric q_{ab} of S and the three components of the shift vector N_a and the lapse function N. Also, the cotriad field e^i_a is transformed to the densitized triad

$$E^a_i = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e^j_b e^k_c = \sqrt{\det(q)}e^a_i, \tag{2.2}$$

which serves as the canonical pair on the gravitational phase space together with the extrinsic curvature,

$$K_{ab} = sgn(\det(e^j_c))K^i_a e^i_b. \tag{2.3}$$

Equations (2.2) and (2.3), together with the connection $A^i_a = \Gamma^i_a + K^i_a$, form the Asthekar-Barbero variables [33–36], where Γ^i_a is the spin connection of e^i_a.

In conjunction with the canonical pair from Yang-Mills theory $(\underline{A}^i_a, \frac{1}{Q^2}\underline{E}^a_i)$, where the first is the above-mentioned G connection and the second the associated electric field, one is set up to start working on $SU(2)\times G$. Because of the gauge fixing dynamically induced by additional matter fields, lapse and shift get frozen to $N=1, N^a=0$, respectively. After performing the Legendre transformation, one finds [14]

$$S_{\mathrm{YM}} = \frac{1}{Q^2}\int_{\mathbb{R}} dt\int_S d^3x\Bigg(\dot{\underline{A}}^I_a\underline{E}^a_I - \Bigg(-\underline{A}^I_t\underline{D}_a\underline{E}^a_I + N^a\underline{F}^I_{ab}\underline{E}^b_I + \frac{q_{ab}}{2\sqrt{\det(q)}}(\underline{E}^a_I\underline{E}^b_I + \underline{B}^a_I\underline{B}^b_I)\Bigg)\Bigg), \tag{2.4}$$

where $\underline{B}^a_I = \epsilon^{abc}\underline{F}^I_{bc}$ and $\underline{D}_a$ acts like the Levi-Civita connection on tensor indices. The contributions to the spatial diffeomorphism constraint and the Hamiltonian can be directly read off: the Hamiltonian is

$$H_{\mathrm{YM}} = \frac{q_{ab}}{2Q^2\sqrt{\det(q)}}(\underline{E}^a_I\underline{E}^b_I + \underline{B}^a_I\underline{B}^b_I). \tag{2.5}$$

B. Quantum Einstein-Yang-Mills theory

In this paper we construct a Hamiltonian for a quantum Einstein-Yang-Mills theory. As already stated, the methods of quantization (2.5) are those of loop quantum gravity. We present the construction separately for the Einstein term and the Yang-Mills term. Finally, we show how the classical Kogut-Susskind Hamiltonian emerges from the theory in the limit of a flat spacetime.

Let us stress again that we are working in the framework of deparametrized models: a suitable gauge fixing leads to a reduced phase spacetime that (when quantized via the methods of LQG) provides a model where all the constraints are solved; all operators are spacetime diffeomorphism invariant and physical states, respectively. In this formulation there is no Hamiltonian constraint, but a Hamiltonian operator [20,25,29,37].

Also, the idea of algebraic quantum gravity (AQG) is used, where we work solely on abstract graphs, which do not care about their embedding. Instead only the nodes and their connection among themselves are of interest. In our case the graphs are of cubic topology (i.e., a general vertex has six edges adjacent to it), which is very like the situation in lattice gauge theory. In this manner we follow the proposal of [28], meaning that physics now happens on such a given graph leaving it invariant, a feature in which AQG differs from the first route of LQG, where there is no Hamiltonian but an infinite number of constraints that must commute with each other on the kernel of the diffeomorphism constraint. The only known way to achieve this without anomalies in this sense is to let the Hamiltonian constraint act by adding new edges. By contrast, with only one Hamiltonian, there is no anomaly to worry about anymore and the quantization of the Hamiltonian can be done in the way that is customary in lattice gauge theory. With every edge e one associates an element $A(e)$ of $SU(2)$ for the gravitational sector and an element $\underline{A}(e)$ of the Yang-Mills gauge group G, as well as elements $E(e)$, $\underline{E}(e)$, respectively, for the corresponding Lie algebra. Hence in both cases there are the following algebraic relations, with Q being the coupling constant and f_{jkl} the structure constant of $SU(2)$ or G respectively:

$$[A(e), A(e')] = 0, \tag{2.6}$$

$$[E_j(e), A(e')] = i\hbar Q^2\delta_{e,e'}\tau_j/2A(e), \tag{2.7}$$

$$[E_j(e), E_k(e')] = i\hbar Q^2\delta_{e,e'}f_{jkl}E_l(e'). \tag{2.8}$$

A nice representation of this algebra is the infinite tensor product Hilbert space $\mathcal{H} = \bigotimes_e \mathcal{H}_e$, where on every edge $\mathcal{H}_e = L^2(G, d\mu_H(G))\otimes L^2(SU(2), d\mu_H(SU(2)))$ [25]. Here $A(e)$ is a unitary matrix valued operator and $E(e)$ an essential self-adjoint derivation operator. So, e.g., the action of $E(e)$ on a function f_e on e is

$$E_j(e)f_e(h) = i\hbar Q^2 \frac{d}{ds}(f_e(e^{s\tau_j/2}h))_{s=0}, \tag{2.9}$$

where τ_j are the generators of the corresponding Lie algebra. This choice gives a parallel to the concept of LQG. And although there is no strict derivation of an algebraic Hamiltonian, it appears sensible to take the quantum version of the operators derived in the LQG framework and use them in AQG. The derivation of those in LQG was first performed in [18,19] for the gravitational sector and in [17] for the Yang-Mills sector).

Considering all this, the gravitational Hamiltonian is set to

$$\hat{H}_{\mathrm{Einstein}}(v) = \hat{S}_E^{(1/2)}(v) - 2(1+\gamma^2)\hat{T}(v) \tag{2.10}$$

with

$$\hat{S}_E^{(r)}(v) = \frac{1}{N_v}\sum_{e_1\cap e_2\cap e_3=v}\frac{\epsilon(e_1,e_2,e_3)}{|L(v,e_1,e_2|}\sum_{\beta\in L(v,e_1,e_2)} \times \mathrm{tr}((\hat{A}(\beta)-\hat{A}(\beta)^{-1})A(e_3)[A(e_3)^{-1},\hat{V}_v^r]), \tag{2.11}$$

$$\hat{T}(v) = \frac{1}{N_v}\sum_{e_1\cap e_2\cap e_3=v}\epsilon(e_1,e_2,e_3)\mathrm{tr}\Big(\hat{A}(e_1)[\hat{A}(e_1)^{-1},\hat{K}] \times \hat{A}(e_2)[\hat{A}(e_2)^{-1},\hat{K}]\hat{A}(e_3)\Big[\hat{A}(e_3)^{-1},\sqrt{\hat{V}}\Big]\Big), \tag{2.12}$$

where $\hat{K} = [\hat{S}_E^{(1)},\hat{V}]$ and $\hat{S}_E^{(1)} = \sum_v \hat{S}_E^{(1)}$, N_v is the number of unordered triples of mutually distinct edges incident at v, and $L(v,e,e')$ is the set of minimal loops. These are all loops, which start at v along e and end at v along $(e')^{-1}$ and are minimal in the sense that there are no other loops with the same restrictions and fewer edges traversed. In our case, where one is restricted to the once and for all fixed cubic graph, the elementary loops are the plaquettes, consisting of four edges. $\hat{V}$ is the algebraic quantum volume operator,

$$\hat{V} = \lim_{N\to\infty}\sum_{I=1}^{N}\sqrt{\left|\frac{1}{3!}\epsilon(a,b,c)\hat{E}_i(S_I^a)\hat{E}_j(S_I^b)\hat{E}_k(S_I^c)\epsilon^{ijk}\right|}, \tag{2.13}$$

where the skew function ϵ is chosen such that it matches that of the embedding dependent Ashtekar-Lewandowski-volume operator of LQG [38] when the algebraic graph is embedded in a generic way (see [28] for further details). One can show that its spectrum has to be discrete and further analysis has been performed in greater detail in [39]. Consequently, the action of the Hamiltonian on an algebraic graph or others is quite involved and the solution of eigenstates cannot be computed analytically; however, it is numerically [40] and semiclassically [30] under good control. Some calculations have been done for the LQG Hamiltonian constraint, which maybe could transfer directly to the algebraic version. For further reading see, e.g., [41,42].

For the Yang-Mills Hamiltonian one sets

$$\hat{H}_{\mathrm{YM}}(v) = \frac{1}{2Q^2}(\hat{H}_E(v)+\hat{H}_B(v)) \tag{2.14}$$

with

$$\hat{H}_E(v) = \frac{1}{P_v}\sum_{e_1\cap e_2=v}\mathrm{tr}\Big(\hat{A}(e_1)\Big[\hat{A}(e_1)^{-1},\sqrt{\hat{V}}]\hat{A}(e_2)\Big[\hat{A}(e_2)^{-1},\sqrt{\hat{V}}\Big]\Big)\underline{\hat{E}}_J(e_1)\underline{\hat{E}}_J(e_2), \tag{2.15}$$

$$\hat{H}_B(v) = \frac{1}{T_v^2}\sum_{e_1\cap e_2\cap e_3=v}\sum_{e_4\cap e_5\cap e_6=v}\frac{\epsilon(e_1,e_2,e_3)}{|L(v,e_2,e_3)|}\frac{\epsilon(e_4,e_5,e_6)}{|L(v,e_5,e_6)|}\sum_{\beta\in L(v,e_2,e_3)}\sum_{\beta'\in L(v,e_5,e_6}} \times \mathrm{tr}\Big(\hat{\tau}_j\hat{A}(e_1)\Big[\hat{A}(e_1)^{-1},\sqrt{\hat{V}}\Big]\Big)\mathrm{tr}\Big(\hat{\tau}_j\hat{A}(e_4)\Big[\hat{A}(e_4)^{-1},\sqrt{\hat{V}}\Big]\Big)\mathrm{tr}(\underline{\hat{\tau}}_J\underline{\hat{A}}(\beta))\mathrm{tr}(\underline{\hat{\tau}}_J\underline{\hat{A}}(\beta')), \tag{2.16}$$

where P_v is the number of all pairs of edges incident at v, T_v is the number of all nontrivial triples of edges incident at v, and the ϵ-term is that of the volume operator. Note that as in the Kogut-Susskind case, while the Hamiltonian expressed in terms of lattice variables has the correct continuum limit when the lattice embedding becomes sufficiently fine, it is but one of infinitely many possible discretizations that have this property. For instance one could consider discretizations that also have next to next neighbor interaction terms.

For the moment one should also notice that the gravitational Gauss constraint and the Yang-Mills Gauss constraint have their algebraic quantum versions as well. Going over to the invariant subspace where these Gauss constraints are solved leads (as in LQG) to the fact that one needs to introduce intertwiners π of both gauge groups respectively on every vertex. The obtained subspace $\mathcal{H}_{\mathrm{kin}}^{\mathcal{G}}$ is commonly referred to in the literature as the space of spin-network functions

$$T_{\gamma,j_e,\pi_v}[A,\underline{A}] = \bigotimes_{v\subset\gamma}\underline{\pi}_v \otimes \pi_v \bigotimes_{e\subset\gamma} h^{j_e}(e) \otimes \underline{h}^{\underline{j}_e}(e), \quad (2.17)$$

where $h^{j_e}(e) = h^{j_e}(e)(A_e)$ corresponds to the irreducible representation of label j_e of the holonomy of $SU(2)$ and $\underline{h}^{\underline{j}_e}(e)$ respectively of the Yang-Mills gauge group G. For more information on these see Sec. III.

To compute the spectrum of the Hamiltonian one has to compute its matrix elements and their calculation is done in Sec. V. In the following the gauge group for the gravitational networks is of course $SU(2)$ and for the Yang-Mills gauge group we pick the case of QCD, i.e., $SU(3)$. This section finishes with a last remark on the Kogut-Susskind Hamiltonian. While there are a lot of ways to derive it from the Wilson action (see, e.g., [1,9]), having this Yang-Mills Hamiltonian of quantum gravity at hand gives an easy derivation of the Kogut-Susskind Hamiltonian, which should be seen as the classical limit of the theory. Hence we replace the general metric with the flat Euclidean one and only quantize the Yang-Mills field. After embedding the graph in Minkowski space with a sufficiently small lattice length ϵ, one arrives, still with only nearest neighbor interactions (as in the case of the Wilson action), indeed at a version of the Kogut-Susskind Hamiltonian,

$$\hat{H}_{KS} = \frac{1}{2Q^2\epsilon}\left(\sum_{e\in\gamma}\hat{E}_J(e)\hat{E}_J(e) + \sum_{\beta,\beta'\in\gamma}\mathrm{tr}(\tau_j\hat{A}(\beta))\mathrm{tr}(\tau_j\hat{A}(\beta'))\right). \quad (2.18)$$

This is not the form generally found in the literature (e.g., [1]), because for the derivation of the LQG version of (2.14) a different approximation scheme for the curvature of the G connection F_{ab} is used. The approximation used in [18,19] is $\mathrm{Im}(A(\beta)) \approx \epsilon^2 F^j_{ab}\tau_j + \mathcal{O}(\epsilon^4)$, while the other one—which is in case of a flat background metric equivalent—is $\mathrm{Re}(A(\beta)) \approx d_n + \epsilon^4 F^i_{ab}F^{ab}_i + \mathcal{O}(\epsilon^6)$. Kogut and Susskind used the latter one; however, in the case of a nontrivial background it is not applicable. In any case this second approximation leads to the addition of a constant, the dimension of the group matrices d_n, which is treated in LQCD as a simple energy shift. Going along this road one obtains

$$\hat{H}_{KS,\mathrm{lit}} = \frac{1}{2Q^2\epsilon}\left(\sum_{e\in\gamma}\hat{E}_J(e)\hat{E}_J(e) + \sum_{\beta\in\gamma}\mathrm{tr}(\hat{A}(\beta)) + \mathrm{tr}(\hat{A}(\beta)^\dagger) - 2d_n\right). \quad (2.19)$$

III. REPRESENTATION THEORY AND GRAPHICAL CALCULUS OF $SU(3)$

Loop quantum gravity and lattice gauge theory both very heavily depend on the representation theory of the corresponding gauge group. [$SU(2)$ for the gravitational sector and for the purpose of this article we restrict ourselves to the $SU(3)$ for the Yang-Mills field.] Brink and Satchler have introduced a formalism called graphical calculus [43] for $SU(2)$, which simplifies the manipulations one wants to perform on the coupled representations of the spin network by suppressing many of indices from the irreducible representations and makes the coupling of different links more obvious. There has also been a proposal for a graphical calculus in [44] for any Lie group but this works only in its defining representation, while for our purpose we want to combine different irreducible representations. The methods we use throughout this paper regarding the computations of the gravitational degrees of freedom have been introduced in [41]. This framework has accomplished the evaluation of the matrix elements of the Euclidian part of the Hamiltonian constraint from [18,19] and the matrix elements of its Lorentzian part in [42]. The matrix elements for the Euclidian and Lorentzian part have been found analytically modulo the matrix elements of the volume operator, which must be determined nonanalytically. To make this paper self-contained we provide a list of the most important identities of this $SU(2)$-related calculus in the appendix. In this chapter we aim at the construction of a similar calculus for the gauge group of $SU(3)$. For this purpose we revisit the representation theory of $SU(3)$ in the following section. The familiar reader may jump forward to Sec. III B.

A. Representation theory of $SU(3)$

In this section, we recall some general properties of the finite dimensional representations of the unitary, compact, and semisimple Lie group $SU(3)$ and construct its Clebsch-Gordan coefficients. We start by choosing a suitable basis for the Lie algebra $su(3)$ as in [45]. This Lie algebra has a real form and we may pick a basis $\{A_{i,k}\}$ (where i, $k = 1, 2, 3$), with the following commutation relations:

$$[A_{i,k}, A_{j,l}] = \delta_{k,j}A_{i,l} - \delta_{i,l}A_{j,k}. \quad (3.1)$$

These are subject to the restriction $A_{11} + A_{22} + A_{33} = 0$ and $A^+_{i,k} = A_{k,i}$, where the adjoint is taken in the respective representation. We now consider representations of these commutation and $*$ relations considered as an abstract Lie algebra. Out of this set one can construct two (so-called) weight operators,

$$H_1 = A_{11} - A_{22}, \tag{3.2}$$

$$H_2 = A_{22} - A_{33}. \tag{3.3}$$

Now given a finite dimensional representation (D, V) over the vector space V of $su(3)$ or equivalently $SU(3)$ [since any representation of $SU(3)$ corresponds to a unique one of $su(3)$ and vice versa, due to $SU(3)$ being simply connected], one can simultaneously diagonalize $D(H_1)$ and $D(H_2)$ as $[H_1, H_2] = 0$. A pair $j = (a, b) \in \mathbb{C}$ is called a weight for D if there exists a $v \neq 0$ in V such that

$$D(H_1)v = av, \tag{3.4}$$

$$D(H_2)v = bv. \tag{3.5}$$

Additionally j is called the highest weight, if for all weights j' of D and $\mu, \nu \geq 0$ holds

$$j - j' = \mu\alpha_1 + \nu\alpha_2, \tag{3.6}$$

where the α_i are roots [a nonzero pair $(\alpha_{i,1}, \alpha_{i,2}) \in \mathbb{C}^2$, such that $[H_j, Z_i] = \alpha_{i,j} Z_i$ with a nonzero $Z_i \in SU(3)$]. In the following the irreducible representation of the highest weight j is denoted by $D^{(j)}$.

According to the theorem of the highest weight [46] the following is true for an irreducible representation D of $SU(3)$.

(1) D is the direct sum of weight spaces.
(2) D has a unique highest weight $j = (a, b)$ with $a, b \in \mathbb{N}^+$.
(3) D and D' are equivalent $\Leftrightarrow j = j'$.

From this we may can also deduce the following: The dimension of the irreducible representation with highest weight $j = (a, b)$ is

$$d_j = \frac{1}{2} \cdot (a + 1)(b + 1)(a + b + 2). \tag{3.7}$$

A proof for this formula can be found, e.g., in [47].

We work with finite dimensional representations of $SU(3)$, which is thus completely reducible [48]. Consequently, the tensor product of these representations can be rewritten as the sum of irreducible representations,

$$D^{(j_1)} \otimes D^{(j_2)} = \sum_j \mu_j D^{(j)}. \tag{3.8}$$

Let the vector spaces on which these act be called V_j and choose orthonormal bases in these spaces. Then a basis for $V_{j_1} \otimes V_{j_2}$ is

$$\{e^{j_1}_{m_1} \otimes e^{j_2}_{m_2}\}$$

and equivalently for V_j $\{e^{j,s}_m\}$, where j labels the weight and $s = 1, \ldots, \mu_j$ is used to distinguish the multiplicities. These bases can be connected by a unitary matrix,

$$e^{j,s}_m = \sum_{m_1,m_2} \langle e^{j_1}_{m_1}, e^{j_2}_{m_2} | e^{j,s}_m \rangle e^{j_1}_{m_1} \otimes e^{j_2}_{m_2}, \tag{3.9}$$

where the entries of the matrix are called the Clebsch-Gordan coefficients of the tensor product. As they are elements of a unitary matrix, the following orthogonality relations hold:

$$\sum_{m_1,m_2} \langle e^{j,s}_m | e^{j_1}_{m_1}, e^{j_2}_{m_2} \rangle \langle e^{j_1}_{m_1}, e^{j_2}_{m_2} | e^{j',s'}_{m'} \rangle = \delta_{j,j'} \delta_{s,s'} \delta_{m,m'}, \tag{3.10}$$

$$\sum_{j,s,m} \langle e^{j_1}_{m_1}, e^{j_2}_{m_2} | e^{j,s}_m \rangle \langle e^{j,s}_m | e^{j_1}_{m'_1}, e^{j_2}_{m'_2} \rangle = \delta_{m_1,m'_1} \delta_{m_2,m'_2}. \tag{3.11}$$

To construct these Clebsch-Gordan coefficients explicitly, we follow the formalism developed by Pluhař *et al.* in [49,50]. It is useful to introduce additional linear combinations of the $A_{i,j}$. In addition to H_1 and H_2 one introduces the following operators: The two Casimir operators

$$F_2 = \frac{3}{2} \sum_{i,k} A_{i,j} A_{j,i}, \tag{3.12}$$

$$F_3 = 9 \sum_{i,j,k} A_{i,j} A_{j,k}, \tag{3.13}$$

which, in the $D^{(j)}$ representation, have the eigenvalues

$$f_2 = (a + b + 3)(a + b) - ab, \tag{3.14}$$

$$f_3 = (a - b)(2a + b + 3)(a + 2b + 3). \tag{3.15}$$

Also, let us look at two subalgebras, one isomorphic to $su(2)$,

$$I_z = \frac{1}{2}(A_{11} - A_{22}), .I_+ = A_{12} \quad \text{and} \quad I_- = A_{21}. \tag{3.16}$$

There exist two eigenvalues for the group $SU(2)$, which we call isospin i (from the total angular momentum operator I^2) and isospin projection i_z (from the operator I_z). Also, there is a different subalgebra isomorphic to $su(2)$,

$$\Lambda_z = A_{11} - A_{33}, .\Lambda_+ = \sqrt{2}(A_{12} - A_{23}) \quad \text{and}$$
$$\Lambda_- = \sqrt{2}(A_{21} + A_{32}), \tag{3.17}$$

the eigenvalues of which are labeled $\lambda_0, \lambda_{0,z}$.

Both subalgebras contain a linear combination of the weight operators. Thus, their quantum numbers i, λ_0 can at most be $i_0 = \frac{1}{2}a$ and $\lambda_0 = a + b$, respectively [49]. The eighth independent operator is

$$Y = \frac{1}{3}(A_{11} + A_{22} - 2A_{33}), \tag{3.18}$$

called the hypercharge operator, whose eigenvalues y can be maximally $y_0 = \frac{1}{3}(a + 2b)$. This operator comes from particle physics where it unifies isospin and flavor into a single charge. Y is just a linear combination of the I_z and Λ_z and thus the group, spanned from the latter operators, is, in principle, redundant. Hypercharge and isospin projection are weight components for $SU(3)$.

Now one has to find how many quantum numbers are needed in general to describe a state in the vector space V of an irreducible highest-weight representation $D^{(j)}$. With $su(n)$ being a complex, semisimple Lie algebra one can do a splitting in the Cartan subalgebra $\mathfrak{h}$, which is the maximal sub-Lie algebra of all Abelian subalgebras, consisting of semisimple elements. Thus,

$$su(n) = \mathfrak{h} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-, \tag{3.19}$$

where $\mathfrak{g}_\pm$ are the subalgebras corresponding to positive/negative roots with respect to a choice of simple positive roots. While $\mathfrak{h}$ has dimension $n-1$, $\mathfrak{g}_\pm$ have dimension $\frac{n(n-1)}{2}$. Every irreducible highest-weight representation is cyclic, i.e., there exists a nontrivial vector $v \in V$, which is a weight vector for j, with $D(\mathfrak{g}_+)v = 0$ and the smallest subspace containing v is all of V. The cyclic highest-weight representation depends on r quantum numbers, where r is the rank of the Lie algebra. These quantum numbers correspond to the highest-weight vector eigenvalues of the Cartan subalgebra generators and the occupation numbers of the generators of $\mathfrak{g}_-$, which are thus $\frac{n(n-1)}{2}$ many.

So now for $n = 3$ one may see that an additional quantum number next to the two weights i_z and y from the Cartan generators I_z and Y is needed. As the Casimir of the $su(2)$-subgroup I^2 commutates with both, it is convenient to use it.

Moreover, for a general rank r semisimple Lie algebra the highest-weight labels (here a, b) are in one-to-one correspondence with the eigenvalues of the r algebraically independent Casimirs of rank $2, .., r+1$ (here F_2, F_3); hence F_2, F_3, I_z, Y, I^2 provides a maximally commuting set of self-adjoint operators characterizing the irreducible representation completely.

Now one labels the basis states of $D^{(j)}$ with hypercharge y, isospin i, and isospin projection i_z as $|(a,b),(y,i,i_z)\rangle \equiv |j,m\rangle$. To reduce the product $D^{(j_1)} \otimes D^{(j_2)}$ one has to deal with the multiplicity factors. These contribute nontrivially here [in contrast to $SU(2)$], as can be seen very easily by looking at the corresponding sets of commutating operators. While there should be ten commutating operators in the representation of $D^{(j_1)} \otimes D^{(j_2)}$, namely, $(F_2, F_3, I_z, Y, I^2)^{(1)}, (F_2, F_3, I_z, Y, I^2)^{(2)}$, after looking at the decomposition into irreducible representations there seem to be only nine commutating ones: $(F_2, F_3, I_z, Y, I^2, F_2^{(1)}, F_3^{(1)}, F_2^{(2)}, F_3^{(2)})$. This strange occurrence is solved by introducing an additional operator S, which is a Casimir operator for the Lie algebra generated by $D^{(j_1)}(X) \otimes 1_{D^{(j_2)}} + 1_{D^{(j_1)}} \otimes D^{(j_2)}(X), X \in su(3)$, and the s-classified reduced states, which are solutions to the eigenvalue problem

$$S(\{A\}_1, \{A\}_2)|(j_1, j_2), j, m, s\rangle = s|(j_1, j_2), j, m, s\rangle, \tag{3.20}$$

where we define

$$S(\{A\}_1, \{A\}_2) = 27\sum_{i,j,k}(A_{i,j;1}A_{j,k;2}A_{k,i;2} - A_{i,j;2}A_{j,k;1}A_{k,i;1}) - 2F_{3;2} + 2F_{3;1}. \tag{3.21}$$

This operator is seen to fulfil some symmetry relations when acting on $D_{j_1} \otimes D_{j_2} \otimes D_{j_3}$,

$$\begin{aligned} S(\{A\}_1, \{A\}_2) &= -S(\{A\}_2, \{A\}_1) \\ &= -S(\{A\}_1, \{A\}_3) \\ &= -S(\{\bar{A}\}_1, \{\bar{A}\}_2), \end{aligned} \tag{3.22}$$

where $D^{(j_3)}$ stands for the coupled representation and the $\bar{A}_{ij} \coloneqq -A_{ij}$ define the generators of the conjugate (i.e., contragredient) representation. Finally, these states have a phase ambiguity that can be resolved by setting

$$\langle j_1, j_2\lambda_{0;2}, \lambda_{0,z;2}|j_1, j_2, j_3, s\rangle > 0. \tag{3.23}$$

It should be noted, however, that the s are in general neither integral nor rational. Pluhař *et al.* [49] have proposed a computational algorithm, where for a given set of highest weights the matrix $S(\{A\}_1, \{A\}_2)$ is finite dimensional. With the last two equations it can be shown that the Clebsch-Gordan coefficients $\langle j_1, m_1, j_2, m_2|(j_1, j_2), j_3, m_3, s\rangle$, which couple the two representations j_1, j_2 to the resulting third j_3, while $m_1 + m_2 = m_3$, fulfil the following symmetry relations [49]:

$$\begin{aligned} &\langle j_1, m_1, j_2, m_2|(j_1, j_2), \bar{j}_3, \bar{m}_3, s\rangle \\ &= \langle j_2, m_2, j_1, m_1|(j_1, j_2), \bar{j}_3, \bar{m}_3, \bar{s}\rangle(-)^{j_1+j_2+j_3} \\ &= \langle(j_1, m_1, j_3, m_3|(j_1, j_2), \bar{j}_2, \bar{m}_2, \bar{s}\rangle(-)^{j_1+m_1}\sqrt{d_{j_2}/\mathrm{d}_{j_3}} \\ &= \langle \bar{j}_1, \bar{m}_1, \bar{j}_2, \bar{m}_2|(j_1, j_2), j_3, m_3, \bar{s}\rangle(-)^{j_1+j_2+j_3}, \end{aligned} \tag{3.24}$$

with $d_j = \dim((a,b))$ being the dimension of the space on which the irreducible representation corresponding to highest weight (a,b) lives. Also, the following abbreviations have been introduced:

$$\bar{j} = (b,a), .\bar{m} = (-y, i, -i_z) \quad \text{and} \quad \bar{s} = -s$$
$$(-)^j = (-1)^{a+b} \quad \text{and} \quad (-)^m = (-1)^{\frac{3}{2}y + i_z}. \tag{3.25}$$

B. Graphical calculus of $SU(3)$

We now develop a method to simplify computations involving the gauge group $SU(3)$. To the best of our knowledge, the graphical calculus developed here for $SU(3)$, while building on the one developed for $SU(2)$, is novel. We start by defining the so-called s-classified $3j$-Wigner symbol, an object that represents the symmetry relations of the Clebsch-Gordan coefficients in an easy way [50],

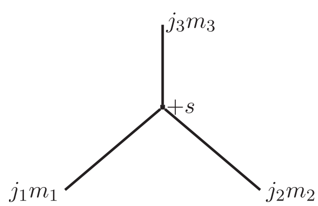
$$\begin{pmatrix} j_1 & j_2 & j_3 & s \\ m_1 & m_2 & m_3 & \end{pmatrix} = \langle j_1, m_1, j_2, m_2 | (j_1, j_2), \bar{j}_3, \bar{m}_3, s \rangle \frac{(-)^{\bar{j}_3 + \bar{m}_3}}{\sqrt{d_{\bar{j}_3}}}. \tag{3.26}$$

The symmetry relations from the last section (3.24) become

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 & s \\ m_1 & m_2 & m_3 & \end{pmatrix} &= \begin{pmatrix} j_2 & j_1 & j_3 & s \\ m_2 & m_1 & m_3 & \end{pmatrix} (-)^{j_1+j_2+j_3} \\ &= \begin{pmatrix} j_1 & j_3 & j_2 & s \\ m_1 & m_3 & m_2 & \end{pmatrix} (-)^{j_1+j_2+j_3} \\ &= \begin{pmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 & \bar{s} \\ \bar{m}_1 & \bar{m}_2 & \bar{m}_3 & \end{pmatrix} (-)^{j_1+j_2+j_3}. \end{aligned} \tag{3.27}$$

From this, it is apparent that the s-classified $3j$ symbols are invariant under even permutations and pick up a sign of $(-)^{j_1+j_2+j_3}$ for odd permutations. The usefulness of this symbol lies in the fact that any coupling of N representations can be expressed via $3j$ symbols. The aim now is to construct a graphical representation that allows one to represent multiple $3j$ symbols and their distinct coupling (e.g., the s-classified $6j$ symbols). We choose our notation such that it closely resembles the established calculus of [43]. The graphical representation of the s-classified Wigner $3j$ symbol is a node, where the three representations are joined in, which are represented as lines,

$$\begin{pmatrix} j_1 & j_2 & j_3 & s \\ m_1 & m_2 & m_3 & \end{pmatrix} =$$



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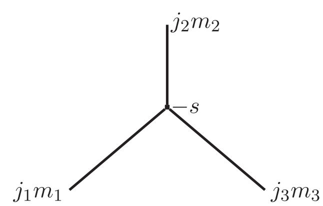
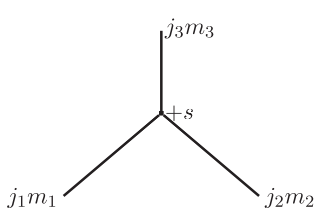
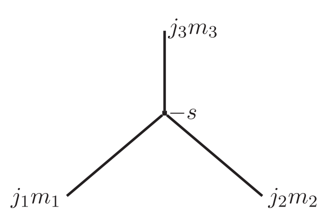


(3.28)

Here the $+$ sign means that the elements of the $3j$ are ordered in an anticlockwise orientation. Equivalently a $-$ sign indicates a clockwise orientation, e.g., a symmetry relation for the $3j$ is



$= (-)^{j_1+j_2+j_3}$

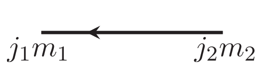


(3.29)

Additionally, arrows are introduced on the lines to indicate the "metric tensor." A line with no arrows means

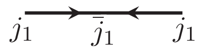
$= \delta_{j_1,j_2}\delta_{m_1,m_2}$

while a line with an arrow denotes the $1j$ symbol,



$$= \delta_{j_1,\bar{j}_2} \begin{pmatrix} j_1 \\ m_1, m_2 \end{pmatrix} = \delta_{\bar{j}_1,j_2}\delta_{\bar{m}_1,m_2}(-)^{j_1+m_1} \tag{3.30}$$

In the following we suppress the magnetic quantum numbers in the pictures. Having multiple arrows on one line, one can realize that (as well as for other orientations of the two arrows)



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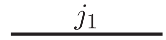
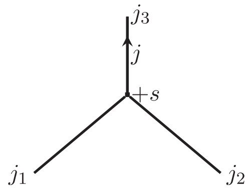


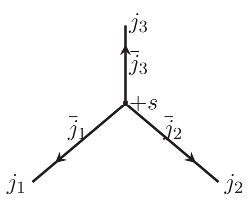
(3.31)

Given all of this we may calculate further: A contraction of $1j$ and $3j$ is

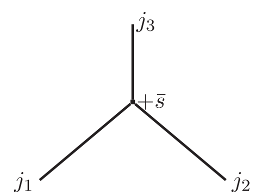


$$= \sum_m \begin{pmatrix} j_1 & j_2 & j & s \\ m_1 & m_2 & m & \end{pmatrix} \begin{pmatrix} j_3 \\ m_3, m \end{pmatrix} \delta_{j_3,\bar{j}}$$
$$= \sum_m \begin{pmatrix} j_1 & j_2 & j & s \\ m_1 & m_2 & m & \end{pmatrix} \delta_{m,\bar{m}_3} \delta_{j_3,\bar{j}} (-)^{j_3+m_3}$$
$$= \begin{pmatrix} j_1 & j_2 & \bar{j}_3 & s \\ m_1 & m_2 & \bar{m}_3 & \end{pmatrix} (-)^{j_3+m_3} \delta_{j_3,\bar{j}} \tag{3.32}$$

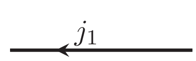
Similarly we can write



$$= \sum_{m_1',m_2',m_3'} \begin{pmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 & s \\ m_1' & m_2' & m_3' & \end{pmatrix} \begin{pmatrix} j_1 \\ m_1, m_1' \end{pmatrix} \begin{pmatrix} j_2 \\ m_2, m_2' \end{pmatrix} \begin{pmatrix} j_3 \\ m_3, m_3' \end{pmatrix}$$
$$= \sum_{m_1',m_2',m_3'} \begin{pmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 & s \\ m_1' & m_2' & m_3' & \end{pmatrix} \delta_{m_1',\bar{m}_1} \delta_{m_2',\bar{m}_2} \delta_{m_3',\bar{m}_3} (-)^{\sum_i j_i+m_i}$$
$$= \begin{pmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 & s \\ \bar{m}_1 & \bar{m}_2 & \bar{m}_3 & \end{pmatrix} (-)^{j_1+j_2+j_3} (-)^{m_1+m_2+m_3} \tag{3.33}$$
$$= \begin{pmatrix} j_1 & j_2 & j_3 & \bar{s} \\ m_1 & m_2 & m_3 & \end{pmatrix} =$$



where we have used that $(-)^{m_1+m_2+m_3} = 0$. In the following one uses the abbreviation,



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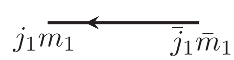

(3.34)

and thus only writes one index to each line from now on. For lines without arrow it indicates the highest weights of its irreducible representation, and if the line has an arrow it indicates the highest weight of the representation where the arrow points towards. Also, the arrows can be changed by dualizing the j,

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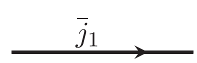
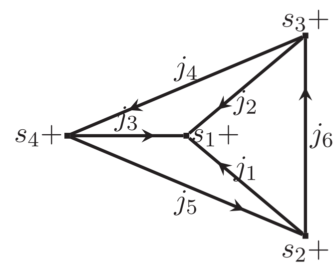


(3.35)

In order to represent more complex structures, lines can be joined as long as they carry the same highest weight. Note that the lines also carry a distinct group element. Joining them means that the magnetic quantum numbers are set equal and summed over. In the following these numbers are omitted in the graphs as already stated. With this definition one is, for example, able to represent the s-classified $6j$ symbol, an object defined in the following way (similar to [50]):

$$\begin{Bmatrix} j_1 & j_2 & j_3 & \\ j_4 & j_5 & j_6 & \\ s_1 & s_2 & s_3 & s_4 \end{Bmatrix} = \sum_{\{m\}} (-)^{\sum_i j_i + m_i} \begin{pmatrix} j_1 & j_2 & j_3 & s_1 \\ m_1 & m_2 & m_3 & \end{pmatrix} \begin{pmatrix} \bar{j}_1 & j_5 & \bar{j}_6 & s_2 \\ \bar{m}_1 & m_5 & \bar{m}_6 & \end{pmatrix} .$$

$$\cdot \begin{pmatrix} \bar{j}_4 & \bar{j}_2 & j_6 & s_3 \\ \bar{m}_4 & \bar{m}_2 & m_6 & \end{pmatrix} \begin{pmatrix} j_4 & \bar{j}_5 & \bar{j}_3 & s_4 \\ m_4 & \bar{m}_5 & \bar{m}_3 & \end{pmatrix} = \tag{3.36}$$



This object has a lot of symmetries at hand, so, e.g., it holds

$$\begin{Bmatrix} j_1 & j_2 & j_3 & \\ j_4 & j_5 & j_6 & \\ s_1 & s_2 & s_3 & s_4 \end{Bmatrix} = \begin{Bmatrix} \bar{j}_2 & \bar{j}_1 & \bar{j}_3 & \\ j_5 & j_4 & j_6 & \\ s_1 & s_3 & s_2 & s_4 \end{Bmatrix} = \begin{Bmatrix} \bar{j}_1 & \bar{j}_3 & \bar{j}_2 & \\ j_4 & j_6 & j_5 & \\ s_1 & s_2 & s_4 & s_3 \end{Bmatrix} = \begin{Bmatrix} j_4 & \bar{j}_5 & \bar{j}_3 & \\ j_1 & \bar{j}_2 & \bar{j}_6 & \\ s_4 & s_3 & s_2 & s_1 \end{Bmatrix} = \begin{Bmatrix} \bar{j}_1 & \bar{j}_2 & \bar{j}_3 & \\ \bar{j}_4 & \bar{j}_5 & \bar{j}_6 & \\ \bar{s}_1 & \bar{s}_2 & \bar{s}_3 & \bar{s}_4 \end{Bmatrix} . \tag{3.37}$$

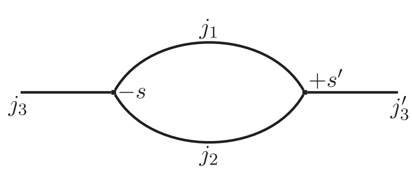
Also, for such a closed diagram (meaning that no open links remain) the object infers the invariance of the change of $+ \leftrightarrow -$, since every link obviously meets exactly two nodes, and $(-)^{2j} = 1$, because, recalling the theorem of the highest weight, $j = (a, b)$ with $a, b \in \mathbb{N}$.

Important relations in the theory of group representations are the two orthogonality relations (3.10) and (3.11) Their form follows from the very definition of the $3j$ symbols and the fact that they are real,

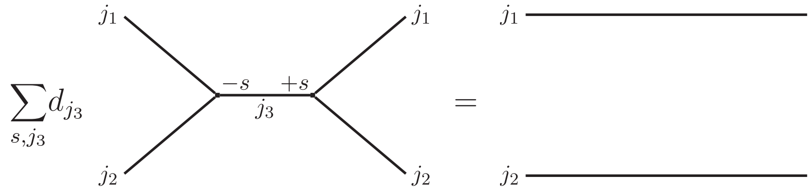
$$\sum_{m_1, m_2} \begin{pmatrix} j_1 & j_2 & j_3 & s \\ m_1 & m_2 & m_3 & \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' & s' \\ m_1 & m_2 & m_3' & \end{pmatrix} = \frac{1}{d_{\bar{j}_3}} \delta_{j_3, j_3'} \delta_{m_3, m_3'} \delta_{s, s'}$$

$$\sum_{j_3, m_3, s} \begin{pmatrix} j_1 & j_2 & j_3 & s \\ m_1 & m_2 & m_3 & \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 & s \\ m_1' & m_2' & m_3 & \end{pmatrix} = \frac{1}{d_{\bar{j}_3}} \delta_{m_1, m_1'} \delta_{m_2, m_2'} .$$

Graphically, these orthogonality relations can be encoded as



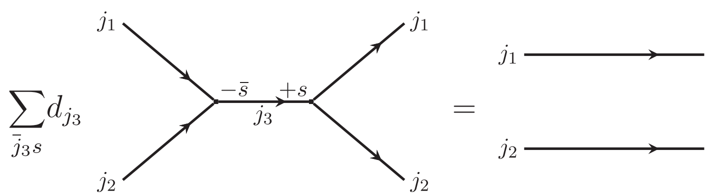
$$= \frac{\delta_{s,s'}}{d_{j_3}} \tag{3.38}$$



(3.39)

It should be noted at this point that the sum over s goes over all the solutions from (3.20) and is highly dependent on the coupled weights j_1, j_2, and j_3. While j_3 itself has to be chosen such that the three representations together form a triad [as for $SU(2)$] [48,51,52], i.e., if j_3 is inside the set $\Pi_{j_1} + j_2$, with Π_{j_1} denoting the set of all weights of the corresponding representation with highest weight j_1.

One can immediately see that the expression of the second orthogonality with arrows on the links is stated as

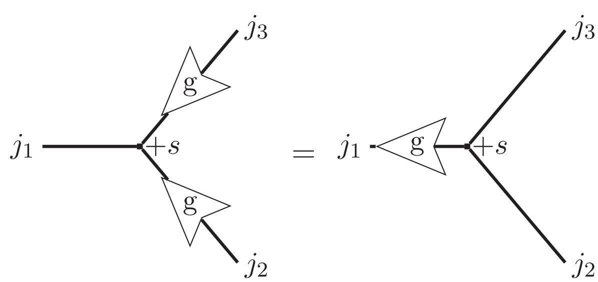


It is now obvious that transforming the algebraic expression of a graph alters its distinct representation, such that there also must exist some rules for transforming the graphs directly. We have already seen that, e.g., the arrows can be changed in their direction, by going from weight $j = (a, b)$ to $\bar{j} = (b, a)$. Also, a line with two arrows is equivalent to a line with no arrows. Furthermore, at a node one can add and remove arrows of the same direction on each line at the same time, while only changing the node internal index $s \to \bar{s}$.

Since one has for any general Lie group [48] that

$$\sum_{m_1' m_2'} \langle e^{j_3,s}_{m_3} | e^{j_1}_{m_1} e^{j_2}_{m_2} \rangle D^{(j_1)}_{m_1' m_1}(g) D^{(j_2)}_{m_2' m_2}(g) = \sum_{m_3'} \langle e^{j_3,s}_{m_3'} | e^{j_1}_{m_1} e^{j_2}_{m_2} \rangle D^{(j_3)}_{m_3' m_3}(g),$$

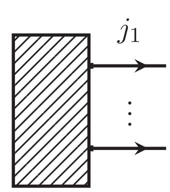
this translates as a transformation rule for our graphical calculus,



(3.40)

We now look at further rules, which change the lines and their coupling itself. For this purpose we define objects equivalent to the $SU(2)$ jm coefficients from [53], which are blocks of connected nodes with an arrow on each line, whose explicit internal structure is of no importance. They have n external lines with label $j_1 \ldots j_n$. Their graphical representation is

$$F_n \begin{pmatrix} j_1 & \ldots & j_n \\ m_1 & \ldots & m_n \end{pmatrix} =$$

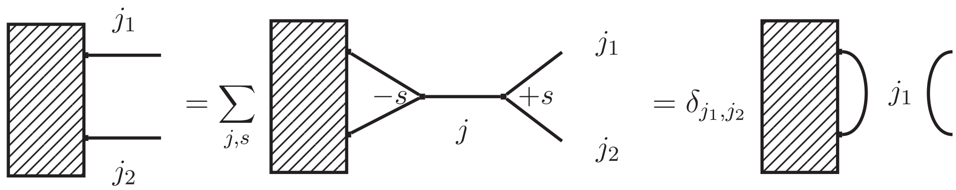


(3.41)

Using the orthogonality relations from above, a lot of manipulation on these external lines can be done. First, one has to notice that a block with only one external line, i.e., $F_1 \begin{pmatrix} j \\ m \end{pmatrix}$, is equivalent to a scalar times a Clebsch-Gordan coefficient with two labels equal to 0 and hence 0 itself, if not $j = m = 0$,

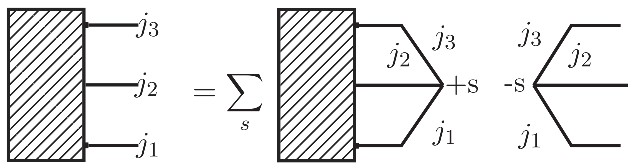
$$F_1\begin{pmatrix} j \\ m \end{pmatrix} = F_1\begin{pmatrix} 0 \\ 0 \end{pmatrix}\delta_{j,0}\delta_{m,0} = \begin{pmatrix} 0 & 0 & j \\ 0 & 0 & m \end{pmatrix}\text{const.} \tag{3.42}$$

This and the second orthogonality relation (3.39) on an F_2 coefficient leads to



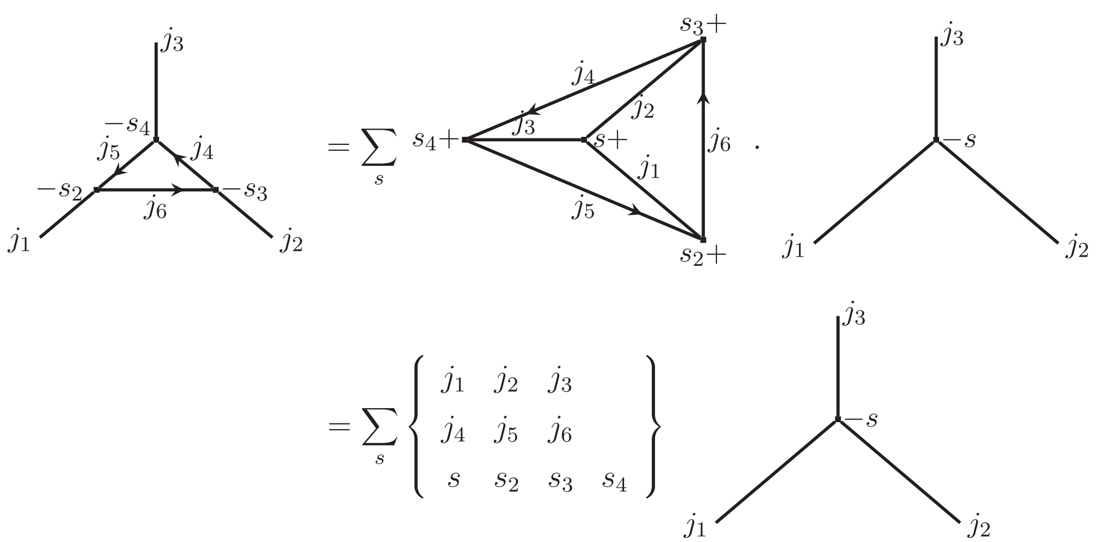
(3.43)

since the one connection link vanishes and the node reduces to a $1j$ symbol and thus the sum over s reduces to a $\delta_{s,2f_3(j_1)}$. With a similar calculation and using (3.43) we arrive at



(3.44)

With this at hand, all the tools of a graphical calculus necessary to simplify calculations involving the gauge group $SU(3)$ are provided. Before we dive into the computations of the matrix elements of the quantum Yang-Mills Hamiltonian, we provide a final example. The following structure is encountered numerous times in the remainder of this article:



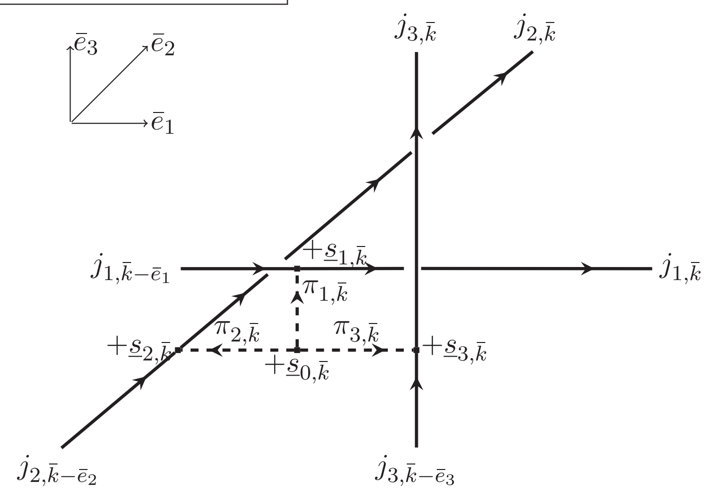
(3.45)

IV. EINSTEIN-YANG-MILLS THEORY IN THE KOGUT-SUSSKIND CASE

In this section we present the results, when applying the developed methods in the case of the background spectrum of the Kogut-Susskind Hamiltonian in flat space. In this work we do not focus on any analytical solvable problem, e.g., the one-plaquette graph, whose eigenstates are given in terms of Mathieu functions [54] in the case of $U(1)$ or $SU(2)$ gauge theory [55–57]. Instead we concentrate on the physically interesting case of multiple-plaquette problems, which so far could be tackled using numerical investigations. A lot of work has been done on this; see, e.g., [58–63] and many more. The most promising approach up to today is still to calculate the matrix elements and continue afterwards with numerical simulations. For this reason this section presents the exact calculation of said matrix elements for further—yet to be done—computations.

The calculation is done in the notation of spin networks, since this basis has certain advantages, e.g., the first term, consisting of the Casimir operators, diagonalizes here and gives the corresponding quadric Casimir $C_2(j)^2$ of the group [59]. Furthermore (hence in the Kogut-Susskind formalism one deals exclusively with it), a three-dimensional spatial cubic lattice is considered. Thus, at each vertex six links meet and the first question to answer is how to choose the intertwiner at this node, which couples all six j's to a resulting seventh that vanishes. There are multiple ways to do this and choosing a specific one corresponds to the choice of a basis. Here we take the pairs of parallel edges (say, e.g., in the $\bar{e}_1$-direction) and couple these to a resulting third (e.g., π_1). At the end we couple all three new representations π_1, π_2, π_3 to a vanishing fourth. This is independent of the gauge group and afterwards one single node looks as follows:

$$|\nu(\{\pi\}_{\bar{k}}\,;\{j\}_{\bar{k}}\,;\{s\}_{\bar{k}})\rangle =$$

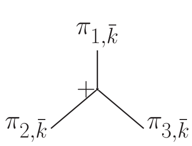


$$= \left|\nu\left(\pi_{1,\bar{k}}, \pi_{2,\bar{k}}, \pi_{3,\bar{k}}; j_{1,\bar{k}}, j_{2,\bar{k}}, j_{3,\bar{k}}, j_{1,\bar{k}-\bar{e}_1}, j_{1,\bar{k}-\bar{e}_2}, j_{1,\bar{k}-\bar{e}_3}; s_{0,\bar{k}}, s_{1.\bar{k}} s_{2,\bar{k}}, s_{3.\bar{k}}\right)\right\rangle \tag{4.1}$$

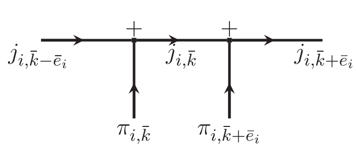
For $SU(2)$ of course all the s vanishes and thus is omitted. Our notation is chosen such that every edge is associated with its direction $\bar{e}_i$ and one point on the lattice $\bar{k}$. In total we write for the corresponding group element $\hat{A}_{i,\bar{k}}$. The group elements themselves however are not written explicitly. If one recalls formula (3.40) one sees that when multiplying two representations of the same group element (as is done when acting with the plaquette part of the Kogut-Susskind Hamiltonian) one can shift it to the coupled representation. In this manner, one sees easily that one always ends up with the same lattice one started with (regarding the group elements), only its distinct irreducible representation will have changed. Since this concept translates to all the following calculations, all the corresponding group elements are obviously omitted in the graphs.

Also, the lines, which are dashed in the picture, are those that are infinitesimally small (like those of $\pi_{i,\bar{k}}$), due to existing only at the vertex itself (and of course not carrying a group element).

To fix the orientation, we choose $\forall i \in \{1,2,3\}$, $\forall\ \bar{k} \in \mathbb{Z}^3$,



and



(4.2)

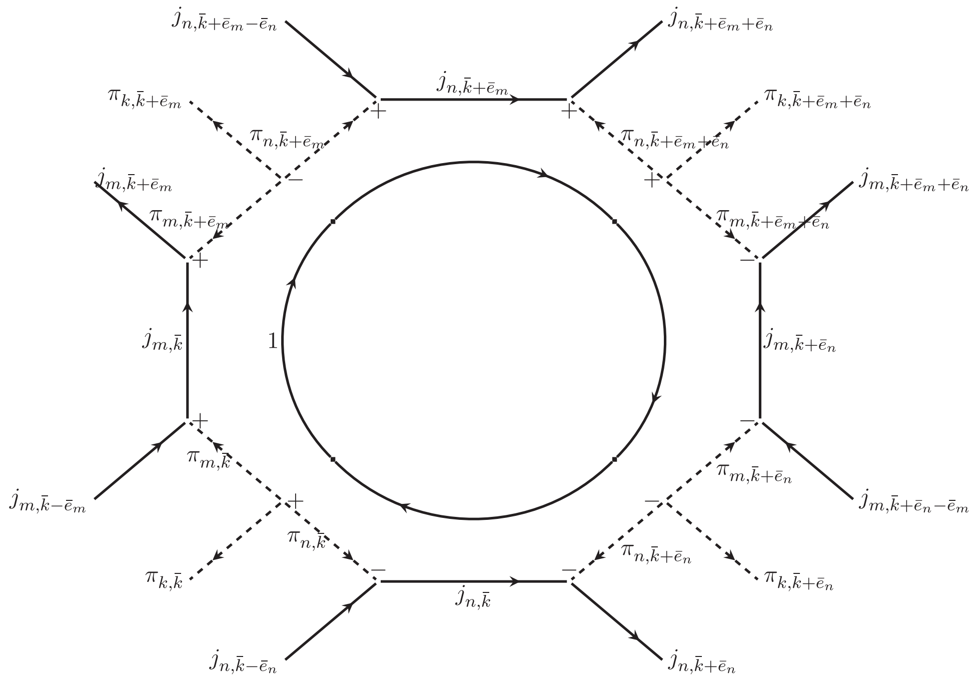
Let Ψ be an arbitrary state of the lattice. As was already stated the electric term is diagonal, so we see immediately that

$$2Q^2\epsilon \hat{H}_{KS,\mathrm{lit}}|\Psi\rangle = \sum_{i,\bar{k}} C_2(j_{i,\bar{k}})^2 + \sum_{\beta} \mathrm{tr}(\hat{A}(\beta)) + \mathrm{tr}(\hat{A}(\beta))^\dagger|\Psi\rangle, \tag{4.3}$$

meaning we can restrict ourselves to the evaluation of the trace over all plaquettes. Furthermore, using that $\hat{A} + \hat{A}^\dagger = 2\mathrm{Re}(\hat{A})$ we focus only on the $\mathrm{tr}(\hat{A}(\beta)$. Given the set $\{k, m, n\}$ as an even permutation of $\{1, 2, 3\}$, one can look without loss of generality at the plaquette in the (m, n)-direction containing amongst others the vertex $\bar{k}$. In this notation the second term of the Hamiltonian is written as

$$\frac{1}{Q^2}\sum_{\bar{k}}\sum_{k=1}^{3} \mathrm{tr}(\hat{A}_{m,\bar{k}}\hat{A}_{n,\bar{k}+\bar{e}_m}\hat{A}^{-1}_{m,\bar{k}+\bar{e}_n}\hat{A}^{-1}_{n,\bar{k}}). \tag{4.4}$$

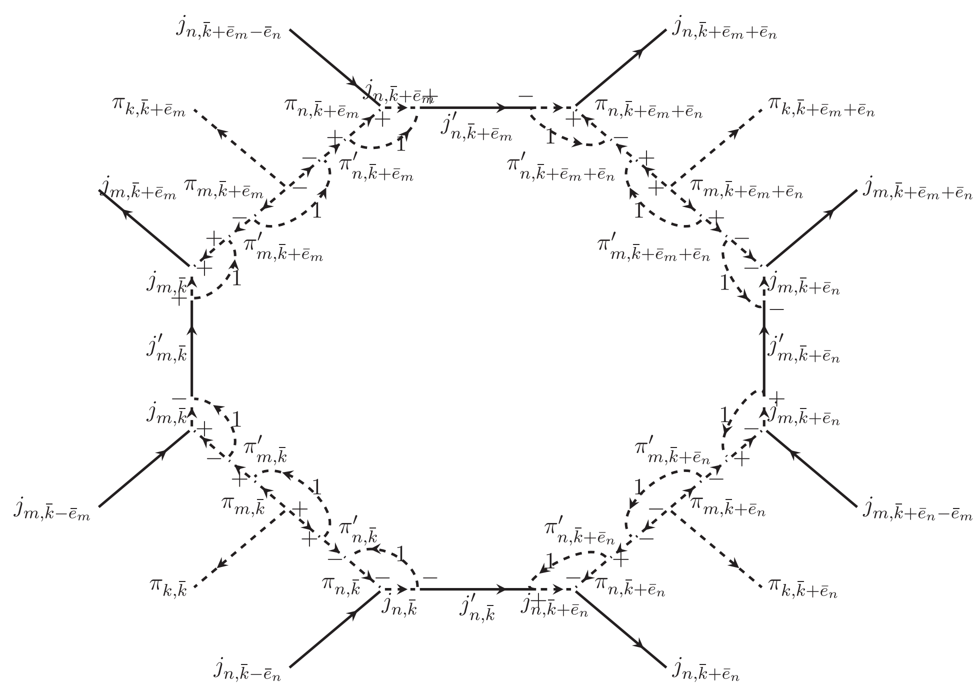
We first present the application of the graphical calculus to evaluate the matrix elements of (4.4) in the case of the gauge group $SU(2)$ and later on state the corresponding results in the case of the $SU(3)$ gauge group. We note in passing that the Kogut-Susskind computation of the magnetic term performed here is the same [for $SU(2)$] as the Euclidian piece of the gravitational contribution to the Hamiltonian, which also has not been done in the nongraph changing setting before, although it was done for its semiclassically valid $U(1)^3$ approximation [29]. The action of the trace on a general graph $|\psi_{\bar{j},\bar{\pi}}\rangle$ is written as



$$= \sum_{\substack{j'_{n,\bar{k}} j'_{m,\bar{k}} \\ j'_{n,\bar{k}+\bar{e}_m},\, j'_{m,\bar{k}+\bar{e}_n}}} \quad \sum_{\substack{\pi'_{n,\bar{k}},\, \pi'_{m,\bar{k}},\, \pi'_{n,\bar{k}+\bar{e}_m},\, \pi'_{n,\bar{k}+\bar{e}_m} \\ \pi'_{n,\bar{k}+\bar{e}_n},\, \pi'_{n,\bar{k}+\bar{e}_n},\, \pi'_{n,\bar{k}+\bar{e}_m+\bar{e}_n},\, \pi'_{n,\bar{k}+\bar{e}_m+\bar{e}_n}}}$$

$$\cdot (-)^{2\left(j'_{m,\bar{k}}+j_{n,\bar{k}}+j_{m,\bar{k}+\bar{e}_n}+j'_{n,\bar{k}+\bar{e}_m}+\pi'_{m,\bar{k}}+\pi_{n,\bar{k}}+\pi_{m,\bar{k}+\bar{e}_m}+\pi'_{n,\bar{k}+\bar{e}_m}+\pi_{m,\bar{k}+\bar{e}_n}+\pi'_{n,\bar{k}+\bar{e}_n}+\pi'_{m,\bar{k}+\bar{e}_m+\bar{e}_n}+\pi_{n,\bar{k}+\bar{e}_m+\bar{e}_n}\right)} .$$

$\cdot$



Now all the $6j$ symbols have to be recoupled. One starts with the bottom left one in the figures [which is the easiest one with the $6j$ being exactly in the form as in Eq. (A1)] and then one brings the orientation of the node back to normal order and continues clockwise. Finally, if we define

$$
\begin{aligned}
&\mathfrak{P}_{SU(2)}(\{\pi\}_{\bar{k}},\{\pi\}_{\bar{k}+\bar{e}_m},\{\pi\}_{\bar{k}+\bar{e}_n},\{\pi\}_{\bar{k}+\bar{e}_m+\bar{e}_n};\{j\};\pi'_{n,\bar{k}},\pi'_{m,\bar{k}},\pi'_{m,\bar{k}+\bar{e}_m},\ldots;j'_{n,\bar{k}},j'_{m,\bar{k}},\ldots)\\
&\equiv\prod_{i=0,1}d_{j'_{m,\bar{k}+i\bar{e}_n}}d_{j'_{n,\bar{k}+i\bar{e}_m}}\prod_{j=0,1}d_{\pi'_{m,\bar{k}+i\bar{e}_m+j\bar{e}_n}}d_{\pi'_{n,\bar{k}+i\bar{e}_m+j\bar{e}_n}}\\
&\cdot(-)^{2(j'_{m,\bar{k}}+j_{n,\bar{k}}+j_{m,\bar{k}+\bar{e}_n}+j'_{n,\bar{k}+\bar{e}_m}+\pi_{k,\bar{k}}+\pi_{k,\bar{k}+\bar{e}_n}+\pi_{k,\bar{k}+\bar{e}_m}+\pi_{k,\bar{k}+\bar{e}_m+\bar{e}_n})+j'_{n,\bar{k}}+j_{n,\bar{k}-\bar{e}_n}+\pi'_{n,\bar{k}}}\begin{Bmatrix} j_{n,\bar{k}-\bar{e}_n} & j_{n,\bar{k}} & \pi_{n,\bar{k}}\\ 1 & \pi'_{n,\bar{k}} & j'_{n,\bar{k}}\end{Bmatrix}\\
&\cdot(-)^{2\pi_{n,\bar{k}}}\begin{Bmatrix} \pi_{k,\bar{k}} & \pi_{n,\bar{k}} & \pi_{m,\bar{k}}\\ 1 & \pi'_{m,\bar{k}} & \pi'_{n,\bar{k}}\end{Bmatrix}(-)^{\pi_{k,\bar{k}}+\pi'_{m,\bar{k}}+\pi'_{n,\bar{k}}}\cdot(-)^{j_{m,\bar{k}}+j_{m,\bar{k}-\bar{e}_m}+\pi_{m,\bar{k}}+2j_{m,\bar{k}-\bar{e}_m}}\begin{Bmatrix} j_{m,\bar{k}-\bar{e}_m} & \pi_{m,\bar{k}} & j_{m,\bar{k}}\\ 1 & j'_{m,\bar{k}} & \pi'_{m,\bar{k}}\end{Bmatrix}\\
&\cdot(-)^{2\pi_{m,\bar{k}+\bar{e}_m}+j'_{m,\bar{k}}+j_{m,\bar{k}+\bar{e}_m}+\pi'_{m,\bar{k}+\bar{e}_m}}\begin{Bmatrix} j'_{m,\bar{k}} & 1 & j_{m,\bar{k}}\\ \pi_{m,\bar{k}+\bar{e}_m} & j_{m,\bar{k}+\bar{e}_m} & \pi'_{m,\bar{k}+\bar{e}_m}\end{Bmatrix}\cdot\begin{Bmatrix} \pi'_{m,\bar{k}+\bar{e}_m} & 1 & \pi_{m,\bar{k}+\bar{e}_m}\\ \pi_{n,\bar{k}+\bar{e}_m} & \pi_{k,\bar{k}+\bar{e}_m} & \pi'_{n,\bar{k}+\bar{e}_m}\end{Bmatrix}\\
&\cdot(-)^{2\pi_{n,\bar{k}+\bar{e}_m}}(-)^{\pi'_{n,\bar{k}+\bar{e}_m}+\pi_{k,\bar{k}+\bar{e}_m}+\pi'_{m,\bar{k}+\bar{e}_m}}\cdot\begin{Bmatrix} \pi'_{n,\bar{k}+\bar{e}_m} & 1 & \pi_{n,\bar{k}+\bar{e}_m}\\ j_{n,\bar{k}+\bar{e}_m} & j_{n,\bar{k}+\bar{e}_m-\bar{e}_n} & j'_{n,\bar{k}+\bar{e}_m}\end{Bmatrix}(-)^{\pi'_{n,\bar{k}+\bar{e}_m}+j_{n,\bar{k}+\bar{e}_m-\bar{e}_n}+j'_{n,\bar{k}+\bar{e}_m}}\\
&\cdot(-)^{j_{n,\bar{k}+\bar{e}_m}+j_{n,\bar{k}+\bar{e}_m+\bar{e}_n}+\pi_{n,\bar{k}+\bar{e}_m+\bar{e}_n}+2j_{n,\bar{k}+\bar{e}_m}}\begin{Bmatrix} j'_{n,\bar{k}+\bar{e}_m} & 1 & j_{n,\bar{k}+\bar{e}_m}\\ \pi_{n,\bar{k}+\bar{e}_m+\bar{e}_n} & j_{n,\bar{k}+\bar{e}_m+\bar{e}_n} & \pi_{n,\bar{k}+\bar{e}_m+\bar{e}_n}{}'\end{Bmatrix}\\
&\cdot(-)^{2\pi_{n,\bar{k}+\bar{e}_m+\bar{e}_n}}\begin{Bmatrix} \pi'_{n,\bar{k}+\bar{e}_m+\bar{e}_n} & 1 & \pi_{n,\bar{k}+\bar{e}_m+\bar{e}_n}\\ \pi_{m,\bar{k}+\bar{e}_m+\bar{e}_n} & \pi_{n,\bar{k}+\bar{e}_m+\bar{e}_n} & \pi'_{m,\bar{k}+\bar{e}_m+\bar{e}_n}\end{Bmatrix}(-)^{\pi'_{n,\bar{k}+\bar{e}_m+\bar{e}_n}+\pi'_{m,\bar{k}+\bar{e}_m+\bar{e}_n}+\pi_{k,\bar{k}+\bar{e}_m+\bar{e}_n}}\\
&\cdot(-)^{2\pi_{m,\bar{k}+\bar{e}_m+\bar{e}_n}}\begin{Bmatrix} \pi'_{m,\bar{k}+\bar{e}_m+\bar{e}_n} & 1 & \pi_{m,\bar{k}+\bar{e}_m+\bar{e}_n}\\ j_{m,\bar{k}+\bar{e}_n} & j_{m,\bar{k}+\bar{e}_m+\bar{e}_n} & j'_{m,\bar{k}+\bar{e}_n}\end{Bmatrix}(-)^{j'_{m,\bar{k}+\bar{e}_n}+\pi'_{m,\bar{k}+\bar{e}_m+\bar{e}_n}+j_{m,\bar{k}+\bar{e}_m+\bar{e}_n}}\\
&\cdot(-)^{j_{m,\bar{k}+\bar{e}_n}+j_{m,\bar{k}-\bar{e}_m+\bar{e}_n}+\pi_{m,\bar{k}+\bar{e}_n}}(-)^{2j_{m,\bar{k}+\bar{e}_n}+2\pi_{m,\bar{k}+\bar{e}_n}}\cdot\begin{Bmatrix} \pi'_{m,\bar{k}+\bar{e}_n} & \pi_{m,\bar{k}+\bar{e}_n} & 1\\ j_{m,\bar{k}+\bar{e}_n} & j'_{m,\bar{k}+\bar{e}_n} & j_{m,\bar{k}-\bar{e}_m+\bar{e}_n}\end{Bmatrix}\\
&\cdot(-)^{2\pi_{n,\bar{k}+\bar{e}_n}}\begin{Bmatrix} \pi'_{n,\bar{k}+\bar{e}_n} & \pi_{n,\bar{k}+\bar{e}_n} & 1\\ \pi_{m,\bar{k}+\bar{e}_n} & \pi'_{m,\bar{k}+\bar{e}_n} & \pi_{k,\bar{k}+\bar{e}_n}\end{Bmatrix}(-)^{\pi_{k,\bar{k}+\bar{e}_n}+\pi'_{m,\bar{k}+\bar{e}_n}+\pi'_{n,\bar{k}+\bar{e}_n}}\\
&\cdot(-)^{j_{n,\bar{k}}+j_{n,\bar{k}+\bar{e}_n}+\pi_{n,\bar{k}+\bar{e}_n}+2j_{n,\bar{k}}}\begin{Bmatrix} j'_{n,\bar{k}} & j_{n,\bar{k}} & 1\\ \pi_{n,\bar{k}+\bar{e}_n} & \pi'_{n,\bar{k}+\bar{e}_n} & j_{n,\bar{k}+\bar{e}_n}\end{Bmatrix}
\end{aligned}
$$

we can write the complete Matrix element for the gauge group $SU(2)$,

$$
\begin{aligned}
\langle\psi_{\bar{j}',\bar{\pi}'}|\hat{H}_{\mathrm{YM}}|\psi_{\bar{j},\bar{\pi}}\rangle=&\frac{1}{2Q^2}\sum_{\bar{k}}j_{\bar{k}}(j_{\bar{k}}+1)+\frac{1}{Q^2}\sum_{\bar{k}}\sum_{m<n}\mathfrak{P}_{SU(2)}(\{\pi\}_{\bar{k}},\{\pi\}_{\bar{k}+\bar{e}_m},\{\pi\}_{\bar{k}+\bar{e}_n},\{\pi\}_{\bar{k}+\bar{e}_m+\bar{e}_n};\\
&\times\{j\};\pi'_{n,\bar{k}},\pi'_{m,\bar{k}},\pi'_{m,\bar{k}+\bar{e}_m},\ldots;j'_{n,\bar{k}},j'_{m,\bar{k}},\ldots).
\end{aligned}\tag{4.5}
$$

A similar calculation with the beforehand established calculus for $SU(3)$ gives us the new plaquette term with $\mathcal{S}_{\mathrm{int}}\coloneqq\{s_{j_m,\bar{k}},s_{j_n,\bar{k}},s_{j_n,\bar{k}+\bar{e}_m},s_{j_m,\bar{k}+\bar{e}_n},s_{\pi_m,\bar{k}},s_{\pi_n,\bar{k}},s_{\pi_n,\bar{k}+\bar{e}_m+\bar{e}_n},s_{\pi_m,\bar{k}+\bar{e}_m+\bar{e}_n},s_{\pi_m,\bar{k}+\bar{e}_m},s_{\pi_n,\bar{k}+\bar{e}_n},s_{\pi_n,\bar{k}+\bar{e}_m},s_{\pi_m,\bar{k}+\bar{e}_n}\}$, which denotes the internal set of multiplicities over which we have to sum this time (in contrast note the absence of an additional sign factor here),

$$\sum_{\mathcal{S}_{\rm int}} \mathfrak{P}_{SU(3)}(\{\pi,s\}_{\bar{k}},\{\pi,s\}_{\bar{k}+\bar{e}_m},\{\pi,s\}_{\bar{k}+\bar{e}_n},\{\pi,s\}_{\bar{k}+\bar{e}_m+\bar{e}_n};\{j\};\pi'_{n,\bar{k}},\pi'_{m,\bar{k}},\pi'_{m,\bar{k}+\bar{e}_m},\ldots;s'_{n,\bar{k}},s'_{0,\bar{k}},s'_{m,\bar{k}},\ldots;j'_{n,\bar{k}},j'_{m,\bar{k}},\ldots)$$

$$\begin{aligned}
&\equiv \sum_{\mathcal{S}_{\rm int}} d_{j_{m,\bar{k}}} d_{j_{n,\bar{k}}} d_{j_{m,\bar{k}+\bar{e}_n}} d_{j_{n,\bar{k}+\bar{e}_m}} \left(\prod_{i,j=0,1} d_{\pi_{m,\bar{k}+i\bar{e}_n+j\bar{e}_m}} d_{\pi_{n,\bar{k}+i\bar{e}_n+j\bar{e}_m}}\right)\\
&\cdot \begin{Bmatrix} j_{n,\bar{k}-\bar{e}_n} & \bar{j}'_{n,\bar{k}} & \pi'_{n,\bar{k}} & \\ 1 & \pi_{n,\bar{k}} & j_{n,\bar{k}} & \\ s'_{n,\bar{k}} & s_{n,\bar{k}} & \bar{s}_{j_n,\bar{k}} & s_{\pi_n,\bar{k}} \end{Bmatrix}
\begin{Bmatrix} \bar{\pi}_{k,\bar{k}} & \bar{\pi}'_{n,\bar{k}} & \bar{\pi}'_{m,\bar{k}} & \\ 1 & \bar{\pi}_{m,\bar{k}} & \pi_{n,\bar{k}} & \\ s'_{0,\bar{k}} & s_{0,\bar{k}} & \bar{s}_{\pi_m,\bar{k}} & \bar{s}_{\pi_n,\bar{k}} \end{Bmatrix}
\begin{Bmatrix} j_{m,\bar{k}-\bar{e}_m} & \pi'_{m,\bar{k}} & \bar{j}'_{m,\bar{k}} & \\ 1 & \bar{j}_{m,\bar{k}} & \bar{\pi}_{m,\bar{k}} & \\ s'_{m,\bar{k}} & s_{m,\bar{k}} & s_{\pi_m,\bar{k}} & s_{j_m,\bar{k}} \end{Bmatrix}\\
&\times \begin{Bmatrix} j'_{m,\bar{k}} & \pi'_{m,\bar{k}+\bar{e}_m} & \bar{j}_{m,\bar{k}+\bar{e}_m} & \\ \pi_{m,\bar{k}+\bar{e}_m} & \bar{j}_{m,\bar{k}} & 1 & \\ s'_{m,\bar{k}+\bar{e}_m} & s_{j_m,\bar{k}} & s_{\pi_m,\bar{k}+\bar{e}_m} & s_{m,\bar{k}+\bar{e}_m} \end{Bmatrix}
\begin{Bmatrix} \bar{\pi}'_{m,\bar{k}+\bar{e}_m} & \bar{\pi}'_{n,\bar{k}+\bar{e}_m} & \bar{\pi}_{k,\bar{k}+\bar{e}_m} & \\ \bar{\pi}_{n,\bar{k}+\bar{e}_m} & \pi_{m,\bar{k}+\bar{e}_m} & 1 & \\ s'_{0,\bar{k}+\bar{e}_m} & \bar{s}_{\pi_m,\bar{k}+\bar{e}_m} & \bar{s}_{\pi_n,\bar{k}+\bar{e}_m} & s_{0,\bar{k}+\bar{e}_m} \end{Bmatrix}\\
&\times \begin{Bmatrix} \pi'_{n,\bar{k}+\bar{e}_m} & \bar{j}'_{n,\bar{k}+\bar{e}_m} & j_{n,\bar{k}+\bar{e}_m-\bar{e}_n} & \\ \bar{j}_{n,\bar{k}+\bar{e}_m} & \bar{\pi}_{n,\bar{k}+\bar{e}_m} & 1 & \\ s'_{n,\bar{k}+\bar{e}_m} & s_{\pi_n,\bar{k}+\bar{e}_m} & \bar{s}_{j_n,\bar{k}+\bar{e}_n} & s_{n,\bar{k}+\bar{e}_m} \end{Bmatrix}
\begin{Bmatrix} \pi'_{m,\bar{k}+\bar{e}_n} & j_{m,\bar{k}+\bar{e}_n-\bar{e}_m} & \bar{j}'_{m,\bar{k}+\bar{e}_n} & \\ j_{m,\bar{k}+\bar{e}_n} & 1 & \pi'_{m,\bar{k}+\bar{e}_n} & \\ s'_{m,\bar{k}+\bar{e}_n} & s_{\pi_m,\bar{k}+\bar{e}_n} & s_{m,\bar{k}+\bar{e}_n} & \bar{s}_{j_m,\bar{k}+\bar{e}_n} \end{Bmatrix}\\
&\times \begin{Bmatrix} \bar{\pi}'_{n,\bar{k}+\bar{e}_n} & \bar{\pi}_{k,\bar{k}+\bar{e}_n} & \bar{\pi}'_{m,\bar{k}+\bar{e}_n} & \\ \pi_{m,\bar{k}+\bar{e}_n} & 1 & \bar{\pi}_{n,\bar{k}+\bar{e}_n} & \\ s'_{0,\bar{k}+\bar{e}_n} & \bar{s}_{\pi_m,\bar{k}+\bar{e}_n} & s_{0,\bar{k}+\bar{e}_n} & \bar{s}_{\pi_m,\bar{k}+\bar{e}_n} \end{Bmatrix}
\begin{Bmatrix} j'_{n,\bar{k}+\bar{e}_m} & \pi'_{n,\bar{k}+\bar{e}_m+\bar{e}_n} & \bar{j}_{n,\bar{k}+\bar{e}_m+\bar{e}_n} & \\ \pi_{n,\bar{k}+\bar{e}_m+\bar{e}_n} & \bar{j}_{n,\bar{k}+\bar{e}_m} & 1 & \\ s'_{n,\bar{k}+\bar{e}_m+\bar{e}_n} & s_{j_m,\bar{k}+\bar{e}_m} & s_{\pi_n,\bar{k}+\bar{e}_m+\bar{e}_n} & s_{n,\bar{k}+\bar{e}_m+\bar{e}_n} \end{Bmatrix}\\
&\times \begin{Bmatrix} j'_{n,k} & \bar{j}_{n,\bar{k}+\bar{e}_n} & \pi'_{n,\bar{k}+\bar{e}_n} & \\ \pi_{n,\bar{k}+\bar{e}_n} & 1 & j_{n,\bar{k}} & \\ s'_{n,\bar{k}+\bar{e}_n} & s_{j_n,\bar{k}} & s_{n,\bar{k}+\bar{e}_n} & s_{\pi_n,\bar{k}+\bar{e}_n} \end{Bmatrix}
\begin{Bmatrix} \bar{\pi}'_{n,\bar{k}+\bar{e}_m+\bar{e}_n} & \bar{\pi}'_{m,\bar{k}+\bar{e}_m+\bar{e}_n} & \bar{\pi}_{k,\bar{k}+\bar{e}_m+\bar{e}_n} & \\ \bar{\pi}_{m,\bar{k}+\bar{e}_m+\bar{e}_n} & \pi_{n,\bar{k}+\bar{e}_m+\bar{e}_n} & 1 & \\ s'_{0,\bar{k}+\bar{e}_m+\bar{e}_n} & \bar{s}_{\pi_m,\bar{k}+\bar{e}_m+\bar{e}_n} & \bar{s}_{\pi_m,\bar{k}+\bar{e}_m+\bar{e}_n} & s_{0,\bar{k}+\bar{e}_m+\bar{e}_n} \end{Bmatrix}\\
&\times \begin{Bmatrix} \pi_{m,\bar{k}+\bar{e}_m+\bar{e}_n} & j'_{m,\bar{k}+\bar{e}_n} & \bar{j}_{m,\bar{k}+\bar{e}_m+\bar{e}_n} & \\ j_{m,\bar{k}+e_n} & \bar{\pi}_{m,\bar{k}+\bar{e}_m+\bar{e}_n} & 1 & \\ s'_{m,\bar{k}+\bar{e}_m+\bar{e}_n} & s_{\pi_m,\bar{k}+\bar{e}_m+\bar{e}_n} & s_{j_m,\bar{k}+\bar{e}_n} & s_{m,\bar{k}+\bar{e}_m+\bar{e}_n} \end{Bmatrix}.
\end{aligned}$$

So the complete matrix element is the same as in (4.5) with this sum over the new plaquette term and the new Casimir. Note that in the action of the Kogut-Susskind Hamiltonian the group elements of the plaquette are in the defining representation. However, the same calculation could be done for an arbitrary m representation. Since this is used later, there have been no simplifications in the above expressions, such that one can easily replace $1 \to m$ and denote the new plaquette term as $\mathfrak{P}(\ldots|m)$ to distinguish it from the QCD case.

V. EINSTEIN-YANG-MILLS THEORY IN QUANTUM GRAVITY

To compute the matrix elements of the full quantum gravity Yang-Mills Hamiltonian, we adopt the same notation as in Sec. II, and denote the gravity quantum numbers with j_i and the Yang-Mills quantum numbers with $\underline{j}_i$, whose gauge group is set to $SU(3)$ for the remainder of this paper. The basis functions Ψ on our cubic graph are labeled by

$$\begin{aligned}
&|\Psi(\{j\})\underline{\Psi}(\{\underline{j}\},\{\underline{\pi}\};\{\underline{s}\})\rangle\\
&\quad = \sum_{\bar{k}\in\mathbb{Z}^3} |\nu(\{\pi\}_{\bar{k}},\{j\}_{\bar{k}})\rangle \otimes |\nu(\{\underline{\pi}\}_{\bar{k}},\{\underline{j}\}_{\bar{k}})\rangle. \qquad (5.1)
\end{aligned}$$

Because of the fact that the result is quite lengthy and splits up into a lot of subcases, we split up this section. The quantum gravity Yang-Mills Hamiltonian

$$\hat{H}_{\rm YM}(v) = \frac{1}{2Q^2}(\hat{H}_E(v) + \hat{H}_B(v))$$

consists of two big parts, the first being the electric term and the second being the magnetic term. For both one can look separately at the gravitational degrees of freedom and at the Yang-Mills degrees of freedom, i.e., the electric fluxes and the plaquette part, respectively. Each of these four parts is calculated in its corresponding subsection below.

A. Gravity part of the electric term

The gravity part of the Yang-Mills Hamiltonian is

$$\mathrm{tr}(\hat{A}_j[\hat{A}_j^{-1},\sqrt{\hat{V}}]\hat{A}_m[\hat{A}_m^{-1},\sqrt{\hat{V}}]). \tag{5.2}$$

Because of the commutators one gets four different parts. The first one is just the definition of the elements of the action of the volume,

$$\hat{V}|\nu(\{\pi\}_{\bar{k}},\{j\}_{\bar{k}})\rangle \equiv \tilde{\sum_{\{\pi\}^2_{\bar{k}}}} V_{\bar{k}}(\{\pi\}_{\bar{k}},\{\pi\}^2_{\bar{k}};\{j_{\bar{k}}\})|\nu(\{\pi\}^2_{\bar{k}},\{j\}_{\bar{k}})\rangle.$$

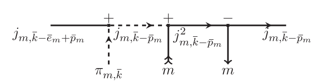
The label $\bar{k}$ is purely of interest for the valency of the vertex (with $\bar{k}\in\mathbb{Z}^3$ there are six edges meeting at the node). Moreover, one realizes that the volume operator only changes the intertwiners, not the graph itself. We have also introduced the weighted sum: $\tilde{\sum}_j = \sum_j d_j$.

For the second one the action of the volume on a nongauge invariant node is needed. The notation here $(\sqrt{V}_{\bar{k}+\bar{e}_j})$ means that on the edge in the j-direction a nongauge invariant edge in the m representation is glued. The additional representation j_j that changes to j_j^2, where one needs to sum over, is also displayed after the first semicolon: $\sqrt{V}_{\bar{k}+\bar{e}_j}(\ldots;j_j,j_j^2;\ldots|m)$. If $j=1,\ 2,\ 3$ only half of the edges are calculated. For the remaining ones, carrying the representation $(j_{j,\bar{k}-\bar{e}_j})$, the calculation broadly remains exactly the same when replacing $j_{j,\bar{k}-\bar{e}_j}\Leftrightarrow j_{j,\bar{k}}$. However, one wants to work on a vertex where all edges are outgoing to maximize the degree of symmetry, which explains the (temporary) additional sign in the second line of the computation. Moreover, one also has to switch the orientation of the vertex itself, since the "+" sign elsewhere becomes "-." To combine both cases in one in the following, we introduce the parameter $p_j\in\{\bar{0},\bar{e}_j\}$, which distinguishes the cases, using $j_{j,\bar{k}-\bar{p}_j}$ and $j_{j,\bar{k}-\bar{e}_j+\bar{p}_j}$. So for one we get a sign of $|\bar{p}_j|(\pi_{j,\bar{k}}+j_{j,\bar{k}}+j_{j,\bar{k}+\bar{e}_j})$ to ensure that the sign at the vertex is always $+$. With all of this the action for the second part is [where one also uses the $SU(2)$ version of the orthogonality relation (3.10) in the last line after having coupled the last holonomy to the graph]

$$tr\left(\sqrt{\hat{V}}\hat{A}_{m,\bar{p}_m}\sqrt{\hat{V}}\hat{A}^{-1}_{m,\bar{p}_m}\right)|\nu\left(\{\pi\}_{\bar{k}};\{j\}_{\bar{k}}\right)\rangle$$

$$= tr\left(\sqrt{\hat{V}}\hat{A}_{m,\bar{p}_m}\sqrt{\hat{V}}\right)(-)^{2\left(j_{1,\bar{k}-\bar{e}_1}+j_{2,\bar{k}-\bar{e}_2}+j_{3,\bar{k}-\bar{e}_3}\right)}(-)^{|\bar{p}_m|\left(\pi_{m,\bar{k}}+j_{m,\bar{k}}+j_{m,\bar{k}+\bar{e}_m}\right)}$$

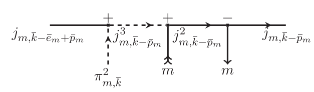
$$\tilde{\sum_{j^2_{m,\bar{k}-\bar{p}_m}}}(-)^{2j^2_{m,\bar{k}-\bar{p}_m}}\Bigg|$$



$$\Bigg\rangle$$

$$= tr\left(\sqrt{\hat{V}}\hat{A}_{m,\bar{p}_m}\right)\sqrt{V}_{\bar{k}+\bar{e}_m-2\bar{p}_m}\left(\{\pi\}_{\bar{k}},\{\pi\}^2_{\bar{k}};j_{m,\bar{k}-\bar{p}_m},j^3_{m,\bar{k}-\bar{p}_m};\ldots j^2_{m,\bar{k}-\bar{p}_m}\ldots\mid m\right)$$

$$(-)^{2j^2_{m,\bar{k}-\bar{p}_m}}\Bigg|$$



$$\Bigg\rangle =$$

$$= \tilde{\sum_{j^2_{m,\bar{k}+\bar{p_m}},\{\pi\}^3_{\bar{k}},\{\pi\}^2_{\bar{k}}}}\sqrt{V}_{\bar{k}+\bar{e}_m-2\bar{p}_m}\left(\{\pi\}_{\bar{k}},\{\pi\}^2_{\bar{k}};j_{m,\bar{k}-\bar{p}_m},j_{m,\bar{k}-\bar{p}_m};\ldots j^2_{m,\bar{k}-\bar{p}_m}\ldots\right)\cdot$$

$$\cdot\sqrt{V}_{\bar{k}}\left(\{\pi\}^2_{\bar{k}},\{\pi\}^3_{\bar{k}};\{j\}_{\bar{k}}\right)\cdot(-)^{|\bar{p}_m|\left(\pi^3_{m,\bar{k}}+\pi_{m,\bar{k}}\right)}\left|\nu\left(\{\pi\}^3_{\bar{k}};\{j\}_{\bar{k}}\right)\right\rangle$$

And correspondingly the third part is

$$\mathrm{tr}\left(\hat{A}_{j,\bar{p}_j}\sqrt{\hat{V}}\hat{A}^{-1}_{j,\bar{p}_j}\sqrt{\hat{V}}\right)|\nu(\{\pi\}_{\bar{k}};\{j\}_{\bar{k}})\rangle = \tilde{\sum_{\{\pi\}^2_{\bar{k}},\{\pi\}^3_{\bar{k}},j^2_{j,\bar{k}-\bar{p}_j}}}(-)^{|\bar{p}_j|(\pi_{j,\bar{k}}-\pi^3_{j,\bar{k}})}\sqrt{V_{\bar{k}}}(\{\pi\}_{\bar{k}},\{\pi\}^2_{\bar{k}};\{j\}_{\bar{k}})\cdot$$

$$\cdot\sqrt{V}_{\bar{k}+\bar{e}_j-2\bar{p}_j}(\{\pi\}^2_{\bar{k}},\{\pi\}^3_{\bar{k}};j_{j,\bar{k}-\bar{p}_j},j_{j,\bar{k}-\bar{p}_j};\ldots j^2_{j,\bar{k}-\bar{p}_j}\ldots|m)|\nu(\{\pi\}^3_{\bar{k}};\{j\}_{\bar{k}})\rangle.$$

The fourth and last part of (5.2) needs some more detailed treatment, since we deal now with two holonomies that are glued to the graph, and that may go in different directions. The term of interest is $\hat{A}_j\sqrt{\hat{V}}\hat{A}_j^-1\hat{A}_m\sqrt{\hat{V}}\hat{A}_m^-1$, where j, m denote the different directions of the glued edges. Summing over all possible combinations of choosing two (possibly the same) edges emanating from one vertex $\bar{k}$, we have 36 combinations, from which many due to symmetry reasons give the same result. In total we have thus only to distinguish three case: Both holonomies may

(i) lie on the same edge ($j_{m,\bar{k}} = j_{j,\bar{k}}$),
(ii) lie on parallel edges ($j_{m,\bar{k}+\bar{e}_m} = j_{j,\bar{k}}$),
(iii) go in different directions.

For (i) it is obvious that the holonomies in the middle cancel, leaving us with a rather simple expression,

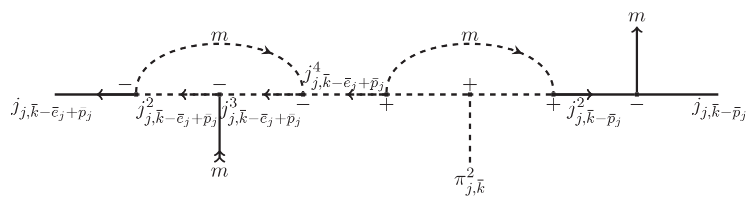
$$\mathrm{tr}(\hat{A}_{j,\bar{p}_j}\hat{V}\hat{A}^{-1}_{j,\bar{p}_j})|\nu(\{\pi\}_{\bar{k}},\{j\}_{\bar{k}})\rangle = \tilde{\sum_{j^2_{j,\bar{k}-\bar{p}_j}\{\pi\}^2_{\bar{k}}}}(-)^{|\bar{p}_j|(\pi_{j,\bar{k}}+\pi^2_{j,\bar{k}})}V_{\bar{k}+\bar{e}_j-2\bar{p}_j}(\{\pi\}_{\bar{k}},\{\pi\}^2_{\bar{k}};j_{j,\bar{k}-\bar{p}_j},j_{j,\bar{k}-\bar{p}_j};\ldots j^2_{j,\bar{k}-\bar{p}_j}\ldots|m)|\nu(\{\pi\}^2_{\bar{k}},\{j\}_{\bar{k}})\rangle.$$

The second part of course incorporates now a change from one link to the other and back to close the trace of the holonomies at the end. As one can easily see, the structures appearing again look similar to Eq. (A1) from the appendix and thus represent $6j$ symbols. Note, moreover, that the open edges in the m representation in the third line denote the open ends of the holonomy. One is attached infinitesimally close to the vertex; hence the action of the volume elements also changes the link between these two, and the other open end (on the $j_{j,\bar{k}-\bar{p}_j}$ edge) is attached after the group element, which we have suppressed and trivially shifted to the $j^2_{j,\bar{k}-\bar{p}_j}$ edge.

$$tr\left(\hat{A}_{j,p_j}\sqrt{\hat{V}}\hat{A}^{-1}_{j,p_j}\hat{A}_{m,p_m}\sqrt{\hat{V}}\hat{A}^{-1}_{m,p_m}\right)|\nu(\{\pi\}_{\bar{k}},\{j\}_{\bar{k}})\rangle = (-)^{2\sum_{i=1}^{3}j_{i,\bar{k}-\bar{e}_i}+|\bar{p}_j|\left(j_{j,\bar{k}-\bar{e}_j}+j_{j,\bar{k}}+\pi_{j,\bar{k}}\right)}\cdot$$

$$\cdot\tilde{\sum_{\substack{j^2_{j,\bar{k}-\bar{e}_j+\bar{p}_j},\{\pi\}^2_{\bar{k}},\\ j^3_{j,\bar{k}-\bar{e}_j+\bar{p}_j},\\ j^2_{j,\bar{k}-\bar{p}_J},j^4_{j,\bar{k}-\bar{e}_j+\bar{p}_j}}}}(-)^{2j^2_{j,\bar{k}-\bar{e}_j+\bar{p}_j}}\sqrt{V}_{\bar{k}-\bar{e}_j+2\bar{p}_j}\left(\{\pi\}_{\bar{k}},\{\pi\}^2_{\bar{k}};j_{j,\bar{k}-\bar{e}_j+\bar{p}_j},j^3_{j,\bar{k}-\bar{e}_j+\bar{p}_j};\ldots j^2_{j,\bar{k}-\bar{e}_j+\bar{p}_j}\ldots|\,m\right)\cdot$$

$$\cdot(-)^{2j^2_{j,\bar{k}-\bar{p}_j}+2j^3_{j,\bar{k}-\bar{e}_j+\bar{p}_j}}\,tr\left(\hat{A}_{j,\bar{p}_j}\sqrt{\hat{V}}\right)\Bigg|$$



$$\Bigg\rangle$$

$$= \tilde{\sum_{\substack{j^{2\ldots5}_{j,\bar{k}-\bar{e}_j+\bar{p}_j},\ j^2_{j,\bar{k}-\bar{p}_J},\\ \{\pi\}^2_{\bar{k}},\ \{\pi\}^3_{\bar{k}},}}}\sqrt{V}_{\bar{k}-\bar{e}_j+2\bar{p}_j}\left(\{\pi\}_{\bar{k}},\{\pi\}^2_{\bar{k}};j_{j,\bar{k}-\bar{e}_j+\bar{p}_j},j^3_{j,\bar{k}-\bar{e}_j+\bar{p}_j};\ldots j^2_{j,\bar{k}-\bar{e}_j+\bar{p}_j}\ldots|\,m\right)$$

$$\sqrt{V}_{\bar{k}-\bar{e}_j+2\bar{p}_j}\left(\{\pi\}^2_{\bar{k}},\{\pi\}^3_{\bar{k}};j^4_{j,\bar{k}-\bar{e}_j+\bar{p}_j},j^5_{j,\bar{k}-\bar{e}_j+\bar{p}_j};\ldots j^2_{j,\bar{k}-\bar{p}_j}\ldots|\,m\right)$$

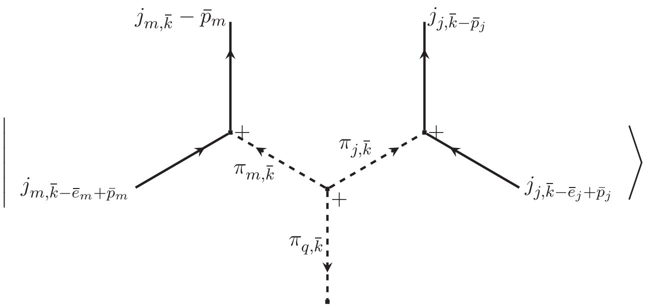
$$(-)^{|\bar{p}_j|\left(\pi_{j,\bar{k}}+\pi^3_{j,\bar{k}}\right)}(-)^{\pi^2_{\bar{k}}-\pi^3_{\bar{k}}}(-)^{2j^5_{j,\bar{k}-\bar{e}_j}+\bar{p}_j}\begin{Bmatrix} j_{j,\bar{k}-\bar{e}_j+\bar{p}_j} & j^2_{j,\bar{k}-\bar{e}_j+\bar{p}_j} & m \\ j^3_{j,\bar{k}-\bar{e}_j+\bar{p}_j} & j^4_{j,\bar{k}-\bar{e}_j+\bar{p}_j} & m \end{Bmatrix}$$

$$\begin{Bmatrix} j^4_{j,\bar{k}-\bar{e}_j+\bar{p}_j} & j^3_{j,\bar{k}-\bar{e}_j+\bar{p}_j} & m \\ j_{j,\bar{k}-\bar{p}_j} & j^2_{j,\bar{k}-\bar{p}_j} & \pi^2_{j\bar{k}} \end{Bmatrix}\begin{Bmatrix} j_{j,\bar{k}-\bar{e}_j+\bar{p}_j} & j^5_{j,\bar{k}-\bar{e}_j+\bar{p}_j} & m \\ j^2_{j,\bar{k}-\bar{p}_j} & j_{j,\bar{k}-\bar{p}_j} & \pi^3_{j,\bar{k}} \end{Bmatrix}\left|\nu\left(\{\pi\}^3_{\bar{k}},\{j\}_{\bar{k}}\right)\right\rangle$$

Note that the additional sign of $\pi^2_{\bar{k}} - \pi^3_{\bar{k}}$ stems from the fact that one has to reorient the vertices in between to act with the second volume operator in the way it was defined on a node with the given orientation. For (iii) again things get more complicated. We have to switch from one edge to another edge, which does not lie in the same direction. Explicitly, we are interested in the action of the holonomy $\hat{A}^{-1}_{j,\bar{p}_j}\hat{A}_{m,\bar{p}_m}$ on a vertex, which we find useful to write in the following form, where σ gives us the sign of the permutation of m, j, q:

$$|\nu(\{\pi\}_{\bar{k}},\{j\}_{\bar{k}})\rangle = (-)^{2\sum_{i=1}^{3} j_{i,\bar{k}-\bar{e}_i}} (-)^{|\bar{p}_j|\left(\pi_{j,\bar{k}}+j_{j,\bar{k}}+j_{j,\bar{k}-\bar{e}_j}\right)} (-)^{(1-|\bar{p}_m|)\left(\pi_{m,\bar{k}}+j_{m,\bar{k}}+j_{m,\bar{k}-\bar{e}_m}\right)}$$

$$(-)^{\sigma(m,j,q)\left(\pi_{j,\bar{k}}+\pi_{m,\bar{k}}+\pi_{q,\bar{k}}\right)} \Bigg|$$



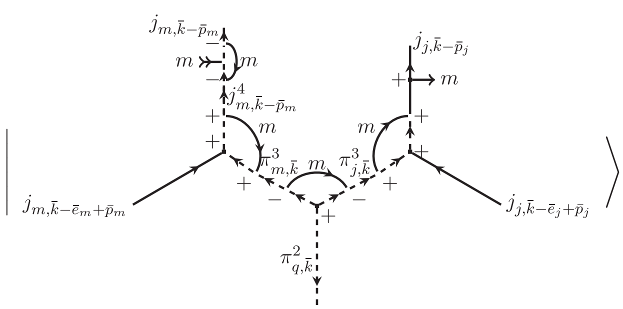
$$\Bigg\rangle$$

Once our Hamiltonian acts on the state, we see that traversing the node results in a couple of $6j$ symbols (four when going from $j_{m,\bar{k}-\bar{p}_m}$ to $j_{j,\bar{k}-\bar{p}_j}$ and three when going back). Remember that in between we have to bring the signs back into an orientation such that $\hat{V}$ can act and after its action we have to restore the given orientation, such that one can close the holonomies. In total one ends up with a fairly complicated expression,

$$tr\left(\hat{A}_{j,\bar{p}_j}\sqrt{\hat{V}}\right) \tilde{\sum_{\substack{j^2_{m,\bar{k}-\bar{p}_m},\, j^3_{m,\bar{k}-\bar{p}_m}\\ \{\pi\}^2_{\bar{k}}}}} (-)^{2j^2_{m,\bar{k}-\bar{p}_m}} (-)^{2\sum_{i=1}^{3} j_{i,\bar{k}-\bar{e}_i}+|\bar{p}_m|\left(j_{m,\bar{k}}+j_{m,\bar{k}-\bar{e}_m}+\pi_{m,\bar{k}}\right)} (-)^{\sigma(m,j,q)\left(\pi^2_{j,\bar{k}}+\pi^2_{m,\bar{k}}+\pi^2_{q,\bar{k}}\right)}$$

$$(-)^{|\bar{p}_j|\left(j_{j,\bar{k}}+j_{j,\bar{k}-\bar{e}_j}+\pi_{j,\bar{k}}\right)} \sqrt{V}_{\bar{k}+\bar{e}_m-2\bar{p}_m}\left(\{\pi\}_{\bar{k}},\{\pi\}^2_{\bar{k}}; j_{m,\bar{k}-\bar{p}_m}, j^3_{m,\bar{k}-\bar{p}_m}; \ldots j^2_{m,\bar{k}-\bar{p}_m}\ldots \mid m\right)$$

$$\cdot \tilde{\sum_{\substack{j^4_{m,\bar{k}-\bar{p}_m},\, j^2_{j,\bar{k}-\bar{p}_j}\\ \{\pi\}^2_{\bar{k}},\, \pi^3_{m,\bar{k}},\, \pi^3_{j,\bar{k}}}}} (-)^{2j^3_{m,\bar{k}-\bar{p}_m}+2\pi^2_{m,\bar{k}}+2\pi^3_{j,\bar{k}}+2j^2_{j,\bar{k}-\bar{p}_j}} \Bigg|$$

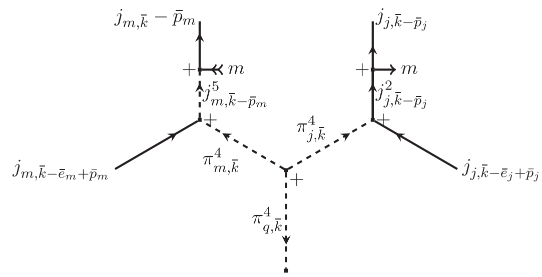


$$\Bigg\rangle$$

$$= tr\left(\hat{A}_{j,\bar{p}_j}\right) \tilde{\sum_{\substack{j^{2\ldots 4}_{m,\bar{k}-\bar{p}_m},\, j^2_{j,\bar{k}-\bar{p}_j}\\ \{\pi\}^2_{\bar{k}},\, \pi^3_{m,\bar{k}},\, \pi^3_{j,\bar{k}}\\ \{\pi\}^4_{\bar{k}},\, j^5_{m,\bar{k}-\bar{p}_m}}}} (-)^{2\sum_{i=1}^{3} j_{i,\bar{k}-\bar{e}_i}} (-)^{\sigma(m,j,q)\left(\pi^2_{j,\bar{k}}+\pi^2_{m,\bar{k}}+2\pi^2_{q,\bar{k}}+\pi^3_{j,\bar{k}}+\pi^3_{m,\bar{k}}+\pi^4_{j,\bar{k}}+\pi^4_{m,\bar{k}}+\pi^4_{q,\bar{k}}\right)}$$

$$(-)^{|\bar{p}_m|\left(j_{m,\bar{k}}+j_{m,\bar{k}-\bar{e}_m}+\pi_{m,\bar{k}}+\pi^3_{m,\bar{k}}+j^4_{m,\bar{k}-\bar{p}_m}+2j_{m,\bar{k}-\bar{e}_m+\bar{p}_m}+j^5_{m,\bar{k}-\bar{p}_m}+\pi^4_{m,\bar{k}}\right)}$$

$$(-)^{|\bar{p}_j|\left(j_{j,\bar{k}}+j_{j,\bar{k}-\bar{e}_j}+\pi_{j,\bar{k}}+\pi^3_{j,\bar{k}}+2j^2_{j,\bar{k}-\bar{p}_j}+2j_{j,\bar{k}-\bar{e}_j+\bar{p}_j}+\pi^4_{j,\bar{k}}\right)} (-)^{m+j_{m,\bar{k}-\bar{p}_m}+j_{m,\bar{k}-\bar{e}_m+\bar{p}_m}+j^2_{j,\bar{k}-\bar{p}_j}+j_{j,\bar{k}-\bar{e}_j+\bar{p}_j}}$$

$$(-)^{\pi^3_{m,\bar k}+\pi^3_{j,\bar k}+\pi^2_{m,\bar k}+\pi^2_{j,\bar k}+\pi^2_{q,\bar k}}\sqrt{V}_{\bar k+\bar e_m-2\bar p_m}\left(\{\pi\}_{\bar k},\{\pi\}^2_{\bar k};j_{m,\bar k-\bar p_m},j^3_{m,\bar k-\bar p_m};\ldots j^2_{m,\bar k-\bar p_m}\ldots\mid m\right)$$
$$\sqrt{V}_{\bar k+\bar e_m-2\bar p_m}\left(\pi^3_{m,\bar k},\pi^3_{j,\bar k},\pi^2_{q,\bar k},\{\pi\}^4_{\bar k};j^4_{m,\bar k-\bar p_m},j^5_{m,\bar k-\bar p_m};\ldots j^2_{j,\bar k-\bar p_m}\ldots\mid m\right)$$
$$\begin{Bmatrix}\pi^3_{j,\bar k} & \pi^2_{j,\bar k} & m\\ j_{j,\bar k-\bar p_j} & j^2_{j,\bar k-\bar p_j} & j_{j,\bar k+\bar e_j-\bar p_j}\end{Bmatrix}\begin{Bmatrix}\pi^2_{q,\bar k} & \pi^2_{j,\bar k} & \pi^2_{m,\bar k}\\ m & \pi^3_{m,\bar k} & \pi^3_{j,\bar k}\end{Bmatrix}\begin{Bmatrix}m & j^3_{m,\bar k-\bar p_m} & j_{m,\bar k-\bar p_m}\\ m & j_{m,\bar k-\bar p_m} & j^4_{m,\bar k-\bar p_m}\end{Bmatrix}$$
$$\begin{Bmatrix}j_{m,\bar k-\bar e_m+\bar p_m} & \pi^2_{m,\bar k} & j^3_{m,\bar k-\bar p_m}\\ m & j^4_{m,\bar k-\bar p_m} & \pi^3_{m,\bar k}\end{Bmatrix}\Bigg|$$


$$\Bigg\rangle$$

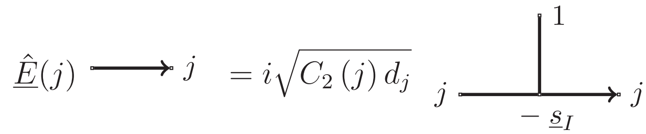
$$= \tilde{\sum_{\substack{j^{2\ldots4}_{m,\bar k-\bar p_m},\,j^2_{j,\bar k-\bar p_j}\\ \{\pi\}^2_{\bar k},\,\pi^3_{m,\bar k},\,\pi^3_{j,\bar k},\,\{\pi\}^4_{\bar k}\\ j^5_{m,\bar k-\bar p_m},\,\pi^5_{m,\bar k},\,\pi^5_{j,\bar k}}}} (-)^{\sigma(m,j,q)\left(\pi^2_{j,\bar k}+\pi^2_{m,\bar k}+2\pi^2_{q,\bar k}+\pi^3_{j,\bar k}+\pi^3_{m,\bar k}+\pi^4_{j,\bar k}+\pi^4_{m,\bar k}+2\pi^4_{q,\bar k}+\pi^5_{m,\bar k}+\pi^5_{j,\bar k}\right)}$$
$$(-)^{|\bar p_m|\left(2j_{m,\bar k-\bar p_m}+j^4_{m,\bar k-\bar p_m}+j^5_{m,\bar k-\bar p_m}+\pi_{m,\bar k}+\pi^3_{m,\bar k}+\pi^4_{m,\bar k}+\pi^5_{m,\bar k}\right)}(-)^{|\bar p_j|\left(\pi_{j,\bar k}+\pi^3_{j,\bar k}+\pi^4_{j,\bar k}+\pi^5_{j,\bar k}+2m\right)}$$
$$(-)^{2\pi^4_{q,\bar k}+\pi^5_{j,\bar k}+\pi^4_{m,\bar k}+\pi^4_{m,\bar k}+\pi^4_{j,\bar k}+\pi^4_{q,\bar k}+\pi^3_{m,\bar k}+\pi^3_{j,\bar k}+\pi^2_{m,\bar k}+\pi^2_{m,\bar k}+\pi^2_{j,\bar k}+\pi^2_{q,\bar k}}$$
$$\sqrt{V}_{\bar k+\bar e_m-2\bar p_m}\left(\{\pi\}_{\bar k},\{\pi\}^2_{\bar k};j_{m,\bar k-\bar p_m},j^3_{m,\bar k-\bar p_m};\ldots j^2_{m,\bar k-\bar p_m}\ldots\mid m\right)$$
$$\sqrt{V}_{\bar k+\bar e_m-2\bar p_m}\left(\pi^3_{m,\bar k},\pi^3_{j,\bar k},\pi^2_{q,\bar k},\{\pi\}^4_{\bar k};j^4_{m,\bar k-\bar p_m},j^5_{m,\bar k-\bar p_m};\ldots j^2_{j,\bar k-\bar p_m}\ldots\mid m\right)$$
$$\begin{Bmatrix}\pi^3_{j,\bar k} & \pi^2_{j,\bar k} & m\\ j_{j,\bar k-\bar p_j} & j^2_{j,\bar k-\bar p_j} & j_{j,\bar k+\bar e_j-\bar p_j}\end{Bmatrix}\begin{Bmatrix}\pi^2_{q,\bar k} & \pi^2_{j,\bar k} & \pi^2_{m,\bar k}\\ m & \pi^3_{m,\bar k} & \pi^3_{j,\bar k}\end{Bmatrix}\begin{Bmatrix}m & j^3_{m,\bar k-\bar p_m} & j_{m,\bar k-\bar p_m}\\ m & j_{m,\bar k-\bar p_m} & j^4_{m,\bar k-\bar p_m}\end{Bmatrix}$$
$$\begin{Bmatrix}j_{m,\bar k-\bar e_m+\bar p_m} & \pi^2_{m,\bar k} & j^3_{m,\bar k-\bar p_m}\\ m & j^4_{m,\bar k-\bar p_m} & \pi^3_{m,\bar k}\end{Bmatrix}\begin{Bmatrix}j_{m,\bar k-\bar e_m+\bar p_m} & \pi^4_{m,\bar k} & j^5_{m,\bar k-\bar p_m}\\ m & j_{m,\bar k-\bar p_m} & \pi^5_{m,\bar k}\end{Bmatrix}\begin{Bmatrix}\pi^5_{m,\bar k} & \pi^4_{m,\bar k} & m\\ \pi^4_{j,\bar k} & \pi^5_{j,\bar k} & \pi^4_{q,\bar k}\end{Bmatrix}$$
$$\begin{Bmatrix}\pi^5_{j,\bar k} & \pi^4_{j,\bar k} & m\\ j^2_{j,\bar k-\bar p_j} & j_{j,\bar k-\bar p_j} & j_{j,\bar k-\bar e_j+\bar p_j}\end{Bmatrix}\left|\nu\left(\pi^5_{m,\bar k},\pi^5_{j,\bar k},\pi^4_{q,\bar k};\{j\}_{\bar k}\right)\right\rangle$$

B. Gluon electric fluxes of the electric term

The electric part of the Hamiltonian is

$$\underline{\hat E}_I(e_1)\underline{\hat E}_I(e_2),$$

where e_1 and e_2 correspond again to all possible tuples of edges incident at a vertex v. The electric fluxes $\underline{\hat E}_I(j)$ themselves are the grasping operators, whose action on a group element has been defined in (2.9). The operator adds a generator of the Lie algebra, which can be viewed as a new intertwiner on the holonomy in the defining (i.e., $\underline{j}=1$) representation. Hence the action is determined up to a normalization factor, which depends on the gauge group and possibly also on the multiplicity factor corresponding to the chosen intertwiner. However, it is easy to check that when choosing an arbitrary $\underline{s}_I$ multiplicity everywhere, the normalization does not depend on it and becomes $N^{(j)} = \sqrt{C_2(j)d_j}$ (the computation for this is, in principle, the same as in [64]). Writing everything down in our graphical calculus,



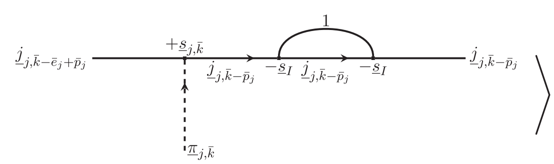
(5.3)

With this at hand we turn again to the three cases (i)–(iii) from Sec. V B. However, due to the nature of the $SU(3)$ gauge group, one cannot obtain a node with all edges outgoing by simply multiplying it with a sign factor. Instead, one now has to take care of the fact that the switched edges carry the dual representation. So one works in the following with an oriented graph, denoted the following way:

$$|\nu_{\text{orient}}(\underline{j}_{1,\bar{k}},\underline{j}_{2,\bar{k}},\underline{j}_{3,\bar{k}},\bar{\underline{j}}_{1,\bar{k}-\bar{e}_1}\bar{\underline{j}}_{2,\bar{k}-\bar{e}_2}\bar{\underline{j}}_{3,\bar{k}-\bar{e}_3},\ldots)\rangle = |\nu(\underline{j}_{1,\bar{k}},\underline{j}_{2,\bar{k}},\underline{j}_{3,\bar{k}},\underline{j}_{1,\bar{k}-\bar{e}_1}\underline{j}_{2,\bar{k}-\bar{e}_2}\underline{j}_{3,\bar{k}-\bar{e}_3},\ldots)\rangle.$$

The first case (i) $(\underline{j}_{j,\bar{k}} = \underline{j}_{m,\bar{k}})$ means that both grasping operators act on the same edge; hence we get twice the square root of the corresponding quadric Casimir and using the orthogonality relation (3.38) one calculates

$$\hat{\underline{E}}(\underline{j}_j)^I\hat{\underline{E}}(\underline{j}_j)^I\left|\nu_{orient}\left(\{\underline{\pi}\}_{\bar{k}};\left\{\underline{j}\right\}_{\bar{k}};\{\underline{s}\}_{\bar{k}}\right)\right\rangle = -C_2\left(\underline{j}_{j,\bar{k}-\bar{p}_j}\right)\Bigg|$$



$$\Bigg\rangle$$

$$= C_2\left(\underline{j}_{j,\bar{k}-\bar{p}_j}\right)\left|\nu_{orient}\left(\{\underline{\pi}\}_{\bar{k}};\left\{\underline{j}\right\}_{\bar{k}};\{\underline{s}\}_{\bar{k}}\right)\right\rangle$$

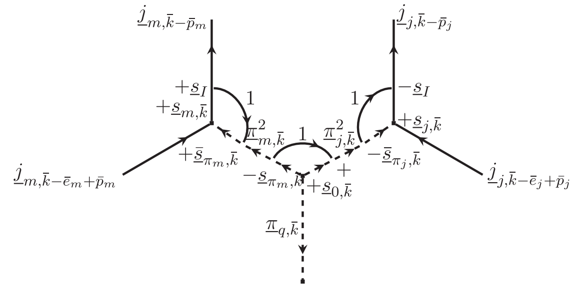
The second case (ii), where the edges in question lie in parallel direction $(\underline{j}_{j,\bar{k}} = \underline{j}_{m,\bar{k}-\bar{e}_m})$, uses again the extraction of the s-classified $3j$ symbol and thus one gets

$$\hat{\underline{E}}(\underline{j}_{j,\bar{k}-\bar{e}_j+\bar{p}_j})^I\hat{\underline{E}}(\underline{j}_{j,\bar{k}-\bar{p}_j})^I|\nu_{\text{orient}}(\{\underline{\pi}\}_{\bar{k}};\{\underline{j}\}_{\bar{k}};\{\underline{s}\}_{\bar{k}})\rangle = \sqrt{C_2(\underline{j}_{j,\bar{k}-\bar{p}_j})C_2(\underline{j}_{j,\bar{k}-\bar{e}_j+\bar{p}_j})}\cdot$$
$$\cdot\sum_{\underline{s}'_{j,\bar{k}}}\begin{Bmatrix} \bar{\underline{j}}_{j,\bar{k}-\bar{e}_i+\bar{p}_j} & \underline{\pi}_{j,\bar{k}} & \bar{\underline{j}}_{j,\bar{k}-\bar{p}_j} & \\ \underline{j}_{j,\bar{k}-\bar{p}_j} & 1 & \bar{\underline{j}}_{j,\bar{k}-\bar{e}_j+\bar{p}_j} & \\ \underline{s}'_{j,\bar{k}} & \underline{s}_I & \underline{s}_{j,\bar{k}} & \underline{s}_I \end{Bmatrix}$$
$$\times|\nu_{\text{orient}}(\{\underline{\pi}\}_{\bar{k}},\{\underline{j}\}_{\bar{k}},\ldots\underline{s}_{j,\bar{k}}{}'\ldots)\rangle.$$

Lastly we look at (iii), where both holonomies go in different directions. With the same strategy as before, we see

$$\underline{\hat{E}}^I_j \underline{\hat{E}}^I_m \left|\nu_{out}\left(\{\underline{\pi}\}_{\bar k};\left\{\underline{j}\right\}_{\bar k};\{\underline{s}\}_{\bar k}\right)\right\rangle = \tilde{\sum_{\substack{\underline{\pi}^2_{m,\bar k}\underline{\pi}^2_{j,\bar k}\\ \underline{s}_{\pi_{j,\bar k}}\underline{s}_{\pi_{m,\bar k}}}}} (-)^{(1-|\bar p_j|)\left(\underline{\pi}_{j,\bar k}+\underline{j}_{j,\bar k}+\underline{j}_{j,\bar k-\bar e_j}\right)+|\bar p_m|\left(\underline{\pi}_{m,\bar k}+\underline{j}_{m,\bar k}+\underline{j}_{m,\bar k-\bar e_m}\right)}.$$

$$(-)^{\sigma(m,j,q)\left(\underline{\pi}_{j,\bar k}+\underline{\pi}_{m,\bar k}+\underline{\pi}_{q,\bar k}\right)}\sqrt{C_2\left(\underline{j}_{m,\bar k-\bar p_m}\right)C_2\left(\underline{j}_{j,\bar k-\bar p_j}\right)}\Bigg|$$



$$\Bigg\rangle$$

$$= \sum_{\substack{\underline{\pi}^2_{m,\bar k}\underline{\pi}^2_{j,\bar k}\\ \underline{s}_{\pi_{j,\bar k}}\underline{s}_{\pi_{m,\bar k}}\\ \underline{s}'_{0,\bar k}\underline{s}'_{m,\bar k}\underline{s}'_{j,\bar k}}} (-)^{\sigma(m,j,q)\left(\underline{\pi}^2_{m,\bar k}+\underline{\pi}^2_{j,\bar k}+\underline{\pi}_{m,\bar k}+\underline{\pi}_{j,\bar k}\right)+(1-|\bar p_j|)\left(\underline{\pi}_{j,\bar k}+\underline{\pi}^2_{j,\bar k}\right)+|\bar p_m|\left(\underline{\pi}_{m,\bar k}+\underline{\pi}^2_{m,\bar k}\right)}(-)^{\underline{\pi}^3_{j,\bar k}+\underline{\pi}_{m,\bar k}+1}$$

$$\sqrt{C_2\left(\underline{j}_{m,\bar k-\bar p_m}\right)C_2\left(\underline{j}_{j,\bar k-\bar p_j}\right)}\begin{Bmatrix} \bar{\underline{j}}_{m,\bar k-\bar e_m+\bar p_m} & \underline{\pi}^2_{m,\bar k} & \bar{\underline{j}}_{m,\bar k-\bar p_m} & \\ 1 & \bar{\underline{j}}_{m,\bar k-\bar p_m} & \underline{\pi}_{m,\bar k} & \\ \underline{s}'_{m,\bar k} & s_{m,\bar k} & \underline{s}_{\pi_m,\bar k} & \underline{s}_I \end{Bmatrix}\begin{Bmatrix} \bar{\underline{\pi}}^2_{m,\bar k} & \bar{\underline{\pi}}_{q,\bar k} & \bar{\underline{\pi}}^2_{j,\bar k} & \\ \underline{\pi}_{j,\bar k} & 1 & \underline{\pi}_{m,\bar k} & \\ \underline{s}'_{0,\bar k} & \underline{s}_{\pi_m,\bar k} & s_{0,\bar k} & s_{\pi_j,\bar k} \end{Bmatrix}$$

$$\begin{Bmatrix} \underline{\pi}^2_{j,\bar k} & \bar{\underline{j}}_{j,\bar k-\bar e_j+\bar p_j} & \bar{\underline{j}}_{\bar k-\bar p_j} & \\ \underline{j}_{j,\bar k-\bar p_j} & 1 & \underline{\pi}_{j,\bar k} & \\ \underline{s}'_{j,\bar k} & \underline{s}_{\pi_j,\bar k} & s_{j,\bar k} & \underline{s}_I \end{Bmatrix}\left|\nu_{orient}\left(\{\underline{\pi}\}_{\bar k};\{\underline{j}\};\underline{s}'_{0,\bar k},\underline{s}'_{j,\bar k},\underline{s}'_{m,\bar k},\underline{s}_{q,\bar k}\right)\right\rangle$$

C. Gravity part of the magnetic term

The gravity part of the magnetic term is

$$\mathrm{tr}(\hat\tau_i\hat A_l[\hat A_l^{-1},\sqrt{\hat V}])\mathrm{tr}(\hat\tau_i\hat A_p[\hat A_p^{-1},\sqrt{\hat V}]).$$

Since there are again two commutators we have, in principle, four different terms to look at. However, three of them vanish trivially. For example, look at the expression, where the $\hat A_p$ cancel,

$$\begin{aligned}&\mathrm{tr}(\hat\tau_i\hat A_l[\hat A_l^{-1},\sqrt{\hat V}])\mathrm{tr}(\hat\tau_i\sqrt{\hat V})|\nu_{\mathrm{out}}(\{\pi\}_{\bar k};\{j\}_{\bar k})\rangle\\ &\quad = tr(\hat\tau_i\hat A_l[\hat A_l^{-1},\sqrt{\hat V}])\mathrm{tr}(\hat\tau_i)\tilde{\sum_{\{\pi\}^2_{\bar k}}}\sqrt{V}_{\bar k}(\{\pi\}_{\bar k},\{\pi\}^2_{\bar k};\{j\}_{\bar k})|\nu_{\mathrm{out}}(\{\pi\}^2_{\bar k};\{j\}_{\bar k})\rangle = 0,\end{aligned}$$

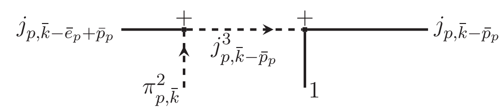
since $\mathrm{tr}(\hat\tau_i)=0$ for $\tau_i\in SU(2)$. The same argument is of course also true in the case of the A_l canceling.

Thus, only the term with both volume operators nested remains. Again we distinguish on which edges the holonomies lie [cases (i)–(iii) from Sec. VA]. Since one has seen that the orientation of the arrows of the edges does not change the result, we suppress this temporary sign from now on and just assume the vertex has been brought in a form such that all links are outgoing. If (i) ($j_{p,\bar k}=j_{l,\bar k}$) then one gets from the first trace a $6j$ symbol and the inserted $\hat\tau_i$ acts like the insertion of an intertwiner in the defining representation, which hence remains open, after closing the first trace. To close the second one, one uses again (3.38) twice. In total one obtains

$$
tr\left(\hat{\tau}_i \hat{A}_p \sqrt{\hat{V}} \hat{A}_p^{-1}\right) tr\left(\hat{\tau}_i \hat{A}_p \sqrt{\hat{V}} \hat{A}_p^{-1}\right) |\nu(\{\pi\}_{\bar{k}}; \{j\}_{\bar{k}})\rangle = tr\left(\hat{\tau}_i \hat{A}_p \sqrt{\hat{V}} \hat{A}_p^{-1}\right) \cdot
$$

$$
\tilde{\sum_{\substack{\{\pi\}^2_{\bar{k}}\, j^2_{p,\bar{k}-\bar{p}_p} \\ j^3_{p,\bar{k}-\bar{p}_p}}}} \sqrt{V}_{\bar{k}-\bar{e}_p+2\bar{p}_p}\left(\{\pi\}_{\bar{k}}, \{\pi\}^2_{\bar{k}}; j_{p,\bar{k}-\bar{p}_p}, j^3_{p,\bar{k}-\bar{p}_p}; \dots j^2_{p,\bar{k}-\bar{p}_p} \dots\right) \left\{\begin{matrix} j^3_{p,\bar{k}-\bar{p}_p} & j^2_{p,\bar{k}-\bar{p}_p} & m \\ m & 1 & j_{p,\bar{k}-\bar{p}_p} \end{matrix}\right\}
$$

$$
(-)^{|\bar{p_p}|\left(\pi^2_{p,\bar{k}} + j^3_{p,\bar{k}-\bar{p}_p} + j_{p,\bar{k}-\bar{e}_p+\bar{p}_p}\right) + 2m + j^2_{p,\bar{k}-\bar{p}_p} + j_{p,\bar{k}-\bar{p}_p} + m + 1} \Big|
$$



$$\Big\rangle$$

$$
= \tilde{\sum_{\substack{\{\pi\}^2_{\bar{k}}\, j^2_{p,\bar{k}-\bar{p}_p,m_1} \\ \{\pi\}^3_{\bar{k}}\, j^3_{p,\bar{k}-\bar{p}_p} \\ j^4_{p,\bar{k}-\bar{p}_p}}}} \sqrt{V}_{\bar{k}-\bar{e}_p+2\bar{p}_p}\left(\{\pi\}_{\bar{k}}, \{\pi\}^2_{\bar{k}}; j_{p,\bar{k}-\bar{p}_p}, j^3_{p,\bar{k}-\bar{p}_p}; \dots j^2_{p,\bar{k}-\bar{p}_p} \cdots \mid m\right)
$$

$$
\sqrt{V}_{\bar{k}-\bar{e}_p+2\bar{p}_p}\left(\{\pi\}^2_{\bar{k}}, \{\pi\}^3_{\bar{k}}; j^3_{p,\bar{k}-\bar{p}_p}, j_{p,\bar{k}-\bar{p}_p}; \dots j^4_{p,\bar{k}-\bar{p}_p} \cdots \mid m\right) (-)^{2m+1+j^2_{p,\bar{k}-\bar{p}_p}+j^4_{p,\bar{k}-\bar{p}_p}}
$$

$$
\left\{\begin{matrix} j^3_{p,\bar{k}-\bar{p}_p} & m & j^2_{p,\bar{k}-\bar{p}_p} \\ m & j^2_{p,\bar{k}-\bar{p}_p} & 1 \end{matrix}\right\} \left\{\begin{matrix} j^3_{p,\bar{k}-\bar{p}_p} & 1 & j_{p,\bar{k}-\bar{p}_p} \\ m & j^4_{,\bar{k}-\bar{p}_p} & m \end{matrix}\right\} \left|\nu\left(\{\pi\}^3_{\bar{k}}; \{j\}_{\bar{k}}\right)\right\rangle
$$

With the same methods as established before, we get for case (ii), meaning both links go in parallel direction ($j_{p,\bar{k}} = j_{l,\bar{k}-\bar{e}_l}$), that

$$
\begin{aligned}
&\mathrm{tr}(\hat{\tau}_i \hat{A}_{p,\bar{p}_p} \sqrt{\hat{V}} \hat{A}^{-1}_{p,\bar{p}_p}) \mathrm{tr}(\hat{\tau}_i \hat{A}_{p,\bar{p}_p-\bar{e}_p} \sqrt{\hat{V}} \hat{A}^{-1}_{p,\bar{p}_p-\bar{e}_p}) |\nu(\{\pi\}_{\bar{k}}; \{j\}_{\bar{k}})\rangle \\
&= \tilde{\sum_{\substack{\{\pi\}^2_{\bar{k}}\{\pi\}^3_{\bar{k}} \\ j^2_{p,\bar{k}-\bar{e}_p+\bar{p}_p} j^3_{p,\bar{k}-\bar{e}_p+\bar{p}_p} \\ j^2_{p,\bar{k}-\bar{p}_p} j^3_{p,\bar{k}-\bar{p}_p}}}} (-)^{|\bar{p}_p|\left(2\pi^3_{p,\bar{k}} + j^3_{p,\bar{k}-\bar{p}_p} + j^5_{p,\bar{k}-\bar{p}_p} + j_{p,\bar{k}-\bar{e}_p+\bar{p}} + j^3_{p,\bar{k}-e_{\bar{p}}+\bar{p}}\right)} (-)^{j^2_{p,\bar{k}-\bar{p}_p} + j_{p,\bar{k}-\bar{e}_p+\bar{p}_p} + j^2_{p,\bar{k}-\bar{e}_p+\bar{p}_p} + j^3_{p,\bar{k}-\bar{e}_p+\bar{p}_p} + \pi^2_{p,\bar{k}} + 1} \\
&\quad \times \left\{\begin{matrix} j^3_{p,\bar{k}-\bar{p}_p} & j_{p,\bar{k}-\bar{p}_p} & 1 \\ m & m & j^2_{p,\bar{k}-\bar{p}_p} \end{matrix}\right\} \left\{\begin{matrix} j_{p,\bar{k}-\bar{e}_p+\bar{p}_p} & j^3_{p,\bar{k}-\bar{e}_p+\bar{p}_p} & 1 \\ j_{p,\bar{k}-\bar{p}_p} & j^3_{p,\bar{k}-\bar{p}_p} & \pi^2_{p,\bar{k}} \end{matrix}\right\} \left\{\begin{matrix} j_{p,\bar{k}-\bar{e}_p+\bar{p}_p} & j^2_{p,\bar{k}-\bar{e}_p+\bar{p}_p} & m \\ m & 1 & j^3_{p,\bar{k}-\bar{e}_p+\bar{p}_p} \end{matrix}\right\} \\
&\quad \times \sqrt{V}_{\bar{k}-2\bar{p}_p}(\{\pi\}_{\bar{k}}, \{\pi\}^2_{\bar{k}}; j_{p,\bar{k}-\bar{e}_p+\bar{p}_p}, j^3_{p,\bar{k}-\bar{e}_p+\bar{p}_p}; \dots j^2_{p,\bar{k}-\bar{e}_p+\bar{p}_p} \dots) \\
&\quad \times \sqrt{V}_{\bar{k}+\bar{e}_p-2\bar{p}_p}(\{\pi\}^2_{\bar{k}}, \{\pi\}^3_{\bar{k}}; j^3_{p,\bar{k}-\bar{p}_p}, j^4_{p,\bar{k}-\bar{p}_p}; \dots j^2_{p,\bar{k}-\bar{p}_p} \dots) |\nu(\{\pi\}^3_{\bar{k}}, \{j\}_{\bar{k}})\rangle.
\end{aligned}
$$

And finally with more suppressed calculation, it follows (iii) (both holonomies go in different directions) that

$$
\begin{aligned}
&\mathrm{tr}(\hat{\tau}_i\hat{A}_l\sqrt{\hat{V}}\hat{A}_l^{-1})\mathrm{tr}(\hat{\tau}_i\hat{A}_p\sqrt{\hat{V}}\hat{A}_p^{-1})|\nu(\{\pi\}_{\bar{k}};\{j\}_{\bar{k}})\rangle \\
&= \tilde{\sum_{\substack{\{\pi\}^2_{\bar{k}}\pi^3_{p,\bar{k}}\pi^3_{l,\bar{k}}\{\pi\}^4_{\bar{k}} \\ j^2_{p,\bar{k}-\bar{p}_p}j^3_{p,\bar{k}-\bar{p}_p}j^2_{l,\bar{k}-\bar{p}_l}j^3_{l,\bar{k}-\bar{p}_l}}}} \sqrt{V}_{\bar{k}+\bar{e}_p-2\bar{p}_p}(\{\pi\}_{\bar{k}},\{\pi\}^2_{\bar{k}};j_{p,\bar{k}-\bar{p}_p},j^3_{p,\bar{k}-\bar{p}_p};\ldots j^2_{p,\bar{k}-\bar{p}_p}\ldots) \\
&\quad\times\sqrt{V}_{\bar{k}-\bar{e}_l-2\bar{p}_l}(\pi^3_{l,\bar{k}},\pi^3_{p,\bar{k}},\pi^2_{q,\bar{k}},\{\pi\}^4_{\bar{k}};j^3_{l,\bar{k}-\bar{p}_l},j_{l,\bar{k}-\bar{p}_l};\ldots j^2_{l,\bar{k}-\bar{p}_l}\ldots)(-)^{\sigma(p,l,q)(\pi^2_{p,\bar{k}}+\pi^2_{l,\bar{k}}+2\pi^2_{q,\bar{k}}+\pi^3_{p,\bar{k}}+\pi^3_{l,\bar{k}})} \\
&\quad(-)^{|\bar{p}_p|(\pi^2_{p,\bar{k}}+j^2_{p,\bar{k}-\bar{p}_p}+2j_{p,\bar{k}-\bar{e}_p+\bar{p}_p}+j_{l,\bar{k}-\bar{p}_l}+\pi^3_{l,\bar{k}})+(1-|\bar{p}_l|)(\pi^2_{l,\bar{k}}+j_{l,\bar{k}-\bar{p}_l}+2j_{l,\bar{k}-\bar{e}_l+\bar{p}_l}+\pi^3_{l,\bar{k}}+j^3_{l,\bar{k}-\bar{p}_l}+\pi^4_{l,\bar{k}}+j_{l,\bar{k}-\bar{p}_l}+j_{l,\bar{k}-\bar{e}_l+\bar{p}_l})} \\
&\quad(-)^{2m+m+j_{p,\bar{k}-\bar{p}_p}+j^2_{p,\bar{k}-\bar{p}_p}+j^3_{p,\bar{k}-\bar{p}_p}+j_{p,\bar{k}-\bar{e}_p+\bar{p}_p}-j_{l,\bar{k}-\bar{e}_l+\bar{p}_l}+j^3_{l,\bar{k}-\bar{p}_l}+\pi^2_{p,\bar{k}}+\pi^3_{p,\bar{k}}+\pi^2_{q,\bar{k}}+\pi^2_{l,\bar{k}}+\pi^3_{l,\bar{k}}} \\
&\quad\times\begin{Bmatrix} 1 & m & m \\ j^2_{p,\bar{k}-\bar{p}_p} & j_{p,\bar{k}-\bar{p}_p} & j^3_{p,\bar{k}-\bar{p}_p}\end{Bmatrix}\begin{Bmatrix} j_{p,\bar{k}-\bar{e}_p+\bar{p}_p} & \pi^2_{p,\bar{k}} & j^3_{p,\bar{k}-\bar{p}_p} \\ 1 & j_{p,\bar{k}-\bar{p}_p} & \pi^3_{p,\bar{k}}\end{Bmatrix} \\
&\quad\times\begin{Bmatrix} \pi^3_{p,\bar{k}} & \pi^2_{p,\bar{k}} & 1 \\ \pi^2_{l,\bar{k}} & \pi^3_{l,\bar{k}} & \pi^2_{q,\bar{k}}\end{Bmatrix}\begin{Bmatrix} \pi^3_{l,\bar{k}} & \pi^2_{l,\bar{k}} & 1 \\ j_{l,\bar{k}-\bar{p}_l} & j^3_{l,\bar{k}-\bar{p}_l} & j_{l,\bar{k}-\bar{e}_l+\bar{p}_l}\end{Bmatrix}\begin{Bmatrix} m_1 & 1 & m \\ j_{l,\bar{k}-\bar{p}_l} & j^2_{l,\bar{k}-\bar{p}_l} & j^3_{l,\bar{k}-\bar{p}_l}\end{Bmatrix}|\nu(\{\pi\}^4_{\bar{k}},\{j\}_{\bar{k}})\rangle.
\end{aligned}
$$

D. Gluon plaquette of the magnetic term

The plaquette part is given by

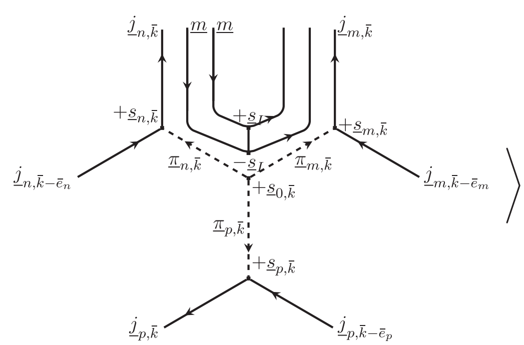
$$\mathrm{tr}(\underline{\hat{\tau}_I}\underline{\hat{A}}_{jk})\mathrm{tr}(\underline{\hat{\tau}}\underline{\hat{A}}_{mn}),$$

which again acts only on the magnetic graph. Each of these two plaquettes, which we add, looks very similar in its structure to Sec. IV. Using this resemblance and inserting again the corresponding plaquette terms $\mathfrak{P}_{SU(3)}$ simplifies the task at hand. Again one has to distinguish different cases, i.e., determined by the possible combinations of j, k, m, and n.

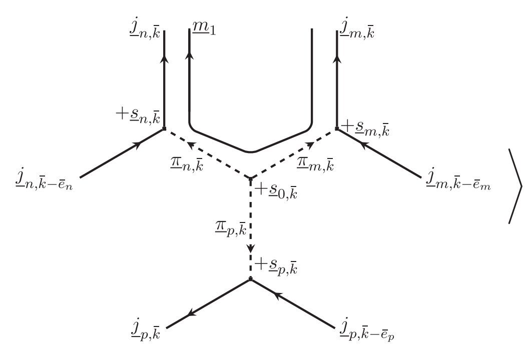
The most simple one is $j = m$ and $k = n$,

$$tr\left(\hat{\underline{\tau}}_I\hat{\underline{A}}_{mn}\right)tr\left(\hat{\underline{\tau}}_I\hat{\underline{A}}_{mn}\right)\left|\nu_{orient}\left(\{\underline{\pi}\}_{\bar{k}};\{\underline{j}\}\right)\right\rangle = (-)^{\sigma(n,m,p)\left(\underline{\pi}_{m,\bar{k}}+\underline{\pi}_{n,\bar{k}}+\underline{\pi}_{p,\bar{k}}\right)}$$

$$(-)^{|\bar{p}_n|\left(\underline{\pi}_{n,\bar{k}}+\underline{j}_{n,\bar{k}-\bar{p}_n}+\underline{j}_{n,\bar{k}-\bar{e}_n+\bar{p}_n}\right)+(1-|\bar{p}_m|)\left(\underline{\pi}_{m,\bar{k}}+\underline{j}_{m,\bar{k}}+\underline{j}_{m,\bar{k}-\bar{e}_m}\right)}\Big|$$



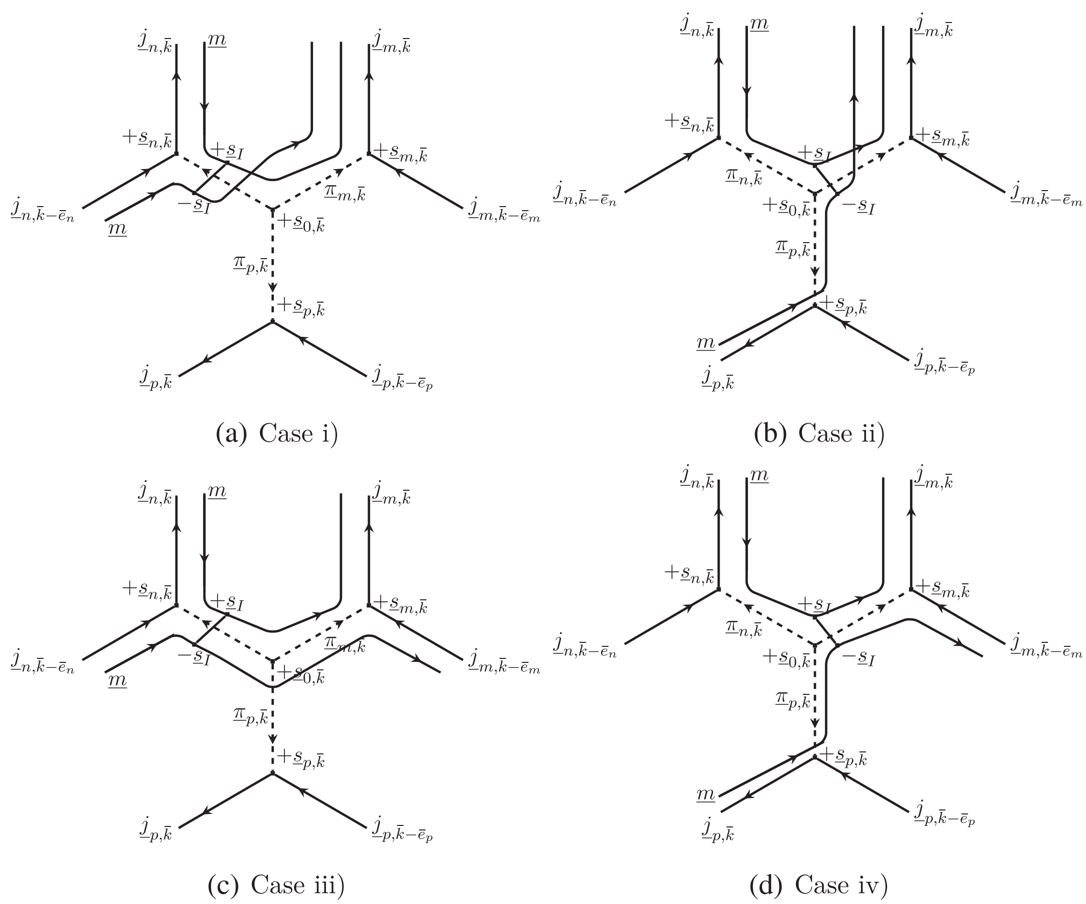
$$= (-)^{\sigma(n,m,p)(\ldots)+|\bar{p}_n|(\ldots)+(1-|\bar{p}_m|)(\ldots)}\sum_{\underline{m}_1,\underline{s}}(-)^1\begin{Bmatrix} \underline{m} & \underline{m} & \underline{\bar{m}}_1 & \\ 1 & \underline{m} & \underline{\bar{m}} & \\ \underline{\bar{s}} & \underline{s}_I & \underline{s}_I & \underline{s}\end{Bmatrix}\Big|$$



Now we have exactly the same plaquette we inserted in the Kogut-Susskind case. To extract exactly the same term again we have to bring the graph in an ordered form, which means we have to take care of the fact that the loop also touches four other nodes. In contrast to the Kogut-Susskind case these signs of the intertwiners now only depend on the chosen permutation of n, m, p, which means that we get a somewhat more complicated sign factor in front,

$$
\begin{aligned}
&= \sum_{\underline{\bar{m}}_1,\mathcal{S}} (-)^{1+\underline{m}_1} \left\{ \begin{matrix} \underline{m} & \underline{m} & \underline{\bar{m}}_1 & \\ 1 & \underline{m} & \underline{\bar{m}} & \\ \underline{\bar{s}} & \underline{s}_I & \underline{s}_I & \underline{s} \end{matrix} \right\} \cdot \\
&(-)^{\sigma(n,m,p)\left(\sum_{i,j=0,1} \pi_{m,\bar{k}+i\bar{e}_m+j\bar{e}_n} + \pi_{n,\bar{k}+i\bar{e}_m+j\bar{e}_n} + 2\pi_{p,\bar{k}+i\bar{e}_m+j\bar{e}_n} + \pi^2_{m,\bar{k}+i\bar{e}_m+j\bar{e}_n} + \pi^2_{n,\bar{k}+i\bar{e}_m+j\bar{e}_n}\right)} \\
&(-)^{|\bar{p}_n|\left(\sum_{i,j=0,1} \pi_{n,\bar{k}+i\bar{e}_n+j\bar{e}_m} + j_{n,\bar{k}-\bar{p}_n+i\bar{e}_m} + 2j_{n,\bar{k}-\bar{e}_n+\bar{p}_n+i\bar{e}_m+2j\bar{e}_n} + j^2_{n,\bar{k}-\bar{p}_n+i\bar{e}_m} + \pi^2_{n,\bar{k}+i\bar{e}_m+j\bar{e}_n}\right)} \\
&(-)^{(1-|\bar{p}_m|)\left(\sum_{i,j=0,1} \pi_{m,\bar{k}+i\bar{e}_m+j\bar{e}_m} + j_{m,\bar{k}-\bar{p}_m+i\bar{e}_n} + 2j_{m,\bar{k}-\bar{e}_m+\bar{p}_m+i\bar{e}_n+2j\bar{e}_m} + j^2_{m,\bar{k}-\bar{p}_m+i\bar{e}_n} + \pi^2_{m,\bar{k}+\bar{e}_i+\bar{e}_j}\right)} \\
&\mathfrak{P}_{SU(3)}(\{\underline{\pi}\}_{\bar{k}} \ldots; \{\underline{j}\}_{\bar{k}}; \{\underline{s}\}_{\bar{k}} \ldots; \underline{\pi}^2_{n,\bar{k}}, \ldots; \underline{j}^2_{n,\bar{k}} \ldots; \underline{s}^2_{0,\bar{k}} \ldots |\underline{\bar{m}}) \\
&|\nu_{\mathrm{orient}}(\underline{\pi}_{p,\bar{k}}, \underline{\pi}^2_{m,\bar{k}}, \underline{\pi}^2_{n,\bar{k}}; \underline{j}^2_{n,\bar{k}}, \underline{j}^2_{m,\bar{k}}, \ldots; \underline{s}^2_{0,\bar{k}}, \underline{s}^2_{m,\bar{k}}, \underline{s}^2_{n,\bar{k}}, \underline{s}_{p,\bar{k}})\rangle,
\end{aligned}
$$

where $\mathcal{S}$ is the set of all new appearing labels in the state, which are the ones one has to sum over.



(a) Case i) (b) Case ii) (c) Case iii) (d) Case iv)

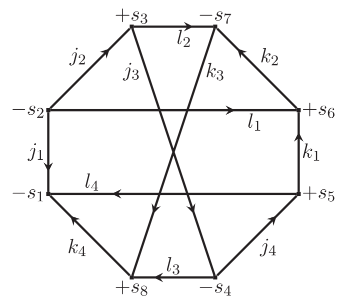
FIG. 1. Different cases of how the holonomies can be oriented. The first plaquette is fixed to be between the m and n direction and the second one can then have four different placements.

There are now four different cases one has to look at left.

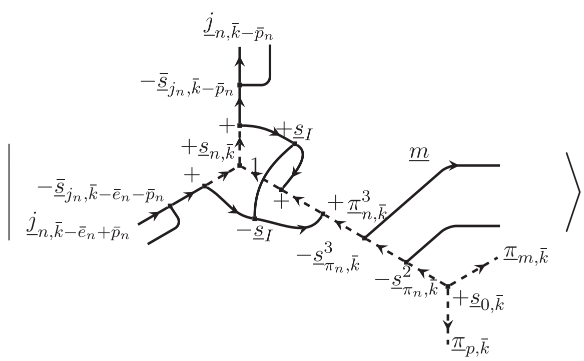
(i) $j = m$ $(p_j = p_m)$ and $k = n$ $(p_k \neq p_n)$,
(ii) $j = m$ $(p_j = p_m)$ and $k \neq n$,
(iii) $j = m$ $(p_j \neq p_m)$ and $k = n$ $(p_k \neq p_n)$,
(iv) $j = m$ $(p_j \neq p_m)$ and $k \neq n$.

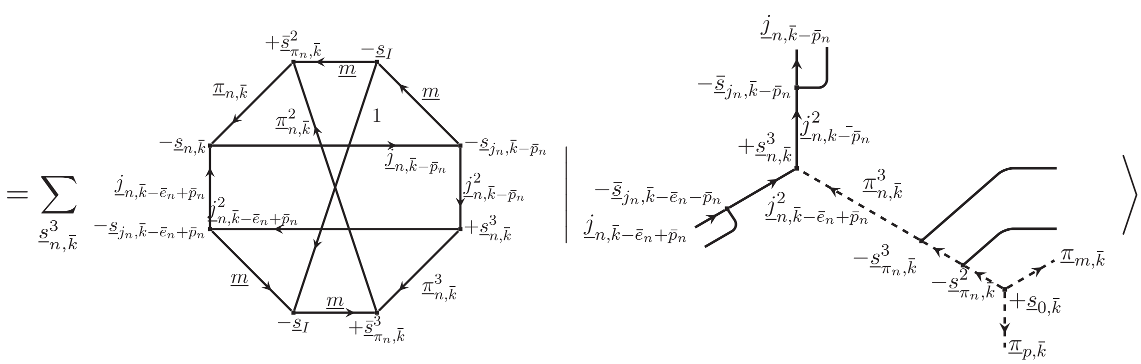
Everything else is (up to relabeling or switching the orientation of the loop) one of theses cases. We could draw them as seen in Fig. 1. Each loop can be recoupled with the previous techniques, giving a $\mathfrak{P}_{SU(3)}(\ldots)$-term up to one $6j$ each, which is due to the coupled $\hat{\tau}_j$. Instead, one gets a 12j symbol, which is defined in the following way:

$$\left\{\begin{matrix} j_1 & j_2 & j_3 & j_4 & \\ l_1 & l_2 & l_3 & l_4 & \\ k_1 & k_2 & k_3 & k_4 & \\ & s_1 & s_2 & s_3 & s_4 \\ & s_5 & s_6 & s_7 & s_8 \end{matrix}\right\} = \tag{5.4}$$



For instance it can be used to recouple the following object:





And this is exactly the nontrivial operation for case (i). So using it we obtain

$$
\begin{aligned}
&\mathrm{tr}(\hat{\underline{\tau}}_I \hat{\underline{A}}_{m,n,p_m,1-p_n})\mathrm{tr}(\hat{\underline{\tau}}_I \hat{\underline{A}}_{m,n})|\nu_{\mathrm{orient}}(\{\underline{\pi}\}_{\bar{k}};\{\underline{j}\}_{\bar{k}};\{\underline{s}\}_{\bar{k}})\rangle \\
&= \sum_{\mathcal{S}}(-)^{(\ldots)}(-)^{\underline{\pi}^3_{n,\bar{k}}+\underline{\pi}^2_{n,\bar{k}}+\underline{m}+j_{n,\bar{k}-\bar{p}_n}+\underline{j}^2_{n,\bar{k}-\bar{p}_n}+\underline{\pi}^3_{n,\bar{k}}+\underline{\pi}^2_{n,\bar{k}}+1}
\left\{\begin{matrix}
\bar{\underline{j}}_{n,\bar{k}-\bar{e}_n+\bar{p}_n} & \bar{\underline{\pi}}_{n,\bar{k}} & \bar{\underline{\pi}}^2_{n,\bar{k}} & \bar{\underline{\pi}}^3_{n,\bar{k}} & \\
\underline{j}_{n,\bar{k}-\bar{p}_n} & \bar{\underline{m}} & \bar{\underline{m}} & \underline{j}^2_{n,\bar{k}-\bar{e}_n+\bar{p}_n} & \\
\bar{\underline{j}}^2_{n,\bar{k}-\bar{p}_n} & \underline{m} & 1 & \bar{\underline{m}} & \\
 & \underline{s}_{j_n,\bar{k}-\bar{e}_n+\bar{p}_n} & \underline{s}_{n,\bar{k}} & \bar{\underline{s}}^2_{\pi_n,\bar{k}} & \bar{\underline{s}}^3_{\pi_n,\bar{k}} \\
 & \underline{s}^3_{n,\bar{k}} & \underline{s}_{j_n,\bar{k}-\bar{p}_n} & \underline{s}_I & \underline{s}_I
\end{matrix}\right\} \\
&\times \sum_{\underline{s}^2_{n,\bar{k}}}
\left\{\begin{matrix}
\bar{\underline{j}}_{n,\bar{k}-\bar{e}_n}+\bar{p}_n & \underline{\pi}^2_{n,\bar{k}} & \bar{\underline{j}}^2_{n,\bar{k}-\bar{p}_n} & \\
\bar{\underline{m}} & \bar{\underline{j}}_{n,\bar{k}-\bar{p}_n} & \bar{\underline{\pi}}_{n,\bar{k}} & \\
\underline{s}^2_{n,\bar{k}} & \underline{s}_{n,\bar{k}} & \underline{s}^2_{\pi_n,\bar{k}} & \underline{s}_{j_n,\bar{k}-\bar{p}_n}
\end{matrix}\right\}^{-1}
\left\{\begin{matrix}
\underline{j}^2_{n,\bar{k}-\bar{e}_n+\bar{p}_n} & \underline{\pi}^3_{n,\bar{k}} & \underline{j}^2_{n,\bar{k}-\bar{p}_n} & \\
\underline{\pi}^2_{n,\bar{k}} & \bar{\underline{j}}_{n,\bar{k}-\bar{e}_n+\bar{p}_n} & \underline{m} & \\
\underline{s}^3_{n,\bar{k}} & \underline{s}_{j_n,\bar{k}-\bar{e}_n+\bar{p}_n} & \underline{s}^3_{\pi_n,\bar{k}} & \underline{s}^2_{n,\bar{k}}
\end{matrix}\right\}^{-1} \\
&\times \mathfrak{P}_{SU(3)}(\underline{\pi}_{n,\bar{k}},\underline{\pi}_{p,\bar{k}},\underline{\pi}_{m,\bar{k}},\{\underline{\pi}\}_{\bar{k}+\bar{e}_n},\ldots;\{\underline{j}\};\underline{s}_{0,\bar{k}},\underline{s}_{m,\bar{k}},\underline{s}_{n,\bar{k}},\underline{s}_{p,\bar{k}},\{\underline{s}\}_{\bar{k}+\bar{e}_n},\ldots; \\
&\times \underline{\pi}^2_{n,\bar{k}},\underline{\pi}^2_{m,\bar{k}},\underline{\pi}^2_{n,\bar{k}+\bar{e}_n},\ldots;\underline{j}^2_{n,\bar{k}-\bar{p}_n},\underline{j}^2_{m,\bar{k}-\bar{p}_m}\ldots;\underline{s}^2_{0,\bar{k}},\underline{s}^2_{m,\bar{k}},\underline{s}^2_{n,\bar{k}},\ldots|\underline{m}) \\
&\times \mathfrak{P}_{SU(3)}(\underline{\pi}^2_{n,\bar{k}},\underline{\pi}^2_{m,\bar{k}},\underline{\pi}_{p,\bar{k}}\ldots;\underline{j}^2_{n,\bar{k}-\bar{p}_n},\underline{j}^2_{m,\bar{k}-\bar{p}_m},\ldots;\underline{s}^2_{0,\bar{k}},\underline{s}_{m,\bar{k}},\underline{s}^2_{n,\bar{k}}\ldots; \\
&\times \underline{\pi}^3_{n,\bar{k}},\underline{\pi}^3_{m,\bar{k}},\ldots;\underline{j}^2_{n,\bar{k}-\bar{e}_n+\bar{p}_n},\underline{j}^3_{m,\bar{k}-\bar{p}_m};\underline{s}^3_{0,\bar{k}},\underline{s}^3_{m,\bar{k}},\underline{s}^3_{n,,\bar{k}},\ldots|\underline{m}) \\
&\times |\nu_{\mathrm{orient}}(\underline{\pi}^3_{n,\bar{k}},\underline{\pi}^3_{m,\bar{k}}\underline{\pi}_{p,\bar{k}},\ldots;\underline{j}^2_{n,\bar{k}},\underline{j}^2_{n,\bar{k}-\bar{e}_n},\underline{j}^3_{m,\bar{k}-\bar{p}_m},\underline{j}_{m,\bar{k}-\bar{e}_m+\bar{p}_m},\ldots;\underline{s}^3_{0,\bar{k}},\underline{s}^3_{n,\bar{k}},\underline{s}^3_{m,\bar{k}}\underline{s}_{p,\bar{k}})\rangle.
\end{aligned}
$$

The additional sign $(-)^{(\ldots)}$ contains again the resulting sign, which stems from the permutation of m, n, p and the choices of $\bar{p}_n$, $\bar{p}_m$. Since its construction is the same as before we refrain from writing it down explicitly. The inverse s-classified $6j$ symbols are chosen in such a way that they cancel the corresponding elements in both $\mathcal{P}_{SU(3)}$ expressions.

For case (ii) we get

$$
\begin{aligned}
&\mathrm{tr}(\hat{\underline{\tau}}_I \hat{\underline{h}}_{m,p})\mathrm{tr}(\hat{\underline{\tau}}_I \hat{\underline{h}}_{m,n})|\nu_{\mathrm{orient}}(\{\underline{\pi}\}_{\bar{k}};\{\underline{j}\}_{\bar{k}};\{\underline{s}\}_{\bar{k}})\rangle \\
&= \sum_{\mathcal{S}}(-)^{(\ldots)}
\left\{\begin{matrix}
\underline{\pi}_{p,\bar{k}} & \underline{\pi}_{m,\bar{k}} & \underline{\pi}^2_{m,\bar{k}} & \underline{\pi}^3_{m,\bar{k}} & \\
\underline{\pi}_{n,\bar{k}} & \bar{\underline{m}} & \bar{\underline{m}} & \bar{\underline{\pi}}^2_{p,\bar{k}} & \\
\bar{\underline{\pi}}^2_{n,\bar{k}} & \underline{m} & 1 & \bar{\underline{m}} & \\
 & \underline{s}^2_{\pi_p,\bar{k}} & \underline{s}_{0,\bar{k}} & \bar{\underline{s}}^2_{\pi_m,\bar{k}} & \bar{\underline{s}}^3_{\pi_m,\bar{k}} \\
 & \underline{s}^3_{0,\bar{k}} & \underline{s}^2_{\pi_n,\bar{k}} & \underline{s}_I & \underline{s}_I
\end{matrix}\right\}(-)^{\underline{\pi}^2_{m,\bar{k}}+\underline{\pi}^3_{m,\bar{k}}+1} \\
&\times
\left\{\begin{matrix}
\bar{\underline{\pi}}_{p,\bar{k}} & \bar{\underline{\pi}}^2_{m,\bar{k}} & \bar{\underline{\pi}}^2_{n,\bar{k}} & \\
\bar{\underline{m}} & \bar{\underline{\pi}}_{n,\bar{k}} & \underline{\pi}_{m,\bar{k}} & \\
\underline{s}^2_{0,\bar{k}} & \underline{s}_{0,\bar{k}} & \underline{s}^2_{\pi_m,\bar{k}} & \underline{s}^2_{\pi_n,\bar{k}}
\end{matrix}\right\}
\left\{\begin{matrix}
\bar{\underline{\pi}}^2_{p,\bar{k}} & \bar{\underline{\pi}}^3_{m,\bar{k}} & \bar{\underline{\pi}}^2_{n,\bar{k}} & \\
\bar{\underline{\pi}}^2_{m,\bar{k}} & \underline{\pi}_{p,\bar{k}} & m & \\
\underline{s}^3_{0,\bar{k}} & \underline{s}^2_{\pi_p,\bar{k}} & \underline{s}^3_{\pi_m,\bar{k}} & \underline{s}^2_{0,\bar{k}}
\end{matrix}\right\} \\
&\times \mathfrak{P}_{SU(3)}(\underline{\pi}_{n,\bar{k}},\underline{\pi}_{p,\bar{k}},\underline{\pi}_{m,\bar{k}},\{\underline{\pi}\}_{\bar{k}+\bar{e}_n},\ldots;\{\underline{j}\};\underline{s}_{n,\bar{k}},\underline{s}_{m,\bar{k}},\underline{s}_{0,\bar{k}},\{\underline{s}\}_{\bar{k}+\bar{e}_n},\ldots; \\
&\times \underline{\pi}_{n,\bar{k}},\underline{\pi}^4_{m,\bar{k}},\underline{\pi}^2_{n,\bar{k}+\bar{e}_n},\ldots;\underline{j}^2_{n,\bar{k}}\ldots;\underline{s}^2_{0,\bar{k}},\underline{s}^2_{n,\bar{k}},\underline{s}^2_{m,\bar{k}},\ldots|\bar{m}) \\
&\times \mathfrak{P}_{SU(3)}(\underline{\pi}^2_{n,\bar{k}},\underline{\pi}_{p,\bar{k}},\underline{\pi}^2_{m,\bar{k}},\{\underline{\pi}\}_{\bar{k}+\bar{e}_p},\ldots;\ldots\underline{j}_{p,\bar{k}},\underline{j}^2_{m,\bar{k}},\ldots;\underline{s}^2_{m,\bar{k}},\underline{s}_{p,\bar{k}},\underline{s}^2_{0,\bar{k}},\{\underline{s}\}_{\bar{k}+\bar{e}_n},\ldots; \\
&\times \underline{\pi}^3_{n,\bar{k}},\underline{\pi}^2_{p,\bar{k}},\ldots;\underline{j}^2_{p,\bar{k}},\underline{j}^3_{m,\bar{k}}\ldots;\underline{s}^3_{0,\bar{k}},\underline{s}^3_{m,\bar{k}},\underline{s}^2_{p,\bar{k}},\ldots|\bar{m}) \\
&\times |\nu_{\mathrm{out}}(\underline{\pi}^2_{n,\bar{k}},\underline{\pi}^3_{m,\bar{k}},\underline{\pi}^2_{p,\bar{k}}\ldots;\underline{j}^2_{n,\bar{k}},\underline{j}^3_{m,\bar{k}},\underline{j}^2_{p,\bar{k}},\ldots;\underline{s}^3_{0,\bar{k}},\underline{s}^3_{m,\bar{k}},\underline{s}^2_{n,\bar{k}},\underline{s}^2_{p,\bar{k}})\rangle.
\end{aligned}
$$

For (iii) one gets almost the same as for (i),

$$
\begin{aligned}
&\mathrm{tr}(\hat{\underline{\tau}}_I \hat{A}_{m,n,1-\bar{p}_m,1-\bar{p}_n})\mathrm{tr}(\hat{\underline{\tau}}_I \hat{\underline{A}}_{m,n})|\nu_{\mathrm{orient}}(\{\underline{\pi}\}_{\bar{k}},\{\underline{j}\};\{\underline{s}\}_{\bar{k}})\rangle \\
&= \sum_{\mathcal{S}} (-)^{(\ldots)}(-)^{\underline{\pi}^3_{n,\bar{k}}+\underline{\pi}^2_{n,\bar{k}}+\underline{m}+j_{n,\bar{k}-\bar{p}_n}+\underline{j}^2_{n,\bar{k}-\bar{p}_n}+\underline{\pi}^3_{n,\bar{k}}+\underline{\pi}^2_{n,\bar{k}}+1}
\left\{\begin{matrix}
\bar{\underline{j}}_{n,\bar{k}-\bar{e}_n+\bar{p}_n} & \bar{\underline{\pi}}_{n,\bar{k}} & \bar{\underline{\pi}}^2_{n,\bar{k}} & \bar{\underline{\pi}}^3_{n,\bar{k}} & \\
\underline{j}_{n,\bar{k}-\bar{p}_n} & \bar{\underline{m}} & \bar{\underline{m}} & \underline{j}^2_{n,\bar{k}-\bar{e}_n+\bar{p}_n} & \\
\bar{\underline{j}}^2_{n,\bar{k}-\bar{p}_n} & \underline{m} & 1 & \bar{\underline{m}} & \\
 & \underline{s}_{j_n,\bar{k}-\bar{e}_n+\bar{p}_n} & \underline{s}_{n,\bar{k}} & \bar{\underline{s}}^2_{\pi_n,\bar{k}} & \bar{\underline{s}}^3_{\pi_n,\bar{k}} \\
 & \underline{s}^3_{n,\bar{k}} & \underline{s}_{j_n,\bar{k}-\bar{p}_n} & \underline{s}_I & \underline{s}_I
\end{matrix}\right\} \\
&\quad\times \sum_{\underline{s}^2_{n,\bar{k}}} \left\{\begin{matrix}
\bar{\underline{j}}_{n,\bar{k}-\bar{e}_n}+\bar{p}_n & \underline{\pi}^2_{n,\bar{k}} & \bar{\underline{j}}^2_{n,\bar{k}-\bar{p}_n} & \\
\bar{\underline{m}} & \bar{\underline{j}}_{n,\bar{k}-\bar{p}_n} & \bar{\underline{\pi}}_{n,\bar{k}} & \\
\underline{s}^2_{n,\bar{k}} & \underline{s}_{n,\bar{k}} & \underline{s}^2_{\pi_n,\bar{k}} & \underline{s}_{j_n,\bar{k}-\bar{p}_n}
\end{matrix}\right\}^{-1}
\left\{\begin{matrix}
\underline{j}^2_{n,\bar{k}-\bar{e}_n+\bar{p}_n} & \underline{\pi}^3_{n,\bar{k}} & \underline{j}^2_{n,\bar{k}-\bar{p}_n} & \\
\underline{\pi}^2_{n,\bar{k}} & \bar{\underline{j}}_{n,\bar{k}-\bar{e}_n+\bar{p}_n} & \underline{m} & \\
\underline{s}^3_{n,\bar{k}} & \underline{s}_{j_n,\bar{k}-\bar{e}_n+\bar{p}_n} & \underline{s}^3_{\pi_n,\bar{k}} & \underline{s}^2_{n,\bar{k}}
\end{matrix}\right\}^{-1} \\
&\quad\times \mathfrak{P}_{SU(3)}(\underline{\pi}_{n,\bar{k}},\underline{\pi}_{p,\bar{k}},\underline{\pi}_{m,\bar{k}},\{\underline{\pi}\}_{\bar{k}+\bar{e}_n},\ldots;\{\underline{j}\};\underline{s}_{m,\bar{k}},\underline{s}_{0,\bar{k}},\underline{s}_{n,\bar{k}},\{\underline{s}\}_{\bar{k}+\bar{e}_n}\ldots; \\
&\quad\times \underline{\pi}^2_{n,\bar{k}},\underline{\pi}^2_{m,\bar{k}},\underline{\pi}^2_{n,\bar{k}+\bar{e}_n},\ldots;\underline{j}^2_{n,\bar{k}-\bar{p}_n},\ldots;\underline{s}^2_{0,\bar{k}},\underline{s}^2_{m,\bar{k}},\underline{s}^2_{n,\bar{k}},\ldots|\bar{m}) \\
&\quad\times \mathfrak{P}_{SU(3)}(\underline{\pi}^2_{n,\bar{k}},\underline{\pi}_{p,\bar{k}},\underline{\pi}^2_{m,\bar{k}},\{\underline{\pi}\}^2_{\bar{k}+\bar{e}_n},\ldots;\ldots\underline{j}^2_{m,\bar{k}-\bar{p}_m},\underline{j}^2_{n,\bar{k}-\bar{p}_n},\ldots;\underline{s}^2_{m,\bar{k}},\underline{s}^2_{0,\bar{k}},\underline{s}^2_{n,\bar{k}},\{\underline{s}\}^2_{\bar{k}+\bar{e}_n}\ldots; \\
&\quad\times \underline{\pi}^3_{n,\bar{k}},\underline{\pi}^3_{m,\bar{k}},\ldots;\underline{j}^2_{n,\bar{k}-\bar{e}_n+\bar{p}_n},\underline{j}^3_{m,\bar{k}-\bar{e}_m+\bar{p}_m},\ldots;\underline{s}^3_{0,\bar{k}},\underline{s}^3_{m,\bar{k}},\underline{s}^3_{n,\bar{k}},\ldots|\bar{m}) \\
&\quad\times |\nu_{\mathrm{orient}}(\underline{\pi}^3_{n,\bar{k}},\underline{\pi}^3_{m,\bar{k}}\underline{\pi}_{p,\bar{k}},\ldots;\underline{j}^2_{n,\bar{k}},\underline{j}^2_{n,\bar{k}-\bar{e}_n},\underline{j}^3_{m,\bar{k}-\bar{p}_m},\underline{j}^2_{m,\bar{k}-\bar{e}_m+\bar{p}_m},\ldots;\underline{s}^3_{0,\bar{k}},\underline{s}^3_{n,\bar{k}},\underline{s}^3_{m,\bar{k}}\underline{s}_{p,\bar{k}})\rangle.
\end{aligned}
$$

For (iv) finally [compare to (ii)]

$$
\begin{aligned}
&\mathrm{tr}(\hat{\underline{\tau}}_I \hat{A}_{m,p,1-\bar{p}_m,\bar{p}_p})\mathrm{tr}(\hat{\underline{\tau}}_I \hat{\underline{A}}_{m,n})|\nu_{\mathrm{orient}}(\{\underline{\pi}\}_{\bar{k}},\{\underline{j}\};\{\underline{s}\}_{\bar{k}})\rangle \\
&= \sum_{\mathcal{S}} (-)^{(\ldots)}
\left\{\begin{matrix}
\underline{\pi}_{p,\bar{k}} & \underline{\pi}_{m,\bar{k}} & \underline{\pi}^2_{m,\bar{k}} & \underline{\pi}^3_{m,\bar{k}} & \\
\underline{\pi}_{n,\bar{k}} & \bar{\underline{m}} & \bar{\underline{m}} & \bar{\underline{\pi}}^2_{p,\bar{k}} & \\
\bar{\underline{\pi}}^2_{n,\bar{k}} & \underline{m} & 1 & \bar{\underline{m}} & \\
 & \underline{s}^2_{\pi_p,\bar{k}} & \underline{s}_{0,\bar{k}} & \bar{\underline{s}}^2_{\pi_m,\bar{k}} & \bar{\underline{s}}^3_{\pi_m,\bar{k}} \\
 & \underline{s}^3_{0,\bar{k}} & \underline{s}^2_{\pi_n,\bar{k}} & \underline{s}_I & \underline{s}_I
\end{matrix}\right\}(-)^{\underline{\pi}^2_{m,\bar{k}}+\underline{\pi}^3_{m,\bar{k}}+1} \\
&\quad\times \left\{\begin{matrix}
\bar{\underline{\pi}}_{p,\bar{k}} & \bar{\underline{\pi}}^2_{m,\bar{k}} & \bar{\underline{\pi}}^2_{n,\bar{k}} & \\
\bar{\underline{m}} & \bar{\underline{\pi}}_{n,\bar{k}} & \underline{\pi}_{m,\bar{k}} & \\
\underline{s}^2_{0,\bar{k}} & \underline{s}_{0,\bar{k}} & \underline{s}^2_{\pi_m,\bar{k}} & \underline{s}^2_{\pi_n,\bar{k}}
\end{matrix}\right\}
\left\{\begin{matrix}
\bar{\underline{\pi}}^2_{p,\bar{k}} & \bar{\underline{\pi}}^3_{m,\bar{k}} & \bar{\underline{\pi}}^2_{n,\bar{k}} & \\
\bar{\underline{\pi}}^2_{m,\bar{k}} & \underline{\pi}_{p,\bar{k}} & m & \\
\underline{s}^3_{0,\bar{k}} & \underline{s}^2_{\pi_p,\bar{k}} & \underline{s}^3_{\pi_m,\bar{k}} & \underline{s}^2_{0,\bar{k}}
\end{matrix}\right\} \\
&\quad\times \mathfrak{P}_{SU(3)}(\underline{\pi}_{n,\bar{k}},\underline{\pi}_{p,\bar{k}},\underline{\pi}_{m,\bar{k}},\{\underline{\pi}\}_{\bar{k}+\bar{e}_n},\ldots;\{\underline{j}\};\underline{s}_{n,\bar{k}},\underline{s}_{m,\bar{k}},\underline{s}_{0,\bar{k}},\{\underline{s}\}_{\bar{k}+\bar{e}_n},\ldots; \\
&\quad\times \underline{\pi}_{n,\bar{k}},\underline{\pi}^4_{m,\bar{k}},\underline{\pi}^2_{n,\bar{k}+\bar{e}_n},\ldots;\underline{j}^2_{n,\bar{k}}\ldots;\underline{s}^2_{0,\bar{k}},\underline{s}^2_{n,\bar{k}},\underline{s}^2_{m,\bar{k}},\ldots|\bar{m}) \\
&\quad\times \mathfrak{P}_{SU(3)}(\underline{\pi}^2_{n,\bar{k}},\underline{\pi}_{p,\bar{k}},\underline{\pi}^2_{m,\bar{k}},\{\underline{\pi}\}_{\bar{k}+\bar{e}_p-2\bar{p}_p}\ldots;\ldots\underline{j}_{p,\bar{k}},\underline{j}^2_{m,\bar{k}},\ldots;\underline{s}^2_{m,\bar{k}},\underline{s}_{p,\bar{k}},\underline{s}_{0,\bar{k}},\{\underline{s}\}_{\bar{k}+\bar{e}_p-2\bar{p}_p},\ldots; \\
&\quad\times \underline{\pi}^3_{m,\bar{k}},\underline{\pi}^2_{p,\bar{k}},\ldots;\underline{j}^2_{p,\bar{k}},\underline{j}^3_{m,\bar{k}-\bar{e}_m+\bar{p}_m},\ldots;\underline{s}^3_{0,\bar{k}},\underline{s}^3_{m,\bar{k}},\underline{s}^2_{p,\bar{k}},\ldots|\bar{m}) \\
&\quad\times |\nu_{\mathrm{out}}(\underline{\pi}^2_{n,\bar{k}},\underline{\pi}^3_{m,\bar{k}},\underline{\pi}^2_{p,\bar{k}}\ldots;\underline{j}^2_{n,\bar{k}},\underline{j}^3_{m,\bar{k}},\underline{j}^2_{p,\bar{k}},\underline{j}^2_{m,\bar{k}-\bar{e}_m}\ldots;\underline{s}^3_{0,\bar{k}},\underline{s}^3_{m,\bar{k}},\underline{s}^2_{n,\bar{k}},\underline{s}^2_{p,\bar{k}})\rangle.
\end{aligned}
$$

VI. CONCLUSION

In this paper we have taken the first steps towards the computation of the fundamental QCD spectrum within the LQG approach to quantum gravity. More precisely, we have computed the matrix elements of the Yang-Mills contribution to the Hamiltonian analytically in closed form as far as the gluon field is concerned, while for the gravitational degrees of freedom a fully analytical analysis is not possible due to the necessity of computing the spectrum of the volume operator, which is known to be possible only numerically. Obviously, more analytical and numerical work is necessary to determine the spectrum with sufficient precision. However, the focus of this paper was not so much on the actual computation of the spectrum, but rather to prepare the necessary analytical tools. The other message we communicate is that the Hamiltonian that we considered in this paper needs to be improved by methods coming from renormalization theory. For this reason, we refrain from investigating more closely the spectrum of the Hamiltonian considered here from [28], but one should rather analyze the improved Hamiltonian. We hope that, once one has found a Hamiltonian description of renormalization, its fixed point Hamiltonian can be used, as this Hamiltonian has minimal if not vanishing discretization errors. Once this point has been understood, we can address the important question of how the picture of the running of the Yang-Mills coupling on a gravitational background is changed in the context of the quantum gravity coupled system. Namely, it transpires that the background dependent Hamiltonian depends on a cutoff while the background independent one does not. Thus, the mechanism for the running of the coupling is very different for these two theories. We reserve this analysis for future research.

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APPENDIX: BRIEF REVIEW OF THE 3j'S AND 6j'S FOR $SU(2)$

For self-containedness some important properties of nj symbols for the group $SU(2)$ are listed here. Introductions to recoupling theory can be found in various textbooks on quantum mechanics and quantum angular momentum, e.g., [43]. For an extensive list of properties of nj symbols see, e.g., [65]

$3j$ symbols

Relation to Clebsh-Gordan coefficients

$$\langle a,\alpha;b,\beta|c,\gamma\rangle = (-)^{b-a+\gamma}\sqrt{2c+1}\begin{pmatrix} a & b & c \\ \alpha & \beta & -\gamma \end{pmatrix},$$

where $|b,\beta;a,\alpha\rangle = |b,\beta\rangle \otimes |a,\alpha\rangle$.

Compatibility criteria

If one (or several) of the following rules is violated, then $\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix}$ is vanishing:

* $a,b,c \in \frac{1}{2}\mathbb{N}, a \pm \alpha \in \mathbb{N}, -a \le \alpha \le a, \cdots,$
* $\alpha + \beta + \gamma = 0,$
* $a+b+c \in \mathbb{N}, |a-b| \le c \le a+b$ (triangle inequality).

Symmetries

$$\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = (-)^{a+b+c}\begin{pmatrix} a & b & c \\ -\alpha & -\beta & -\gamma \end{pmatrix} = (-)^{a+b+c}\begin{pmatrix} b & a & c \\ \beta & \alpha & \gamma \end{pmatrix} = \begin{pmatrix} b & c & a \\ \beta & \gamma & \alpha \end{pmatrix}.$$

$6j$ symbols

Definition in terms of $3j$'s

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \sum_{\mu_1,\ldots,\mu_6} (-)^{\sum_{i=1}^{6}(j_i-\mu_i)} \begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & -\mu_3 \end{pmatrix} \times \begin{pmatrix} j_1 & j_5 & j_6 \\ -\mu_1 & \mu_5 & \mu_6 \end{pmatrix}\begin{pmatrix} j_4 & j_5 & j_3 \\ \mu_4 & -\mu_5 & \mu_3 \end{pmatrix} \times \begin{pmatrix} j_4 & j_2 & j_6 \\ -\mu_4 & -\mu_2 & -\mu_6 \end{pmatrix}.$$

Symmetries

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Bmatrix} b & a & c \\ e & d & f \end{Bmatrix} = \begin{Bmatrix} b & c & a \\ e & f & d \end{Bmatrix} = \begin{Bmatrix} d & e & c \\ a & b & f \end{Bmatrix} = \begin{Bmatrix} d & b & f \\ a & e & c \end{Bmatrix} = \begin{Bmatrix} a & e & f \\ d & b & c \end{Bmatrix}.$$

Compatibility

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = 0,$$

unless the triangle inequalities hold for $\{a,b,c\}$, $\{a,e,f\}, \{d,b,f\}$, and $\{d,e,c\}$.

Orthogonality

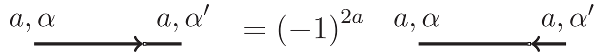
$$\sum_x d_x \begin{Bmatrix} a & b & x \\ d & e & c \end{Bmatrix}\begin{Bmatrix} a & b & x \\ d & e & c' \end{Bmatrix} = \delta_{c,c'}\frac{1}{d_c}$$

if the compatibility requirements are fulfilled.

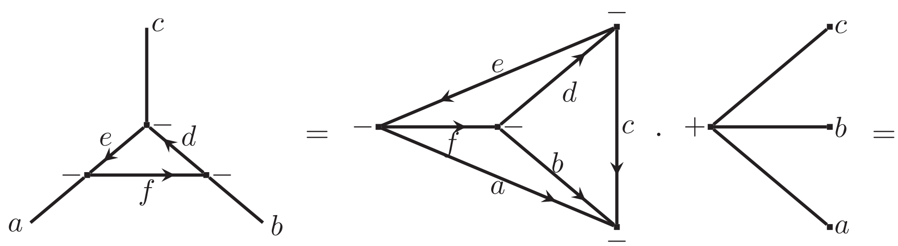
Graphical calculus of $SU(2)$

The definitions of the basic objects in this graphical calculus are the same as in [41,42] and thus reduce to the same labeling as has been done for the $SU(3)$ case.

Some of the rules for changing the graphs however have altered, e.g., since the magnetic numbers are now in $\frac{1}{2}\mathbb{N}$ an arrow may change its direction by adding a sign factor of $(-)^{2a}$,

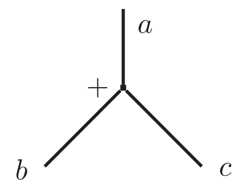


This changes some of the more complex recoupling schemes (for a full list see [43]), e.g., the extraction of a $6j$ symbol,



(A1)

$$\begin{Bmatrix} a & f & e \\ d & c & b \end{Bmatrix} \cdot$$



[1] J. Kogut and L. Susskind, Hamiltonian formulation of Wilson's lattice gauge theories, Phys. Rev. D **11**, 395 (1975).

[2] K. Osterwalder and R. Schrader, Axioms for Euclidean Green's functions, Commun. Math. Phys. **31**, 83 (1973).

[3] J. Glimm and A. Jaffe, *Quantum Physics—A Functional Integral Point of View* (Springer-Verlag, New York, 1987).

[4] T. Balaban, Large field renormalization I. The basic step of the R operation, Commun. Math. Phys. **122**, 207 (1989).

[5] T. Balaban, Large field renormalization II. Localization, exponentation, and bounds for the R operation, Commun. Math. Phys. **122**, 355 (1989).

[6] J. Froehlich, *An Introduction to Some Topics in Constructive QFT* (Springer-Verlag, New York, 1978).

[7] V. Rivasseau, Constructive field theory and applications: Perspectives and open problems, J. Math. Phys. (N.Y.) **41**, 3764 (2000).

[8] C. Gattringer and C. B. Lang, *Quantum Chromodynamics on the Lattice* (Springer-Verlag, New York, 2010).

[9] M. Creutz, *Quarks, Gluons, and Lattices* (Cambridge University Press, Cambridge, 1983).

[10] R. Brunetti, K. Fredenhagen, and R. Verch, The generally covaraiant locality principle—A new paradigm for local quantum field theory, Commun. Math. Phys. **237**, 31 (2003).

[11] D. Buchholz, *Algebraic Quantum Field Theory: A Status Report* (2000).

[12] K. Fredenhagen, K.-H. Rehren, and E. Seiler, Quantum field theory: Where we are, Lect. Notes Phys. **721**, 61 (2007).

[13] A. Jaffe and G. Ritter, *Quantum Field Theory on Curved Backgrounds. II. Spacetime Symmetries* (2007).

[14] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, 2007).

[15] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, 2004).

[16] K. Giesel and H. Sahlmann, From classical to quantum gravity: Introduction to loop quantum gravity, *Proc. Sci.* QGQGS2011 (2011) 002.

[17] T. Thiemann, Quantum spin dynamics V: Quantum gravity as the natural regulator of matter quantum field theories, Classical Quantum Gravity **15**, 1281 (1998).

[18] T. Thiemann, Quantum spin dynamics (QSD), Classical Quantum Gravity **15**, 839 (1998).

[19] T. Thiemann, Quantum spin dynamics (QSD) II, Classical Quantum Gravity **15**, 875 (1998).

[20] K. Giesel and T. Thiemann, Scalar material reference systems and loop quantum gravity, Classical Quantum Gravity **32**, 135015 (2015).

[21] M. Domagala, K. Giesel, W. Kaminski, and J. Lewandowski, Gravity quantized, Phys. Rev. D **82**, 104038 (2010).

[22] V. Husain and T. Pawlowski, Time and a Physical Hamiltonian for a Quantum Gravity, Phys. Rev. Lett. **108**, 141301 (2012).

[23] V. Husain and T. Pawloski, A computable framework for loop quantum gravity, arXiv:1305.5203.
[24] K. V. Kuchar and C. G. Torre, World sheet diffeomorphisms and the canonical string, J. Math. Phys. (N.Y.) **30**, 1769 (1989).
[25] K. Giesel and T. Thiemann, Algebraic quantum gravity (AQG) IV. Reduced phase space quantization of loop quantum gravity, Classical Quantum Gravity **27**, 175009 (2010).
[26] J. Lewandowski, A. Okolow, H. Sahlmann, and T. Thiemann, Uniqueness of diffeomorphism invariant states on holonomy-flux algebras, Commun. Math. Phys. **267**, 703 (2006).
[27] C. Fleischhack, Quantization and Analysis on Symmetric Spaces, *Proceedings of the ESI workshops* (Wien, 2006), pp. 55–63.
[28] K. Giesel and T. Thiemann, Algebraic quantum gravity (AQG) I. Conceptual setup, Classical Quantum Gravity **24**, 2465 (2007).
[29] K. Giesel and T. Thiemann, Algebraic quantum gravity (AQG) II. Semiclassical analysis, Classical Quantum Gravity **24**, 2499 (2007).
[30] K. Giesel and T. Thiemann, Algebraic quantum gravity (AQG) III. Semiclassical perturbation theory, Classical Quantum Gravity **24**, 2499 (2007).
[31] H. Sahlmann, T. Thiemann, and O. Winkler, Coherent states for canonical quantum general relativity and the infinte tensor product extension, Nucl. Phys. **B606**, 401 (2001).
[32] R. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[33] A. Ashtekar, *Lectures on Nonperturbative Canonical Gravity* (Word Scientific, Singapore, 1991).
[34] J. Fernando Barbero G., A real polynomial formulation of general relativity in terms of connection, Phys. Rev. D **49**, 6935 (1994).
[35] J. Fernando Barbero G., Real Ashtekar variables for Lorentzian signature spacetimes, Phys. Rev. D **51**, 5507 (1995).
[36] J. Fernando Barbero G., From Euclidean to Lorentzian general relativity: The real way, Phys. Rev. D **54**, 1492 (1996).
[37] K. Giesel, S. Hofmann, T. Thiemann, and O. Winkler, Manifestly gauge-invariant general relativistic perturbation theory: I. Foundations, Classical Quantum Gravity **27** (2010).
[38] A. Ashtekar and J. Lewandowski, Quantum theory of geometry II: Volume operators, Adv. Theor. Math. Phys. **1**, 388 (1997).
[39] J. Brunnemann and T. Thiemann, Simplification of the spectral analysis of the volume operator in loop quantum gravity, Classical Quantum Gravity **23**, 1289 (2006).
[40] J. Brunnemann and D. Rideout, Spectral analysis of the volume operator in loop quantum gravity, arXiv:gr-qc/0612147.
[41] E. Alesci, T. Thiemann, and A. Zipfel, Linking covariant and canonical LQG: New solutions to the Euclidean scalar constraint, Phys. Rev. D **86**, 024017 (2012).
[42] E. Alesci, K. Liegener, and A. Zipfel, Matrix elements of Lorentzian Hamiltonian constraint in LQG, Phys. Rev. D **88**, 084043 (2013).
[43] D. M. Brink and G. R. Satchler, *Angular Momentum* (Clarendon Press Oxford, Oxford, 1968).
[44] P. Cvitanović, *Group Theory: Birdtracks, Lie's, and Exceptional Groups* (Princeton University Press, Princeton, 2008).
[45] M. Grigorescu, $SU(3)$ Clebsch-Gordan coefficients, Stud. Cercetari Fiz. **36**, 3 (1984).
[46] B. C. Hall, *Lie Groups, Lie Algebras, and Representations* (Springer-Verlag, New York, 2003).
[47] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory* (Springer-Verlag, New York, 1972).
[48] N. Ja. Vilenkin and A. U. Klimyk, *Representations of Lie Groups and Special Functions, Recent Advances* (Kluwer Academic Publishers, Dordrecht, 1995).
[49] Z. Pluhar, Yu F. Smirnov, and V. N. Tolstoy, Clebsch-Gordan coefficients of $SU(3)$ with simple symmetry properties, J. Phys. A **19**, 29 (1986).
[50] Z. Pluhar, L. J. Weigert, and P. Holan, Symmetry properties of s-classified $SU(3)$ $3j$, $6j$, and $9j$ symbols, J. Phys. A **19**, 29 (1986).
[51] J.-R. Derome and W. T. Sharp, Racah algebra for an arbitrary group, J. Math. Phys. (N.Y.) **6**, 1584 (1965).
[52] N. Ja. Vilenkin and A. U. Klimyk, *Representations of Lie Groups and Special Functions*, Simplest Lie Groups, Special Functions, and Integral Transforms Vol. 1 (Kluwer Academic Publishers, Dordrecht, 1991).
[53] A. Yutsis, I. Levinson, and V. Vanagas, *The Theory of Angular Momentum* (Israel Program for Scientific Translations, Jerusalem, 1962).
[54] J. Meixner and F. W. Schaefke, *Mathieu Functions and Sphaeroid Functions* (Springer-Verlag, Berlin, 1954).
[55] D. Robson and D. M. Webber, Gauge theories on a small lattice, Z. Phys. C **7**, 53 (1980).
[56] D. Robson and D. M. Webber, The gauge boson sector of U(1) lattice theories, Z. Phys. C **10**, 159 (1981).
[57] D. Robson and D. M. Webber, Gauge covariance in lattice field theories, Z. Phys. C **15**, 199 (1982).
[58] G. Burgio, R. De Pietri, H. A. Morales-Tecotl, L. F. Urrutia, and J. D. Vergara, Matrix elements of the plaquette operator of lattice gauge theory, arXiv:hep-lat/9911019.
[59] G. Burgio, R. De Pietri, H. A. Morales-Tecotl, L. F. Urrutia, and J. D. Vergara, The basis of the physical Hilbert space of lattice gauge theory, Nucl. Phys. **B566**, 547 (2000).
[60] M. Mathur, Loop states in lattice gauge theories, Phys. Lett. B **640**, 292 (2006).
[61] H. Kroeger, A. Hosseinizadeh, J. F. Laprise, and J. Kroeger, Spectrum and wave functions of exicted states in lattice gauge theory, *Proc. Sci.* LATTICE2008 (2008) 235 [arXiv:0902.2944].
[62] J.-W. Choe, A. Duncan, and R. Roskies, Lanczos calculation of the spectrum of Hamiltonian lattice gauge theory, Phys. Rev. D **37**, 472 (1988).
[63] A. Duncan and R. Roskies, Variational estimates for spectra in lattice Hamiltonian theories, Phys. Rev. D **31**, 2 (1985).
[64] E. Alesci and C. Rovelli, The complete LQG propagator I. Difficulties with the Barrett-Crane vertex, Phys. Rev. D **76**, 104012 (2007).
[65] http://functions.wolfram.com/HypergeometricFunctions/ThreeJSymbol/.