Geometry of the quantum Hall effect: An effective action for all dimensions

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We present a general formula for the topological part of the effective action for integer quantum Hall systems in higher dimensions, including fluctuations of the gauge field and metric around background fields of a specified topological class. The result is based on a procedure of integrating up from the Dolbeault index density which applies for the degeneracies of Landau levels, combined with some input from the standard descent procedure for anomalies. Features of the topological action in $(2+1)$, $(4+1)$, $(6+1)$ dimensions, including the contribution due to gravitational anomalies, are discussed in some detail.

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I. INTRODUCTION

The quantum Hall effect (QHE) has long been a fascinating phenomenon, both experimentally and theoretically [1]. On the theoretical side, there has been the immensely successful description based on wave functions. It was also realized fairly early that the Chern-Simons action provides an effective description of many of the features of the quantum Hall effect. For a quantum Hall droplet, i.e., for a finite system with boundary, the effective description must also include the action for a chiral boson theory. This is needed to render the Chern-Simons action fully gauge invariant and provides a description for the edge excitations of the droplet. The Chern-Simons action involves the electromagnetic vector potential and hence pertains to the electromagnetic response of the QHE system; in fact, the effective action incorporates the relevant transport coefficient, namely, the Hall conductivity. Some of the other transport coefficients of interest, such as the Hall viscosity, correspond to the response of the Hall system to perturbations of the background metric [2,3]. Further, considerations of the quantum Hall effect on spaces of nontrivial topology can give insights into the physics of the problem, even though experimentally we may only be interested in spaces of trivial topology [4,5]. As a result, the response of QHE systems to changes in the background metric, captured via an effective action on spaces of different geometry and topology, has become the focus of many recent studies [6–15]. The mathematical structures underlying the quantum Hall effect have also generated much interest in their own right, giving further impetus to such studies.

Another branch of interesting generalizations of the QHE has been to higher dimensions [16–26]. The Landau problem has been analyzed and the wave functions and effective actions have been obtained for a number of different spaces such as the four-sphere, complex projective spaces, etc. In higher dimensions, the background gauge field can be Abelian or non-Abelian. As in the $(2+1)$-dimensional case, one can consider a bulk effective action which captures the response to fluctuations of the gauge field. The topological part of this effective action is a generalization of the Chern-Simons action to higher dimensions [24–26]. Also in analogy with the lower dimensional case, one can consider a quantum Hall droplet which would then allow for edge excitations, even when the background gauge field is fixed, i.e., nonfluctuating. The effective action for this has also been obtained in the case of integer filling fraction $\nu = 1$; it is a generalization of the Wess-Zumino-Witten action [18–20,23]. Once fluctuations in the gauge field are also introduced, the calculated bulk and boundary actions were shown to be consistent with the mutual cancellation of anomalies in the gauge symmetry [24].[1] The complete effective action captures the response of the system to various gauge field perturbations and edge fluctuations of the droplet.

The natural question which arises from the juxtaposition of the two lines of development outlined above would be: What is the effective action for QHE systems in higher dimensions, including the response to gravitational fields? This is the subject of the present paper. We will start with an index theorem for the degeneracy of the quantum Hall states at various Landau levels on a Kähler manifold. This degeneracy is also the total charge (for the relevant gauge field) of the fully occupied Landau level and hence the response of the system to changes in the electrostatic-type component (or time component) of the corresponding vector potential. The effective action can then be constructed, in essence, by integrating this response with respect to the vector potential and making the result covariant. We will use complex projective spaces to

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[1]This is a generalization of the well-known similar structure in two dimensions [27]; the cancellation of anomalies between the bulk and boundary terms is traceable to [28].

illustrate various aspects of these considerations, but the result is general and applies to any Kähler manifold.

As mentioned above, many aspects of the effective action in $(2+1)$-dimensions with nontrivial geometry and topology have been considered by several authors. In [8], local Galilean invariance is used to elucidate features of the effective action. Effective actions, including gravitational contributions, are obtained in [9,11] from microscopic dynamics. Geometric adiabatic transport has been considered in [12–14]. In [14], the effective action is discussed from the point of view of an index density. There are many points of concordance with these papers, when we specialize our general effective action to $(2+1)$ dimensions; these will be referred to as the occasion arises.

This paper is organized as follows. In the next section, we start with the Landau problem on complex projective spaces and the degeneracy of the quantum Hall states at various Landau levels. In Sec. III, we consider the degeneracy in terms of the relevant index theorem. The index density can be identified as the charge density or the response to the time component of an Abelian gauge field. We can then write down the gauge-field dependent terms of the topological part of the effective action. This is the action which would correspond to what is obtained by integrating out the fermions occupying the lowest Landau levels. Generalization to higher Landau levels is taken up in Sec. IV. In Sec. V, we consider general gauge and gravitational fields and write down the effective action for higher dimensions. This action, given in (41), is the main result of this paper. The gauge-field dependent terms in this action are simplified in Sec. VI for $(4+1)$ dimensions, working out special cases in some detail; Sec. VII addresses the same in $(6+1)$ dimensions. The contribution from the terms related to the gravitational anomaly is considered in Sec. VIII, with details worked out for $(2+1)$, $(4+1)$ and $(6+1)$ dimensions explicitly. A discussion section compares our results with the existing literature. There is a short Appendix on some basic features and geometry of $\mathbb{CP}^k$ spaces.

For clarification we emphasize that, in this paper, we consider fully filled Landau levels (integer QHE) on manifolds without a boundary. What is obtained is the topological part of the bulk effective action. Fully filled Landau levels on manifolds with a boundary, droplets of finite size (with possible edge excitations) and the corresponding bulk and boundary actions are important issues. These will be left to future work.

II. LANDAU LEVELS AND DEGENERACY

As mentioned in the Introduction, we will be concerned with the QHE and Landau levels on spacetimes of the form $\mathbb{R}\times K$, where the spatial manifold K is a complex manifold. We will consider the case of K being Kähler to begin with, where the background magnetic field can be taken as the Kähler two-form, up to a constant of proportionality, with the Hamiltonian being proportional to the Laplace operator. The states of the lowest Landau level (LLL) correspond to wave functions which satisfy a holomorphicity condition. More precisely, the wave functions will be sections of an appropriate power of the line bundle with the background field as the curvature. In higher dimensions, non-Abelian background fields are possible, so a slight generalization is needed. Further, wave functions for the higher Landau levels can be considered as the wave functions of the lowest Landau level of an equivalent problem where the charged particles carry an appropriate amount of spin. These statements are somewhat abstract and it is illuminating to have an explicit construction. For most of the explicit examples, we will consider the case of K being a complex projective space of complex dimension k, i.e, $K=\mathbb{CP}^k$. So we start by setting up the framework for QHE on $\mathbb{CP}^k$.

Since $\mathbb{CP}^k$ is the coset space $SU(k+1)/U(k)$, the discussion is most easily carried out following the group theoretic analysis given in [17–19]. The group $SU(k+1)$ is the full group of continuous isometries of $\mathbb{CP}^k$, with $U(k)$ as the isotropy group at each point. Thus the representation of $U(k)$ for any field is the specification of its spin. Further, the curvatures on the manifold take values in the Lie algebra of $U(k)$. In particular, they are constant in the tangent frame basis. (Explicit formulas for the curvatures on $\mathbb{CP}^k$ are given in the Appendix.) It is then possible to consider additional "constant" gauge background fields which are proportional to these curvatures; more explicitly, we can have an Abelian background corresponding to the $U(1)$ part of $U(k)\sim U(1)\times SU(k)$ and a non-Abelian background corresponding to the $SU(k)$ part. This gives a well-posed Landau problem of particle motion in a constant background field.

Let t_A, $A=1,2,\ldots,k^2+2k$, denote a basis of Hermitian $(k+1)\times(k+1)$-matrices viewed as the fundamental representation of the Lie algebra of $SU(k+1)$. We choose the normalization by $\mathrm{Tr}(t_A t_B)=\frac{1}{2}\delta_{AB}$. The Lie algebra commutation rules, when needed, will be taken to be of the form $[t_A,t_B]=if_{ABC}t_C$, with structure constants f_{ABC}. The generators corresponding to the $SU(k)$ part of $U(k)\subset SU(k+1)$ will be denoted by t_a, $a=1,2,\ldots,k^2-1$ and the generator for the $U(1)$ direction of the subgroup $U(k)$ will be denoted by t_{k^2+2k}.

The Landau level wave functions can be considered as functions on $SU(k+1)$ which have a specific transformation property under the $U(k)\subset SU(k+1)$. A basis of functions on the group $SU(k+1)$ is given by the matrices corresponding to the group elements in a representation, or the so-called Wigner $\mathcal{D}$-functions, which are defined as

$$\mathcal{D}^{(J)}_{\mathfrak{l};\mathfrak{r}}(g)=\langle J,\mathfrak{l}|g|J,\mathfrak{r}\rangle, \tag{1}$$

where $\mathfrak{l}$, $\mathfrak{r}$ stand for two sets of quantum numbers specifying the states within the representation. There is a natural left and right action on an element $g \in SU(k+1)$, defined by

$$\hat{L}_A g = T_A g, \qquad \hat{R}_A g = g T_A, \tag{2}$$

where T_A are the $SU(k+1)$ generators in the representation to which g belongs.

There are $2k$ right generators of $SU(k+1)$ which are not in the algebra of $U(k) \subset SU(k+1)$; these can be separated into T_{+i}, $i = 1, 2, \ldots, k$, which are of the raising type and T_{-i} which are of the lowering type. These generate translations while $U(k)$ generates rotations at a point. We can thus define the covariant derivatives on $\mathbb{CP}^k$ in terms of the right translation operators on g as

$$D_{\pm i} = i \frac{\hat{R}_{\pm i}}{r}, \tag{3}$$

where r is a parameter with the dimensions of length. (The volume of the manifold will be proportional to r^{2k}.) Since the strength of the gauge field is given by the commutator of covariant derivatives, we can then specify the background magnetic field for our problem by specifying the action of $U(k)$ on the wave functions; this is so because the commutators of $\hat{R}_{+i}$ and $\hat{R}_{-i}$ are in the Lie algebra of $U(k)$. The constant background field is given by the conditions

$$\hat{R}_a \Psi^J_{m;\alpha}(g) = (T_a)_{\alpha\beta} \Psi^J_{m;\beta}(g) \tag{4}$$

$$\hat{R}_{k^2+2k} \Psi^J_{m;\alpha}(g) = -\frac{nk}{\sqrt{2k(k+1)}} \Psi^J_{m;\alpha}(g), \tag{5}$$

where $m = 1, \ldots, \dim J$ counts the degeneracy of the Landau level. Equation (4) shows that the wave functions $\Psi^J_{m;\alpha}$ transform, under right rotations, as a representation $\tilde{J}$ of $SU(k)$. Here $(T_a)_{\alpha\beta}$ are the representation matrices for the generators of $SU(k)$ in the representation $\tilde{J}$, and n is an integer characterizing the Abelian part of the background field. α, β label states within the $SU(k)$ representation $\tilde{J}$ [which is itself contained in the representation J of $SU(k+1)$]. The index α carried by the wave functions $\Psi^J_{m;\alpha}(g)$ is basically the gauge index. The wave functions are sections of a $U(k)$ bundle on $\mathbb{CP}^k$.

The Hamiltonian H for the Landau problem is proportional to the covariant Laplacian on $\mathbb{CP}^k$; explicitly the action of H on wave functions is given by

$$\begin{aligned} H\Psi &= -\frac{1}{4m}(D_{+i}D_{-i} + D_{-i}D_{+i})\Psi \\ &= \frac{1}{2mr^2}\left[\hat{R}_{+i}\hat{R}_{-i} + \frac{1}{2}(if_{-i,+i,a}\hat{R}_a \right. \\ &\quad \left. + if_{-i,+i,k^2+2k}\hat{R}_{k^2+2k})\right]\Psi \\ &= \frac{1}{2mr^2}\left[\hat{R}_{+i}\hat{R}_{-i} + \frac{i}{2}f_{-i,+i,a}T_a \right. \\ &\quad \left. + \frac{i}{2}f_{-i,+i,k^2+2k}\left(-\frac{nk}{\sqrt{2k(k+1)}}\right)\right]\Psi. \end{aligned} \tag{6}$$

We see that H is proportional to $\sum_i \hat{R}_{+i}\hat{R}_{-i}$, apart from additive constants. Thus the lowest Landau level should satisfy, in addition to the requirements (4) and (5), the condition

$$\hat{R}_{-i}\Psi = 0. \tag{7}$$

This is the holomorphicity condition on the lowest Landau level wave functions. Thus the values of the background fields are specified or chosen by (4) and (5), which correspondingly set the choice of the states $|J, \mathfrak{r}\rangle \equiv |J, \alpha, w\rangle$ in (1), where $w = -nk/\sqrt{2k(k+1)}$ is the eigenvalue of $\hat{R}_{k^2+2k}$, and the lowest Landau level wave functions are holomorphic as in (7).

The degeneracy of the lowest Landau level for $\mathbb{CP}^k$ may be obtained easily from group theory. The relevant conditions are (4), (5), and (7), or in terms of the state $|J, \alpha, w\rangle$,

$$\hat{R}_{-i}|J, \alpha, w\rangle = 0 \tag{8}$$

$$\begin{aligned} \hat{R}_a|J, \alpha, w\rangle &= (T_a)_{\alpha\beta}|J, \beta, w\rangle, \\ \hat{R}_{k^2+2k}|J, \alpha, w\rangle &= -\frac{nk}{\sqrt{2k(k+1)}}|J, \alpha, w\rangle. \end{aligned} \tag{9}$$

The state $|J, \alpha, w\rangle$ must be a lowest weight state in the representation J according to (8). The weight vector of this state itself is specified by (9). Thus the representation J is fixed by (8), (9), and its dimension will give the degeneracy. Explicit formulas for the degeneracy of the quantum Hall states on $\mathbb{CP}^k$ for arbitrary Landau levels have been derived in [19].

III. THE INDEX THEOREM AND THE EFFECTIVE ACTION FOR LLL

There is another way to think about the degeneracy. The holomorphicity condition (8) shows that the degeneracy, which is the number of normalizable solutions to (8), may be obtained from the index theorem for the Dolbeault complex [29]. Since the wave functions respond to the background gauge fields as in (9), we need a version of the index theorem in the presence of gauge fields; this is given by the twisted Dolbeault complex [29]. This index theorem is given as

$$\mathrm{Index}(\bar{\partial}_V) = \int_K \mathrm{td}(T_cK) \wedge \mathrm{ch}(V), \tag{10}$$

where $\mathrm{td}(T_cK)$ is the Todd class on the complex tangent space of K and $\mathrm{ch}(V)$ is the Chern character of the vector bundle V (given in terms of traces of powers of the curvature of the vector bundle which is also referred to as the field strength of the gauge field). Explicitly, the Todd class has the expansion

$$\begin{aligned}\mathrm{td} = 1 &+ \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \\ &+ \frac{1}{720}(-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4) + \cdots,\end{aligned} \tag{11}$$

where c_i are the Chern classes. For any vector bundle with curvature $\mathcal{F}$, these are given by

$$\det\left(1 + \frac{i\mathcal{F}}{2\pi}t\right) = \sum_i c_i t^i. \tag{12}$$

The Todd class may also be represented, via the splitting principle, in terms of a generating function as

$$\mathrm{td} = \prod_i \frac{x_i}{1 - e^{-x_i}}, \tag{13}$$

where x_i represent the "eigenvalues" of the curvature in a suitable canonical form (diagonal or the canonical antisymmetric form for real antisymmetric $i\mathcal{F}$).

The first few Chern classes for the complex tangent space can be explicitly written, using (12), as

$$\begin{aligned}c_1(T_cK) &= \mathrm{Tr}\frac{iR}{2\pi} \\ c_2(T_cK) &= \frac{1}{2}\left[\left(\mathrm{Tr}\frac{iR}{2\pi}\right)^2 - \mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2\right] \\ c_3(T_cK) &= \frac{1}{3!}\left[\left(\mathrm{Tr}\frac{iR}{2\pi}\right)^3 - 3\mathrm{Tr}\frac{iR}{2\pi}\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2 \right. \\ &\quad \left. + 2\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^3\right] \\ c_4(T_cK) &= \frac{1}{4!}\left[\left(\mathrm{Tr}\frac{iR}{2\pi}\right)^4 - 6\left(\mathrm{Tr}\frac{iR}{2\pi}\right)^2\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2 \right. \\ &\quad + 8\mathrm{Tr}\frac{iR}{2\pi}\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^3 + 3\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2 \\ &\quad \left. - 6\mathrm{Tr}\left(\frac{iR}{2\pi}\right)^4\right],\end{aligned} \tag{14}$$

where R is the curvature for T_cK. The Chern character, which is needed in (10), is defined by

$$\begin{aligned}\mathrm{ch}(V) &= \mathrm{Tr}(e^{i\mathcal{F}/2\pi}) \\ &= \dim V + \mathrm{Tr}\frac{i\mathcal{F}}{2\pi} + \frac{1}{2!}\mathrm{Tr}\frac{i\mathcal{F}\wedge i\mathcal{F}}{(2\pi)^2} + \cdots,\end{aligned} \tag{15}$$

where $\dim V$ is the dimension of the bundle V. (For now, $\mathcal{F}$ can be taken as F, the field strength due to the external gauge field. Later, we will include the curvature of the spin bundle in $\mathcal{F}$ as well).

Since these classes are expressed in terms of the curvatures R and F, the index theorem gives a more general counting of states. The curvatures do not have to be the fixed, background values used in the group theoretic analysis, fluctuations of the metric and gauge fields are automatically included. For example, when K is two dimensional, the index reduces to

$$\begin{aligned}\mathrm{Index}(\bar{\partial}_V) &= \int_K \left[\mathrm{Tr}\frac{iF}{2\pi} + \dim V\frac{c_1(T_cK)}{2}\right] \\ &= \int_K \left[\frac{iF}{2\pi} + \frac{iR}{4\pi}\right].\end{aligned} \tag{16}$$

For $\mathbb{CP}^1 = SU(2)/U(1) \sim S^2$, only Abelian gauge fields are allowed, so $\dim(V) = 1$. Further the corresponding background curvatures are (see the Appendix)

$$\bar{F} = -in\Omega, \qquad \bar{R}|_{T_cK} = -i2\Omega, \tag{17}$$

where Ω is the Kähler two-form on $\mathbb{CP}^1$. From now on we will denote the constant background fields by an overbar, while the unbarred quantities include fluctuations. Further, we take all connections and curvatures to be anti-Hermitian.

For spinless charged fields (i.e., $\dim V = 1$) and small fluctuations around the background fields given in (17) the index works out to be

$$\mathrm{Index}(\bar{\partial}_V) = (n+1)\int \frac{\Omega}{2\pi} = n + 1. \tag{18}$$

From the point of view of group theory, the conditions (8) and (9) tell us that the lowest Landau level states form an $SU(2)$ representation with spin $j = \frac{1}{2}n$, giving the degeneracy $2j + 1 = n + 1$, in agreement with (18).

The index theorem, however, gives the degeneracy for any general choice of curvatures, of which (17) are only a special case. *We can therefore use the index density to construct an effective action with an arbitrary metric and gauge field. This will be our basic strategy.* (But K should still remain a complex manifold for us to be able to use the Dolbeault index).

Continuing with the two-dimensional case, for a fully filled Landau level, the number of states is identical to the total charge if we assign a unit charge to each particle. Since the degeneracy of the lowest Landau level is given by the Dolbeault index, we can identify the corresponding index density with the charge density J_0 up to a total derivative term, i.e.,

$$J_0 = \frac{iF}{2\pi} + \frac{iR}{4\pi} + dM, \tag{19}$$

where M is one-form. [It will be a $(2k-1)$-form in general.] Further, the charge density J_0 is also the functional derivative of the effective action with respect to the time component (A_0) of the $U(1)$ gauge field,

$$\frac{\delta S_{\rm eff}}{\delta A_0} = J_0. \tag{20}$$

Thus the effective action involving gauge fields can be obtained by "integrating" the index density with respect to A_0, in other words, finding an $S_{\rm eff}$ such that

$$\begin{aligned}\delta S_{\rm eff} &= \int (i\delta A_0 dx^0)\wedge\left(\frac{iF}{2\pi}+\frac{iR}{4\pi}\right) + id(\delta A_0 dx^0)\wedge M \\ &= \delta\left[\frac{i^2}{4\pi}\int A(F+R)\right] + \delta\tilde{S}.\end{aligned} \tag{21}$$

[We use anti-Hermitian components for the gauge fields, including the time component, which explains the additional factors of i in (21).] The effective action can thus be taken to be

$$S_{3d}^{\rm LLL} = \frac{i^2}{4\pi}\int A(F+R) + S_{\rm grav} + \tilde{S}. \tag{22}$$

There is some explanation needed for the steps leading to (22). First of all, the Chern-Simons form involves terms with the time derivatives of the spatial components of the gauge potential, such as, for example, $A\partial_0 A$. Our argument does not directly give these terms since there is no A_0 in such terms. For the topological part of the action, our strategy is to complete by covariance the result obtained from (21) to arrive at (22). Second, there could be purely gravitational terms which cannot be determined from (21) since they are not A_0-dependent. The most important such terms have to do with possible gravitational anomalies. These will be taken up later; for the moment, $S_{\rm grav}$ in (22) signifies such terms. Finally, since the charge density is specified as the index density only up to an additive total derivative, as in (19), there can be additional terms of the form $\tilde{S}$ in (22) whose variation gives $i(\delta A_0 dx^0 M)$. The term dM in (19) integrates to zero since we consider manifolds without boundary. Thus the physics of a term like $\tilde{S}$ will involve dipole and higher moments of the charge distribution of the filled Landau level. Therefore, we can expect them to be subdominant in a derivative expansion of the effective action. Generically, they will also involve the metric and hence would not qualify as topological terms. In $(2+1)$ dimensions such terms have been derived under the assumption of local Galilean invariance [8] and explicitly calculated from the microscopic theory [9,11,14].

We can now easily generalize these results to write down the topological bulk effective action describing the dynamics of the lowest Landau level with Abelian gauge fields for a complex space of arbitrary even spatial dimensions $2k$:

$$\begin{aligned}S_{2k+1}^{\rm LLL} =& \int\left\{\left[1+\frac{1}{2}c_1+\frac{1}{12}(c_1^2+c_2)+\frac{1}{24}c_1c_2+\cdots\right]_{T_cK}\wedge\right.\\ &\left.\times\left[iA+\frac{i^2}{2(2\pi)}AF+\cdots+\frac{i^{l+1}}{(l+1)!(2\pi)^l}AF^l+\cdots\right]\right\}_{2k+1}\\ &+S_{\rm grav}+\tilde{S},\end{aligned} \tag{23}$$

where the differential form of dimension $2k+1$ should be picked up in the integrand. Expression (23) can be further generalized to include non-Abelian gauge fields.

The general expression for the $(2k+1)$-dimensional Chern-Simons term (including Abelian and non-Abelian connections) can be written in the form

$$\begin{aligned}(CS)_{2k+1}(A) &= \frac{i}{k!}\int_0^1 d\tau {\rm Tr}\left[A\left(\frac{iF_\tau}{2\pi}\right)^k\right],\\ F_\tau &= \tau dA + \tau^2 A^2.\end{aligned} \tag{24}$$

One can check that its variation is of the form

$$\delta(CS)_{2k+1} = \frac{i}{k!}{\rm Tr}\left[\delta A\left(\frac{iF}{2\pi}\right)^k\right]. \tag{25}$$

Following similar reasoning as before, we can now write down the general bulk effective action for the lowest Landau level for any odd dimensional spacetime, for which the spatial part admits a complex structure:

$$S_{2k+1}^{\rm LLL} = \int\left[{\rm td}(T_cK)\wedge\sum_p (CS)_{2p+1}(A)\right]_{2k+1} + S_{\rm grav} + \tilde{S}. \tag{26}$$

IV. EFFECTIVE ACTIONS FOR HIGHER LANDAU LEVELS

So far we have considered the lowest Landau level. The wave functions for the higher Landau levels do not satisfy a holomorphicity condition like (8), so we cannot directly use the Dolbeault index. However, we can use a simple trick to transform this to a lowest Landau problem for a charged particle carrying an appropriate amount of spin. For this, let us first consider the sth Landau level on $\mathbb{CP}^1$. The wave functions are given by

$$\Psi_m(g) \sim \left\langle J, m|g|J, -\frac{1}{2}n \right\rangle, \qquad J = \frac{1}{2}n + s \tag{27}$$

which has $R_3\Psi = -\frac{1}{2}n\Psi$ as required by (9) but does not satisfy the holomorphicity condition (7). The states (27) are however in the same representation as

$$\tilde{\Psi}_m(g) \sim \left\langle J, m|g|J, -\frac{1}{2}n - s \right\rangle \tag{28}$$

for which the holomorphicity condition is satisfied, $R_-\tilde{\Psi} = 0$. We now consider a field ϕ which has $U(1)$ charge equal to 1 and which has $U(1)$ spin s. Such a field couples to the background field

$$\bar{\mathcal{F}} = -i(n+2s)\Omega = \bar{F} + s\bar{R} = \bar{F} + \bar{\mathcal{R}}_s. \tag{29}$$

This comes about because the chosen background $U(1)$ gauge field is proportional to the spin connection on $\mathbb{CP}^1$ (see the Appendix). The lowest Landau level for this field will obey a holomorphicity condition and, in fact, the wave functions are given by $\tilde{\Psi}_m(g)$. So the degeneracy for LLL of the field ϕ is the same as the degeneracy for the sth Landau level for a spinless field with $U(1)$ gauge charge 1, which is the original field of interest. Thus for the counting of states, we can now use the Dolbeault index for the lowest Landau level for ϕ (which is possible by virtue of the holomorphicity condition). Our strategy is to use this equality of degeneracies to formulate the effective action in terms of the index density for ϕ.

The Dolbeault index is now written as

$$\begin{aligned}\mathrm{Index}(\bar{\partial}_V) &= \int_K \left[\mathrm{Tr}\frac{i(F+\mathcal{R}_s)}{2\pi} + \dim V \frac{c_1(T_c K)}{2}\right] \\ &= \int_K \left[\frac{iF}{2\pi} + \left(s + \frac{1}{2}\right)\frac{iR}{2\pi}\right].\end{aligned} \tag{30}$$

For the particular values of F, R as in (29), this index counts correctly the degeneracy of the states in the sth Landau level to be $n+1+2s$. In (30), we can allow fluctuations in the fields, so that F is the $U(1)$ magnetic field, R is the curvature and $\mathcal{R}_s = sR$ is the curvature of the spin bundle, all including fluctuations. [The choice of specific background values, as in (29), will be indicated by barred quantities].

Using (30) and repeating the steps going from (16) to (22), we find the bulk effective action for the filled sth Landau level as

$$S^{(s)}_{3d} = \frac{i^2}{4\pi}\int A[F + (2s+1)R] + S_{\mathrm{grav}} + \tilde{S}. \tag{31}$$

The second term in (31) arises from the coupling to gravity as discussed by [6] and [7] and is often referred to as the Wen-Zee term. For us, $s = 0$ corresponds to the lowest Landau level, so if we have N filled Landau levels, the result would be

$$S = \sum_{s=0}^{N-1} S^{(s)}. \tag{32}$$

It is worth recapitulating the basic argument we have used. Instead of dealing directly with the quantum Hall system in a higher Landau level, which we cannot do because of the lack of holomorphicity, we consider a mock system made of particles with a suitably chosen value of spin, such that the lowest Landau level of the mock system has wave functions in the same multiplet as the original system at the required higher Landau level. Since the degeneracies of the two systems are the same, and since, at least for the $(2+1)$-dimensional case, the topological part of the response of the Hall system depends only on the degeneracies or the index density, we can use the mock system to obtain the topological part of the effective action. This is the basic strategy we are using.

V. GENERAL FIELDS AND HIGHER DIMENSIONS

We can now extend these results to higher dimensional cases with $U(k)$ gauge fields and higher Landau levels, and gravitational fields, guided by the discussion of the $\mathbb{CP}^1$ case. For $\mathbb{CP}^k$, the field ϕ, mentioned after (28), couples to the constant background field,

$$\bar{\mathcal{F}} = -i(n\Omega\mathbf{1} + s\bar{R}^0\mathbf{1} + \bar{R}^a T_a) = \bar{F} + \bar{\mathcal{R}}_s, \tag{33}$$

where $\bar{R}^0$, $\bar{R}^a$ are the curvature components defined in (A11) and T_a, $\mathbf{1}$ are $U(k)$ matrices in the appropriate spin representation. With the addition of spin, the vector bundle whose Chern character enters the definition of the index in (10) is the tensor product of the spin bundle and the vector bundle for the internal gauge field. (By spin bundle, we do not necessarily mean the spinor bundle, but rather the bundle carrying a representation of the isotropy group of the manifold. Also, for many examples, we will use the spin as a trick to get the action for higher Landau levels, but we emphasize that this is not the only case of interest. One may also consider the Hall effect for the lowest Landau level for particles of higher intrinsic spin. Our considerations apply to such cases as well, with the suitable identification of the various gauge fields and spin connections involved.) Thus $V \to S \otimes V$. The Chern character obviously splits into a product $\mathrm{ch}(S)\wedge\mathrm{ch}(V)$,

$$\mathrm{ch}(S\otimes V) = \mathrm{Tr}(e^{i(\mathcal{R}_s+F)/2\pi}) = \mathrm{ch}(S)\wedge\mathrm{ch}(V). \tag{34}$$

In (34), $\mathcal{R}_s$ is the curvature R in the representation appropriate to the chosen spin and the trace is over the spin module. and F is in the representation for the (gauge) charge rotations of the field ϕ. The spin connection which leads to $\mathcal{R}_s$ will be denoted by ω_s which will be valued in the Lie algebra of $U(k)$. The connection for the bundle $S \otimes V$ is thus $\omega_s \otimes 1 + 1 \otimes A$ which we will often abbreviate as $\omega_s + A$.

The index theorem now becomes[2]

$$\mathrm{Index}(\bar{\partial}_V) = \int_K \mathrm{td}(T_c K) \wedge \mathrm{ch}(S \otimes V). \qquad (35)$$

Upon taking the index density and following the steps which led from (19) to (22), we can obtain an effective action in $(2k+1)$ dimensions. More directly, we can now introduce the Chern-Simons forms by noting that

$$\delta\left[\sum_p \{(CS)_{2p+1}(\omega_s + A) - (CS)_{2p+1}(\omega_s)\}\right] = \delta A \wedge \mathrm{ch}(S \otimes V), \qquad (36)$$

where δA is the variation of the Abelian $U(1)$ component of the gauge field. Since the term involving only ω_s in the expansion of $(CS)(\omega_s + A)$ does not contribute in the variation, we have subtracted it out on the left-hand side of (36). Such a term will contribute to the gravitational anomaly and will be discussed shortly. The effective action can now be written as

$$\begin{aligned} S^{(s)}_{2k+1} &= \int \left[\mathrm{td}(T_c K) \wedge \sum_p [(CS)_{2p+1}(\omega_s + A) - (CS)_{2p+1}(\omega_s)]\right]_{2k+1} + S_{\mathrm{grav}} + \tilde{S} \\ &= \int \left[\mathrm{td}(T_c K) \wedge \sum_p (CS)_{2p+1}(\omega_s + A)\right]_{2k+1} - \int \left[\mathrm{td}(T_c K) \wedge \sum_p (CS)_{2p+1}(\omega_s)\right]_{2k+1} \\ &\quad + S_{\mathrm{grav}} + \tilde{S}. \end{aligned} \qquad (37)$$

There are several observations to be made about this action. This action is in agreement with the well-known descent method used for anomalies [30]. Focusing first on just the gauge field dependent terms, and using

$$\frac{1}{2\pi} d(CS)_{2p+1} = \frac{1}{(p+1)!} \mathrm{Tr}\left(\frac{iF}{2\pi}\right)^{p+1}, \qquad (38)$$

we see that the purely gauge field dependent part of the action (37) may be considered as arising from the index density in $(2k+2)$ dimensions as

$$S = 2\pi \int \Omega_{2k+1} + \cdots, \qquad [\text{index density}]_{2k+2} = d\Omega_{2k+1}. \qquad (39)$$

This relates our bottom-up approach of starting in $2k$ spatial dimensions to the descent approach used for the $(2+1)$-dimensional case in [14]. If we restrict the integration region in (37), i.e., to a droplet, the action (37) will not be gauge invariant; the lack of gauge invariance is expressed as a boundary term. This boundary term will be canceled by the anomaly of the $(2k-1,1)$-dimensional theory of the edge excitations. The anomaly of this $(2k-1,1)$-dimensional theory is related to the index density in $(2k+2)$ dimensions in the standard descent procedure for anomalies. The action (37) is in accord with these expectations.

Such a descent method is known to apply to all anomalies, including the gravitational ones [31] as well as the mixed gauge-gravity anomalies. The mixed terms are already apparent in (37). To include the purely gravitational part and identify S_{grav} in (37), we note that the gravitational anomaly can be obtained from the index density in $(2k+2)$ dimensions from the appropriate terms in $\mathrm{td}(T_c K) \wedge \mathrm{ch}(S)$ [31]. Using the definition of the Chern character in (15), Eq. (38) and the fact that $d[\mathrm{td}(T_c K)] = 0$, we can write $\mathrm{td}(T_c K) \wedge \mathrm{ch}(S)$ as the exterior derivative of a $(2k+1)$-form as follows:

$$[\mathrm{td}(T_c K) \wedge \mathrm{ch}(S)]_{2k+2} = d\Omega^{\mathrm{grav}}_{2k+1} + \frac{1}{2\pi} d\left[\mathrm{td}(T_c K) \wedge \sum_p (CS)_{2p+1}(\omega_s)\right]_{2k+1}. \qquad (40)$$

Here $d\Omega^{\mathrm{grav}}_{2k+1}$ gives the $(2k+2)$-form in $\mathrm{td}(T_c K)$, namely $[\mathrm{td}(T_c K)]_{2k+2} = d\Omega^{\mathrm{grav}}_{2k+1}$. Adding this term to (37), we see that the effective action becomes

$$S^{(s)}_{2k+1} = \int \left[\mathrm{td}(T_c K) \wedge \sum_p (CS)_{2p+1}(\omega_s + A)\right]_{2k+1} + 2\pi \int \Omega^{\mathrm{grav}}_{2k+1} + \tilde{S}. \qquad (41)$$

In this action we have gathered together the contributions from both gauge and gravitational fields. This result gives all

[2]The zero modes of the $\bar{\partial}_V$ operator are also the lowest Landau levels as in (6) and (7). Thus the Dolbeault index is what is relevant for us. In [22], the zero modes of the Laplacian were analyzed by relating them to the zero modes of the Dirac operator for a specific choice of the gauge potential being proportional to the spin connection. For this choice, the index theorem for the Dirac operator can be written entirely in terms of the Chern classes for the gauge field. While this is adequate for evaluating the degeneracy, and response of the system to a limited variation in the fields which preserves the proportionality of gauge potential and spin connection, we are interested in considering arbitrary and independent fluctuations for the gauge and gravitational fields, so that an effective action for the response of the system to either or both can be obtained. So a more general setup is needed.

the topological terms in the bulk effective action, encoding the response of the system to gauge and gravitational fluctuations in arbitrary dimensions.

Finally, we note that in starting with the index density in $2k$ dimensions and interpreting it as the charge density for the Abelian field, there is an ambiguity in writing down the effective action. This is because several terms which only involve non-Abelian fields, such as, for example, $(CS)_{2p+1}(A)$ where A is in $\underline{SU(k)}$ do not contribute to the index and hence the question of whether they are to be included in the effective action or not is not settled by the index in $2k$ dimensions. [The underlining of $SU(k)$ denotes the Lie algebra of the group.] However, we know that there should be terms like $(CS)(\omega_s)$ in the contribution due to the gravitational anomaly. Further, it is the $S \otimes V$ bundle which is relevant and hence there is some equivalence between the A's and the ω's once we restrict to the background fields. For this reason, we should also have the purely non-Abelian A-dependent terms in (37) and (41).

To recapitulate, (41) gives the bulk effective action for the sth higher Landau level for any odd dimensional spacetime, for which the spatial part admits a complex structure, with $U(k)$ gauge fields. (As mentioned, it can also be used for the Hall effect in the lowest Landau level for particles of arbitrary spin, with the suitable identification of the fields.) As always, for the topological terms, the differential form of the appropriate dimension, namely $(2k+1)$, must be picked out from the integrand in (37) or (41); this is indicated by the subscript. While the topological terms follow from the index theorem, there can be nonuniversal, metric dependent corrections which are indicated by $\tilde{S}$ in (37) and (41).

The effective action (41) is the main result of this paper. Since it is still in rather cryptic form, we will now consider working out the details of this action for some special cases and for certain choices of dimensions. We will first consider the gauge-field dependent terms, since these are the ones relevant for the counting of states. The terms which depend only on the gravitational fields will be taken up in Sec. VIII.

VI. 4 + 1 DIMENSIONS: GAUGE FIELD DEPENDENT TERMS

In the $(4+1)$-dimensional case, the part of the effective action depending on the gauge fields reduces to

$$\begin{aligned} S_{\text{gauge}} &= \int \left[\frac{\dim S}{12}(c_1^2+c_2)_{(T_cK)} + \frac{1}{2}c_1(T_cK)\wedge \text{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right) + \frac{1}{2}\text{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\wedge\frac{i\mathcal{R}_s}{2\pi}\right)\right]\wedge (CS)_1(A) \\ &\quad + \int\left[\frac{\dim S}{2}c_1(T_cK) + \text{Tr}\frac{i\mathcal{R}_s}{2\pi}\right]\wedge (CS)_3(A) + \dim S\int (CS)_5(A) \\ &= \frac{i^2}{(2\pi)^2}\int\left[\frac{\dim S}{24}(3(\text{Tr}R)^2 - \text{Tr}(R^2)) + \frac{1}{2}(\text{Tr}R)\wedge(\text{Tr}\mathcal{R}_s) + \frac{1}{2}\text{Tr}(\mathcal{R}_s)^2\right]\wedge (CS)_1(A) \\ &\quad + \frac{i}{2\pi}\int\left[\frac{\dim S}{2}\text{Tr}R + \text{Tr}\mathcal{R}_s\right]\wedge (CS)_3(A) + \dim S\int (CS)_5(A), \end{aligned} \tag{42}$$

where, in the second expression, we have written out the characteristic classes explicitly. The Chern-Simons terms are

$$(CS)_1 = i\text{Tr}(A), \qquad (CS)_3 = \frac{i^2}{4\pi}\text{Tr}\left[AdA + \frac{2}{3}A^3\right]$$
$$(CS)_5 = \frac{i^3}{3!(2\pi)^2}\text{Tr}\left[AdAdA + \frac{3}{2}A^3dA + \frac{3}{5}A^5\right] \tag{43}$$

and

$$R = -i[R^0\mathbf{1} + R^a t_a] \quad \mathcal{R}_s = -i[sR^0\mathbf{1} + R^a T_a] \tag{44}$$

with t^a, T^a being $SU(2)$ matrices in the fundamental and $j = s/2$ representation, respectively. The action (42) is general, just restricting (41) to $4+1$ dimensions. The rest of this section will be devoted to verifying that this is consistent with the expected degeneracies for various special cases.

The index theorem which is associated with the action (42) is

$$\begin{aligned} \text{Index}(\bar{\partial}_V) = \int_K \dim V\Bigg[&\frac{\dim S}{12}(c_1^2+c_2)_{T_cK} \\ &+\frac{1}{2}c_1(T_cK)\wedge\text{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right) + \frac{1}{2}\text{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\wedge\frac{i\mathcal{R}_s}{2\pi}\right)\Bigg] \\ &+\left[\frac{\dim S}{2}c_1(T_cK) + \text{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right)\right]\wedge\text{Tr}\frac{iF}{2\pi} \\ &+\frac{\dim S}{2}\text{Tr}\left(\frac{iF}{2\pi}\wedge\frac{iF}{2\pi}\right). \end{aligned} \tag{45}$$

Our purpose will be to consider the index theorem for some special cases to show that the counting agrees with what is obtained by explicit calculation of wave functions. This will justify the use of the index density as the charge density for a $U(1)$ background and hence justify the effective action (42).

A. $\mathbb{CP}^2$ with $U(1)$ gauge fields, lowest Landau level

As the first special case, we take K to be $\mathbb{CP}^2 = SU(3)/U(2)$. In this case, uniform background magnetic fields taking values in the Lie algebra of $U(2) \sim SU(2) \times U(1)$ are possible. As a first example then, we take the case of a magnetic field which is Abelian, corresponding to the $U(1)$ subgroup of $U(2) \subset SU(3)$. Further, we will consider a spinless field in the lowest Landau level, so that $c_1(S) = 0$, $c_2(S) = 0$. The vector bundle is one dimensional, so $\dim V = 1$. Using the specific values of constant background fields for $\mathbb{CP}^2$ from (A7)–(A17), we find

$$\begin{aligned} \mathrm{Tr}\frac{i\bar{R}}{2\pi} &= 3\frac{\Omega}{2\pi} \\ \mathrm{Tr}\left(\frac{i\bar{R}}{2\pi}\wedge\frac{i\bar{R}}{2\pi}\right) &= 3\left(\frac{\Omega}{2\pi}\right)^2 \\ \int_{\mathbb{CP}^2}\frac{1}{12}(c_1^2 + c_2)|_{T_cK} &= 1. \end{aligned} \tag{46}$$

The last line in (46) holds, of course, even when fluctuations around the constant background values are included. The background magnetic field is given by

$$\bar{F} = -in\Omega. \tag{47}$$

The index theorem now gives

$$\begin{aligned} \mathrm{Index}(\bar{\partial}_V) &= \frac{1}{12}\int (c_1^2 + c_2)|_{T_cK} + \frac{1}{2}\int c_1(T_cK)\wedge\frac{iF}{2\pi} \\ &\quad + \frac{1}{2}\int\frac{iF\wedge iF}{(2\pi)^2} \\ &= 1 + \frac{3n}{2} + \frac{n^2}{2} = \frac{(n+1)(n+2)}{2}. \end{aligned} \tag{48}$$

We can check this against the group theoretic derivation of the wave functions, which are proportional to $\langle J, \mathfrak{l}|g|J, \mathfrak{r}\rangle$. A representation of $SU(3)$ may be taken to be of the (p, q)-type corresponding to states of the form $|J,{}^{j_1 j_2\cdots j_p}_{i_1 i_2\cdots i_q}\rangle$ where each index (each of the i's and the j's) can take values 1,2,3. The upper indices transform as the $\mathbf{3}^*$-representation, while the lower ones correspond to the $\mathbf{3}$-representation. The states $|J,{}^{j_1 j_2\cdots j_p}_{i_1 i_2\cdots i_q}\rangle$ are symmetric in all p indices $i_1 \cdots i_p$, symmetric in all q indices $j_1 \cdots j_q$ and traceless. The state $|J, \mathfrak{r}\rangle$ must be a lowest weight state with $\hat{R}_8|J, \mathfrak{r}\rangle = -(n/\sqrt{3})|J, \mathfrak{r}\rangle$, $\hat{R}_{-i}|J, \mathfrak{r}\rangle = 0$. This identifies the required representation as $(n, 0)$ with the state $|J, \mathfrak{r}\rangle = |J,{}_{33\cdots 3}\rangle$ [19]. The dimension of the representation is thus $\frac{1}{2}(n+1)(n+2)$, verifying (48).

B. $\mathbb{CP}^2$ with $U(1)$ gauge fields, sth Landau level

Consider now the higher Landau levels, say, the sth level, taking $s = 0$ as the lowest level. In this case, the required state is of the $(n+s, s)$ type with $|J, \mathfrak{r}\rangle = |J,{}^{3\cdots 3}_{33\cdots 3}\rangle$. This is not the lowest weight state; the lowest weight state in the same representation is of the form $|J,{}^{i_1 i_2\cdots i_s}_{33\cdots 3}\rangle$ where the upper indices take values 1,2. We can view this as the lowest Landau level of a field with spin, specifically, with $SU(2)$ spin $j = \frac{1}{2}s$ (hence $\dim S = 2j + 1 = s + 1$), $U(1)$ spin equal to s, and electric charge 1, coupling to the background field

$$\bar{\mathcal{F}} = -i(n\Omega\mathbf{1} + s\bar{R}^0\mathbf{1} + \bar{R}^a T_a) = -i(\bar{F}\mathbf{1} + \bar{\mathcal{R}}_s). \tag{49}$$

For such a field, $|J,{}^{i_1 i_2\cdots i_s}_{33\cdots 3}\rangle$ would be the lowest Landau level, satisfying the holomorphicity condition. With these spin assignments, in addition to the Chern classes in (46) and (47), we find

$$\begin{aligned} \mathrm{Tr}\left(\frac{i\bar{\mathcal{R}}_s}{2\pi}\right) &= \frac{3s(s+1)}{2}\frac{\Omega}{2\pi} \\ \mathrm{Tr}\frac{i\bar{\mathcal{R}}_s\wedge i\bar{\mathcal{R}}_s}{(2\pi)^2} &= s(s+1)\left(2s - \frac{1}{2}\right)\left(\frac{\Omega}{2\pi}\right)^2. \end{aligned} \tag{50}$$

It is now easy to check that the index becomes

$$\begin{aligned} \mathrm{Index}(\bar{\partial}_V) &= (s+1)\left[\frac{n^2}{2} + \frac{3n}{2}(s+1) + (s+1)^2\right] \\ &= \frac{(s+1)(n+s+1)(n+2s+2)}{2}. \end{aligned} \tag{51}$$

Group theoretically, the dimension of the $SU(3)$ $(n+s, s)$ representation is the same as (51) [19], justifying the use of the index density (45) in constructing the effective action (42).

C. $\mathbb{CP}^2$ with non-Abelian gauge fields, lowest Landau level

As we mentioned before in the case of $\mathbb{CP}^k$, $k \geq 2$, there is a possibility of non-Abelian background gauge fields. In the case of $\mathbb{CP}^2$, the lowest Landau level states belong to a representation of $SU(3)$ with a lowest weight state which transforms nontrivially under $SU(2)$, as a representation $\tilde{J}$, as in (4). It was further shown in [19] that allowed $\tilde{J}$'s must correspond to integer values of the spin $\tilde{j}$.

The background field is now purely of gauge nature (no coupling to spin connection), given by

$$\bar{\mathcal{F}} = -i(n\Omega\mathbf{1} + \bar{R}^a T_a), \tag{52}$$

where T_a are $(2\tilde{j}+1)\times(2\tilde{j}+1)$ matrices. Since there is no coupling to the spin connection, $\mathcal{R}_s$ can be set to zero in (42). The index theorem (45) now gives

$$\begin{aligned}\mathrm{Index}(\bar{\partial}_V) &= \frac{\dim V}{12}\int (c_1^2+c_2)|_{T_cK} \\ &\quad + \frac{1}{2}\int c_1(T_cK)\wedge \mathrm{Tr}\frac{iF}{2\pi} + \frac{1}{2}\int \mathrm{Tr}\frac{iF\wedge iF}{(2\pi)^2} \\ &= (2\tilde{j}+1)\left[1+\frac{3}{2}n+\frac{1}{2}n^2-\frac{1}{2}\tilde{j}(\tilde{j}+1)\right] \\ &= \frac{(2\tilde{j}+1)(n+\tilde{j}+2)(n-\tilde{j}+1)}{2}. \qquad (53)\end{aligned}$$

This again agrees with the degeneracy of the lowest Landau level which is the dimension of the $SU(3)$ representation of the type $(p=n-\tilde{j}, q=2\tilde{j})$ [19].

D. $\mathbb{CP}^2$ with non-Abelian gauge fields and higher Landau levels

There are some intricacies when we consider a non-Abelian background gauge field and higher Landau levels.

The wave functions at the sth Landau level form an $SU(3)$ representation of the (p,q) type with $J=(p,q)=(n+s-\tilde{j}, s+2\tilde{j})$. They are of the form $\langle J,\mathfrak{l}|g|J,\mathfrak{r}\rangle$, with

$$|J,\mathfrak{r}\rangle = |J_{,33\cdots3}^{\;33\cdots3;l_1l_2\cdots l_{2\tilde{j}}}\rangle. \qquad (54)$$

There are s upper 3's and $n+s-\tilde{j}$ lower 3's here. The l indices indicate the non-Abelian gauge degrees of freedom. This corresponds to a state with an eigenvalue of $\hat{R}^8$ equal to $-n/\sqrt{3}$ (as required) and transforming as the spin-$\tilde{j}$ representation of $SU(2)$. (We also need $\tilde{j}$ to be an integer [17,19]; this is related to the fact that $\mathbb{CP}^2$ does not admit spinors.) The dimension of the representation is given by

$$\dim J = \frac{1}{2}(n+2s+\tilde{j}+2)(n+s-\tilde{j}+1)(2\tilde{j}+s+1). \qquad (55)$$

As mentioned earlier these wave functions do not satisfy the holomorphicity condition. In order to be able to use the Dolbeault index as before, we convert this to a problem of lowest Landau level of a higher spin field. We consider the states $\tilde{\Psi}=\langle J,\mathfrak{l}|g|J,\hat{\mathfrak{r}}\rangle$ where

$$|J,\hat{\mathfrak{r}}\rangle = |J_{,33\cdots3}^{\;i_1i_2\cdots i_s;l_1l_2\cdots l_{2\tilde{j}}}\rangle, \qquad (56)$$

where there are $n+s-\tilde{j}$ lower 3's. The indices i now indicate the spin and l the gauge degrees of freedom. This state has $\hat{R}^8$ equal to $-n/\sqrt{3}$ (as required) and it is a lowest weight state. The representation it belongs to has dimension equal to (55) assuming that the indices i, l in (56) are fully symmetrized.

The corresponding field ϕ couples to the constant background field,

$$\bar{\mathcal{F}} = -i(n\Omega\mathbf{1}+s\bar{R}^0\mathbf{1}+\bar{R}^aT_a), \qquad (57)$$

where T_a are $(2j+s+1)\times(2j+s+1)$ matrices. Fluctuations are then introduced as

$$\mathcal{F} = -i((n\Omega+\delta F)\mathbf{1}+s(\bar{R}^0+\delta R^0)\mathbf{1}+(\bar{R}^a+\delta R^a)T_a). \qquad (58)$$

There is an ambiguity though of how to interpret the fluctuations δR^a. These can be thought of as either fluctuations of the non-Abelian gauge field or fluctuations of the non-Abelian spin curvature. In other words one can think of the field ϕ coupling to an Abelian gauge field and a $U(2)$ spin connection $(s,\tilde{j}+s/2)$ or coupling to a $U(2)$ non-Abelian gauge field and a $U(1)$ spin connection with spin s. Depending on the choice though, the effective action (42) will have a different field content. In particular the response to the metric will be different. On the other hand, the index (45) evaluated for the background (57) will be exactly the same in both cases and equal to (55).

This ambiguity in constructing an effective action for a quantum Hall system with non-Abelian gauge fields at higher Landau levels has to do with the following. For the case of $\mathbb{CP}^2$, for example, recall that, for a field with spin which carries a nontrivial $SU(2)$ gauge charge, the commutator of the covariant derivatives has the form

$$\begin{aligned}[D_\mu, D_\nu]\phi = -i(&F_{\mu\nu}\mathbf{1}+sR^0_{\mu\nu}\mathbf{1}+F^a_{\mu\nu}t_a\otimes 1 \\ &+R^a_{\mu\nu}1\otimes T_a)\phi, \qquad (59)\end{aligned}$$

where $F_{\mu\nu}$ is the $U(1)$ gauge field, $R^0_{\mu\nu}$ is the $U(1)$ spin curvature, $\{t_a\}$ are in the representation of ϕ corresponding to the gauge group action (say, $\tilde{j}$), $\{T_a\}$ are in the representation corresponding to the spin of ϕ [say, $s/2$ of $SU(2)$]. In the Landau problem, we choose the background value for the gauge field as $\bar{F}^a_{\mu\nu}=\bar{R}^a_{\mu\nu}$, where $\bar{R}^a_{\mu\nu}$ is the standard curvature of $\mathbb{CP}^2$. Thus, on the right-hand side of (59), we have the combination $\bar{R}^a_{\mu\nu}(t_a\otimes 1+1\otimes T_a)$. The group transformations generated separately by the t_a and T_a are not important, only the group action corresponding to the combination $(t_a\otimes 1+1\otimes T_a)$ is relevant. The wave functions which transform under the product of the two $SU(2)$'s corresponding to the gauge group and spin, namely, as $\tilde{j}\otimes s/2$, can be reduced to irreducible components for the action of the combination $(t_a \otimes 1 + 1 \otimes T_a)$. The sth Landau level problem corresponds to a particular irreducible representation $(\tilde{j} + s/2)$, in the reduction of $\tilde{j} \otimes s/2$. [This corresponds to the full symmetrization of the indices $i_1 \cdots i_s, l_1 \cdots j_{2\tilde{j}}$ in (56)].

When we consider perturbations of the metric and the gauge field, we then have two cases worthy of being distinguished. If we consider perturbations which preserve the combination $(t_a \otimes 1 + 1 \otimes T_a)$, then the effective action can be obtained as the action with an Abelian gauge field and a curvature coupling for a spin corresponding to the representation $\tilde{j} + s/2$, or as the effective action with an Abelian gauge field and Abelian spin curvature and a non-Abelian gauge field of strength given by the representation $\tilde{j} + s/2$. These two actions are not equivalent to each other although they give rise to the same index. However, such perturbations are not the most general perturbations of the metric and the gauge field. A general perturbation would consider independent values for $F^a_{\mu\nu} = \bar{F}^a_{\mu\nu} + \delta F^a_{\mu\nu}$ and $R^a_{\mu\nu} = \bar{R}^a_{\mu\nu} + \delta R^a_{\mu\nu}$. In this case, we can no longer classify wave functions under the combined $SU(2)$. The perturbations couple different irreducible representations of the combined $SU(2)$. In this case, we cannot sensibly consider integrating out one Landau level (i.e. one irreducible representation in the reduction of $\tilde{j} \otimes s/2$) to obtain an effective action. One must consider all irreducible representations resulting from a given spin and given gauge group representation. This corresponds to the case of lowest Landau level for a field with intrinsic spin and gauge degrees of freedom with a Hamiltonian proportional to the covariant $\bar{\partial}$ operator. Such a field would couple to

$$\mathcal{F} = -i((n\Omega + \delta F)\mathbf{1} + s(\bar{R}^0 + \delta R^0)\mathbf{1} + (\bar{R}^a + \delta F^a)t_a + (\bar{R}^a + \delta R^a)T_a), \tag{60}$$

where t_a is in the $\tilde{j}$ and T_a in the $s/2$ representation. We can now evaluate the index (53) for this background and we find it to be

$$\text{Index} = (2j+1)(s+1)\left[\frac{n^2}{2} + \frac{3n}{2}(s+1) + (s+1)^2 - \frac{1}{2}j(j+1)\right]. \tag{61}$$

As mentioned earlier when $\delta F^a = \delta R^a$, the states can be classified into multiplets corresponding to irreducible representations of the combined $SU(2)$ (of t_a and T_a). These have spin values given by $J_i = \tilde{j} + \frac{s}{2} - i$, $i = 1, \ldots, s$. The dimension for each of these multiplets is given by (55), where $\tilde{j} \to \tilde{j} - i$,

$$\dim J_i = \frac{1}{2}(n + 2s + \tilde{j} - i + 2)(n + s - \tilde{j} + i + 1) \times (2\tilde{j} - 2i + s + 1). \tag{62}$$

It is straightforward to verify that summing over all these representations will produce the index in (61),

$$\begin{aligned}\dim &= \sum_{i=0}^{s}(2J_i+1)\left[1 + \frac{3}{2}\left(n + \frac{3}{2}s\right) + \frac{1}{2}\left(n + \frac{3}{2}s\right)^2 - \frac{1}{2}J_i(J_i+1)\right] \\ &= (2j+1)(s+1)\left[\frac{n^2}{2} + \frac{3n}{2}(s+1) + (s+1)^2 - \frac{1}{2}j(j+1)\right].\end{aligned} \tag{63}$$

To briefly recapitulate the discussion in this subsection, when we have a higher Landau level for, say, a spinless field, but with a non-Abelian gauge field background, we cannot directly use the index theorem as we do not have holomorphicity for the wave functions. Translating the problem to a lowest Landau level problem for a field with spin, we get fields of a certain spin as well as the non-Abelian charges. The original Landau level of interest is one representation in the reduction of the product of the spin representation and the gauge group representation of the field. However, if we allow arbitrary fluctuations of the gauge field and the spin connection, all representations in the reduction of the product mentioned above can occur. Hence it is not possible to obtain an effective action for the original problem, i.e., just for the higher Landau level of interest, by this method. However, one can consider different but related physical situations. One can write the action for the field with spin and gauge charges (in the lowest Landau level), from which we can obtain the response of such a system to arbitrary independent variations of the gauge field and the gravitational fields. Or one can write an action for the restricted case of identical fluctuations for the non-Abelian gauge field and the spin connection. In this case, the response functions are also thus restricted.

E. $S^2 \times S^2$, arbitrary Landau levels

As another example, consider $K = S^2 \times S^2$. In this case,

$$R(T_c K) = \begin{bmatrix} R & 0 \\ 0 & \tilde{R} \end{bmatrix}, \tag{64}$$

where R refers to the (anti-Hermitian) curvature of the first S^2 and $\tilde{R}$ to the second. Notice that $\mathrm{Tr}(R \wedge R) = 0$ for dimensional reasons, so that $c_2(T_cK) = \frac{1}{2}c_1^2$. Considering Landau levels (s_1, s_2) corresponding to the two S^2's, we have

$$\frac{i\bar{\mathcal{R}}_s}{2\pi} = \frac{s_1\bar{R} + s_2\bar{\bar{R}}}{2\pi} = 2\frac{s_1\Omega + s_2\tilde{\Omega}}{2\pi}$$
$$\frac{i\bar{F}}{2\pi} = \frac{n_1\Omega + n_2\tilde{\Omega}}{2\pi}. \tag{65}$$

The index theorem can now be verified to be

$$\text{Index} = (n_1 + 2s_1 + 1)(n_2 + 2s_2 + 1). \tag{66}$$

In all these cases, namely the $\mathbb{CP}^2$ examples and the $S^2 \times S^2$ example, we see that the index density from (45) does indeed reproduce the correct counting of states and hence we can use it to construct the effective action, which, of course, agrees with (42).

We close this section with a note about the normalization of the gauge fields. We have taken the charge carried by the matter fields for the Abelian gauge fields as unity, so that the number of states (which is what the index theorem gives us) is equal to the integral of the charge density. But in writing the action, it is possible to use other normalizations. For example, one might consider the Chern-Simons action for the $U(k)$ gauge fields with the normalization of the $U(k)$ Lie algebra matrices fixed by their embedding in $SU(k+1)$. While there is no particular motivation to do so, it may be useful if one considers dimensional reduction of effective actions from a higher dimension to a lower dimension. The $U(1)$ charges in such a choice would not be unity, so the normalization of the Chern-Simons term would be different from what is given in (37) or (42). The appropriate normalization will follow from tracking the $U(1)$ charges of the relevant matter fields of the Landau problem.

VII. 6 + 1 DIMENSIONS: GAUGE FIELD DEPENDENT TERMS

In $(6+1)$ dimensions, the part of the effective action which depends on the gauge fields is

$$\begin{aligned} S_{\text{gauge}} = &\int \left[\frac{\dim S}{24} c_1 c_2 + \frac{(c_1^2 + c_2)}{12} \wedge \text{Tr}\frac{i\mathcal{R}_s}{2\pi} + \frac{c_1}{2} \wedge \frac{1}{2}\text{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right)^2 + \frac{1}{3!}\text{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right)^2\right] \wedge (CS)_1(A) \\ &+ \int \left[\frac{\dim S}{12}(c_1^2 + c_2) + \frac{1}{2}c_1 \wedge \text{Tr}\frac{i\mathcal{R}_s}{2\pi} + \frac{1}{2}\text{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right)^2\right] \wedge (CS)_3(A) \\ &+ \int \left[\frac{\dim S}{2}c_1 + \text{Tr}\frac{i\mathcal{R}_s}{2\pi}\right] \wedge (CS)_5(A) + \dim S \int (CS)_7(A) + S_{\text{grav}} + \tilde{S}. \end{aligned} \tag{67}$$

Using the formulas for the Chern classes, this can be written more explicitly as

$$\begin{aligned} S_{\text{gauge}} = &\frac{i^3}{(2\pi)^3}\int \left[\frac{\dim S}{48}((\text{Tr}R)^3 - \text{Tr}R\,\text{Tr}(R^2)) + \frac{1}{24}(3(\text{Tr}R)^2 - \text{Tr}(R^2)) \wedge (\text{Tr}\mathcal{R}_s) \right. \\ &\left. + \frac{1}{4}\text{Tr}R \wedge \text{Tr}(\mathcal{R}_s)^2 + \frac{1}{3!}\text{Tr}(\mathcal{R}_s)^3\right] \wedge (CS)_1(A) \\ &+ \frac{i^2}{(2\pi)^2}\int \left[\frac{\dim S}{24}(3(\text{Tr}R)^2 - \text{Tr}(R^2)) + \frac{1}{2}\text{Tr}R \wedge (\text{Tr}\mathcal{R}_s) + \frac{1}{2}\text{Tr}(\mathcal{R}_s)^2\right] \wedge (CS)_3(A) \\ &+ \frac{i}{2\pi}\int \left[\frac{\dim S}{2}\text{Tr}R + \text{Tr}\mathcal{R}_s\right] \wedge (CS)_5(A) + \dim S \int (CS)_7(A) + S_{\text{grav}} + \tilde{S}, \end{aligned} \tag{68}$$

where

$$R = -i[R^0\mathbf{1} + R^a t_a] \quad \mathcal{R}_s = -i[sR^0\mathbf{1} + R^a T_a] \tag{69}$$

with t_a, T_a being $SU(3)$ matrices in the fundamental and appropriate spin representation respectively. The index associated with this action is

$$\begin{aligned}\mathrm{Index}(\bar\partial_V)_{6d} &= \int \dim V\left[\frac{\dim S}{24}c_1c_2 + \frac{1}{12}(c_1^2+c_2)\wedge \mathrm{Tr}\frac{i\mathcal{R}_s}{2\pi} + \frac{1}{2}c_1\wedge\frac{1}{2}\mathrm{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right)^2 + \frac{1}{3!}\mathrm{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right)^2\right]\\ &\quad + \int\left[\frac{\dim S}{12}(c_1^2+c_2) + \frac{1}{2}c_1\wedge \mathrm{Tr}\frac{i\mathcal{R}_s}{2\pi} + \frac{1}{2}\mathrm{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right)^2\right]\wedge\mathrm{Tr}\frac{iF}{2\pi}\\ &\quad + \int\left[\frac{\dim S}{2}c_1 + \mathrm{Tr}\frac{i\mathcal{R}_s}{2\pi}\right]\wedge\frac{1}{2}\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^2 + \frac{\dim S}{3!}\int\mathrm{Tr}\left(\frac{iF}{2\pi}\right)^3. \end{aligned} \tag{70}$$

As a check on the effective action, we can evaluate the index for a case for which the degeneracy of the Landau level is known. Specifically, we will consider the special case corresponding to the QHE on $\mathbb{CP}^3$ with Abelian magnetic field at Landau level s ($s = 0$ corresponds to the lowest Landau level). The following relations are useful in evaluating the index:

$$\begin{aligned} \dim S &= \frac{(s+1)(s+2)}{2}\\ \bar F &= -in\Omega, \qquad \mathrm{Tr}\frac{i\bar R}{2\pi} = 4\frac{\Omega}{2\pi}, \qquad \mathrm{Tr}\left(\frac{i\bar R}{2\pi}\right)^2 = 4\left(\frac{\Omega}{2\pi}\right)^2\\ \mathrm{Tr}\frac{i\bar{\mathcal{R}}_s}{2\pi} &= \frac{s(s+1)(s+2)}{2}\frac{4}{3}\frac{\Omega}{2\pi}\\ \mathrm{Tr}\left(\frac{i\bar{\mathcal{R}}_s}{2\pi}\right)^2 &= \frac{(s+1)(s+2)}{2}\frac{(5s^2-1)}{3}\left(\frac{\Omega}{2\pi}\right)^2\\ \mathrm{Tr}\left(\frac{i\bar{\mathcal{R}}_s}{2\pi}\right)^3 &= \frac{(s+1)(s+2)}{2}\left(2s^3 - s^2 + \frac{s}{3}\right)\left(\frac{\Omega}{2\pi}\right)^3. \end{aligned} \tag{71}$$

Using (71) we find that the index can be written as

$$\mathrm{Index}(\bar\partial_V)_{6d} = \frac{(s+1)(s+2)}{2}\frac{(n+2s+3)(n+s+1)(n+s+2)}{3!}. \tag{72}$$

This is exactly the dimension of the $(n+s,s)$ $SU(4)$ representation which gives the degeneracy of the sth Landau level for the Abelian $\mathbb{CP}^3$ QH states [19].

VIII. FULL EFFECTIVE ACTION INCLUDING GRAVITATIONAL TERMS

We now turn to the details of the terms in the effective action related to the gravitational anomaly in $(2+1)$, $(4+1)$ and $(6+1)$ dimensions. We will first consider these terms separately, then combine them with the gauge field dependent terms discussed in the previous sections to obtain the full effective action. The result will, of course, correspond to the expansion of the full action (41) for the appropriate dimension.

A. $(2+1)$-dimensional case

In this case, we need those terms in the index density in four dimensions which involve only the gravitational fields. This is given by

$$\begin{aligned}\text{Index density}(\bar\partial) &= \frac{\dim S}{12}(c_1^2+c_2)_{(T_cK)}\\ &\quad + \frac{1}{2}c_1(T_cK)\wedge\mathrm{Tr}\frac{i\mathcal{R}_s}{2\pi} + \frac{1}{2}\mathrm{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right)^2. \end{aligned} \tag{73}$$

This follows from (45) upon setting $\dim V = 1$ and $F = 0$. Also, although we have spin s, since we are interested in two dimensions eventually, we should keep in mind that the fields have only one component; thus we can set $\dim S = 1$. The various characteristic classes are

$$\begin{aligned} (c_1^2+c_2)(T_cK) &= \frac{i^2}{(2\pi)^2}(d\omega)^2\\ \mathrm{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right)^2 &= \frac{i^2}{(2\pi)^2}s^2(d\omega)^2\\ c_1(T_cK)\wedge\mathrm{Tr}\frac{i\mathcal{R}_s}{2\pi} &= \frac{i^2}{(2\pi)^2}s(d\omega)^2, \end{aligned} \tag{74}$$

where ω is the spin connection, $R = d\omega$. The index density (73) reduces to

$$\text{Index density}(\bar{\partial}) = \frac{i^2}{2(2\pi)^2}\left(s^2 + s + \frac{1}{6}\right) d(\omega d\omega). \tag{75}$$

The purely gravitational part of the topological effective action in (31) for $(2+1)$ dimensions is thus given by

$$S_{\rm grav} = \frac{i^2}{4\pi}\left[\left(s+\frac{1}{2}\right)^2 - \frac{1}{12}\right] \int \omega d\omega. \tag{76}$$

Combining with the gauge-field part in (31), the full topological bulk effective action for the sth Landau level in $(2+1)$ dimensions is

$$\begin{aligned} S^{(s)}_{3d} &= \frac{i^2}{4\pi}\left[\int A\left(dA + 2\left(s+\frac{1}{2}\right)d\omega\right) \right.\\ &\quad \left. + \left(\left(s+\frac{1}{2}\right)^2 - \frac{1}{12}\right)\int \omega d\omega\right] \\ &= \frac{i^2}{4\pi}\int\left\{\left[A + \left(s+\frac{1}{2}\right)\omega\right] d\left[A + \left(s+\frac{1}{2}\right)\omega\right] \right. \\ &\quad \left. - \frac{1}{12}\omega d\omega\right\}. \end{aligned} \tag{77}$$

This result agrees with [9] and [14]. (In our case $s = 0$ corresponds to the lowest Landau level).

B. (4 + 1)-dimensional case

We now turn to the case of $(4+1)$ dimensions. The six-form index density for the gravitational fields is easily worked out from (35) as

$$\begin{aligned} \text{Index density}(\bar{\partial}) &= \dim V\left[\frac{\dim S}{24} c_1 c_2 + \frac{c_1^2 + c_2}{12}\text{ch}_1(S) \right. \\ &\quad \left. + \frac{c_1}{2}\text{ch}_2(S) + \text{ch}_3(S)\right] \\ \text{ch}_k(S) &= \frac{1}{k!}\text{Tr}\left(\frac{i\mathcal{R}_s}{2\pi}\right)^k. \end{aligned} \tag{78}$$

For the four-dimensional K, the holonomy group being $U(2)$, the curvatures take values in the Lie algebra of $U(2)$, so R is of the form

$$R(T_c K) = -i(R^0 \mathbf{1} + t_a R^a) \equiv d\omega^0 + \tilde{R}, \tag{79}$$

where t_a are the $SU(2)$ generators in the fundamental representation, $\mathbf{1}$ is the 2×2 identity matrix, ω^0 is the $U(1)$ connection and $\tilde{R}$ is the $SU(2)$ curvature. The curvature for the spin bundle is

$$\mathcal{R}_s = -i(sR^0\mathbf{1} + R^a T_a), \tag{80}$$

where T_a is in some spin j representation of $SU(2)$ and 1 is the $(2j+1)\times(2j+1)$ identity matrix. For generality we can keep s, j independent from each other. In the particular case where we want to write down the effective action for spinless charged particles for the sth Landau level of $K = \mathbb{CP}^2$, we need to identify $j = \frac{1}{2}s$.

The index density works out to be

$$\begin{aligned} \text{Index density}(\bar{\partial}) &= \frac{i^3}{(2\pi)^3}\frac{(\dim V)(2j+1)(s+1)}{12} \\ &\quad \times\left[(2s^2+4s+1)(d\omega^0)^3 \right. \\ &\quad \left. + \frac{8j(j+1)-1}{4} d\omega^0 \wedge(-iR^a)\wedge(-iR^a)\right]. \end{aligned} \tag{81}$$

We then identify the gravitational contribution to the effective action as

$$\begin{aligned} S_{\rm grav} &= \frac{i^3}{(2\pi)^2}(\dim V)(2j+1)(s+1) \\ &\quad \times\left[\frac{1}{6}\left(\left(s+1\right)^2 - \frac{1}{2}\right)\int \omega^0 (d\omega^0)^2 \right. \\ &\quad \left. + \left(\frac{1}{3}j(j+1) - \frac{1}{24}\right)\int \omega^0 \text{Tr}(\tilde{R}\wedge\tilde{R})\right], \end{aligned} \tag{82}$$

where $\tilde{R}$ indicates the $SU(2)$ curvature and $\text{Tr}(\tilde{R}\wedge\tilde{R}) = \frac{1}{2}(-iR^a)\wedge(-iR^a)$. There are alternate ways to write this. For example, in the last term, we can replace the integral by a partial integration as

$$\int \omega^0 \text{Tr}(\tilde{R}\wedge\tilde{R}) = \int d\omega^0 \text{Tr}\left(\tilde{\omega} d\tilde{\omega} + \frac{2}{3}\tilde{\omega}^3\right), \tag{83}$$

where ω^0, $\tilde{\omega}$ are the connections for the $U(1)$ and $SU(2)$ curvatures. Since we are considering manifolds without boundary, these different forms are equivalent. (The boundary at the limits of the time integration is not null, and so these different ways would correspond to different ways of writing the symplectic form, if one proposes to set up a Hamiltonian version of the effective action).

One can now combine (42) and (82) to write down the full topological action in $(4+1)$ dimensions. [This is, of course, equivalent to the $(4+1)$-form from the action (41).] For simplicity, we will only consider an Abelian gauge field now. The gauge part of the action in (42) can then be written as

$$S_{\rm gauge} = \frac{i^3(2j+1)}{(2\pi)^2}\int\Big\{\frac{1}{2}\left[(s+1)^2 - \frac{1}{6}\right]A(d\omega^0)^2 + \left[\frac{1}{3}j(j+1) - \frac{1}{24}\right]A{\rm Tr}(\tilde{R}\wedge\tilde{R}) + \frac{(s+1)}{2}AdAd\omega^0 + \frac{1}{3!}A(dA)^2\Big\} \quad (84)$$

$$= \frac{i^3(2j+1)}{(2\pi)^2}\int\Big\{\frac{1}{3!}(A+(s+1)\omega^0)[d(A+(s+1)\omega^0)]^2 - \frac{(s+1)^3}{3!}\omega^0(d\omega^0)^2 - \frac{1}{12}A(d\omega^0)^2 + \left(\frac{1}{3}j(j+1) - \frac{1}{24}\right)A{\rm Tr}(\tilde{R}\wedge\tilde{R})\Big\}. \quad (85)$$

The first four terms in (84) constitute the analog of the Wen-Zee term in $(4+1)$ dimensions while the last term is the gauge Chern-Simons term. Combining (85) and (82) and setting $\dim V = 1$ we find the full topological action

$$S^{(s)}_{5d} = \frac{i^3(2j+1)}{(2\pi)^2}\int\Big\{\frac{1}{3!}(A+(s+1)\omega^0)[d(A+(s+1)\omega^0)] - \frac{1}{12}(A+(s+1)\omega^0)\Big[(d\omega^0)^2 - \left[(4j(j+1) - \frac{1}{2}\right]{\rm Tr}(\tilde{R}\wedge\tilde{R})\Big]\Big\}. \quad (86)$$

Further setting $j = s/2$ in (86) will give the bulk topological action for the sth Landau level QHE on $\mathbb{CP}^2$ with Abelian magnetic fields. Notice that an interesting effect of the gravitational interaction is to replace $A \to A + (s+1)\omega^0$ in (86). The analog effect in the case of $\mathbb{CP}^1$ was $A \to A + (s+\frac{1}{2})\omega$ as in (77).

C. (6 + 1)-dimensional case

In $(6+1)$ dimensions we need to evaluate the eight-form index density. Again, for simplicity we will consider the case of Abelian magnetic fields ($\dim V = 1$); we will also consider only the case of spin zero fields, $s = 0$, $\mathcal{R}_s = 0$, $\dim S = 1$ (lowest Landau level). The corresponding index density involving gravitational fields is

$$\text{Index Density}(\bar{\partial}) = \frac{1}{720}(-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4). \quad (87)$$

Using the expressions for the characteristic classes in (15) we find

$$\text{Index density}(\bar{\partial}) = \frac{1}{720}\Big\{\frac{15}{8}\left({\rm Tr}\frac{iR}{2\pi}\right)^4 - \frac{15}{4}\left({\rm Tr}\frac{iR}{2\pi}\right)^2{\rm Tr}\left(\frac{iR}{2\pi}\right)^2 + \frac{5}{8}\left[{\rm Tr}\left(\frac{iR}{2\pi}\right)^2\right]^2 + \frac{1}{4}{\rm Tr}\left(\frac{iR}{2\pi}\right)^4\Big\}, \quad (88)$$

where

$$R = -i(R^0\mathbf{1} + R^at_a) \equiv d\omega^0 + \tilde{R}, \quad (89)$$

where t_a are the $SU(3)$ generators in the fundamental representation, $\mathbf{1}$ is the 3×3 identity matrix, ω^0 is the $U(1)$ spin connection and $\tilde{R}$ is the $SU(3)$ curvature.

From (89) we find that the purely gravitational contribution to the topological action in $(6+1)$ dimensions is

$$S_{\rm grav} = \frac{1}{(2\pi)^3}\frac{1}{720}\int\Big\{57\omega^0d\omega^0\left[(d\omega^0)^2 - \frac{1}{2}{\rm Tr}(\tilde{R}\wedge\tilde{R})\right] + \omega^0{\rm Tr}(\tilde{R}\wedge\tilde{R}\wedge\tilde{R})\Big\} + \frac{1}{120}\int(CS)_7(\tilde{\omega}), \quad (90)$$

where $\tilde{\omega}$ is the $SU(3)$ spin connection and

$$CS_7(\tilde{\omega}) = \frac{1}{4!(2\pi)^3}{\rm Tr}\left[\tilde{\omega}(d\tilde{\omega})^3 + \frac{12}{5}\tilde{\omega}^3(d\tilde{\omega})^2 + 2\tilde{\omega}^5(d\tilde{\omega}) + \frac{4}{7}\tilde{\omega}^7\right]. \quad (91)$$

The gauge contribution to the topological action (68) for an Abelian magnetic field and spin zero fields (LLL) is

$$S_{\rm gauge} = \frac{1}{(2\pi)^3}\int\Big\{\frac{1}{4!}\left(A + \frac{3}{2}\omega^0\right)\left[d\left(A + \frac{3}{2}\omega^0\right)\right]^3 - \frac{1}{16}\left(A + \frac{3}{2}\omega^0\right)d\left(A + \frac{3}{2}\omega^0\right) \times \left[(d\omega^0)^2 + \frac{1}{3}{\rm Tr}(\tilde{R}\wedge\tilde{R})\right] - \frac{9}{128}\omega^0d\omega^0\left[(d\omega^0)^2 - \frac{2}{3}{\rm Tr}(\tilde{R}\wedge\tilde{R})\right]\Big\}. \quad (92)$$

Adding (90) and (92) we get the bulk topological action for the lowest Landau level of $\mathbb{CP}^3$ with Abelian magnetic fields. The full action is

$$S^{\mathrm{LLL}}_{7d} = \frac{1}{(2\pi)^3}\int\left\{\frac{1}{4!}\left(A+\frac{3}{2}\omega^0\right)\left[d\left(A+\frac{3}{2}\omega^0\right)\right]^3 - \frac{1}{16}\left(A+\frac{3}{2}\omega^0\right)d\left(A+\frac{3}{2}\omega^0\right)\left[(d\omega^0)^2 + \frac{1}{3}\mathrm{Tr}(\tilde{R}\wedge\tilde{R})\right] + \frac{1}{1920}\omega^0 d\omega^0[17(d\omega^0)^2 + 14\mathrm{Tr}(\tilde{R}\wedge\tilde{R})] + \frac{1}{720}\omega^0\mathrm{Tr}(\tilde{R}\wedge\tilde{R}\wedge\tilde{R})\right\} + \frac{1}{120}\int (CS)_7(\tilde{\omega}). \tag{93}$$

Again, this corresponds to the appropriate simplification of the general action (41). In (93) we see again the shift $A \to A + \frac{3}{2}\omega^0$ in the presence of gravitational interactions.

D. Comments

It is worth pointing out a couple of interesting features of the gravitational contributions. First of all, we notice that the $U(1)$ part of the spin connection combines with the $U(1)$ gauge field, as $A + (s + \frac{1}{2}k)\omega^0$, for $\mathbb{CP}^k$. This is explicitly seen for the cases we have considered, namely, for $k = 1$, 2 for arbitrary s and for $k = 3$ with $s = 0$. We expect this to be true in general. This is seemingly related to the metaplectic correction in geometric quantization, something we plan to address in more detail in a future publication.

Second, if we consider $2n$-manifolds with the full $SO(2n)$ holonomy, we do not expect purely gravitational anomalies except for $2n = 4k + 2$, $k = 0$, 1, 2, etc. This is because the index density from which the anomaly is descended, namely, $\mathrm{Tr}R^{n+1}$ vanishes by virtue of the antisymmetry of R as an element of the algebra of $SO(2n)$. In our case, we consider the restriction to holonomies in $U(k) \subset SO(2k)$, so we do not have the transformations which can combine the $U(k)$-valued curvatures into a real antisymmetric matrix in $SO(2k)$.

The existence of the purely gravitational contributions is related to the fact that the Dolbeault index is nonzero for even dimensions, in a way similar to the argument given in [31] for fermions. For the gravitational anomaly for fermions in a general dimension $2n$, one can consider the compactification of the manifold as $M_2 \times M_{2n-2}$, where M_2 is two dimensional and M_{2n-2} is taken to be compact. One can then consider the anomaly for Lorentz transformations (or diffeomorphisms) on M_2. The effect of the remaining $(2n - 2)$ dimensions is a multiplicative factor corresponding to the number of zero modes of the relevant kinetic operator, i.e., the Dirac operator, on M_{2n-2}. The anomaly in two dimensions, namely on M_2, then implies a nonzero anomaly on M_{2n} if the Dirac operator has a nonzero index on M_{2n-2}. This is possible for fermions only if $2n - 2 = 4k$. This reasoning works because the anomaly may be viewed as a short distance effect arising from issues of regularization and hence the compactification does not affect the final answer. For the case of interest to us, the Dolbeault operator has a nonzero index generically for any even dimension, in particular on M_{2k-2}. Thus we should expect a gravitational anomaly with the Dolbeault index density in $2k + 2$ dimensions as the starting point for the descent procedure.

However, we may note that, although we do have a nonzero gravitational contribution, there is a remnant in the final expressions from the vanishing of $\mathrm{Tr}R^{n+1}$ due to the antisymmetry property of R (if it has values in $SO(2k)$. Once we have combined A with ω^0 as in $A + (s + \frac{1}{2}k)\omega^0$, there is a left-over purely gravitational piece in some cases. In $(2 + 1)$ dimensions, this is given by the last term in the braces in (77). This has been interpreted as what is needed to cancel the gravitational anomaly due to the chiral field on the edge in the case of a finite droplet. In $(2 + 1)$ dimensions, the chiral field on the edge lives in $1 + 1$ dimensions, and produces an anomaly for the Lorentz connection ω. For the $(4 + 1)$-dimensional case, the edge field is in $3 + 1$ dimensions. The gravitational fields are valued in $SO(3,1)$ or $SO(4)$ after a Euclidean continuation. A chiral field would couple to one of the chiral components in the splitting $SO(4) \sim SO(3)_L \otimes SO(3)_R$. In this case, there is no Lorentz anomaly by the same reasoning as related to the antisymmetry of R with values in the algebra of the orthogonal group. Thus we should expect no purely non-Abelian gravitational part in the action. This is in agreement with what we find in (86), where there is no purely non-Abelian gravitational term.

IX. DISCUSSION

In this paper, we have given a general expression (41) for the topological part of the bulk effective action for quantum Hall systems in arbitrary even spatial dimensions. Explicit detailed formulas for the action are given in $(2 + 1)$, $(4 + 1)$ and $(6 + 1)$ dimensions. The background metric and gauge field can be arbitrary in the sense that fluctuations of the metric and the gauge field around a given background, but which do not change the topological class of the background, are included. This action thus yields the topological terms in the response of the system (or correlation functions of the source currents) to changes in the gauge and gravitational fields. The terms which involve only the gauge field had been obtained earlier for the lowest Landau level in a large N simplification, where N denotes the degeneracy of the Landau level [24–26]. Terms which involve both the gauge and the gravitational fields provide a generalization of the well-known Wen-Zee term in the $(2 + 1)$-dimensional case. Since these are subdominant in N, they were not evident in the leading large N calculations. (Some metric-dependent subdominant terms, including some gauge-gravity mixing terms, were already in [24–26], but they were not explicitly stated in terms of the curvatures, since a fixed gravitational background was used.)

The main justification for the effective action (41) is that the current densities obtained from it correctly reproduce the degeneracies of the Landau levels via the Dolbeault index theorem. In $(2+1)$ dimensions, our results agree with the effective action which has been obtained by other authors by different techniques. The approach in [14] uses a Dolbeault index density as well. However, the starting point there is the index density in four dimensions. A path in this space is considered as the time direction and a descent procedure from four dimensions to the $(2+1)$-dimensional world of this line and the two-dimensional transverse space is used. There are other important considerations in [14], including going beyond the topological terms, but on questions for which our work has overlap with this paper, the results agree.

More generally, for the effective action in $(2k+1)$ dimensions, there are two index densities we can consider, in $2k$ dimensions and in $(2k+2)$ dimensions. The first one is relevant for the degeneracy and can be used to obtain many of the terms in the effective action. However, as explained after (37), we may think of the action as also obtained via the descent procedure from the Dolbeault index density in $(2k+2)$ dimensions. The latter can be used to identify the purely gravitational terms related to gravitational anomalies and to clarify terms involving non-Abelian gauge fields. It is useful to consider both index densities together as they highlight complementary aspects of the problem.

We have considered only fully filled Landau levels on manifolds without boundary. The case of quantum Hall droplets, the action for the edge excitations which exist in such cases and the interplay between the bulk and boundary actions are clearly the next set of interesting questions, to be taken up in the future. Also, beyond the milieu of exploring the geometry of the quantum Hall effect in arbitrary dimensions, geometry and topology, we may note that quantum Hall effect in higher dimensions has been of interest for spin Hall effect and for considerations on gravity. The results of this paper may therefore be of specific interest in such contexts as well.

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APPENDIX: BASIC FEATURES AND GEOMETRY OF $\mathbb{CP}^k$ SPACES

Let t_A denote the generators of $SU(k+1)$ as matrices in the fundamental representation, normalized so that $\mathrm{Tr}(t_A t_B) = \frac{1}{2}\delta_{AB}$. These generators are classified into three groups. The ones corresponding to the $SU(k)$ part of $U(k)\subset SU(k+1)$ will be denoted by t_a, $a = 1,2,\ldots,k^2-1$ while the generator for the $U(1)$ direction of the subgroup $U(k)$ will be denoted by t_{k^2+2k}. The $2k$ remaining generators of $SU(k+1)$ which are not in $U(k)$ are the coset generators, denoted by t_α, $\alpha = k^2,\ldots,k^2+2k-1$. The coset generators can be further separated into the raising and lowering type $t_{\pm i} = t_{k^2+2i-2} \pm i t_{k^2+2i-1}$, $i = 1,\ldots,k$.

We can now use a $(k+1)\times(k+1)$ matrix g in the fundamental representation of $SU(k+1)$ to parametrize $\mathbb{CP}^k$, by making the identification $g \sim gh$, where $h \in U(k)$. We can use the freedom of h transformations to write g as a function of the real coset coordinates x^I, $I = 1,\ldots,2k$. The relation between the complex coordinates $z^i, \bar{z}^i$ in (16) and x^I is the usual one,

$$z^i = x^{2i-1} + i x^{2i}, \qquad \bar{z}^i = x^{2i-1} - i x^{2i}, \qquad i = 1,\ldots,k. \tag{A1}$$

We can write

$$g^{-1}dg = (-iE^{k^2+2k}t_{k^2+2k} - iE^a t_a - iE^\alpha t_\alpha). \tag{A2}$$

E^α are one-forms corresponding to the frame fields in terms of which the Cartan-Killing metric on $\mathbb{CP}^k$ is given by

$$ds^2 = g_{ij}dx^i dx^j = E_i^\alpha E_j^\alpha dx^i dx^j. \tag{A3}$$

The Kähler two-form on $\mathbb{CP}^k$ is written as

$$\begin{aligned}\Omega &= -i\sqrt{\frac{2k}{k+1}}\mathrm{tr}(t_{k^2+2k}g^{-1}dg\wedge g^{-1}dg)\\ &= -\frac{1}{4}\sqrt{\frac{2k}{k+1}}f_{(k^2+2k)\alpha\beta}E^\alpha\wedge E^\beta = -\frac{1}{4}\epsilon_{\alpha\beta}E^\alpha\wedge E^\beta\end{aligned} \tag{A4}$$

f_{ABC} are the $SU(k+1)$ structure constants, where $[t_A, t_B] = i f_{ABC} t_C$. In deriving the last line we used the fact that $f_{(k^2+2k)\alpha\beta} = \sqrt{\frac{k+1}{2k}}\epsilon_{\alpha\beta}$, where $\epsilon_{\alpha\beta} = 1$ if $\alpha = 2i-1, \beta = 2i, i = 1,\ldots,k$.

The Kähler two-form Ω can also be written in terms of the local complex coordinates in the more familiar form

$$\Omega = i\left[\frac{dz\cdot d\bar{z}}{1+z\cdot\bar{z}} - \frac{\bar{z}\cdot dz z\cdot d\bar{z}}{(1+z\cdot\bar{z})^2}\right], \tag{A5}$$

where

$$\begin{aligned} g_{i,k+1} &= \frac{z_i}{\sqrt{1+z\cdot\bar{z}}}, \qquad i = 1,\ldots,k\\ g_{k+1,k+1} &= \frac{1}{\sqrt{1+z\cdot\bar{z}}}\end{aligned} \tag{A6}$$

was used in (A4).

The volume of $\mathbb{CP}^k$ is normalized so that

$$\int_{\mathbb{CP}^k} \left(\frac{\Omega}{2\pi}\right)^k = 1. \tag{A7}$$

The Maurer-Cartan identity along with (A2) leads to

$$\begin{aligned} dE^{k^2+2k} &= -\frac{1}{2} f^{(k^2+2k)\alpha\beta} E^\alpha \wedge E^\beta = 2\sqrt{\frac{k+1}{2k}}\Omega \\ dE^a + \frac{1}{2} f^{abc} E^b \wedge E^c &= -\frac{1}{2} f^{a\alpha\beta} E^\alpha \wedge E^\beta \\ dE^\alpha &= -f^{\alpha A\beta} E^A \wedge E^\beta. \end{aligned} \tag{A8}$$

Combining the $2k$ frame fields E^α into holomorphic combinations and using (A8) we can identify the spin connection for the complex cotangent space $T_c^* K$,

$$\begin{aligned} &\mathcal{E}^I \equiv E^{2I-1} + iE^{2I}, \quad d\mathcal{E}^I + \omega_*^{IJ}\mathcal{E}^J = 0, \quad I = 1, \ldots, k \\ &\omega_* = -i\left(\sqrt{\frac{k+1}{2k}} E^{k^2+2k}(-\mathbf{1}) + E^a(-t_a)^T\right), \end{aligned} \tag{A9}$$

where $\mathbf{1}$ is the $k \times k$ identity matrix and t_a are the $SU(k)$ matrices in the fundamental representation and the superscript T on t_a indicates the transpose. A basis for the tangent space $T_c K$ is given by vector fields dual to $\mathcal{E}^I$. By differentiating the relation $\mathcal{E}_a^I (\mathcal{E}^{-1})_I^b = \delta_a^b$, we can identify the spin connection for $T_c K$ as

$$\omega = -i\left(\sqrt{\frac{k+1}{2k}} E^{k^2+2k}\mathbf{1} + E^a t_a\right). \tag{A10}$$

Notice that $\mathbf{1} \to (-\mathbf{1})$ and $t_a \to (-t_a)^T$ appearing in (A9) correspond to the conjugation operation for the Lie algebra. Thus the Lie algebra conjugation operation ingoing from the cotangent space to the tangent space is exactly as expected.

Using (A8) we can also derive the curvature two-form as

$$\begin{aligned} R &= d\omega + \omega \wedge \omega \\ &= -i\left(\frac{k+1}{k}\Omega\mathbf{1} - \frac{1}{2} f^{a\alpha\beta} E^\alpha \wedge E^\beta t_a\right) \\ &= -i(R^0\mathbf{1} + R^a t_a), \end{aligned} \tag{A11}$$

where $R^0 = \frac{k+1}{k}\Omega$ and $R^a = -\frac{1}{2} f^{a\alpha\beta} E^\alpha \wedge E^\beta$.

For $\mathbb{CP}^k$ spaces

$$\int_{\mathbb{CP}^k} \mathrm{td}(T_c K)|_{2k} = 1, \tag{A12}$$

where $\mathrm{td}(T_c K)$ is the Todd class in the complex tangent space and in (A12) the $2k$-form is selected as the integrand. Explicitly, the Todd class has the expansion given in (11) as

$$\begin{aligned} \mathrm{td} &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1 c_2 \\ &\quad + \frac{1}{720}(-c_4 + c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4) + \cdots, \end{aligned} \tag{A13}$$

where c_i are the Chern classes. The first few Chern classes can be easily evaluated using (12) as

$$\begin{aligned} c_1 &= \mathrm{Tr}\frac{iR}{2\pi} = (k+1)\frac{\Omega}{2\pi} \\ c_2 &= \frac{1}{2}\left[\left(\mathrm{Tr}\frac{iR}{2\pi}\right)^2 - \mathrm{Tr}\left(\frac{iR}{2\pi}\right)^2\right] = \frac{1}{2}k(k+1)\left(\frac{\Omega}{2\pi}\right)^2. \end{aligned} \tag{A14}$$

In deriving the expression for c_2 we used the fact that

$$\begin{aligned} R^a \wedge R^a &= \frac{1}{4} f^{a\alpha\beta} f^{a\gamma\delta} E^\alpha E^\beta E^\gamma E^\delta = -2\frac{k+1}{k}\Omega^2 \\ \mathrm{Tr}[iR \wedge iR] &= k(R^0)^2 + \frac{1}{2}(R^a)^2 = (k+1)\Omega^2. \end{aligned} \tag{A15}$$

More generally the Chern classes for $\mathbb{CP}^k$ can be written as

$$c_i = \frac{k!}{i!(k-i)!}\left(\frac{\Omega}{2\pi}\right)^i. \tag{A16}$$

Using (A7) and (A14), we can easily check the validity of (A12) for $\mathbb{CP}^1$, $\mathbb{CP}^2$ and $\mathbb{CP}^3$, the needed integrals being

$$\begin{aligned} \int_{\mathbb{CP}^1} c_1 &= 2\int \frac{\Omega}{2\pi} = 2 \\ \int_{\mathbb{CP}^2} c_1^2 + c_2 &= (3^2 + 3)\int \left(\frac{\Omega}{2\pi}\right)^2 = 12 \\ \int_{\mathbb{CP}^3} c_1 c_2 &= 4 \times 6 \int \left(\frac{\Omega}{2\pi}\right)^3 = 24. \end{aligned} \tag{A17}$$

In formulating QHE on $\mathbb{CP}^k$, we choose $U(1)$ and $SU(k)$ background gauge fields proportional to $E_i^{k^2+2k}$ and E_i^a. In particular

$$\begin{aligned} A^{k^2+2k} &= -in\sqrt{\frac{2k}{k+1}}\mathrm{tr}(t_{k^2+2k} g^{-1}dg) = \frac{n}{2}\sqrt{\frac{2k}{k+1}} E^{k^2+2k} \\ A^a &= E^a = 2i\mathrm{Tr}(t^a g^{-1}dg). \end{aligned} \tag{A18}$$

The corresponding $U(1)$ and $SU(k)$ background field strengths are

$$\begin{aligned} F &= n\Omega = -\frac{n}{4}\sqrt{\frac{2k}{k+1}} f^{(k^2+2k)\alpha\beta} E^\alpha \wedge E^\beta \\ F^a &= -f^{a\alpha\beta} E^\alpha \wedge E^\beta. \end{aligned} \tag{A19}$$

Notice that F^a in (A19) does not depend on n, while the Abelian field is proportional to n. We see from (A19) that the background field strengths are constant in the appropriate frame basis, proportional to the $U(k)$ structure constants. It is in this sense that the field strengths in (A19) correspond to uniform magnetic fields appropriate in defining QHE.

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