PHYSICAL REVIEW D **94,** 024017 (2016)

Functional renormalization group analysis of tensorial group field theories on $\mathbb{R}^d$

Joseph Ben Geloun,[1,2,*] Riccardo Martini,[3,1,†] and Daniele Oriti[1,‡]

[1] *Max Planck Institute for Gravitational Physics, Albert Einstein Institute, Am Mühlenberg 1, 14476 Potsdam, Germany*

[2] *International Chair in Mathematical Physics and Applications, ICMPA-UNESCO Chair, 072BP50 Cotonou, Benin*

[3] *Alma Mater Studiorum, Università di Bologna, via Zamboni 33, 40126 Bologna, Italy*

(Received 15 March 2016; published 6 July 2016)

Rank-d tensorial group field theories are quantum field theories (QFTs) defined on a group manifold $G^{\times d}$, which represent a nonlocal generalization of standard QFT and a candidate formalism for quantum gravity, since, when endowed with appropriate data, they can be interpreted as defining a field theoretic description of the fundamental building blocks of quantum spacetime. Their renormalization analysis is crucial both for establishing their consistency as quantum field theories and for studying the emergence of continuum spacetime and geometry from them. In this paper, we study the renormalization group flow of two simple classes of tensorial group field theories (TGFTs), defined for the group $G = \mathbb{R}$ for arbitrary rank, both without and with gauge invariance conditions, by means of functional renormalization group techniques. The issue of IR divergences is tackled by the definition of a proper thermodynamic limit for TGFTs. We map the phase diagram of such models, in a simple truncation, and identify both UV and IR fixed points of the RG flow. Encouragingly, for all the models we study, we find evidence for the existence of a phase transition of condensation type.

DOI: 10.1103/PhysRevD.94.024017

I. INTRODUCTION

Group field theories (GFTs) [1] are a new type of quantum field theories characterized by a peculiar nonlocal pattern of pairings of field arguments in the interactions. The domain of definition of the fields is, for the most studied models, a (Lie) group manifold, hence the name of the formalism. The first consequence of the nonlocality of the GFT interactions is that their quantum states can be associated with graphs (or networks), while the Feynman diagrams arising in the GFT perturbative expansion are dual to cellular complexes. These graphs and cellular complexes are then decorated by group-theoretic data, corresponding to the degrees of freedom associated with the GFT fields. This implies that a number of standard QFT techniques have to be adapted to this new context, and that a host of new mathematical structures can be explored by such field theoretic means. This formalism finds its historic roots, and main applications, at present, as a promising framework for quantum gravity. From this more physical perspective, GFTs are a tentative definition of the microstructure of quantum spacetime and of its fundamental quantum dynamics. The decorated graphs, in this interpretation, are the fundamental quantum structures from which a continuum spacetime and geometry should emerge in the appropriate regime of approximation. In fact, group field theories were first proposed [2] as an enrichment, by the addition of group-theoretic data, of tensor models [3] (in turn a generalization of matrix models for two-dimensional (2D) quantum gravity [4] to higher dimensions), with the main goal being to obtain Feynman amplitudes of the form of state sum models of topological field theories. The link with loop quantum gravity (LQG) [5] became quickly clear [6]: group field theories and loop quantum gravity share the same type of quantum states, i.e., spin networks. It is then in the context of loop quantum gravity and state sum models, called spin foam models [7] and developed as a covariant definition of loop quantum gravity, that most subsequent work has been done, once it was understood [8] that the correspondence between group field theory and spin foam amplitudes is completely general. Finally, the relation between group field theory and lattice quantum gravity, already evident in their origin in tensor models, became stronger because of the appearance of the Regge action in semiclassical analyses of spin foam amplitudes (see, for example, [9]), and, more recently, of the general possibility to recast group field theory amplitudes as (noncommutative) simplicial gravity path integrals [10]. It is now clear that group field theories sit at the crossroads of several approaches to quantum gravity, as a second quantized framework for loop quantum gravity degrees of freedom [11] as well as an enrichment of tensor models. The quantum field theory framework they provide for the candidate fundamental degrees of freedom of

*jbengeloun@aei.mpg.de
†riccardo.martini@studio.unibo.it
‡daniele.oriti@aei.mpg.de

quantum spacetime is then crucial for tackling the open issues of these approaches. In particular, it makes it possible to take on a condensed matterlike perspective, making precise the idea of "atoms of space," to study from this perspective the emergence of continuum spacetime [12], and to use powerful renormalization group techniques for the analysis of their quantum dynamics. The renormalization group analysis of GFT models has two main goals: establishing their perturbative renormalizability and exploring the continuum phase diagram. The first goal is all the more important because these models are initially defined and studied in perturbative expansion around the trivial vacuum, and it is in this expansion that their relation with loop quantum gravity and lattice quantum gravity, as well as their quantum geometric content, is more apparent. Establishing their perturbative renormalizability amounts then to establishing the consistency of this definition, and it also serves the purpose of constraining quantization ambiguities (the GFT counterpart of those arising in the canonical loop quantum gravity formulation) as well as model building. The second goal is the most important open issue in all these related quantum gravity approaches: their continuum limit, i.e., the macroscopic, collective dynamics of their microscopic degrees of freedom, and the possibility of spacetime and geometry emerging from a phase transition of the same degrees of freedom [12], as it has been proposed also in related approaches [13–17]. It also amounts to controlling the full GFT expansion in terms of the sum over cellular complexes and spin foam histories; thus it can be seen as solving, by QFT techniques, the problem of the continuum limit in both dynamical triangulations and spin foam models (for which alternative strategies have also been explored [18]).

GFT renormalization is, in fact, one of the most rapidly developing research directions in this area, and it has benefited greatly from concurrent developments in tensor models [3], which provides analytic tools and many insights concerning the combinatorics and the topology of GFT Feynman diagrams [19,20] as well as the possible definitions of the theory space to focus on [21,22]. Indeed, most of the work in GFT renormalization has concerned a class of GFTs, called tensorial group field theories (TGFTs), in which tensorial structures are prominent. Several interesting TGFT models have been proven to be renormalizable [23–28], and their RG flow has also been studied, mainly in the vicinity of the UV fixed point [25,29–32], showing that asymptotic freedom is a very general feature of TGFT models [33]. This work has encompassed Abelian as well as non-Abelian models, and both models with and without the additional gauge invariance properties characterize GFTs for topological BF theory and four-dimensional (4D) gravity, by giving their Feynman amplitudes the structure of lattice gauge theories. The same analysis has also been extended to models defined not by groups but by homogeneous spaces [34]. More recently, nonperturbative GFT renormalization has been tackled as well. Some work [35,36] has been based on the Polchinski equation and on the analysis of the Schwinger-Dyson equations (see also [37]). Most work has, however, been framed in the language of the functional renormalization group approach to QFTs, first adapted to TGFTs in [38], after the initial steps taken in [39–42] for matrix models. The first model being studied [38] was an Abelian rank-3 one on $U(1)$, and this analysis was quickly extended to the noncompact case in [43]. A model in rank-6 and again based on $U(1)$ was instead analyzed in [44], this time incorporating gauge invariance. All these models were analyzed in a fourth order truncation in the number of fields. In all these cases, not only was it possible to confirm the asymptotic freedom of the models in the UV, but it was also possible to identify IR fixed points and to provide hints of a phase transition. The IR fixed points resemble Wilson-Fisher fixed points for ordinary scalar field theories, and the phase transition appears to separate a symmetric and a broken or condensate phase, with a nonzero expectation value for the TGFT field operator. With a different perspective, the existence of a phase transition has been proven for quartic tensor models in [45,46] with a characterization of the related phases and also for GFT models related to topological BF theory, in any dimension [47]. In models more directly related to loop quantum gravity and lattice quantum gravity, this type of phase transition was suggested to govern the emergence of an effective cosmological dynamics from such quantum gravity models [48].

In this paper, we generalize the analysis of Abelian models on $\mathbb{R}$ performed in [43], in two main ways: we compute and study the RG flow of models of arbitrary rank, and we perform the same analysis also for gauge invariant models, again in arbitrary rank. In both cases, we then specialize the results to ranks 3, 4, and 5, identify the UV and IR fixed points, and describe the resulting phase diagram. We still work, though, in a fourth order truncation of the number of fields.

The plan of this paper is as follows. Section II reviews the functional renormalization group applied to group field theories following [38]. In Sec. III, we describe the analysis of the simplest class of noncompact models, without gauge invariance, for arbitrary rank. We also complete the analysis given in [43] by providing the solution of the system of the β-function at second order around the Gaussian fixed point, provide details on the neighborhood of that trivial fixed point. As in that previous work, the analysis of such a noncompact model requires IR regularization, as we will discuss in detail. The key point of our regularization scheme is the introduction of a new parameter representing the dependence of couplings on the volume of the direct space. In Sec. IV, we repeat the analysis for another interesting class of models obtained introducing an additional gauge invariance in the amplitudes, by means of suitable projector operators inserted in the GFT action.

After appropriate regularization, we can again study the RG flow of these models. In Sec. V we give a summary of our results and list some important open problems for this approach. Appendixes A and B provide more details of our calculations, and Appendix C deals with the issue of scaling dimensions in the TGFTs.

II. THE FUNCTIONAL RENORMALIZATION GROUP FOR TGFTS: AN OVERVIEW

In this section we first review the basic ingredients of the (tensorial) group field theory formalism, in its covariant functional integral formulation. Then, we present the functional renormalization approach, as it has been adapted and applied to TGFTs in [38].

A. Tensorial group field theories

Let us introduce the special class of GFTs we will work with in the following, known as tensorial group field theories [23–32,49–51].

Consider a field ϕ defined over d-copies of a group manifold G, ϕ: $G^{\times d} \to \mathbb{C}$. For the moment, we assume G to be a compact Lie group. Without assuming any symmetry under permutations of field labels and, using the Peter-Weyl theorem (or its counterpart for a noncompact group, e.g., a Plancherel decomposition for the Lorentz group), the field decomposes in group representations as follows:

$$\phi(g_1, \ldots, g_d) = \sum_{\mathbf{P}} \phi_{\mathbf{P}} \prod_{i=1}^{d} D^{p_i}(g_i), \tag{1}$$

with $\mathbf{P} = (p_1, \ldots, p_d)$, $g_i \in G$ and where the functions $D^{p_i}(g_i)$ form a complete orthonormal basis of functions on the group characterized by the labels p_i. In a TGFT model, we require fields to have tensorial properties under basis changes. We define a rank-d covariant complex tensor $\phi_{\mathbf{P}}$ to transform through the action of the tensor product of unitary representations of the group $\bigotimes_{i=1}^{d} U^{(i)}$, each of them acting independently over the indices of field labels,

$$\phi_{p'_1,\ldots,p'_d} = \sum_{\mathbf{P}} U^{(1)}_{p'_1,p_1} \cdots U^{(d)}_{p'_d,p_d} \phi_{p_1,\ldots,p_d}. \tag{2}$$

The complex conjugate field will then be the contravariant tensor transforming as

$$\bar{\phi}_{p'_1,\ldots,p'_d} = \sum_{\mathbf{P}} (U^\dagger)^{(d)}_{p'_d,p_d} \cdots (U^\dagger)^{(1)}_{p'_1,p_1} \bar{\phi}_{p_1,\ldots,p_d}. \tag{3}$$

TGFT interactions are defined by "trace invariants" built out of ϕ and $\bar{\phi}$, which allow a strong control on the combinatorial structure of field convolutions, and are thus relevant for the construction of renormalizable TGFT actions. Tensorial trace invariants generalize invariant traces over matrices, which indeed are classical unitary invariants. They are obtained contracting pairwise the indices with the same position of covariant and contravariant tensors and saturating all of them. In this way, they always involve the same number of ϕ and $\bar{\phi}$. A simple example is the following:

$$\mathrm{Tr}(\phi\bar{\phi}) = \sum_{\mathbf{P},\mathbf{Q}} \phi_{\mathbf{P}} \bar{\phi}_{\mathbf{Q}} \prod_{i=1}^{d} \delta_{p_i,q_i}. \tag{4}$$

Considering that $\phi_{\mathbf{P}}$ ($\bar{\phi}_{\mathbf{P}}$) transforms as a complex vector (1-form) under the action of the unitary representations of G on one single index, the fundamental theorem on classical invariants for U on each index entails that all invariant polynomials in field entries can be written as a linear combination of trace invariants [52]. This formulation of tensor models can be adapted to the real field case, where the unitary group is replaced by the orthogonal one [53].

An interesting feature, which becomes an important computational tool, is that tensor invariants can be given a graphical representation as bipartite colored graphs, and, in fact, they are in one-to-one correspondence with them. A tensor ϕ is represented by a (white) node with d labeled half-lines outgoing from it. Its complex conjugate is a similar d-valent node with a different color (black). A tensor contraction is represented then by joining the half-lines, equally labeled, of two nodes of different color.

Trace invariants can be generalized to convolutions where the contractions are made by operators different from the delta distribution, i.e., by nontrivial kernels. In this case, the resulting object is not guaranteed to be a unitary invariant.

We write a generic action for a TGFT model symbolically as

$$\begin{aligned} S[\phi,\bar{\phi}] &= \mathrm{Tr}(\bar{\phi}\cdot\mathcal{K}\cdot\phi) + S^{\mathrm{int}}[\phi,\bar{\phi}], \\ \mathrm{Tr}(\bar{\phi}\cdot\mathcal{K}\cdot\phi) &= \sum_{\mathbf{P},\mathbf{Q}} \bar{\phi}_{\mathbf{P}} \mathcal{K}(\mathbf{P};\mathbf{Q}) \phi_{\mathbf{Q}}, \\ S^{\mathrm{int}}[\phi,\bar{\phi}] &= \sum_{\{n_b\}} \lambda_{n_b} \mathrm{Tr}(\mathcal{V}_{n_b}\cdot\phi^n\cdot\bar{\phi}^n). \end{aligned} \tag{5}$$

Here $\mathcal{K}$ and $\mathcal{V}_n$ are kernels implementing the convolutions in the kinetic and interaction terms, respectively, where n indicates the numbers of covariant and contravariant fields appearing in the vertices, b labels the combinatorics of convolutions (i.e., corresponds to some given bipartite d-colored graph), and λ_{n_b} is a coupling constant for the interaction n_b.

The formalism can easily be generalized to a TGFT based on a noncompact group manifold G, and in this case the Plancherel decomposition into (unitary) representations replaces the Peter-Weyl one to decompose fields, and the

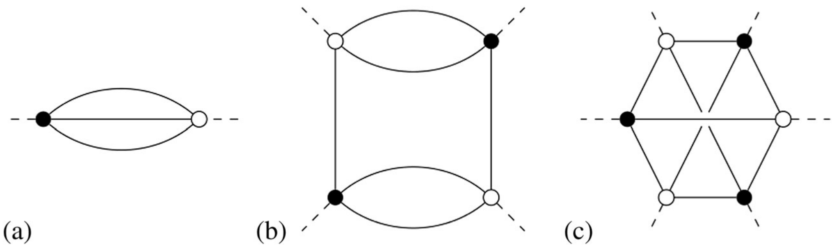


FIG. 1. Three examples of Feynman graphs for a rank-3 TGFT. The trace invariants used to build the interactions are the following: (a) $\mathrm{Tr}(\phi\bar{\phi})$, (b) an example of $\mathrm{Tr}(\phi\bar{\phi}\phi\bar{\phi})$, and (c) an example of $\mathrm{Tr}(\phi\bar{\phi}\phi\bar{\phi}\phi\bar{\phi})$.

definition of the trace over representation labels involves, in general, also integrals over continuous variables.

Given an action $S[\phi,\bar{\phi}]$, the partition function is defined as usual,

$$\mathcal{Z}[J,\bar{J}] = e^{W[J,\bar{J}]} = \int d\phi d\bar{\phi} e^{-S[\phi,\bar{\phi}]+\mathrm{Tr}(J\cdot\bar{\phi})+\mathrm{Tr}(\bar{J}\cdot\phi)}, \qquad (6)$$

where J is a rank-d complex source term and $\mathrm{Tr}(J\cdot\bar{\phi})$ is defined in (4).

The partition function can be expanded in perturbation theory around a Gaussian distribution and expressed as a (formal) sum over Feynman diagrams. Feynman diagrams of a rank-d TGFT are obtained by attaching, to the bipartite graph corresponding to a trace invariant defining each interaction vertex, a propagator (dashed line) for each field obtaining a $(d+1)$-colored graph (some examples are depicted in Fig. 1).

B. FRG formulation for TGFTs

The generalization of the FRG formalism [54–58] to TGFTs is straightforward and was first provided in [38]. Given a partition function of the type (6), we choose a UV cutoff M and a IR cutoff N.[1] Adding to the action a regulator term of the form

$$\Delta S_N[\phi,\bar{\phi}] = \mathrm{Tr}(\bar{\phi}\cdot R_N\cdot\phi) = \sum_{\mathbf{P},\mathbf{P}'}\bar{\phi}_{\mathbf{P}} R_N(\mathbf{P};\mathbf{P}')\phi_{\mathbf{P}'}, \qquad (7)$$

we can perform the usual splitting in high and low modes. In particular, given an action with a generic kernel depending on the derivative of the fields $\mathcal{K}(\nabla\phi)$ and a generalized Fourier transform $\mathcal{F}$, if we choose R_N to be of the specific form

$$R_N(\mathbf{P};\mathbf{P}') = N\delta_{\mathbf{P},\mathbf{P}'} R\left(\frac{\mathcal{F}(\mathcal{K}_{\mathbf{P}})}{N}\right), \qquad (8)$$

[1] We adopt a standard QFT terminology for field modes, even if no spacetime interpretation should be attached to them, at this stage.

we need to impose on the profile function $R(z)$ the following conditions:

(i) positivity $R(z)\geq 0$, to indeed suppress and not enhance modes outside of the domain of the regulator function;
(ii) monotonicity $\frac{d}{dz}R(z)\leq 0$, so that high modes will not be suppressed more that low modes;
(iii) $R(0)>0$ and $\lim_{z\to+\infty}R(z)=0$ to exclude functions with constant profile.

The last requirement, together with the form (8), guarantees that the regulator is removed for $Z\to 0$. In accordance with the usual FRG procedure, we define the scale dependent partition function as

$$\mathcal{Z}_N[J,\bar{J}] = e^{W_N[J,\bar{J}]} = \int d\phi d\bar{\phi} e^{-S[\phi,\bar{\phi}]-\Delta S_N[\phi,\bar{\phi}]+\mathrm{Tr}(J\cdot\bar{\phi})+\mathrm{Tr}(\bar{J}\cdot\phi)}, \qquad (9)$$

and the generating functionals of one-particle irreducible (1PI) correlation functions after Legendre transform are given in terms of the average field $\varphi = \langle\phi\rangle$ as

$$\Gamma_N[\varphi,\bar{\varphi}] = \sup_{J,\bar{J}}\{\mathrm{Tr}(J\cdot\bar{\varphi}) + \mathrm{Tr}(\bar{J}\cdot\varphi) - W_N[J,\bar{J}] - \Delta S_N[\varphi,\bar{\varphi}]\}. \qquad (10)$$

Given the above definitions, the Wetterich equation takes the form

$$\partial_t\Gamma_N[\varphi,\bar{\varphi}] = \overline{\mathrm{Tr}}(\partial_t R_N\cdot[\Gamma_N^{(2)}+R_N]^{-1}), \qquad (11)$$

where $t=\log N$, so that $\partial_t = N\partial_N$, and the "supertrace" symbol $\overline{\mathrm{Tr}}$ means that we are summing over all mode labels. More explicitly, the functional trace reads

$$\sum_{\mathbf{P},\mathbf{P}'}\partial_t R_N(\mathbf{P};\mathbf{P}')[\Gamma_N^{(2)}+R_N]^{-1}(\mathbf{P}';\mathbf{P}). \qquad (12)$$

The presence of the $\partial_t R_N$ in the Wetterich equation for TGFTs, enforces the trace to be UV finite if the profile function and its derivative go fast enough to 0, as $z\to+\infty$. In this way, we can basically forget about the UV cutoff M. In any case, we need an initial condition of the type

$$\Gamma_{N=M}[\varphi,\bar{\varphi}] = S[\varphi,\bar{\varphi}], \tag{13}$$

for some scale M. The problem of solving the full quantum theory is now phrased as the one of pushing the initial condition to infinity, which usually requires the existence of a UV fixed point, and solving the Wetterich equation with such an initial condition. The full quantum field theory will then be defined by the corresponding solution, i.e., by the full RG trajectory.

The Wetterich equation has a 1-loop structure, and since no (perturbative) approximation is required to obtain it, it is an exact functional equation. However, although we have expressed the problem of extracting the flow of the theory in terms of a partial differential equation in one single variable, we still have the issue that all possible (i.e., compatible with symmetry requirements and field content) couplings are allowed in Γ_k, which is thus expressible as an infinite sum of monomials in the field (and its conjugate). If we want to perform practical computations, we need some approximation scheme for the form of the free energy. Usually, this is obtained by truncating Γ_k to a maximal power in the fields and in their derivatives. It is then a truncation in theory space, which maintains the nonperturbative character of the RG equation.

What is peculiar, and interesting, about the application of FRG to TGFTs, is that $\Gamma_N^{(2)}$ carries inside the Wetterich equation information about the combinatorial nonlocality of the theory, i.e., the intricate combinatorics of TGFT interactions. In the case we consider here, that of a noncompact group manifold, this will also backreact at the level of the β-functions, with the fact that, depending on the combinatorics of the interaction, the volume contributions appearing in (11) will be not homogeneous and, in general, a natural definition of an effective local potential does not exist. Let us explain this key point, which we will deal with in detail in the following.

In its usual form, namely when applied to a standard, local quantum field theory (see, for instance, in [56]), the Wetterich equation shows pathological IR divergences due to the presence of $\delta(0)$ arising from the two-point Green's function computed at a single point $G_k^{(2)}(q,q)$. In the local field theory case, these divergent delta functions are homogeneous and proportional to the total volume of the system, namely, the domain manifold of the fields. A particular approximation procedure allows one to cure this problem, and it is called the local potential approximation (LPA) [56]. This procedure cannot be applied, at least not in the same straightforward way, to combinatorially nonlocal theories as TGFTs. One reason is that, in such nonlocal theories, the same type of IR divergence arises, in general, in a nonhomogeneous combination of $\delta(0)$ which are strictly dependent on the combinatorics of the interaction. We will discuss this and several other issues characterizing TGFTs as QFTs of an interesting new kind.

III. RANK-d TENSORIAL GROUP FIELD THEORY ON $\mathbb{R}$

As discussed in the Introduction, the first model studied within the FRG framework for TGFTs, already in [38], was a rank-3 model with compact group manifold $U(1)$, and subsequently, we have studied a noncompact counterpart of the same model, i.e., a rank-3 TGFT on $\mathbb{R}$ [43]. New issues concerning the thermodynamic limit but also more compelling hints for the existence of UV and IR fixed points, and of condensation phase transitions, were found.

We now extend the analysis and results of the latter work to arbitrary rank (as well as analyzing in more detail in the rank-3 model), showing how those intriguing hints are actually confirmed in a more general case. In the following section, we will analyze a modification of the same type of TGFT models which includes a gauge invariance property of fields and amplitudes, thus moving closer to full-fledged TGFT models for quantum geometry and discrete quantum gravity, as well as related to loop quantum gravity.

We start by introducing the class of TGFT models we will analyze.

A. The model

The TGFTs we work with have "melonic" interactions (in correspondence with d-colored graphs called "melons") [59–61]. Such melons are dual to special triangulations of the d-ball [51] and, of course, correspond also to trace invariants of the type introduced in Sec. II A.

We consider a rank-d model with complex field, ϕ: $\mathbb{R}^d \to \mathbb{C}$, defined by the following action:

$$\begin{aligned} S[\phi,\bar{\phi}] = {}& (2\pi)^d \int_{\mathbb{R}^{\times d}} [dx_i]_{i=1}^d \bar{\phi}(x_1,\ldots,x_d)\left(-\sum_{s=1}^d \triangle_s + \mu\right)\phi(x_1,\ldots,x_d) \\ & + \frac{\lambda}{2}(2\pi)^{2d} \int_{\mathbb{R}^{\times 2d}} [dx_i]_{i=1}^d [dx'_j]_{j=1}^d [\phi(x_1,x_2,\ldots,x_d)\bar{\phi}(x'_1,x_2,\ldots,x_d)\phi(x'_1,x'_2,\ldots,x'_d)\bar{\phi}(x_1,x_2{}',\ldots,x'_d) \\ & + \mathrm{sym}\{1,2,\ldots,d\}], \end{aligned} \tag{14}$$

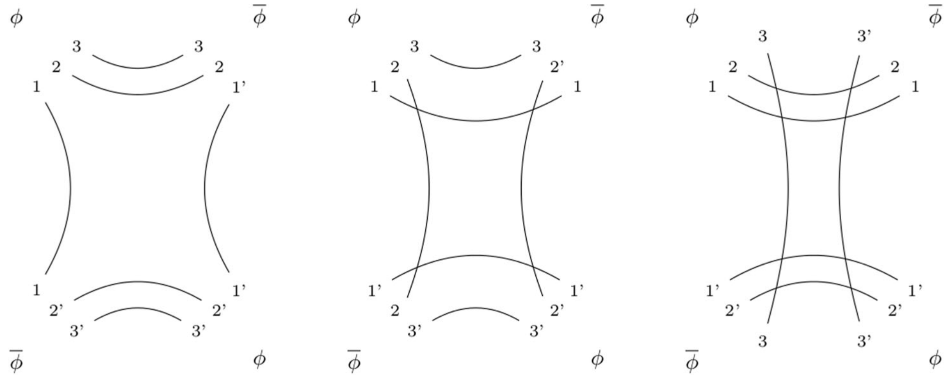


FIG. 2. Colored symmetric interaction terms in rank $d = 3$.

where 2π factors have been conveniently introduced in the definition of the Fourier transform, the symbol sym $\{\cdot\}$ represents the rest of the colored symmetric terms in the interaction (see Fig. 2 for a graphical representation[2] in rank $d = 3$), and μ and λ are coupling constants. As it is easy to see, because of the structure of the interaction kernels, the interaction fully depends on all the $2d$ coordinates, and this makes it nonlocal from the combinatorial point of view. Note that we have introduced a unique coupling constant for the d colored interactions. This is because at this point we have no criterion to distinguish them, as we do not associate a direct physical meaning to the coloring. A different choice is to introduce a different coupling for each color term. The FRG analysis of a model of this type, called anisotropic in [29], is slightly more cumbersome (at the end, for instance, the theory flow in a multidimensional coupling space must be drawn using sections or projections) but could in practice be deduced from the calculations performed here. We plan to analyze this anisotropic model in a subsequent work.

After Fourier transform, we write the action in momentum space as

$$S[\phi,\bar{\phi}] = \int_{\mathbb{R}^{\times d}} [dp_i]_{i=1}^{d} \bar{\phi}_{12\cdots d}\left(\sum_{s=1}^{d} p_s^2 + \mu\right)\phi_{12\cdots d} + \frac{\lambda}{2}\int_{\mathbb{R}^{\times 2d}} [dp_i]_{i=1}^{d}[dp'_j]_{j=1}^{d} \times [\phi_{12\cdots d}\bar{\phi}_{1'2\cdots d}\phi_{1'2'\cdots d'}\bar{\phi}_{12'\cdots d'} + \mathrm{sym}\{1,2,\ldots,d\}], \tag{15}$$

where we use the conventions

$$\phi_{12\cdots d} = \phi_{p_1,p_2,\ldots,p_d} = \phi(\mathbf{p}) = \int_{\mathbb{R}^{\times d}} [dx_i]_{i=1}^{d}\phi(x_1,x_2,\ldots,x_d)e^{-i\sum_i p_i x_i}, \tag{16}$$

$$\phi(x_1,x_2,\ldots,x_d) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{\times d}} [dp_i]_{i=1}^{d}\phi_{12\cdots d}e^{i\sum_i p_i x_i}. \tag{17}$$

We represent the propagator as a stranded line made with d segments (strands). See Fig. 3 for the case $d = 3$. The combinatorics of the interaction is preserved by the Fourier transform.

B. Dimensions

We can now proceed with the dimensional analysis to fix the dimensions of the coupling constants. To make sense of the exponentiation of the action in the partition function, we must set $[S] = 0$. Furthermore, we fix the dimensions in units of the momentum, i.e., $[p] = [dp] = 1$.[3] Now, for consistency we must have $[\mu] = 2$. This leads us to the following equations:

$$d + 2[\phi] + 2 = 0 \Rightarrow [\phi] = -\frac{d+2}{2}, \tag{18}$$

$$[\lambda] + 2d + 4[\phi] = 0 \Rightarrow [\lambda] = 4, \tag{19}$$

which fix the dimension of the TGFT fields depending on the rank-d of the model. Note that the above dimensions are fixed with respect to the Fourier components of the field. In the direct space, the field has a dimension $[\phi] = (d-2)/2$, in units $[x] = -1$. In addition, as one directly realizes from (19), the notion of dimension of a coupling does not depend on the dimension d of the manifold of the theory.

It is important to stress that, in this (T)GFT context, the canonical dimension differs from the scaling dimension of the coupling constants, which instead depends on the rank-d, and reflects the power counting of the given model. In general, the notions of canonical or scaling dimensions or power counting must be studied with special care because

[2]As a remark, in the following subsections, illustrations and figures are made in the case $d = 3$ because the general case can easily be recovered from that case.

[3]Notice that the physical dimension of such momentum variables, if any, is not especially relevant in this context; what matters is the relative dimension of the various ingredients entering the TGFT action.

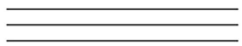
$\bar{\phi}$  ϕ $= \left(\sum_s p_s^2 + \mu\right)^{-1}$

FIG. 3. Feynman rule for the propagator at $d = 3$.

they do not reflect the conventional wisdom from usual (local) quantum field theories.

We now address in some detail the issue of scaling dimensions of the couplings, which in turn gives the first set of information about relevant, marginal, and irrelevant directions of the RG flow and then also a preliminary indication about the reliability of any given truncation of the effective action.

As in standard QFT (see Appendix C 1), we need to estimate the power counting corresponding to various interactions, in order to determine their proper scaling dimension. The general idea is that the scaling dimension is chosen so that the most divergent interactions are regularized, the RG system is made autonomous, and other interactions are correspondingly suppressed. And as in standard QFT, we do this by using the power counting in the UV (which sets the initial conditions of our GR flow equations), and only around the putative Gaussian fixed point. This is necessary because it is only at the perturbative level around the free theory that we have some control over such power counting. For this reason, however, one should be careful in taking the result of this analysis as giving any general indication about relevant and irrelevant directions, and about the reliability of a given truncation. The latter, in fact, can only be studied by detailed analysis of appropriate extensions of the same truncation.

The power counting of an amplitude has been treated in full-fledged form using multiscale analysis in [25] (Proposition 2 therein). Note that the power counting is universal and independent of the regularization scheme. For a graph $\mathcal{G}$, we write the amplitude $A_{\mathcal{G}}$ after imposing some cutoff in the momentum space involving some scale k as

$$|A_{\mathcal{G}}| \propto \left|\left(\prod_{v\in\mathcal{V}} \lambda_v\right)\right| k^{\omega_{\rm div}(\mathcal{G})}, \tag{20}$$

where v runs over the set $\mathcal{V}$ of vertices of $\mathcal{G}$ and λ_v is the coupling associated with v, and where $\omega_{\rm div}(\mathcal{G})$ is the divergence degree of the graph $\mathcal{G}$ giving the highest power involving the scale k. The integer $|\mathcal{V}|$ determines the order of the perturbation in which this amplitude has been evaluated. It remains to interpret and restrict this formula according to our present context. We must first stress that this counting was performed on compact group manifolds [namely $G_D = U(1)^D$ or $SU(2)^D$]. In our situation, the counting using the group $G_D = U(1)$ (fixing in [25] $D = 1$) must coincide with what we obtain before taking the thermodynamic limit. For simplicity, we focus on connected diagrams since we will be interested in a truncation containing only connected interactions (such that the number of connected components is $C_\partial = 1$ in the same reference; in fact, the below reasoning can easily be extended to the more general case $C_\partial \geq 1$). In such circumstances, the degree of divergence is of the form

$$\omega_{\rm div}(\mathcal{G}) = -2L + F_{\rm int} = -\Omega_{\mathcal{G}} - \frac{1}{2}[(d-3)n - 2(d-1)] + \frac{1}{2}[(d-3)l\cdot - 2(d-1)]V, \tag{21}$$

where L is the number of propagator lines, and $F_{\rm int}$ is the number of closed loops (internal faces) or momenta integrated. Then, the degree is further expanded in terms of $\Omega_{\mathcal{G}}$ related to the Gurau degree of the graph [19] which is a positive quantity (its explicit expression is not required in the following), n (written $N_{\rm ext}$ in [25]) is the number of external legs of the diagram, $l\cdot V \coloneqq \sum_{l=2}^{k_{\max}} lV_l$ is the number of exiting half-lines of all vertices, $V = \sum_l V_l$ is the total number of vertices, and V_l is the number of vertices having degree l.

Consider now a graph $\mathcal{G}_0$ having n_0 external legs, $n_0 \in [\![2, k_{\max}]\!]$. We have provided in Appendix C 1 the standard picture in scalar field theory of the extraction of the scaling dimension of some coupling λ_{n_0} just from the knowledge of the number of external legs n_0 using some power counting arguments. The idea is straightforward: the RG equation describing the evolution of a coupling $\lambda_{n_0;k}$ can be expanded perturbatively, and several terms $A_{\mathcal{G};k}$ contribute to that equation. One must equate the scaling of $A_{\mathcal{G};k}$ and that of $\lambda_{n_0;k}$. A set of constraints is then generated and allows one to fix the scaling $\{\lambda_{n_0;k}\}$. However, observables in the tensor case have a lot more structure: they are sorted by a large N expansion (they have different power counting), i.e., by their UV behavior, and distinguished by their boundary data (including the number of external legs n_0). We report the corresponding analysis in Appendix C 2. The result of that analysis is quite involved in the generic case. Using the large N expansion, we can, however, find couplings with explicit expressions for scaling dimensions.

Consider the set of graphs $\mathcal{G}$ with the same set of external data associated with a particular pattern b characterizing an interaction $\lambda_{n_0;b}{\rm Tr}_{n_0;b}((\phi\cdot\bar{\phi})^{n_0/2})$. There are choices for b such that $\mathcal{G}$ is dominant in power counting such that the quantity $\Omega_{\mathcal{G}} = 0$, and such that the scaling dimension of $\lambda_{n_0;b}$ can be evaluated as (see Appendix C 2)

$$\{\lambda_{n_0;b}\} = -\frac{1}{2}[(d-3)n_0 - 2(d-1)]. \tag{22}$$

Remarkably, the condition imposes that the scaling dimension becomes actually independent from the precise pattern b (at given polynomial order in the fields).

The scaling dimension analysis of the gauge invariant TGFT, as treated in Sec. IV, can be performed in a similar way (see Appendix C 2). In the end, the scaling dimensions of particular couplings with the same boundary conditions as above are again of the form (22) with a shift in the dimension $d \to d-1$.

In the following, we will present the general analysis of RG flow equations for TGFT models for arbitrary rank-d, and then specialize it, giving more details, in the cases $d = 3, 4, 5$, in order to cover a certain variety of cases.

In these different cases, the above analysis of scaling dimensions, in the TGFT models without gauge invariance, as defined by (14), indicates that

(i) at rank $d = 3$, there is an infinite tower of relevant couplings with an arbitrary valence of the interaction [25]; this is an analogue of the superrenormalizable $P(\phi)_2$-scalar model in dimension 2.

(ii) at rank $d = 4$, mass and ϕ^4 are relevant operators, and the model includes two different marginal higher order ϕ^6-terms;

(iii) at rank $d = 5$, mass and ϕ^4 terms are relevant and marginal directions in the UV, respectively.

We will discuss relevant and marginal operators for the class of TGFT with gauge invariance in Sec. IV. More extensively, in the following, we will comment on the reliability of the truncation of the effective action we use (including only a mass term and ϕ^4-interaction) in characterizing the RG flow of the various models.

C. Effective action and Wetterich equation

To proceed with the functional renormalization group analysis, following the general template described in the previous section, we introduce an IR cutoff k and a UV cutoff Λ. We need to perform a truncation on the form of the effective action. A possible choice, compatible with the condition (13), is to truncate the effective action to be of the same form of the action itself for any value of the cutoffs, that is,

$$\Gamma_k[\varphi,\bar\varphi] = \int_{\mathbb{R}^{\times d}} [dp_i]_{i=1}^d \bar\varphi_{12\cdots d}\left(Z_k \sum_s p_s^2 + \mu_k\right)\varphi_{12\cdots d} + \frac{\lambda_k}{2}\int_{\mathbb{R}^{\times 2d}} [dp_i]_{i=1}^d [dp'_j]_{j=1}^d [\varphi_{12\cdots d}\bar\varphi_{1'2\cdots d}\varphi_{1'2'\cdots d'}\bar\varphi_{12'\cdots d'} + \mathrm{sym}\{1,2,\ldots,d\}], \tag{23}$$

where $\varphi = \langle\phi\rangle$.

According to the analysis of Sec. III B, the truncation (23) includes

(i) at rank $d = 3$, two relevant directions out of an infinite tower of relevant others with arbitrary valence of the interaction [25];

(ii) at rank $d = 4$, all relevant directions but does not include marginal higher order φ^6-terms;

(iii) at rank $d = 5$, all the relevant and marginal directions.

Therefore, *a priori*, within this φ^4-truncation, the model at rank $d = 5$ containing all relevant and marginal directions will have the most reliable results in terms of qualitative stability of the flow with respect to the inclusion of higher order couplings. However, this being said, the results for the other ranks $d = 3$ and, especially, 4 might still turn out to be qualitatively stable, and one cannot prove or disprove this possibility without a detailed analysis of the improved RG flow.

As we have already stressed, this is a nonperturbative truncation of the theory, and any of the ensuing results should then be tested by extending this truncation, including more invariants [including other types of $\mathrm{Tr}(\phi^4)$ invariants, i.e., with different combinatorics, as well as higher order terms $\mathrm{Tr}(\phi^{2n})$, $n \geq 3$; in general, one should include also disconnected invariants such as multitraces, $\mathrm{Tr}(\phi^{2n})\mathrm{Tr}(\phi^{2m})\ldots$] and checking for (qualitative) convergence. Enlarging the theory space is postponed for future investigations, but it should be obvious that, even in the truncation given by (23), the calculations and the outcome of the present analysis remain highly nontrivial.

From the dimensional analysis of the previous section and from the fact that $[\Gamma_k] = 0$ and $[\varphi] = [\phi]$, one infers $[Z_k] = 0$, $[\mu_k] = [\mu] = 2$, and $[\lambda_k] = [\lambda] = 4$.

We introduce a regulator kernel of the following form [62,63]:

$$R_k(\mathbf{p},\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')Z_k\left(k^2 - \sum_s p_s^2\right)\theta\left(k^2 - \sum_s p_s^2\right), \tag{24}$$

where θ stands for the Heaviside step function. This form of the regulator is convenient because it allows one to solve analytically many spectral sums. It is easy to show that R_k satisfies the minimal requirements for a regulator kernel:

(i) as a consequence of the fact that $\theta(-|x|) = 0$, we have

$$R_{k=0}(\mathbf{p},\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')Z_k\left(-\sum_s p_s^2\right)\theta\left(-\sum_s p_s^2\right) = 0; \tag{25}$$

(ii) at the scale $k = \Lambda$, the regulator takes the form

$$R_{k=\Lambda}(\mathbf{p},\mathbf{p}') = \delta(\mathbf{p}-\mathbf{p}')Z_\Lambda\left(\Lambda^2 - \sum_s p_s^2\right)\theta\left(\Lambda^2 - \sum_s p_s^2\right), \tag{26}$$

which at the first order gives $R_{k=\Lambda} \simeq Z_\Lambda \Lambda^2$;

(iii) for $k \in [0, \Lambda]$, we also have

$$R_k(\mathbf{p}, \mathbf{p}') = 0, \quad \forall\ \mathbf{p}, \mathbf{p}',\ \text{such that } |\mathbf{p}|, |\mathbf{p}'| > k, \tag{27}$$

$$R_k(\mathbf{p}, \mathbf{p}') \simeq Z_k k^2, \quad \forall\ \mathbf{p}, \mathbf{p}',\ \text{such that } |\mathbf{p}|, |\mathbf{p}'| < k. \tag{28}$$

The derivative of the regulator kernel with respect to the logarithmic scale $t = \log k$, entering in the Wetterich equation, evaluates as

$$\partial_t R_k(\mathbf{p}, \mathbf{p}') = \theta(k^2 - \Sigma_s p_s^2)[\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]\delta(\mathbf{p} - \mathbf{p}'). \tag{29}$$

One notes that R_k and $\partial_t R_k$ are both symmetric kernels, which is important in evaluating the convolutions induced by the Wetterich equation.

Computing the 1PI 2-point function yields

$$\begin{aligned}\Gamma_k^{(2)}(\mathbf{q}, \mathbf{q}') &= \left(Z_k \sum_s q_s^2 + \mu_k\right)\delta(\mathbf{q} - \mathbf{q}') + \lambda_k \left[\int_{\mathbb{R}} dp_1 \varphi_{p_1 q_2' \dots q_d'} \bar{\varphi}_{p_1 q_2 \dots q_d} \delta(q_1 - q_1') \right.\\ &\quad \left. + \int_{\mathbb{R}^{\times(d-1)}} [dp_i]_{i=2}^d \varphi_{q_1' p_2 \dots p_d} \bar{\varphi}_{q_1 p_2 \dots p_d} \left[\prod_{i=2}^d \delta(q_i - q_i')\right] + \mathrm{sym}\{1, 2, \dots, d\}\right]\\ &= \left(Z_k \sum_s q_s^2 + \mu_k\right)\delta(\mathbf{q} - \mathbf{q}') + F_k(\mathbf{q}, \mathbf{q}').\end{aligned} \tag{30}$$

There is a simple graphical way to picture the various terms contributing to F_k. Each summed index can be represented by a segment and each fixed index (not summed) by a dot. As an example in rank $d = 3$, Fig. 4 displays two terms coming from the second variation of the interaction labeled by color 1 [the ones which appear explicitly in (30)]. The other terms appearing in $\mathrm{sym}\{\cdot\}$ can be inferred by color permutation.

Defining the operator P_k with kernel

$$P_k(\mathbf{p}, \mathbf{p}') = R_k(\mathbf{p}, \mathbf{p}') + \left(Z_k \sum_s p_s^2 + \mu_k\right)\delta(\mathbf{p} - \mathbf{p}'), \tag{31}$$

the Wetterich equation can be recast as

$$\partial_t \Gamma_k = \mathrm{Tr}[\partial_t R_k \cdot (P_k + F_k)^{-1}]. \tag{32}$$

The right-hand side (RHS) of (32) generates an infinite series of terms with convolutions involving an arbitrary number of fields. To compare the two sides of (32), we must therefore perform a truncation in this series to match with the left-hand side (LHS) of that equation. This may be achieved expanding (32) in powers of $F_k \cdot (P_k)^{-1}$, that is, in powers of $\varphi\bar{\varphi}$, and considering only the terms up to the power 2,

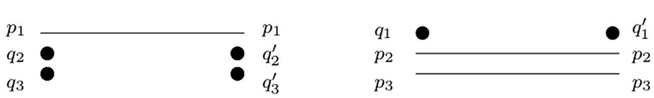


FIG. 4. Terms of the second variation of Γ_k at rank $d = 3$.

$$\begin{aligned}\partial_t \Gamma_k &= \mathrm{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot (1 + F_k \cdot (P_k)^{-1})^{-1}]\\ &= \mathrm{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot (1 - F_k \cdot (P_k)^{-1}\\ &\quad + F_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1}) + o((\varphi\bar{\varphi})^3)].\end{aligned} \tag{33}$$

The vacuum term proportional to the zeroth order in the above expansion will be discarded because it does not reflect any term in the LHS of (32). As an explicit example, the trace at linear order takes the form

$$\partial_t \Gamma_k^{\mathrm{kin}} = \int_{\mathbb{R}^{\times 12}} \partial_t R_k(\mathbf{p}, \mathbf{p}')(P_k)^{-1}(\mathbf{p}', \mathbf{q}) F_k(\mathbf{q}, \mathbf{q}')(P_k)^{-1}(\mathbf{q}', \mathbf{p}). \tag{34}$$

Already, from the structure of the operators, $\partial_t R_k$, P_k, and F_k, we expect the presence of singular δ-functions which need to be regularized. Indeed, the appearance of $\delta(0)$-terms reflects the fact that we have infinite volume effects which have to be treated. The presence of such infinities, as we have anticipated above, is not a specific feature of TGFTs, as it also arises in standard QFT. What is peculiar in TGFTs is the fact that, due to the combinatorics of the vertex operators, these divergences cannot be addressed by projection on the constant fields. Roughly speaking, in ordinary (local) field theories projecting on constant fields allows one to factorize out the full volume of the space entering some given power of $\delta(0)$, and depending only on the order of the field interaction. Such a procedure cannot be applied in the present setting, the main reason being that the order of volume divergences depends not only on the order of field interactions but also on their precise convolution pattern. This would be entirely lost in a constant field projection and must instead be checked term-by-term in the expansion of (32). The best way to tackle these divergences is to resort to a compactification of configuration space, corresponding to a discretization in the conjugate space, and define an appropriate thermodynamic limit. This is explained in the next section.

D. IR divergences and thermodynamic limit

To regularize volume divergences, we perform a compactification of the direct space and a lattice regularization in the conjugate space, following the conventions of [64], and generalizing to arbitrary rank the procedure adopted in [43]. Defining the model (14) over a compact set $D \subset \mathbb{R}^{\times d}$ with volume $L^d = (2\pi r)^d$, and taking a Fourier transform, the domain of integration (actually, summation) of the effective action, in momentum space, becomes the lattice

$$D^* = \left(\frac{2\pi}{L}\mathbb{Z}\right)^{\times d} = \left(\frac{1}{r}\mathbb{Z}\right)^{\times d} \coloneqq \left(l\mathbb{Z}\right)^{\times d}, \tag{35}$$

so that we have, for any function $F(\mathbf{p})$,

$$\int_{D^*} [dp_i]_{i=1}^d F(\mathbf{p}) = l^d \sum_{p_1, p_2, \ldots, p_d \in D^*} F(\mathbf{p}). \tag{36}$$

We define the delta distribution in D^* in momentum space as

$$\delta_{D^*}(\mathbf{p}, \mathbf{q}) = l^{-d} \delta_{\mathbf{p},\mathbf{q}}, \tag{37}$$

with $\delta_{\mathbf{p},\mathbf{q}} = \prod_s \delta_{p_s, q_s}$, the Kronecker delta. Choosing an orthonormal basis $(e_{\mathbf{p}})_{\mathbf{p}\in D^*}$ for the space of fields such that $e_{\mathbf{p}}(\mathbf{q}) = \delta_{D^*}(\mathbf{p}, \mathbf{q})$, we have

$$\phi(\mathbf{p}) = \langle e_{\mathbf{p}}, \phi \rangle_{D^*}. \tag{38}$$

For a generic observable A, we then have

$$\begin{aligned}(A\phi)(\mathbf{p}) &= \int_{D^*} [dq_i]_{i=1}^d A(\mathbf{q}, \mathbf{p}) \phi(\mathbf{p}) \\ &= \int_{D^*} [dq_i]_{i=1}^d \langle e_{\mathbf{q}}, A e_{\mathbf{p}} \rangle_{D^*} \phi(\mathbf{p}).\end{aligned} \tag{39}$$

Whenever A is invertible, then the inverse operator satisfies

$$\int_{D^*} [dr_i]_{i=1}^d A(\mathbf{p}, \mathbf{r}) A^{-1}(\mathbf{r}, \mathbf{q}) = \delta_{D^*}(\mathbf{p}, \mathbf{q}). \tag{40}$$

We also define the regularized functional derivative as

$$\frac{\delta}{\delta\phi(\mathbf{p})} = l^{-d} \frac{\partial}{\partial\phi(\mathbf{p})}, \tag{41}$$

so that the following relations hold:

$$\begin{aligned}&\frac{\delta}{\delta\phi(\mathbf{p})} \phi(\mathbf{q}) = \delta_{D^*}(\mathbf{p}, \mathbf{q}), \\ &\frac{\delta}{\delta J(\mathbf{p})} e^{\langle J, \phi \rangle_{D^*}} = J(\mathbf{p}) e^{\langle J, \phi \rangle_{D^*}}.\end{aligned} \tag{42}$$

This set of conventions is, of course, consistent with the continuous version of field theory, where δ_{D^*} becomes the Dirac δ-distribution and the derivative (41) becomes the standard functional derivative.

Using this regularization prescription, the effective action of the model takes the form

$$\begin{aligned}\Gamma_k[\varphi, \bar{\varphi}; l] = {}& l^d \sum_{\mathbf{p}\in D^*} \bar{\varphi}_{12\cdots d} \left(Z_k \sum_s p_s^2 + \mu_k \right) \varphi_{12\cdots d} \\ &+ l^{2d} \frac{\lambda_k}{2} \sum_{\mathbf{p},\mathbf{p}' \in D^*} [\varphi_{12\cdots d} \bar{\varphi}_{1'2\cdots d} \varphi_{1'2'\cdots d'} \bar{\varphi}_{12'\cdots d'} \\ &+ \mathrm{sym}\{1, 2, \ldots, d\}],\end{aligned} \tag{43}$$

where, using the same notation φ for the field and its Fourier transform, one has

$$\begin{aligned}\varphi(x_1, x_2, \ldots, x_d) &= (2\pi)^{-d} l^d \sum_{\mathbf{p}\in D^*} e^{i \sum_i p_i x_i} \varphi(\mathbf{p}), \\ \varphi(\mathbf{p}) &= \int_D [dx_i]_{i=1}^d e^{-i \sum_i p_i x_i} \varphi(x_1, x_2, \ldots, x_d).\end{aligned} \tag{44}$$

Now we use the relations (44) to transform δ_{D^*} and obtain

$$(2\pi)^{-d} l^d \sum_{\mathbf{p}\in D^*} \delta_{D^*}(\mathbf{p}, \mathbf{q}) e^{i \sum_i p_i x_i} = (2\pi)^{-d} e^{i \sum_i q_i x_i}. \tag{45}$$

Thus, an integral representation of the delta distribution over D^* can be consistently defined as

$$\delta_{D^*}(\mathbf{p}, \mathbf{q}) = (2\pi)^{-d} \int_D [dx_i]_{i=1}^d e^{-i \sum_i (p_i - q_i) x_i}. \tag{46}$$

As a final result, we have

$$\delta_{D^*}(\mathbf{p}, \mathbf{p}) = \frac{(2\pi r)^d}{(2\pi)^d} = r^d = \frac{1}{l^d}. \tag{47}$$

From these formulas, the continuum description will be recovered in the thermodynamic limit $l \to 0$.

This procedure makes the dependence on the volume of the direct space explicit. We can then also rescale the coupling constants of the model to incorporate in their definition a dependence on the same volume. Then, we can use this dependence in such a way that the noncompact (thermodynamic) limit of the theory becomes well defined and all divergences are consistently removed.

E. β-functions and RG flows

We introduce a regularization as outlined in Sec. III D and write the regularized effective action as

$$\Gamma_k[\varphi,\bar{\varphi}] = \int_{D^*} [dp_i]_{i=1}^d \bar{\varphi}_{12\cdots d}\left(Z_k \sum_s p_s^2 + \mu_k\right)\varphi_{12\cdots d} + \frac{\lambda_k}{2}\int_{D^{*\times 2}} [dp_i]_{i=1}^d [dp'_j]_{j=1}^d [\varphi_{12\cdots d}\bar{\varphi}_{1'2\cdots d}\varphi_{1'2'\cdots d'}\bar{\varphi}_{12'\cdots d'} + \mathrm{sym}\{1,2,\ldots,d\}]. \tag{48}$$

We can study the Wetterich equation corresponding to the action (48), incorporating a dependence on the volume in the coupling constants, and perform a thermodynamic limit at the end of the computation to extract the coefficients valid in the noncompact case.

The set of β-functions that we obtain from the discretized model is[4]

$$\begin{cases} \beta(Z_k) = \frac{\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[2(d-1)\frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)}\frac{k^{d-1}}{l^{d-1}}\right] + 2Z_k\left[(d-1)\frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d-1}{2}\right)}\frac{k^{d-1}}{l^{d-1}}\right]\right\} \\ \beta(\mu_k) = -\frac{d\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[\frac{4}{3}\frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)}\frac{k^{d+1}}{l^{d-1}}\right] + 2Z_k\left[2\frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)}\frac{k^{d+1}}{l^{d-1}}\right]\right\} \\ \beta(\lambda_k) = \frac{2\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left\{\partial_t Z_k\left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)}\frac{k^{d+1}}{l^{d-1}} + \frac{4(2d-1)}{3}\frac{k^3}{l} + 2\delta_{d,3}k^2\right]\right. \\ \qquad\left. + 2Z_k\left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)}\frac{k^{d+1}}{l^{d-1}} + 2(2d-1)\frac{k^3}{l} + 2\delta_{d,3}k^2\right]\right\} \end{cases}. \tag{49}$$

It must be stressed that the coefficients appearing in (49) are computed with integrals as in the continuous setup. This is, however, not an issue, once the volume dependence has been factored out; the order of taking the limit and performing the integral does not matter.

Some interesting features of the system (49) must be stressed. At this intermediate step (the limit $\lim_{l\to 0}$ still has to be taken), this is a nonautonomous system, and it involves terms of different powers in the cutoff k (we refer to this feature as "nonhomogeneity" in k). Nonautonomous systems are known to occur in other contexts, for example, quantum field theory at finite temperature [58], or on a curved [65] and noncommutative spacetime [66]. The nonhomogeneity in k of the system signals the presence of an external scale, for the system: here, the radius of the compactified configuration space. The specific form of the terms appearing in this case is an effect of the particular combinatorics of the vertices of the theory which, after differentiation, yields the 1PI 2-point function with terms with different volume contributions. If the l parameter is kept finite, we see two different systems arising in the UV and IR limits, coming from different leading terms. Such a feature has been found in previous work [38], and both the two limits and the intermediate regime were investigated. In the two limits one can compute the analogue of fixed points, which, however, cannot be straightforwardly interpreted as such.

On the other hand, if one tries to proceed in the usual way, extracting the dimensions of the coupling constants using one parameter (k or l), one obtains a set of β-functions which are either trivial or still divergent in the limit. Hence, in the end the nonlocal combinatorics of the TGFT interactions requires a revision of conventional procedures of local QFTs. As we now show, the correct way of proceeding in the TGFT case requires taking advantage of the presence of both the two parameters (k,l) when defining the scaling of the couplings.

To make sense of the above system, consider the following ansatz:

$$Z_k = \bar{Z}_k l^\chi k^{-\chi}, \qquad \mu_k = \bar{\mu}_k \bar{Z}_k l^\chi k^{2-\chi}, \qquad \lambda_k = \bar{\lambda}_k \bar{Z}_k^2 l^\xi k^\sigma, \tag{50}$$

where $[\bar{Z}_k] = [\bar{\mu}_k] = [\bar{\lambda}_k] = 0$, $[\varphi] = -\frac{d+2}{2}$, and $\xi + \sigma = 4$.

The ordinary coarse graining picture, adapted to a finite size volume, applies equally in our TGFT context. In ordinary local QFT, one regularizes volume divergences just as is performed here, by introducing an external length/volume scale; in the local QFT case, this external scale

[4] Important steps of the calculation are detailed in Appendix A.

(which can just be set to the identity, of course) enters the couplings just as their canonical dimension would suggest, and the same happens for their k-dependence, i.e., their scaling dimension. In other words, the fact that no real advantage is obtained by keeping the external volume scale explicit is due to the identity between canonical and scaling dimension. In the TGFT case, by following the very same procedure, we realize that canonical and scaling dimensions of the couplings are different. Keeping the external scale explicit allows us to distinguish more easily the contribution to the k-dependence of the various couplings that is due to their physical scaling (coarse graining), from the contribution that only indicates their canonical dimension, because the latter is directly reflected in the dependence on the external volume scale.

We look for the scaling of dimensionless coupling constants, i.e., for dimensionless β-functions. From (50), and using the convention $\eta_k = \partial_t \ln \bar{Z}_k$, one finds

$$\begin{aligned} \eta_k &= \frac{1}{\bar{Z}_k}\beta(\bar{Z}_k) = \frac{1}{Z_k}\beta(Z_k) + \chi, \\ \beta(\bar{\mu}_k) &= \frac{1}{\bar{Z}_k l^\chi k^{2-\chi}}\beta(\mu_k) - \eta_k\bar{\mu}_k - (2-\chi)\bar{\mu}_k, \\ \beta(\bar{\lambda}_k) &= \frac{1}{l^\xi k^\sigma \bar{Z}_k^2}\beta(\lambda_k) - 2\eta_k\bar{\lambda}_k - \sigma\bar{\lambda}_k, \end{aligned} \tag{51}$$

and inserts this in (49) to reach the following expressions:

$$\begin{cases} \eta_k = \frac{\bar{\lambda}_k l^\xi k^\sigma}{l^{2\chi}k^{2(2-\chi)}(1+\bar{\mu}_k)^2}\left\{(\eta_k-\chi)\left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)}\frac{k^{d-1}}{l^{d-1}} + 2(d-1)\frac{k}{l}\right] + 2\left[(d-1)\frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d-1}{2}\right)}\frac{k^{d-1}}{l^{d-1}}\right]\right\} + \chi, \\ \beta(\bar{\mu}_k) = -\frac{d\bar{\lambda}_k l^\xi k^\sigma}{l^{2\chi}k^{6-2\chi}(1+\bar{\mu}_k)^2}\left\{(\eta-\chi)\left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)}\frac{k^{d+1}}{l^{d-1}} + \frac{4}{3}\frac{k^3}{l}\right] + 2\left[2\frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)}\frac{k^{d+1}}{l^{d-1}}\right]\right\} - \eta_k\bar{\mu}_k - (2-\chi)\bar{\mu}_k, \\ \beta(\bar{\lambda}_k) = \frac{2\bar{\lambda}_k^2 l^\xi k^\sigma}{l^{2\chi}k^{6-2\chi}(1+\bar{\mu}_k)^3}\left\{(\eta-\chi)\left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+3}{2}\right)}\frac{k^{d+1}}{l^{d-1}} + \frac{4(2d-1)}{3}\frac{k^3}{l} + 2\delta_{d,3}k^2\right]\right. \\ \qquad \left. + 2\left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)}\frac{k^{d+1}}{l^{d-1}} + 2(2d-1)\frac{k^3}{l} + 2\delta_{d,3}k^2\right]\right\} - 2\eta_k\bar{\lambda}_k - \sigma\bar{\lambda}_k. \end{cases} \tag{52}$$

To make the noncompact limit regular, we must solve the system in the variables ξ and χ by requiring that the highest degree of divergence (highest negative power of l) is regularized and all the subleading infinities are sent to zero. This is achieved by solving, for any $d \geq 3$,

$$\xi - 2\chi - (d-1) = 0. \tag{53}$$

We make a natural choice $\chi = 0$ (thus implying that Z_k is dimensionless) and obtain

$$(\chi = 0, \xi = d-1) \Rightarrow \sigma = 4 - \xi = 5 - d. \tag{54}$$

Let us come back to the issue of scaling and canonical dimensions. We realize that the scaling dimension can be expanded in a form similar to (54) as

$$\begin{aligned} \{\lambda_{n;b}\} &= (d-1) - (d-3)\frac{n}{2} = n - \xi_n(d) = \sigma_n(d), \\ \xi_n(d) &:= (d-1)\left(\frac{n}{2} - 1\right), \end{aligned} \tag{55}$$

where, we recall, n is the number of external legs of the diagram. Hence for a tensor interaction with degree n associated with a melonic boundary graph, the notion of canonical dimension, namely (19), $[\lambda_n] = \xi_n + \sigma_n = \xi_n + \{\lambda_n\} = n$, does not match with scaling dimension $\{\lambda_n\} = \sigma_n$. We apply the above formula at $n = 4$ and recover $\sigma_4(d) = (d-1) - (d-3)2 = d - 1 - 2d + 6 = 5 - d$ and $\xi_4(d) = d - 1$. Thus the truncation of the effective action must be restricted to relevant and marginal directions, in terms of their scaling dimension $\sigma_n \geq 0$, as discussed in Sec. III C after (23).

With the above tools, we now comment on the term $t_{\max}$ of maximal power $k^{d+1}/(k^{2(e+1)}l^{d-1})$ appearing in the system of β-functions (52). Noting that the Wetterich equation is a one-loop equation, convoluting all $(\partial_t R \cdot P^{-1}) \cdot (F \cdot P^{-1})^e$, we note that the number $F_{\rm int}$ of internal momenta integrated cannot exceed $d - 1$. The case when $F_{\rm int} = d - 1$ coincides precisely with a maximal degree of divergence corresponding to a boundary melonic invariant. Then the scaling of this integration is k^{d-1+2}; the extra k^2 comes from the regulator $\partial_t R \sim k^2$. Note that this case only occurs when one term of F, the term including the maximal number of δ's, is convoluted

$(d-1)$-times with itself. At the end, one gets $1/l^{d-1} = 1/l^{\xi_4(d)}$ as a volume factor for the term $t_{\max}$. At order e of the expansion, the number of propagators is precisely e plus the extra propagator included in the regulator $\partial_t R \cdot P^{-1}$; this gives a $1/k^{2(e+1)}$. Therefore, the power counting associated with this term respects $\omega_{\rm div}(\mathcal{G}) = -2e + d - 1$.

After reducing to dimensionful quantities, the resulting system of equations for the theory becomes

$$\begin{cases} \eta_k = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d-1}{2}\right)} \frac{\bar{\lambda}_k}{(1+\bar{\mu}_k)^2}\left[\frac{\eta_k}{d-1}+1\right] \\ \beta(\bar{\mu}_k) = \frac{-2d\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{\bar{\lambda}_k}{(1+\bar{\mu}_k)^2}\left[\frac{\eta_k}{d+1}+1\right] - \eta_k\bar{\mu}_k - 2\bar{\mu}_k \\ \beta(\bar{\lambda}_k) = \frac{4\pi^{\frac{d-1}{2}}}{\Gamma_E\left(\frac{d+1}{2}\right)} \frac{\bar{\lambda}_k^2}{(1+\bar{\mu}_k)^3}\left[\frac{\eta_k}{d+1}+1\right] - 2\eta_k\bar{\lambda}_k - (5-d)\bar{\lambda}_k \end{cases} \tag{56}$$

which defines an autonomous system of coupled differential equations describing the flow of dimensionless coupling constants.

These equations hold for generic rank-d. They could be solved at the same level of generality, in principle, but we specialize the analysis for various interesting choices of rank, so that the results can be reported in more explicit terms. Specifically, we study the above system of equations when restricted to the first nontrivial rank situations at $d = 3, 4, 5$. These models have been proved perturbatively renormalizable; at rank $d = 5$, the φ^4-truncation is special because it includes all relevant and marginal terms. Because the key results of the following analysis are, in fact, valid for any of the ranks 3, 4, and 5, we provide the treatment of the rank $d = 3$ in all details, and we will simply report the key results in higher ranks $d = 4, 5$.

F. Rank $d=3$

At rank $d = 3$, the system (56) reduces to

$$\begin{cases} \eta_k = \frac{\pi\bar{\lambda}_k}{(1+\bar{\mu}_k)^2}(\eta_k+2) \\ \beta(\bar{\mu}_k) = -\frac{3\pi\bar{\lambda}_k}{(1+\bar{\mu}_k)^2}\left(\frac{\eta_k}{2}+2\right) - \eta_k\bar{\mu}_k - 2\bar{\mu}_k\,. \\ \beta(\bar{\lambda}_k) = \frac{\pi\bar{\lambda}_k^2}{(1+\bar{\mu}_k)^3}(\eta_k+4) - 2\eta_k\bar{\lambda}_k - 2\bar{\lambda}_k \end{cases} \tag{57}$$

Before proceeding with the standard analysis, which consists in finding fixed points of the flow and studying the linearized equations around them, we point out that, because of the nonlinear nature of the β-functions, we have a singularity at $\bar{\mu} = -1$ and $\bar{\lambda} = (1+\bar{\mu})^2/\pi$. This is a common feature in dealing with a truncated Wetterich equation. In a neighborhood of those singularities, we do not trust the linear approximation, and being interested in the part of the RG flow connected with the Gaussian fixed point, we will not study the flow beyond the mentioned divergence of the β-functions.

By numerical integration, we find a Gaussian fixed point and three non-Gaussian fixed points in the plane $(\bar{\mu}, \bar{\lambda})$ at

$$\begin{aligned} {}_{d=3}P_1 &= (8.619, -47.049), \\ {}_3P_2 &= 10^{-1}(-6.518, 0.096), \\ {}_3P_3 &= 10^{-1}(-8.010, 0.212). \end{aligned} \tag{58}$$

A quick inspection proves that ${}_3P_3$ lies in the sector disconnected from the origin, so we will not perform any analysis around it.

We linearize the system of equations by evaluating the stability matrix around the other three fixed points,

$$\begin{pmatrix} \beta(\bar{\mu}_k) \\ \beta(\bar{\lambda}_k) \end{pmatrix} = \begin{pmatrix} \partial_{\bar{\mu}_k}\beta(\bar{\mu}_k) & \partial_{\bar{\lambda}_k}\beta(\bar{\mu}_k) \\ \partial_{\bar{\mu}_k}\beta(\bar{\lambda}_k) & \partial_{\bar{\lambda}_k}\beta(\bar{\lambda}_k) \end{pmatrix}_{\rm F.P.} \begin{pmatrix} \bar{\mu}_k \\ \bar{\lambda}_k \end{pmatrix}. \tag{59}$$

In a neighborhood of the Gaussian fixed point, the stability matrix is of the form

$$(\beta^*_{ij})|_{\rm GFP} \coloneqq \begin{pmatrix} -2 & -6\pi \\ 0 & -2 \end{pmatrix}, \tag{60}$$

which has an eigenvalue with algebraic multiplicity 2, corresponding to the canonical scaling dimensions of the couplings λ_k and μ_k: ${}_{d=3}\theta_0 = -2$. The geometric multiplicity of ${}_3\theta_0$ is 1; hence, the matrix of the linearized system turns out to be not diagonalizable and has a single eigenvector ${}_3\mathbf{v}_0 = (1, 0)$.

In a neighborhood of the non-Gaussian fixed points (NGFP) we have

$${}_3P_1 \quad {}_3\theta_{11} \sim 0.351 \quad \text{for } {}_3\mathbf{v}_{11} \sim 10^{-1}(0.65, -9.98), \tag{61}$$

$${}_3P_1 \quad {}_3\theta_{12} \sim 2.548 \quad \text{for } {}_3\mathbf{v}_{12} \sim 10^{-1}(-6.88, 7.26), \tag{62}$$

$${}_3P_2 \quad {}_3\theta_{21} \sim 10.066 \quad \text{for } {}_3\mathbf{v}_{21} \sim 10^{-1}(9.996, -0.269), \tag{63}$$

$${}_3P_2 \quad {}_3\theta_{22} \sim -1.988 \quad \text{for } {}_3\mathbf{v}_{22} \sim 10^{-1}(9.987, 0.506). \tag{64}$$

Because of the difference in their magnitudes (distance from the origin), it becomes difficult to plot the two NGFPs simultaneously with enough precision in their vicinity. We plot two sectors of the RG flow in the plane $(\bar{\mu}_k, \bar{\lambda}_k)$ (see Fig. 5).

In the vicinity of a fixed point, we define as relevant directions those eigendirections that are UV attractive with respect to the cutoff, while we call irrelevant the UV repulsive eigendirections. Marginal directions can be

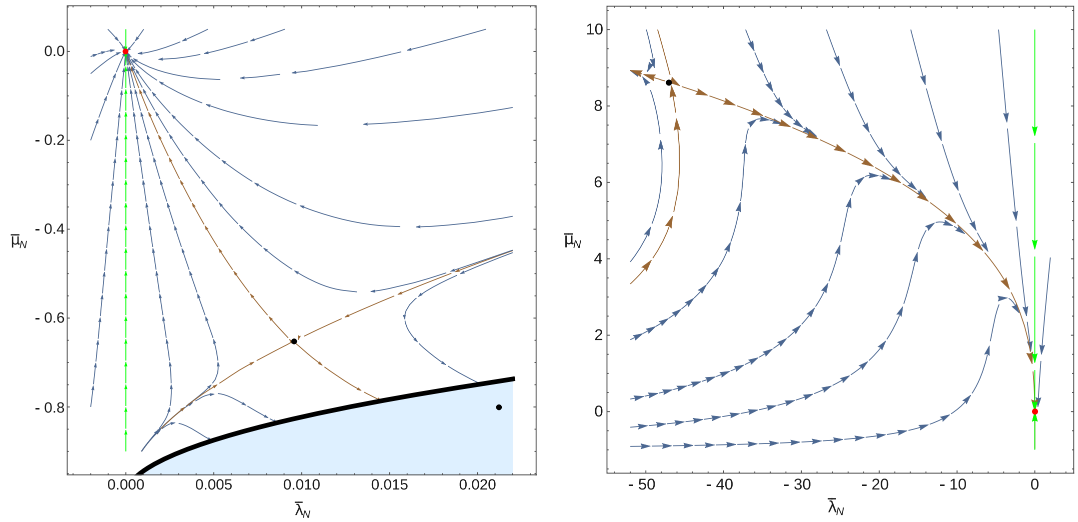


FIG. 5. Flow of the theory. The red and blue lines represent, respectively, the zeros of $\beta(\bar{\mu}_k)$ and $\beta(\bar{\lambda}_k)$, the brown arrows are the eigenperturbations of the non-Gaussian fixed points (represented in black), and the green ones those of the Gaussian fixed point (in red). Arrows point in the UV direction. The thick black line is the singularity of the flow.

attractive or repulsive depending on the initial condition of the trajectory. The origin is a *great* attractor and has one relevant direction connecting it to the other two fixed points. The absence of a second eigenvector for the stability matrix around the Gaussian fixed point requires an approximation beyond the linear order, when the flow is studied analytically, and is a signal of the presence of a marginal perturbation. We can instead integrate numerically the flow, and we find that this marginal direction will still be UV attractive, which means that it corresponds to a marginally relevant direction. The fact that the GFP is a sink for the flow means that this model is asymptotically free with respect to the cutoff. Both non-Gaussian fixed points have one relevant and one irrelevant direction. The eigendirections connecting the three fixed points turn out to be stable under RG transformations, and they are characterized by an effect known as the *large river effect* [56]. This signifies that all the RG trajectories in a neighborhood of these eigendirections get closer and closer to them while pointing in the UV. This effect shows a splitting of the space of coupling into two regions not connected by any RG trajectory. Thus, the relevant directions for the Gaussian fixed point reflect the properties of a critical surface and suggest the presence of phase transitions in the model. In the $\bar{\lambda}_k > 0$ plane, the flow is similar to the one of standard local scalar field theory in a neighborhood of the Wilson-Fisher fixed point, but the presence of a second non-Gaussian fixed point in the $\bar{\lambda}_k < 0$ plane makes the theory quite different. Nevertheless, the properties of this second NGFP are basically the same as the former one. Another particularity of the second fixed point is that it lies at a negative value of $\bar{\lambda}$. This might lead to unstable and unbounded action in this truncation. However, we know already that this truncation should be completed with other relevant terms $\phi^{2n\geq 6}$, so we must check whether this non-Gaussian fixed point persists in such extended truncations. It can also be stressed that this feature has been found in different tensorial models. Indeed, in [44], one finds that, by tuning the rank $4 \leq d \leq 6$, the fixed points with a negative value of the coupling merge with the Gaussian fixed point. In another context, namely, colored tensor models [46], the search for instantons led also to negative valued couplings and have raised the issue of constructing a new type of renormalizable tensorial theories with interactions of the "wrong" sign. At this stage, we simply report here the existence of such a fixed point which certainly deserves to be further investigated.

In this context (and we emphasize that the same picture can be drawn in rank $d = 4, 5$, and that in the last case it is fully reliable), therefore, we have hints of a phase transition with two phases: a symmetric and a broken one. The spontaneous symmetry breaking would happen while crossing the critical surface, generating a condensed state of the TGFT field (nonzero expectation value of the field operator). This is interesting from a physical perspective, because, in more involved models defined in a simplicial gravity or LQG context, this kind of phase transition has been suggested to relate to the emergence of a geometric spacetime from the theory [12], and the corresponding condensate states have been shown to admit a cosmological interpretation [48]. To confirm this condensate interpretation of the broken phase, one should change parametrization for the effective potential and study the theory around the new (degenerate) ground state solving the equation of motion in the saddle point approximation. This (complicated) analysis of our TGFT model is left for future work. Here we only notice that, in the constant modes approximation, which forgets about the peculiar combinatorial nonlocality of our interactions, and whose results should therefore be taken with great care, we find an algebraic equation of Ginsburg-Landau type for a ϕ^4 scalar complex theory, which indeed describes this type of condensate phase transitions. Once more, however, the rank $d=3$ model possesses an infinite tower of relevant directions, and thus our conclusions could be confirmed only after studying the higher order truncations and testing the regulator dependence on the results. At rank $d=4$, which we discuss in the next section, we still have to include marginal ϕ^6 couplings. On the other hand, the similar results at rank $d=5$, which we discuss in the next section, are trustable, because our truncation already exhausts all marginal and relevant directions.

Finally, we ought to mention that, at a more elaborate and technical level as the analysis performed in [67], the values of critical exponents and anomalous dimension at the NGFPs may be used to test the viability of the truncation. That analysis can be, in principle, extended to our present system and will be addressed in future work.

G. Rank $d=4$, 5

We now give a streamlined analysis of the flow in the case of rank $d=4$, which is very similar to the case $d=3$, and the rank $d=5$, which shares similarities but also a few differences that we will list.

Writing the system in rank $d=4$ as

$$\begin{cases} \eta_k = \frac{4\pi}{3}\frac{\bar{\lambda}_k}{(1+\bar{\mu}_k)^2}(\eta_k+2) \\ \beta(\bar{\mu}_k) = \frac{-32\pi}{3}\frac{\bar{\lambda}_k}{(1+\bar{\mu}_k)^2}\left[\frac{\eta_k}{5}+1\right] - \eta_k\bar{\mu}_k - 2\bar{\mu}_k\,, \\ \beta(\bar{\lambda}_k) = \frac{16\pi}{3}\frac{\bar{\lambda}_k^2}{(1+\bar{\mu}_k)^3}\left[\frac{\eta_k}{5}+1\right] - 2\eta_k\bar{\lambda}_k - \bar{\lambda}_k \end{cases} \quad (65)$$

we find, in addition to the Gaussian fixed point, the following NGFPs:

$$\begin{aligned} {}_4P_1 &= 10^{-1}(-6.402, 0.058), \\ {}_4P_2 &= (1.612, -0.496482), \\ {}_4P_3 &= 10^{-1}(-8.452, 0.112). \end{aligned} \quad (66)$$

As in the case $d=3$, the fixed point ${}_4P_3$ lies beyond the singularity. The eigenvalues and eigenvectors in the vicinity of the GFP and of ${}_4P_1$ and ${}_4P_2$ are given in the following table:

$$\mathrm{GFP}_4 \quad {}_4\theta_0^+ = -2 \quad \text{for } {}_4\mathbf{v}_0^+ = (1,0), \quad (67)$$

$$\mathrm{GFP}_4 \quad {}_4\theta_0^- = -1 \quad \text{for } {}_4\mathbf{v}_0^- = \left(-\frac{32\pi}{3}, 1\right) \quad (68)$$

$${}_4P_1 \quad {}_4\theta_{11} \sim 7.899 \quad \text{for } {}_4\mathbf{v}_{11} \sim 10^{-1}(10, -0.106), \quad (69)$$

$${}_4P_1 \quad {}_4\theta_{12} \sim 1.570 \quad \text{for } {}_4\mathbf{v}_{12} \sim 10^{-1}(10, 0.279), \quad (70)$$

$${}_4P_2 \quad {}_4\theta_{21} \sim 3.082 \quad \text{for } {}_4\mathbf{v}_{21} \sim 10^{-1}(-10, 0.521), \quad (71)$$

$${}_4P_2 \quad {}_4\theta_{22} \sim 0.439 \quad \text{for } {}_4\mathbf{v}_{22} \sim 10^{-1}(8.193, -5.733), \quad (72)$$

Negative eigenvalues at the vicinity of the GFP show that its eigendirections are all relevant. The NGFPs have a relevant and an irrelevant direction.

In rank $d=5$, on the other hand, the system (56) specializes as

$$\begin{cases} \eta_k = \frac{\pi^2}{2}\frac{\bar{\lambda}_k}{(1+\bar{\mu}_k)^2}(\eta_k+2) \\ \beta(\bar{\mu}_k) = -5\pi^2\frac{\bar{\lambda}_k}{(1+\bar{\mu}_k)^2}\left[\frac{\eta_k}{6}+1\right] - \eta_k\bar{\mu}_k - 2\bar{\mu}_k\,. \\ \beta(\bar{\lambda}_k) = 2\pi^2\frac{\bar{\lambda}_k^2}{(1+\bar{\mu}_k)^3}\left[\frac{\eta_k}{6}+1\right] - 2\eta_k\bar{\lambda}_k \end{cases} \quad (73)$$

Here, along with the GFP, we identify two NGFPs as

$$\begin{aligned} {}_5P_1 &= \left(\frac{-23+\sqrt{34}}{33}, \frac{4(191-4\sqrt{34})}{11979\pi^2}\right) \\ &= 10^{-1}(-5.202, 0.056), \\ {}_5P_2 &= 10^{-1}(-8.736, 0.072). \end{aligned} \quad (74)$$

Again, one of them, ${}_5P_2$, is beyond the singularity so we will skip its analysis. We list eigenvalues and eigenvectors in the vicinity of the GFP and ${}_5P_1$ as follows:

$$\mathrm{GFP}_5 \quad {}_5\theta_0^+ = -2 \quad \text{for } {}_5\mathbf{v}_0^+ = (1,0), \quad (75)$$

$$\mathrm{GFP}_5 \quad {}_5\theta_0^- = 0 \quad \text{for } {}_5\mathbf{v}_0^- = \left(-\frac{5\pi^2}{2}, 1\right), \quad (76)$$

$${}_5P_1 \quad {}_5\theta_1 \sim 2.947 \text{ for } {}_5\mathbf{v}_1 \sim (-249.652, 1), \quad (77)$$

$${}_5P_1 \quad {}_5\theta_2 \sim -0.843 \quad \text{for } {}_5\mathbf{v}_2 \sim (66.431, 1), \quad (78)$$

The GFP has one relevant eigendirection (corresponding to the mass coupling) and a marginal one. The numerical integration of the flow shows that this direction is marginally relevant for positive λ but becomes irrelevant for a negative λ. Similar to the previous case, the NGFP ${}_5P_1$ has one relevant and one irrelevant eigendirection.

The models at rank-4 and rank-5 are very similar to the previous rank-3 case. Hence, similar conclusions concerning the analysis of their flow hold, in particular the separation of the space of couplings in regions which are not connected by any RG trajectories which again suggest of phase transition. The numerical flow of the rank-4 and rank-5 models have been given in Fig. 6. Note that we did not display the NGFP ${}_4P_2$ of the rank-4 model which should be similar to the second fixed point of rank-3.

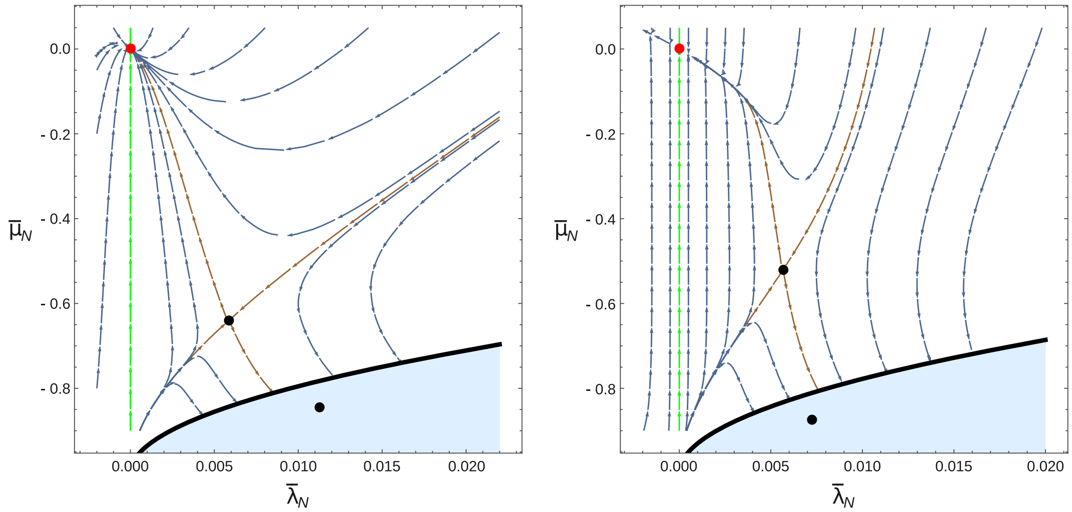


FIG. 6. Flow at rank $d = 4$ (left) and 5 (right).

IV. GAUGE INVARIANT RANK-*d* TENSORIAL GROUP FIELD THEORY ON $\mathbb{R}$

We now proceed to analyze a modified version of the TGFT models studied in the previous section, in which an additional gauge invariance condition is included in the model. These models define topological lattice gauge theories of BF type for the gauge group G at the level of their Feynman amplitudes. The first model of this type has been studied in [44] in rank-6 for the group $G = U(1)$. We therefore extend these first results by working with a noncompact group manifold, albeit still Abelian, keeping the rank arbitrary. As for the previous model, we first introduce the gauge-invariant model, then proceed with the FRG analysis in the general case, and finally specialize to interesting choices of rank to show explicitly the results of our analysis.

A. The gauge projection

We work with rank-d fields over the group manifold G satisfying the gauge invariance condition

$$\phi(g_1, g_2, \dots, g_d) = \phi(g_1 h, g_2 h, \dots, g_d h), \quad \forall\, h \in G. \tag{79}$$

This invariance condition can be imposed directly at the level of the space of fields or as a condition on the dynamics, which then restricts indirectly the field degrees of freedom. In both cases, this translates into a modification of the action (14). This modification can take different forms and should be implemented with some care. A possible (formal) way to implement it would be to allow only the propagation of modes satisfying (79) by inserting in the kinetic kernel a projector on the space of these modes. Defining the projector P and a kinetic kernel $\mathcal{K}$, one may encounter some ambiguity. A proper inspection shows that in our case, where the kinetic term has the form of a Laplacian plus constant, $\mathcal{K}$ and P commute. We choose to implement the kinetic term of the action in the following form:

$$S[\phi, \bar\phi] = \int_{G^{\times d}} [dg_i]_{i=1}^d [dg'_i]_{i=1}^d \bar\phi(g_1, g_2, \dots, g_d)(P \cdot \mathcal{K}) \times (\{g_i\}_{i=1}^d ; \{g'_i\}_{i=1}^d)\phi(g'_1, g'_2, \dots, g'_d) + \mathcal{V}[\phi, \bar\phi], \tag{80}$$

where $\mathcal{V}$ is the interaction term. The main issue of this formulation is that a projector is by definition not invertible; thus, a kinetic kernel built out of such an operator cannot, in general, define a covariance of a field theory measure. We partially avoid this problem by inverting the kinetic kernel in the operatorial sense, in such a way that the same constraint will define the covariance itself. In other words, also the propagator is defined as $P \cdot \mathcal{K}^{-1}$.

Now we restrict our description to the case of the Abelian additive group $\mathbb{R}$ and consider $\mathcal{V}$ with the same combinatorics used in Sec. III,

$$S_1[\phi, \bar\phi] = (2\pi)^{d-1} \int_{\mathbb{R}^{\times(2d+1)}} d\mathbf{x} d\mathbf{y} dh \bar\phi(\mathbf{x}) \prod_{i=1}^d \delta(x_i + h - y_i)\left(-\sum_{s=1}^d \Delta_{y_s} + \mu\right)\phi(\mathbf{y})$$
$$+ \frac{\lambda}{2}(2\pi)^{2d} \int_{\mathbb{R}^{\times 2d}} d\mathbf{x} d\mathbf{x}' [\phi(x_1, x_2, \dots, x_d)\bar\phi(x'_1, x_2, \dots, x_d)\phi(x'_1, x'_2, \dots, x'_d)\bar\phi(x_1, x'_2, \dots, x'_d)$$
$$+ \mathrm{sym}\{1, 2, \dots, d\}], \tag{81}$$

where $\mathbf{x} = (x_i)$, $\mathbf{x}' = (x'_i)$, and $\mathbf{y} = (y_i)$ are vectors in $\mathbb{R}^d$, and $h \in \mathbb{R}$.

We expect that the Wetterich equation will exhibit IR divergences of the same type encountered in the nonprojected model, although the gauge invariance conditions relate in a nontrivial way the arguments of the fields entering the interactions, and therefore modifies the combinatorics of the same; as a result, we expect a different degree of IR divergences with respect to the case we have treated in the previous section. In any case, we introduce again a regularization scheme. We consider a compact subset D of $\mathbb{R}$ homeomorphic to S^1 and write a regularized action as

$$\begin{aligned} S_1[\phi,\bar{\phi}] = (2\pi)^{d-1} \int_{D^{\times(2d+1)}} d\mathbf{x} d\mathbf{y} dh \bar{\phi}(\mathbf{x}) \prod_{i=1}^{d} \delta(x_i + h - y_i) \left(-\sum_{s=1}^{d} \Delta_{y_s} + \mu \right) \phi(\mathbf{y}) \\ + \frac{\lambda}{2} (2\pi)^{2d} \int_{D^{\times 2d}} d\mathbf{x} d\mathbf{x}' [\phi(x_1, x_2, \ldots, x_d) \bar{\phi}(x'_1, x_2, \ldots, x_d) \phi(x'_1, x'_2, \ldots, x'_d) \bar{\phi}(x_1, x'_2, \ldots, x'_d) \\ + \mathrm{sym}\{1, 2, \ldots, d\}], \end{aligned} \tag{82}$$

where we used the same notations introduced in Sec. III D.

The computation will be performed in momentum space. Using again the same notation for the lattice as $D^* = \mathcal{D}^{\times d}$, and denoting the gauge invariance constraint on the corresponding lattice as $\delta_{\mathcal{D}}(X) \coloneqq \delta_{\mathcal{D}}(X, 0)$, the Fourier series of the model (82) reads

$$S_1[\phi,\bar{\phi}] = l^d \sum_{\mathbf{p} \in D^*} \bar{\phi}(\mathbf{p}) [\Sigma_s p_s^2 + \mu] \phi(\mathbf{p}) \delta_{\mathcal{D}}(\Sigma p) + \frac{\lambda}{2} l^{2d} \sum_{\mathbf{p},\mathbf{p}' \in D^*} [\phi_{12\ldots d} \bar{\phi}_{1'2\ldots d} \phi_{1'2'\ldots d'} \bar{\phi}_{12'\ldots d'} + \mathrm{sym}\{1, 2, \ldots, d\}]. \tag{83}$$

The general FRG formalism introduced in Sec. II B applies to this model as to the one in the previous section. In particular, the regulator kernel will incorporate the same gauge constraint appearing in the kinetic term. The Wetterich equation has the same structure as well and expands again as (33).

We choose to truncate the effective action as

$$\begin{aligned} \Gamma^1_k[\varphi,\bar{\varphi}] = l^d \sum_{\mathbf{p} \in D^*} \bar{\varphi}(\mathbf{p}) [Z_k \Sigma_s p_s^2 + \mu_k] \varphi(\mathbf{p}) \delta_{\mathcal{D}}(\Sigma p) \\ + \frac{\lambda_k}{2} l^{2d} \sum_{\mathbf{p},\mathbf{p}' \in D^*} [\varphi_{12\ldots d} \bar{\varphi}_{1'2\ldots d} \varphi_{1'2'\ldots d'} \bar{\varphi}_{12'\ldots d'} + \mathrm{sym}\{1, 2, \ldots, d\}], \end{aligned} \tag{84}$$

and, then, we introduce the kernels [using the same notation as (33)],

$$R_k(\mathbf{q}, \mathbf{q}^{'}) = \Theta(k^2 - \Sigma_s q_s^2) Z_k (k^2 - \Sigma_s q_s^2) \delta_{\mathcal{D}}(\Sigma q) \prod \delta_{\mathcal{D}}(\mathbf{q}, \mathbf{q}^{'}), \tag{85}$$

$$F^1_k(\mathbf{q}, \mathbf{q}') = \frac{\delta^2}{\delta \bar{\varphi}_{\mathbf{q}'} \delta \varphi_{\mathbf{q}}} \mathcal{V}^1[\varphi, \bar{\varphi}], \tag{86}$$

where $\mathcal{V}^1_k$ refers to the interaction part of Γ^1_k. This is a natural choice following directly from a straightforward FRG formulation of (82). Performing the computation of the Wetterich equation, however, one realizes that this proposal drastically fails: the delta's enforcing the gauge constraints do not convolute properly with the TGFT fields. This is due to the fact that, if one evaluates (33) using (85) and (86), the fields appearing in the RHS come from the F^1_k operator, while the constraints always come from the masslike terms. The comparison of the two sides of the Wetterich equation for this model then would lead to all β-functions being trivial.

A moment of reflection shows that another way of choosing the interaction term produces a more sensible result. We simply insert gauge projections also in all fields in the interaction. An interaction satisfying this requirement is expressed as

$$\begin{aligned}\mathcal{V}[\phi,\bar{\phi}] &= \frac{\lambda_k}{2}(2\pi)^{2d-4}\int_{D^{\times(6d+4)}}\{d\mathbf{w}^i\}_{i=1}^4 d\mathbf{x}d\mathbf{x}'\{dh_j\}_{j=1}^4\phi(\mathbf{w}^1)\bar{\phi}(\mathbf{w}^2)\phi(\mathbf{w}^3)\bar{\phi}(\mathbf{w}^4)\\ &\quad\times\delta(x_1+h_1-w_1^1)\delta(x_2+h_1-w_2^1)\ldots\delta(x_d+h_1-w_d^1)\\ &\quad\times\delta(x_1'+h_2-w_1^2)\delta(x_2+h_2-w_2^2)\ldots\delta(x_d+h_2-w_d^2)\\ &\quad\times\delta(x_1'+h_3-w_1^3)\delta(x_2'+h_3-w_2^3)\ldots\delta(x_d'+h_3-w_d^3)\\ &\quad\times\delta(x_1+h_4-w_1^4)\delta(x_2'+h_4-w_2^4)\ldots\delta(x_d'+h_4-w_d^4)\\ &\quad+\mathrm{sym}\{1,2,\ldots,d\}\\ &= \frac{\lambda_k}{2}l^{2d}\sum_{\mathbf{p},\mathbf{p}'}\phi_{12\ldots d}\bar{\phi}_{1'2\ldots d}\phi_{1'2'\ldots d'}\bar{\phi}_{12'\ldots d'}\delta_{\mathcal{D}}(\Sigma p)\delta_{\mathcal{D}}(\Sigma p')\\ &\quad\times\delta_{\mathcal{D}}(p_1'+p_2+\cdots+p_d)\delta_{\mathcal{D}}(p_1+p_2'+\cdots+p_d')+\mathrm{sym}\{1,2,\ldots,d\}. \end{aligned} \tag{87}$$

Hence, restarting the analysis from the beginning, we define a model with gauge constraints on both the kinetic and interaction kernels via the action

$$\begin{aligned}S[\phi,\bar{\phi}] &= l^{2d}\sum_{\mathbf{p}}\bar{\phi}(\mathbf{p})[\Sigma_s p_s^2+\mu]\phi(\mathbf{p})\delta_{\mathcal{D}}(\Sigma p)\\ &\quad+\frac{\lambda_k}{2}l^{2d}\sum_{\mathbf{p},\mathbf{p}'}\phi_{12\ldots d}\bar{\phi}_{1'2\ldots d}\phi_{1'2'\ldots d'}\bar{\phi}_{12'\ldots d'}\delta_{\mathcal{D}}(\Sigma p)\delta_{\mathcal{D}}(\Sigma p')\delta_{\mathcal{D}}(p_1'+p_2+\cdots+p_d)\delta_{\mathcal{D}}(p_1+p_2'+\cdots+p_d')\\ &\quad+\mathrm{sym}\{1,2,\ldots,d\}, \end{aligned} \tag{88}$$

with the corresponding continuous model defined by

$$\begin{aligned}S[\phi,\bar{\phi}] &= \int d\mathbf{p}\bar{\phi}(\mathbf{p})[\Sigma_s p_s^2+\mu]\phi(\mathbf{p})\delta(\Sigma p)\\ &\quad+\frac{\lambda_k}{2}\int d\mathbf{p}d\mathbf{p}'\phi_{12\ldots d}\bar{\phi}_{1'2\ldots d}\phi_{1'2'\ldots d'}\bar{\phi}_{12'\ldots d'}\delta_{\mathcal{D}}(\Sigma p)\delta_{\mathcal{D}}(\Sigma p')\delta_{\mathcal{D}}(p_1'+p_2+\cdots+p_d)\delta_{\mathcal{D}}(p_1+p_2'+\cdots+p_d')\\ &\quad+\mathrm{sym}\{1,2,\ldots,d\}. \end{aligned} \tag{89}$$

In fact, with hindsight, one realizes that this result could have been guessed from a more general consideration. Even if the perturbative quantum amplitudes of the theory do not depend on whether the gauge projection appears in the kinetic term, in the interaction, or in both, and only gauge invariant degrees of freedom have nontrivial Feynman amplitudes (spin foam models), the nonperturbative analysis is, of course, radically different. From a nonperturbative point of view one is suggested to simply project the model to the space of gauge invariant fields, and thus insert projections in all elements of the TGFT action. From this point of view, a model which presents this constraint in only one of the two terms cannot be consistent. This directly reflects in the analysis we just presented.

At the same time, notice that inserting gauge projections on all fields in the action, both in kinetic and interaction terms, results in a trivial overall divergence equal to the volume of the domain, due to the fact that the combinatorics of field pairings is such that imposing gauge invariance on all but one field in each polynomial automatically implies the gauge invariance of the last one. We can easily remove this trivial divergence, therefore, by removing one gauge projection from one of the fields in each polynomial term in the action. The above prescription of the effective action together with (32) coincides with the Wetterich equation as formulated in [44] (albeit the formalism differs by the nature of the field background).

We can now proceed further using the model (89).

B. Effective action and Wetterich equation

Having defined the main ingredients of the model, we are in position to analyze its FRG equation. We shall again restrict to a simple truncation of the effective action for the model (88), which reads

$$\Gamma_k[\varphi,\bar\varphi] = l^d \sum_{\mathbf{p}} \bar\varphi(\mathbf{p})[Z_k \Sigma_s p_s^2 + \mu_k]\varphi(\mathbf{p})\delta_{\mathcal{D}}(\Sigma p)$$
$$+ \frac{\lambda_k}{2} l^{2d} \sum_{\mathbf{p},\mathbf{p}'} \varphi_{12\cdots d}\bar\varphi_{1'2\cdots d}\varphi_{1'2'\cdots d'}\bar\varphi_{12'\cdots d'}\delta_{\mathcal{D}}(\Sigma p)\delta_{\mathcal{D}}(\Sigma p')\delta_{\mathcal{D}}(p'_1 + p_2 + \cdots + p_d)\delta_{\mathcal{D}}(p_1 + p'_2 + \cdots + p'_d)$$
$$+ \mathrm{sym}\{1,2,\ldots,d\}. \tag{90}$$

Considering that $[\delta_{\mathcal{D}}(p)] = -1$, the dimensional analysis for the coupling constants gives different results from the model of Sec. III. We have

$$[Z_k] = 0 \Rightarrow [\mu_k] = 2,$$
$$2[\varphi] + d + 2 - 1 = 0 \Rightarrow [\varphi] = -\frac{d+1}{2},$$
$$[\lambda_k] + 2d + 4[\varphi] - 4 = 0 \Rightarrow [\lambda_k] = 6, \tag{91}$$

where, again, we set the canonical dimensions by requiring $[S] = [\Gamma_k] = 0$ and $[dp] = 1$. Once again, it must be stressed that the notions of canonical dimension and of scaling dimension are different in this context. Using the treatment of Appendix C 2, and for particular boundary data b, one associates the scaling dimension $\{\lambda_{n;b}\}$ with a given coupling $\lambda_{n;b}$ by taking

$$\{\lambda_{n;b}\} = -\frac{1}{2}[(d-4)n - 2(d-2)]. \tag{92}$$

The truncation of the effective action will be performed using positive scaling dimensions. Fixing $n = 4$ as given in the effective action, therefore the effective action contains

(i) at $d = 3$ and $d = 4$, two out of an infinite tower of relevant couplings; at $d = 4$, all couplings have, however, a fixed scaling dimension equal to 2;
(ii) at $d = 6$, one relevant (the mass) and one marginal coupling, and this exhausts the number of relevant and marginal couplings.

We introduce

$$R_k(\mathbf{q},\mathbf{q}') = \Theta(k^2 - \Sigma_s q_s^2) Z_k (k^2 - \Sigma_s q_s^2)\delta_{\mathcal{D}}(\Sigma q)\delta_{D^*}(\mathbf{q},\mathbf{q}'), \tag{93}$$

$$\partial_t R_k(\mathbf{q},\mathbf{q}') = \Theta(k^2 - \Sigma_s q_s^2)[\partial_t Z_k (k^2 - \Sigma_s q_s^2) + 2k^2 Z_k]\delta_{\mathcal{D}}(\Sigma q)\delta_{D^*}(\mathbf{q},\mathbf{q}'), \tag{94}$$

$$F_k(\mathbf{q},\mathbf{q}') = \lambda_k \Bigg[l^{d-1} \sum_{m_i} \varphi_{q'_1 m_2\ldots m_d}\bar\varphi_{q_1 m_2\ldots m_d}\delta_{\mathcal{D}}(\Sigma q)\delta_{\mathcal{D}}(q'_1 + m_2 + \cdots + m_d)$$
$$\times \delta_{\mathcal{D}}(q'_1 + q_2 + \cdots + q_d)\delta_{\mathcal{D}}(q_1 + m_2 + \cdots + m_d)\delta_{\mathcal{D}}(q_2 - q'_2)\ldots\delta_{\mathcal{D}}(q_d - q'_d)$$
$$+ l \sum_{m_1} \varphi_{m_1 q'_2\ldots q'_d}\bar\varphi_{m_1 q_2\ldots q_d}\delta_{\mathcal{D}}(\Sigma q)\delta_{\mathcal{D}}(m_1 + q'_2 + \cdots + q'_d)$$
$$\times \delta_{\mathcal{D}}(m_1 + q_2 + \cdots + q_d)\delta_{\mathcal{D}}(q_1 + q'_2 + \cdots + q'_d)\delta_{\mathcal{D}}(q_1 - q'_1)$$
$$+ \mathrm{sym}\{1,2,\ldots,d\}\Bigg], \tag{95}$$

$$P_k(\mathbf{q},\mathbf{q}') = R_k(\mathbf{q},\mathbf{q}') + \left(Z_k \sum_s q_s^2 + \mu_k\right)\delta_{\mathcal{D}}(\Sigma q)\delta_{D^*}(\mathbf{q},\mathbf{q}'). \tag{96}$$

This leads to the Wetterich equation,

$$\partial_t \Gamma_k = \mathrm{Tr}[\partial_t R_k \cdot (P_k + F_k)^{-1}]$$
$$= l^{2d} \sum_{\mathbf{p},\mathbf{p}'} \partial_t R_k(\mathbf{p},\mathbf{p}')(P_k + F_k)^{-1}(\mathbf{p}',\mathbf{p}). \tag{97}$$

On the LHS, as in Sec. III, we truncate at the level of the quartic interactions. This gives then, for the RHS of the Wetterich equation, the same expansion shown in (33), where now the operators involved are given by (94), (95), and (96).

An extra subtlety must be paid attention to, however, in extracting the β-functions of this model. The δ's implementing the convolutions which appear in the P_k operators can be inverted using (40), and summing over their indices we do not modify the dimensions of the whole expression. This is, however, not true for the δ's coming from the gauge constraints because they are not summed, so we need to keep them in the denominator. In any case, these constraints turn out to be redundant with other delta functions coming from the F_k and $\partial_t R_k$ operators, in such a way that their contribution, because of the regularization, is equivalent to some power of l, and it is naturally well defined.

C. β-functions and RG flows

Expanding the FRG equation (97), we find the following system of dimensionful β-functions (the main steps of the calculations are given in Appendix B):

$$\begin{cases} \beta_{d\neq 4}(Z_k) = \frac{d\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[\frac{\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}}\Gamma_E\left(\frac{d}{2}\right)}\frac{k^{d-2}}{l^d}+\frac{1}{l^2}\right]+\frac{2\pi^{\frac{d-2}{2}}Z_k}{(d-1)^{\frac{3}{2}}\Gamma_E\left(\frac{d-2}{2}\right)}\frac{k^{d-2}}{l^d}\right\} \\ \beta_{d\neq 4}(\mu_k) = -\frac{d\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[\frac{k^d}{l^d}\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma_E\left(\frac{d+2}{2}\right)}+\frac{k^2}{l^2}\right]+2Z_k\left[\frac{k^d}{l^d}\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma_E\left(\frac{d}{2}\right)}+\frac{k^2}{l^2}\right]\right\} \\ \beta_{d\neq 4}(\lambda_k) = \frac{2\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left\{\partial_t Z_k\left[\frac{2\pi^{\frac{d-2}{2}}}{d\sqrt{d-1}\Gamma\left(\frac{d}{2}\right)}\frac{k^d}{l^d}+(2[d+\delta_{d,3}]-1)\frac{k^2}{l^2}\right]+2Z_k\left[\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma\left(\frac{d}{2}\right)}\frac{k^d}{l^d}+(2[d+\delta_{d,3}]-1)\frac{k^2}{l^2}\right]\right\}, \end{cases} \tag{98}$$

and, at $d=4$, we have

$$\begin{cases} \beta_{d=4}(Z_k) = \frac{\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[\frac{\pi}{\sqrt{3}}\frac{k^2}{l^4}+\frac{4}{l^2}\right]+\frac{2\pi}{\sqrt{3}}\frac{k^2}{l^4}Z_k\right\} \\ \beta_{d=4}(\mu_k) = -\frac{4\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[\frac{\pi}{2\sqrt{3}}\frac{k^4}{l^4}+\frac{k^2}{l^2}\right]+2Z_k\left[\frac{\pi}{\sqrt{3}}\frac{k^4}{l^4}+\frac{k^2}{l^2}\right]\right\} \\ \beta_{d=4}(\lambda_k) = \frac{2\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left\{\partial_t Z_k\left[\frac{2\pi}{4\sqrt{3}}\frac{k^4}{l^4}+7\frac{k^2}{l^2}\right]+2Z_k\left[\frac{\pi}{\sqrt{3}}\frac{k^4}{l^4}+7\frac{k^2}{l^2}\right]\right\} \end{cases}. \tag{99}$$

To obtain a well defined noncompact limit of the model, we use a modified ansatz (different from the one of Sec. III E),

$$Z_k = \bar{Z}_k k^{-\chi} l^{\chi}, \qquad \mu_k = \bar{\mu}_k \bar{Z}_k k^{2-\chi} l^{\chi}, \qquad \lambda_k = \bar{\lambda}_k \bar{Z}_k^2 k^{6-\xi} l^{\xi}, \tag{100}$$

from which we obtain the dimensionless β-functions according to the following calculation:

$$\begin{aligned} \eta_k &= \frac{1}{\bar{Z}_k}\beta(\bar{Z}_k) = \frac{k^{\chi} l^{-\chi}}{\bar{Z}_k}\beta(Z_k)+\chi, \\ \beta(\bar{\mu}_k) &= \frac{k^{\chi-2} l^{-\chi}}{\bar{Z}_k}\beta(\mu_k)-\eta_k\bar{\mu}_k+(\chi-2)\bar{\mu}_k, \\ \beta(\bar{\lambda}_k) &= \frac{k^{\xi-6} l^{-\xi}}{\bar{Z}_k^2}\beta(\lambda_k)-2\eta_k\bar{\lambda}_k+(\xi-6)\bar{\lambda}_k. \end{aligned} \tag{101}$$

Inserting the above in (98), we deduce the equations for the dimensionless coupling constants,

$$\begin{aligned} \eta_k &= \frac{d\bar{\lambda}_k k^{2-\xi+2\chi} l^{\xi-2\chi}}{(1+\bar{\mu}_k)^2}\left\{(\eta_k-\chi)\left[\frac{\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}}\Gamma_E(\frac{d}{2})}\frac{k^{d-2}}{l^d}+\frac{1}{l^2}\right]+\frac{2\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}}\Gamma_E(\frac{d-2}{2})}\frac{k^{d-2}}{l^d}\right\}+\chi, \\ \beta_{d\neq 4}(\bar{\mu}_k) &= -\frac{d\bar{\lambda}_k k^{-\xi+2\chi} l^{\xi-2\chi}}{(1+\bar{\mu}_k)^2}\left\{(\eta_k-\chi)\left[\frac{k^d}{l^d}\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma_E(\frac{d+2}{2})}+\frac{k^2}{l^2}\right]+2\left[\frac{k^d}{l^d}\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma_E(\frac{d}{2})}+\frac{k^2}{l^2}\right]\right\}-\eta_k\bar{\mu}_k+(\chi-2)\bar{\mu}_k, \\ \beta_{d\neq 4}(\bar{\lambda}_k) &= \frac{2\bar{\lambda}_k^2 k^{2\chi-\xi} l^{\xi-2\chi}}{(1+\bar{\mu}_k)^3}\left\{(\eta_k-\chi)\left[\frac{2\pi^{\frac{d-2}{2}}}{d\sqrt{d-1}\Gamma(\frac{d}{2})}\frac{k^d}{l^d}+(2[d+\delta_{d,3}]-1)\frac{k^2}{l^2}\right]\right. \\ &\quad\left.+2\left[\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma(\frac{d}{2})}\frac{k^d}{l^d}+(2[d+\delta_{d,3}]-1)\frac{k^2}{l^2}\right]\right\}-2\eta_k\bar{\lambda}_k+(\xi-6)\bar{\lambda}_k. \end{aligned} \tag{102}$$

As in Sec. III E, the system of β-functions is nonautonomous in the IR cutoff k, as long as l is kept finite. We also notice a different dependence on the parameters k and l with respect to (52). The difference is, of course, a consequence of the presence of the delta functions which, having nontrivial dimensions, change both the canonical and scaling dimensions of couplings and fields, and remove degrees of freedom from the space of dynamical fields by imposing the gauge invariance constraints. Concerning this, we point out that, had we introduced one delta for each field appearing in both the kinetic and interaction kernels, this operation would have caused some extra divergences, but it would have also allowed us to absorb, from the point of view of the dimensions, the contribution of deltas inside a redefinition of the fields. In that case we would expect the couplings to have the same (canonical) dimensions of those appearing in the previous model. Finally, we can also note that the system might be reexpressed in terms of a shifted anomalous dimension $\eta_k \to \eta_k - \chi$; thus it could be defined up to constant χ. In the following, we have set $\chi = 0$.

To get an autonomous system in the limit of the regulator being removed, we set

$$\xi - 2\chi - d = 0, \tag{103}$$

and fixing $\chi = 0$, we come to $\xi = d$. In the thermodynamic limit, for $d \neq 4$, we obtain the autonomous system,

$$\begin{cases} \eta_k = \frac{d\bar{\lambda}_k}{(1+\bar{\mu}_k)^2} \frac{\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}}} \left\{ \eta_k \frac{1}{\Gamma_E\left(\frac{d}{2}\right)} + \frac{2}{\Gamma_E\left(\frac{d-2}{2}\right)} \right\} \\ \beta_{d\neq 4}(\bar{\mu}_k) = -\frac{d\bar{\lambda}_k}{(1+\bar{\mu}_k)^2} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}} \left\{ \eta_k \frac{1}{\Gamma_E\left(\frac{d+2}{2}\right)} + \frac{2}{\Gamma_E\left(\frac{d}{2}\right)} \right\} - (\eta_k + 2)\bar{\mu}_k \\ \beta_{d\neq 4}(\bar{\lambda}_k) = \frac{2\bar{\lambda}_k^2}{(1+\bar{\mu}_k)^3} \frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}} \left\{ \eta_k \frac{1}{\Gamma_E\left(\frac{d+2}{2}\right)} + \frac{2}{\Gamma_E\left(\frac{d}{2}\right)} \right\} - 2\eta_k\bar{\lambda}_k + (d-6)\bar{\lambda}_k \end{cases} . \tag{104}$$

In passing, we observe that at $d = 6 = \xi$, the coupling λ_k becomes marginal.

The same analysis performed at $d = 4$ yields

$$\begin{cases} \eta_k = \frac{\bar{\lambda}_k}{(1+\bar{\mu}_k)^2} \frac{\pi}{\sqrt{3}} (\eta_k + 2) \\ \beta_{d=4}(\mu_k) = -\frac{4\bar{\lambda}_k}{(1+\bar{\mu}_k)^2} \frac{\pi}{\sqrt{3}} \left(\frac{1}{2}\eta_k + 2\right) - (\eta_k + 2)\bar{\mu}_k \\ \beta_{d=4}(\lambda_k) = \frac{2\bar{\lambda}_k^2}{(1+\mu_k)^3} \frac{\pi}{\sqrt{3}} \left(\frac{1}{2}\eta_k + 2\right) - 2(\eta_k + 1)\bar{\lambda}_k \end{cases} . \tag{105}$$

D. Rank $d = 3$, 4

We can now fix the rank-d to be able to explicitly compute the flow.

We start with the case $d = 3$. The dependence in χ can be reabsorbed by a redefinition $\eta_k \to \eta_k - \chi$ (and the resulting variable is called again η_k). We therefore finally have a system of dimensionless β-functions given by

$$\begin{cases} \eta_k = \frac{3\bar{\lambda}_k}{\sqrt{2}(1+\bar{\mu}_k)^2 - 3\bar{\lambda}_k} \\ \beta(\bar{\mu}_k) = -\frac{6\bar{\lambda}_k\sqrt{2}}{(1+\bar{\mu}_k)^2} \left(\frac{\eta_k}{3} + 1\right) - \eta_k\bar{\mu}_k - 2\bar{\mu}_k \\ \beta(\bar{\lambda}_k) = \frac{4\bar{\lambda}_k^2\sqrt{2}}{(1+\bar{\mu}_k)^3} \left(\frac{\eta_k}{3} + 1\right) - 2\eta_k\bar{\lambda}_k - 3\bar{\lambda}_k \end{cases} . \tag{106}$$

As in the model without gauge projection, the system presents a divergence in the flow due to the truncation scheme. Here the singularity occurs at $\bar{\mu} = -1$ and $\bar{\lambda} = \frac{\sqrt{2}}{3}(1 + \bar{\mu})^2$. In the plane $(\bar{\mu}, \bar{\lambda})$, we find four fixed points, the Gaussian and three non-Gaussian fixed points at

$$\begin{aligned} {}_3P_1 &= (10)^{-1}(-7.083, 0.154), \\ {}_3P_2 &= 10^{-1}(-7.935, 0.273), \\ {}_3P_3 &= (-12.809, 169.635). \end{aligned} \tag{107}$$

Both ${}_3P_2$ and ${}_3P_3$ lie in the sector disconnected from the origin; therefore, we restrict the analysis and linearize the system only around ${}_3P_1$ and the Gaussian fixed point. The following eigenvalues and eigenvectors can be found by calculation from the stability matrix:

$$\mathrm{GFP}_3 \quad {}_3\theta_0^+ = -2 \quad \text{for } {}_3\mathbf{v}_0^+ = (1, 0), \tag{108}$$

$$\mathrm{GFP}_3 \quad {}_3\theta_0^- = -3 \quad \text{for } {}_3\mathbf{v}_0^- = (6\sqrt{2}, 1), \tag{109}$$

$${}_3P_1 \quad {}_3\theta_1 \sim 14.47 \quad \text{for } {}_3\mathbf{v}_1 \sim 10^{-1}(9.986, -0.529), \tag{110}$$

$${}_3P_1 \quad {}_3\theta_2 \sim -2.29 \quad \text{for } {}_3\mathbf{v}_2 \sim 10^{-1}(9.948, 1.022). \tag{111}$$

Negative eigenvalues represent UV-attractive eigendirections, while positive eigenvalues correspond to UV-repulsive eigendirections. From the plot in Fig. 7, we see that the Gaussian fixed point, where we have two negative eigenvalues corresponding to the scaling dimensions of the couplings, is a UV-attractor and has two relevant directions. Thus, we infer that the model is asymptotically free in the UV. Meanwhile, the NGFP has one relevant direction and one irrelevant direction. In this model, there are no marginal directions in the flow, and, qualitatively, the structure of the plot is again

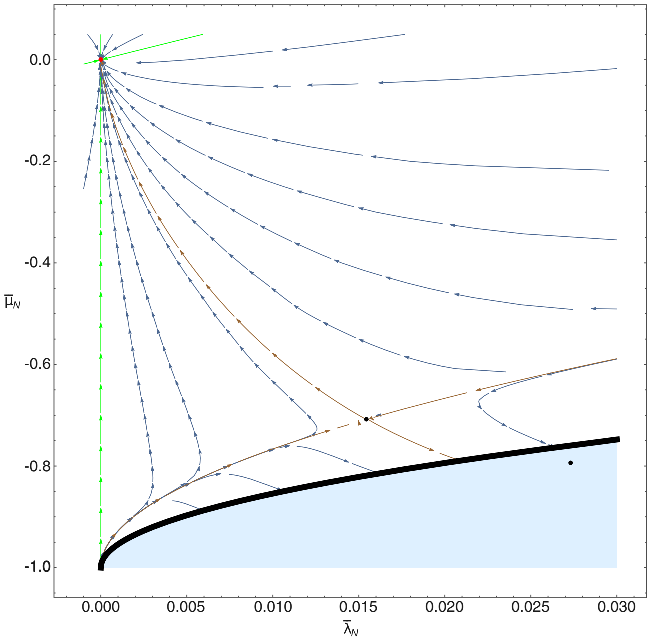


FIG. 7. Flow for the rank-3 gauge invariant model. Brown arrows represent the eigendirections of the NGFP (in black), while green arrows are the eigendirections of GFP (in red). The thick black line indicates the singularity of the system.

reminiscent of the Wilson-Fisher fixed point in standard scalar field theory in three dimensions. This is again suggestive of a phase transition between a symmetric and a broken phase, interpreted as a condensate phase labeled by a nonzero expectation value of the TGFT field operator. We again stress that the above claims could be trustable only after testing their validity by extending the truncation and changing the regulator.

Comparing this model with the one studied in Sec. III F, we can list some similarities, as well as the differences that follow then directly from the new gauge invariance imposition.

From the computational point of view, there are no fundamental differences. The presence of the gauge constraints influences the end result for what concerns the exact dependence of the FRG equations on the parameters k and l. The way the thermodynamic limit turns the regularized system of RG equations into an autonomous one is similar, but results from different canonical dimensions attributed to the various elements of the theory. For example, the canonical dimension of the ϕ^4-coupling changes from one model to the other. We claim that these models are not in the same universality class.

From a qualitative point of view, we find in both models the same number of non-Gaussian fixed points, but their distribution in the plane $(\bar{\mu},\bar{\lambda})$ is different. The TGFT model without gauge projection has two interesting NGFPs in the region of the plane $(\bar{\mu},\bar{\lambda})$ connected to the origin, whereas the gauge projected model has a unique NGFP lying in the same region. Also, the linearized theory around the Gaussian fixed point turns out to be slightly different. While in the previous section we have found a non-diagonalizable stability matrix with only one strictly relevant direction, for the gauge invariant model we have two relevant directions and the eigenperturbations form indeed a basis for the linearized system. On the other hand, the GFPs of both models are sinks, and so both models are asymptotically free in this truncation.

In rank $d=4$, the results are very similar to the above rank $d=3$. We obtain, in addition to the Gaussian fixed point, the fixed points

$$
\begin{aligned}
{}_4P_1 &= (10)^{-1}(-7.05, 0.093),\\
{}_4P_2 &= 10^{-1}(-8.465, 0.228),\\
{}_4P_3 &= (10.051, -97.962). \qquad (112)
\end{aligned}
$$

${}_4P_2$ which stands below the singularity will be not further analyzed. We will focus on the rest of the fixed points and perform a linearization around those.

Around the Gaussian fixed point the stability matrix becomes

$$
(\beta^*_{ij})|_{\rm GFP} := \begin{pmatrix} -2 & -\frac{8\pi}{\sqrt{3}} \\ 0 & -2 \end{pmatrix}, \qquad (113)
$$

which has an eigenvalue ${}_4\theta_0 = -2$ with multiplicity 2 with a single eigenvector ${}_4\mathbf{v}_0 = (1,0)$. We cannot diagonalize it and will integrate numerically the flow around this point.

We have the following critical exponents:

$$
{\rm GFP}_4 \quad {}_4\theta_0 = -2 \quad \text{for } {}_4\mathbf{v}_0 = (1,0), \qquad (114)
$$

$$
{}_4P_1 \quad {}_4\theta_{11} \sim 11.819 \quad \text{for } {}_4\mathbf{v}_{11} \sim 10^{-1}(10,-0.225), \qquad (115)
$$

$$
{}_4P_1 \quad {}_4\theta_{12} \sim -2.158 \quad \text{for } {}_4\mathbf{v}_{12} \sim 10^{-1}(10,0.624), \qquad (116)
$$

$$
{}_4P_3 \quad {}_4\theta_{31} \sim -2.654 \quad \text{for } {}_4\mathbf{v}_{31} \sim 10^{-1}(-3.891,9.211), \qquad (117)
$$

$$
{}_4P_3 \quad {}_4\theta_{32} \sim 0.624 \quad \text{for } {}_4\mathbf{v}_{32} \sim 10^{-1}(0.316,-10). \qquad (118)
$$

Both NGFPs have one relevant and one irrelevant direction. The analysis of perturbations around the fixed points leads to the phase diagram and RG flow presented in Fig. 8. From the numerical integration, we observe that the second eigendirection of the GFP is marginally relevant. We represent the phase diagram in Fig. 6.

We see once more RG trajectories indicating asymptotic freedom in the UV, and the presence of a phase transition between a symmetric and a broken phase in the IR (this must, however, be confirmed by extending the truncation and testing the dependence of the results on the regulator function).

E. Rank $d = 6$

Another interesting case to look at in more detail is the one for $d = 6$. For this rank, the model has one marginal direction around the GFP as the scaling dimension of the coupling λ vanishes. In this case, in fact, we can compare our results directly with the ones obtained in [44]. This comparison has two aspects. At the regularized level, with the system restricted to (six copies of) the compact domain S^1, we expect our RG equations to match the ones found in [44], up to normalizations. This can indeed be verified, but we do not report on it. On the other hand, by studying the RG flow in the thermodynamic limit, we will then be able to check how the phase diagram we obtain compares with the limiting cases studied for the compact model, expecting a qualitative agreement with the results found there in the UV approximation.

In rank $d = 6$, we have the following fixed points alongside the Gaussian fixed point:

$$ {}_6P_{\pm} = \left(\frac{1}{234}(-175 \pm \sqrt{1141}), \frac{\sqrt{5}(43309 \mp 79\sqrt{1141})}{1067742\pi^2}\right). \tag{119} $$

The NGFP ${}_6P_-$ is below the singularity. We focus on the Gaussian FP and ${}_6P_+$ which gives

$$ \mathrm{GFP}_6 \quad {}_6\theta_0 = -2 \quad \text{for } {}_6\mathbf{v}_0^+ = (1, 0), \tag{120} $$

$$ \mathrm{GFP}_6 \quad {}_6\theta_0 = -2 \quad \text{for } {}_6\mathbf{v}_0^- = \left(-\frac{3\pi^2}{\sqrt{5}}, 1\right), \tag{121} $$

$$ {}_6P_+ \quad {}_6\theta_1 \sim 4.589 \quad \text{for } {}_6\mathbf{v}_1 \sim (-185.549, 1), \tag{122} $$

$$ {}_6P_+ \quad {}_6\theta_2 \sim -0.9 \quad \text{for } {}_6\mathbf{v}_2 \sim 10^{-1}(31.289, 1), \tag{123} $$

The GFP has one relevant (mass) direction, and one marginally relevant direction for positive λ, which signals asymptotic freedom. Notice that for negative λ we do not expect the theory to be nonperturbatively well defined. On the other hand, the NGFP has a relevant and an irrelevant direction and shares a similar structure as the Wilson-Fisher FP. The analysis of perturbations around the fixed points in this case, then, leads to the phase diagram and RG flow presented in Fig. 9. The same conclusions discussed so far hold again in the present rank-6.

After the following change of normalization $\lambda \to 2\lambda$, the NGFP ${}_6P_+$, its critical exponents, and those of the GFP match with the results in rank $d = 6$ in the large mode limit of [44]. The RG flow lines are also very similar. Interestingly, at least for this model at rank-6, this coincidence means that the large radius sphere limit of the TGFT corresponds to our thermodynamic limit with our particular choice of scaling the coupling including both IR cutoff scaling and lattice spacing scaling. In fact, we expect this to be true more generally (for instance, at any rank-d or for any background of the fields).

As pointed out in [44], the presence of both an attractive UV fixed point and an IR Wilson-Fisher fixed point seems to be a general feature of TGFTs. While many other (local) quantum field theories present just one of these results (this is the case of QCD for just asymptotic freedom and of scalar field theory for the IR fixed point), the nonlocal models that we studied appear to always have a well defined behavior in both the limits. Moreover, there is

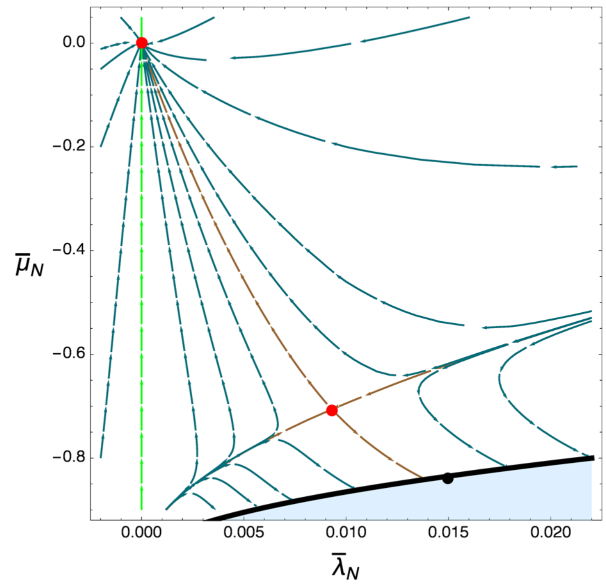


FIG. 8. Flow of the gauged model at rank $d = 4$.

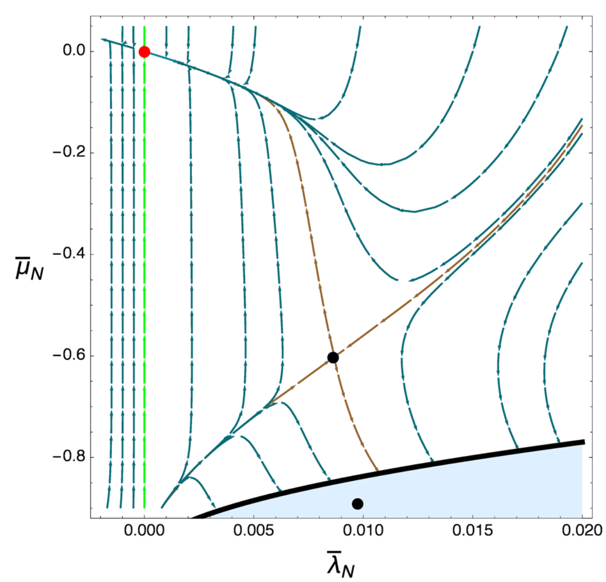


FIG. 9. Flow of the gauged model at rank-6.

another important property which might be interesting and fertile for future developments. All the models that we studied also present a second IR fixed point lying beyond the singularity of the flow. Even if we said that the presence of the anomalous dimension as a parameter in our effective action generates a divergence which prevents us from trusting in the flow across itself, we should remember that far from the infinite values of the flow the computation is probably correct. In other words, given initial conditions in the sector connected with the origin, we are not able to integrate the RG equation beyond the singularity but, had we given initial conditions in the other sector, the situation would be the opposite. Even if we cannot reconnect the flows over all the space of couplings, there are hints that other fixed points could arise and, with them, there is the possibility to find new (nontrivial) UV attractors. If this is confirmed by further investigations, TGFTs would also show asymptotic safety in the UV, in some regions of parameter space, and for specific models at least. If reproduced for 4D gravity models with more quantum geometric structure, this result would be in agreement with the hypothesis of asymptotic safety proposed by Weinberg and Reuter for quantum gravity theories [68]. However, it is not immediate to match TGFT results of this type with the asymptotic safety program for quantum gravity, since this is based directly on quantum Einstein gravity, and thus quantum field theories on spacetime involving directly a metric field, while TGFTs aim to be models of the microscopic constituents of spacetime and geometry itself. Still, it may be taken to suggest a nice convergence of results from different directions.

V. CONCLUSION

We have undertaken the functional renormalization group analysis of two classes of tensorial group field theories, as a further application of the formalism first studied in [38].

The models are defined on the noncompact group manifold $\mathbb{R}$ and for arbitrary tensor rank. They are endowed with melonic combinatorial interactions and distinguished from the presence (or absence) of a projection on the gauge invariant dynamics under the diagonal group action on the field arguments.

Both classes of models are simplified with respect to full-fledged TGFT models for quantum gravity, usually based on the group manifolds $SU(2)$ or $SL(2,\mathbb{C})$, and characterized by additional conditions on the dynamics, in addition to the gauge invariance models. However, they may capture many of their relevant features, and they are in any case of great interest from a more technical/mathematical point of view; the FRG analysis is a further step toward controlling and understanding this new type of quantum field theories. More generally, any GFT defines a sum over cellular complexes, which can be interpreted as a discrete definition of the covariant path integral for quantum gravity (with the details of the interpretation depending, of course, on the details of the amplitudes of the model), of the same type as those defining the dynamical triangulations approach to quantum gravity. The FRG analysis has the main objective of probing their continuum limit and phase structure, which would be, for quantum gravity models, a continuum limit for the pre-geometric, discrete, and quantum building blocks of spacetime. The search for a continuum geometric phase governed by a general relativistic dynamics is, in fact, the main outstanding open issue of these quantum gravity theories.

At a more technical level, the specific aim of our study was to obtain a picture of the fixed points and phase diagram, while enlightening the peculiarities coming from the noncompactness of the underlying group manifold, and thus comparing these results to previous work on TGFTs based on Abelian compact groups [38,44].

The main new issue posed by the noncompactness of the group manifold is the presence of IR divergences in the expansion of the Wetterich equation, which cannot be dealt with in the same way in which one removes simple infinite volume factors in local field theories, due to the particular combinatorics of TGFT interaction terms. We have shown, generalizing the previous work [43], how to regularize, first, and then remove these divergences using the appropriate thermodynamic limit. In particular, a comparison with [44] and the verified matching of critical exponents and scaling dimensions suggests a new concept of scaling dimension for this class of theories. While in the previous work the dimensional analysis leading to scaling dimensions was based on a perturbative approach and on the analysis of n-loops Green functions, at this nonperturbative level we find it more appropriate to rely on the order of divergences that need to be regularized to make the theory consistent in the noncompact limit. In this limit, all the models we study define a well-posed autonomous system of RG equations for the coupling constants, and we then proceed to solve numerically for various interesting values of the rank, in a simple truncation of the effective action.

In this simple truncation, and for all models considered, we identify UV and IR fixed points, study the perturbations around them, and obtain the corresponding phase diagram. In all these models, we find indications of the following: (1) asymptotic freedom in the UV; (2) a number of non-Gaussian fixed points in the IR; (3) a phase transition similar to the Wilson-Fisher type, between a symmetric and a broken (or condensate) phase with a nonzero expectation value of the TGFT field operator. These points must, however, be checked and consolidated by extending the truncation of the effective action (by including more involved invariants) and checking their dependence on the regulator function.

The first point is interesting because it confirms, by different means, the apparently generic asymptotic freedom of TGFT models, due to the dominance of wave function renormalization over coupling constant renormalization [33]. The last point, on the other hand, is important because phase transitions (in particular, of condensation type) have been suggested to mark the emergence of spacetime and geometry in GFT models of 4D quantum gravity [12,69], and because GFT condensate states have, in fact, been used to extract effective cosmological dynamics directly from the microscopic GFT quantum dynamics [48].

However, more work is certainly needed to further corroborate these findings.

Even for this simple class of TGFT models, one would need to improve the truncation scheme to include more terms in the effective action entering the Wetterich equation and change also the regulator to test the validity of the results about the overall phase diagram. And, concerning the study of the phase transition, a clear understanding of the different phases require at least solving the equations of motion (thus a mean field analysis), which is highly nontrivial due to the combinatorial structure of the TGFT interactions and the integrodifferential nature of the equations, and a change of parametrization for the effective potential (see the discussion in [38]).

And of course, we need to proceed toward the FRG analysis of more involved models, investigating how different groups and more involved forms of interaction kernels affect the results, and especially toward models with a more complete quantum geometric interpretation, and stronger links with simplicial quantum gravity and loop quantum gravity. The road ahead is long but promising.

ACKNOWLEDGMENTS

We thank Dario Benedetti for many helpful comments, and an anonymous referee for useful remarks and for suggesting to analyze in more detail the scaling dimensions of the TGFT couplings. DO acknowledges financial support from Templeton Foundation through Grant No. ID44100

APPENDIX A: EVALUATION OF β-FUNCTIONS IN RANK-d

In this appendix, we provide the detailed calculation of the β equations and emphasize its particularities. Note that this computation of the β-functions is performed in the regularized framework and only, at the end, do we take the thermodynamic limit. The system of equations that we obtain is an autonomous system in a continuous noncompact space.

Notations.— Given the regularization prescription introduced in Sec. III D, we set the notation $\delta_{D^*}(\mathbf{p},\mathbf{q}) = \delta(\mathbf{p}-\mathbf{q})$ not to be confused with the continuous Dirac delta that we do not use in this appendix. We also define $\mathcal{D}$ to be the one-dimensional lattice, that is, the domain of a single component of objects in D^*. We have $D^* = \mathcal{D}^{\times d}$ so that

$$l\sum_{p_i} = \int_{\mathcal{D}} dp_i. \tag{A1}$$

A change of notation helps during the calculation,

$$\begin{aligned}\mathbf{q} = (q_1, q_2, \ldots, q_d) &\Rightarrow q_1 \coloneqq q_1;\\ \mathbf{q}_1^{(d-1)} &\coloneqq (q_2, q_3, \ldots, q_d);\\ q_1^{(d-1)} &\coloneqq \sqrt{q_2^2 + q_3^2 + \ldots, q_d^2},\end{aligned}$$

for a generic d-dimensional momentum $\mathbf{q}$. When there is no possible confusion, we will simply forget the subscript 1 of $\mathbf{q}_1^{(d-1)}$ and $q_1^{(d-1)}$, and use $\mathbf{q}^{(d-1)}$ and $q^{(d-1)}$, respectively.

Let us recall the second variation of the effective action (30) in these new notations,

$$\begin{aligned}\Gamma_k^{(2)} &= \left(Z_k\sum_s p_s^2 + \mu_k\right)\delta(\mathbf{p}-\mathbf{p}') + \lambda_k\Bigg[\int_{\mathcal{D}^{\times d-1}} dq_2\ldots dq_d \varphi_{p_1' q_2\ldots q_d}\bar{\varphi}_{p_1 q_2\ldots q_d}\prod_{i=2}^d \delta(p_i - p_i')\\ &\quad + \int_{\mathcal{D}} dq_1 \varphi_{q_1 p_2'\ldots p_d'}\bar{\varphi}_{q_1 p_2\ldots p_d}\delta(p_1 - p_1') + \mathrm{sym}\{1,2,\ldots,d\}\Bigg]\\ &= \left(Z_k\sum_s p_s^2 + \mu_k\right)\delta(\mathbf{p}-\mathbf{p}') + F_k(\mathbf{p},\mathbf{p}'),\end{aligned}$$

and choose a regulator of the form (24) where θ is now replaced by $\Theta(f(\mathbf{p}))$ the discrete step function. This implies

$$\partial_t R_k = \delta(\mathbf{p}-\mathbf{p}')\Theta\left(k^2 - \sum_s p_s^2\right)\left[\partial_t Z_k\left(k^2 - \sum_s p_s^2\right) + Z_k 2k^2\right].$$

Defining $P_k(\mathbf{p},\mathbf{p}')$ like (31), with appropriate replacements, we expand and truncate the Wetterich equation as (33). The zeroth order of the previous expansion is the vacuum term and does not provide us any useful information. On the other hand, the first and the second orders will provide us with the flow of the kinetic (φ^2) and interaction (φ^4) couplings, respectively, namely, the β-functions for the couplings μ_k, Z_k, and λ_k.

1. φ^2-terms

To compute the flow of couplings of the quadratic terms of Γ_k, in other words, the β-functions for μ_k and Z_k, we focus on the first order of (33). To have more compact notations, let us introduce the first convolution appearing in the expansion,

$$
\begin{aligned}
\tilde{\partial}_t R_k(\mathbf{p},\mathbf{p}'') &= \int_{D^*} d\mathbf{p}' \partial_t R_k(\mathbf{p},\mathbf{p}')(P_k)^{-1}(\mathbf{p}',\mathbf{p}'') \\
&= \int_{D^*} d\mathbf{p}' \delta(\mathbf{p}-\mathbf{p}')\delta(\mathbf{p}'-\mathbf{p}'')\Theta\left(k^2 - \sum_s p_s^2\right) \frac{\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k}{Z_k(k^2 - \sum_s p_s'^2)\Theta(k^2 - \sum_s p_s'^2) + Z_k \sum_s p_s'^2 + \mu_k} \\
&= \delta(\mathbf{p}-\mathbf{p}'')\Theta\left(k^2 - \sum_s p_s^2\right) \frac{\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k}{(Z_k k^2 + \mu_k)},
\end{aligned}
$$

where we used the fact that, after integration, the two Θ's appearing in the expression are redundant.

Thus, calling $(I)_W$ the first order of the Wetterich equation, we write

$$
\begin{aligned}
-(I)_W = \overline{\mathrm{Tr}}[\tilde{\partial}_t R_k \cdot F_k \cdot (P_k)^{-1}] &= \int_{D^{*\times 2}} d\mathbf{p} d\mathbf{p}' \tilde{\partial}_t R_k(\mathbf{p},\mathbf{p}') \int_{D^*} d\mathbf{q} F_k(\mathbf{p}',\mathbf{q})(P_k)^{-1}(\mathbf{q},\mathbf{p}) \\
&= \int_{D^*} d\mathbf{p} \Theta\left(k^2 - \sum_s p_s^2\right) \frac{\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k}{(Z_k k^2 + \mu_k)^2} F_k(\mathbf{p},\mathbf{p}).
\end{aligned}
$$

To simplify the computation, we split the integral in two pieces, namely,

$$
\begin{aligned}
A &= \frac{\partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} \Theta\left(k^2 - \sum_s p_s^2\right)\left(\sum_s p_s^2\right) F_k(\mathbf{p},\mathbf{p}), \\
B &= \frac{k^2(2+\partial_t) Z_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} \Theta\left(k^2 - \sum_s p_s^2\right) F_k(\mathbf{p},\mathbf{p}),
\end{aligned}
\tag{A2}
$$

having $(I)_W = A - B$. Let us treat the first term and recall that $\delta_{\mathcal{D}}(0) = \delta(0) = \frac{1}{l}$,

$$
\begin{aligned}
A &= \frac{\lambda_k \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} \Theta\left(k^2 - \sum_s p_s^2\right)\left(\sum_s p_s^2\right) \\
&\quad \times \left[\frac{1}{l^{d-1}} \int_{\mathcal{D}^{\times d-1}} dq_2 \cdots dq_d |\varphi_{p_1 q_2 \cdots q_d}|^2 + \frac{1}{l} \int_{\mathcal{D}} dq_1 |\varphi_{q_1 p_2 \cdots p_d}|^2 + \mathrm{sym}\{1,2,\ldots,d\}\right] \\
&= \frac{\lambda_k \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \left\{\frac{1}{l^{d-1}} \int_{D^*} dp_1 dq_2 \cdots dq_d |\varphi_{p_1 q_2 \cdots q_d}|^2 \int_{\mathcal{D}^{\times d-1}} dp_2 \cdots dp_d \Theta[(k^2 - p_1^2) - \Sigma_{i=2}^d p_i^2][\Sigma_{i=2}^d p_i^2 + p_1^2] \right. \\
&\quad \left. + \frac{1}{l} \int_{D^*} dq_1 dp_2 \cdots dp_d |\varphi_{q_1 p_2 \cdots p_d}|^2 \int_{\mathcal{D}} dp_1 \Theta[(k^2 - \Sigma_{i=2}^d p_i^2) - p_1^2][\Sigma_{i=2}^d p_i^2 + p_1^2]\right\} \\
&\quad + \mathrm{sym}\{1,2,\ldots,d\}.
\end{aligned}
$$

Now we perform the continuum limit $l \to \infty$, and this corresponds to

$$
\int_{\mathcal{D}} \to \int_{\mathbb{R}}, \qquad \Theta \to \theta. \tag{A3}
$$

The negative powers of l appearing in the expressions keep track of the former IR divergences of the continuous model. Extracting an l dependence from the couplings, we will address them at the end. To simplify the notation, we drop the limit symbol $\lim_{l\to\infty}$ and get

$$
\begin{aligned}
A = {} & \frac{\lambda_k \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l} \int_{\mathbb{R}^d} dq_1 dp_2 \cdots dp_d \theta(k^2 - \Sigma_{i=2}^d p_i^2) |\varphi_{q_1 p_2 \cdots p_d}|^2 \int_{-\sqrt{k^2 - \Sigma_{i=2}^d p_i^2}}^{\sqrt{k^2 - \Sigma_{i=2}^d p_i^2}} dp_1 [\Sigma_{i=2}^d p_i^2 + p_1^2] \right. \\
& \left. + \frac{1}{l^{d-1}} \int_{\mathbb{R}^d} dp_1 dq_2 \cdots dq_d \theta(k^2 - p_1^2) |\varphi_{p_1 q_2 \cdots q_d}|^2 \int d\Omega_{d-1} \int_0^{\sqrt{k^2 - p_1^2}} dr r^{d-2} [r^2 + p_1^2] \right\} \\
& + \mathrm{sym}\{1, 2, \ldots, d\} \\
= {} & \frac{\lambda_k \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l} \int_{\mathbb{R}^d} dq_1 dp_2 \cdots dp_d \theta(k^2 - \Sigma_{i=2}^d p_i^2) \left[2(\Sigma_{i=2}^d p_i^2) \sqrt{k^2 - \Sigma_{i=2}^d p_i^2} + \frac{2}{3} (k^2 - \Sigma_{i=2}^d p_i^2)^{3/2} \right] |\varphi_{q_1 p_2 \cdots p_d}|^2 \right. \\
& \left. + \frac{1}{l^{d-1}} \int_{\mathbb{R}^d} dp_1 dq_2 \cdots dq_d \theta(k^2 - p_1^2) \left[\frac{(k^2 - p_1^2)^{\frac{d+1}{2}}}{d+1} + \frac{p_1^2}{d-1} (k^2 - p_1^2)^{\frac{d-1}{2}} \right] \Omega_{d-1} |\varphi_{p_1 q_2 \cdots q_d}|^2 \right\} \\
& + \mathrm{sym}\{1, 2, \ldots, d\},
\end{aligned}
$$

where in the first passage we changed the variable to the $(d-1)$-dimensional spherical coordinates and introduced the following notation:

$$
\Omega_d = \int d\Omega_d = \prod_{i=1}^{d-2} \left[\int_0^\pi d\alpha_i \sin^{d-1-i}(\alpha_i) \right] \int_0^{2\pi} d\alpha_{d-1} = \frac{2\pi^{d/2}}{\Gamma_E(\frac{d}{2})}, \tag{A4}
$$

with Γ_E the Euler gamma function. Expanding the term B, we find

$$
\begin{aligned}
B = {} & \lambda_k \frac{k^2 (2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^2} \int_{D^*} d\mathbf{p} \Theta\left(k^2 - \sum_s p_s^2 \right) \left[\frac{1}{l^{d-1}} \int_{\mathcal{D}^{\times d-1}} dq_2 \cdots dq_d |\varphi_{p_1 q_2 \cdots q_d}|^2 + \frac{1}{l} \int_{\mathcal{D}} dq_1 |\varphi_{q_1 p_2 \cdots p_d}|^2 \right] \\
& + \mathrm{sym}\{1, 2, \ldots, d\},
\end{aligned} \tag{A5}
$$

which, in the limit, gives

$$
\begin{aligned}
B = {} & \lambda_k \frac{k^2 (2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l^{d-1}} \int_{\mathbb{R}^d} dp_1 dq_2 \cdots dq_d \theta(k^2 - p_1^2) |\varphi_{p_1 q_2 \cdots q_d}|^2 \Omega_{d-1} \int_0^{\sqrt{k^2 - p_1^2}} dr r^{d-2} \right. \\
& \left. + \frac{1}{l} \int_{\mathbb{R}^d} dq_1 dp_2 \cdots dp_d \theta(k^2 - \Sigma_{s=2}^d p_s^2) |\varphi_{q_1 p_2 \cdots p_d}|^2 \int_{-\sqrt{k^2 - \Sigma_{s=2}^d p_s^2}}^{\sqrt{k^2 - \Sigma_{s=2}^d p_s^2}} dp_1 \right\} \\
& + \mathrm{sym}\{1, 2, \ldots, d\} \\
= {} & \lambda_k \frac{k^2 (2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l^{d-1}} \int_{\mathbb{R}^d} dp_1 dq_2 \cdots dq_d \theta(k^2 - p_1^2) \Omega_{d-1} \frac{(k^2 - p_1^2)^{\frac{d-1}{2}}}{d-1} |\varphi_{p_1 q_2 \cdots q_d}|^2 \right. \\
& \left. + \frac{2}{l} \int_{\mathbb{R}^d} dq_1 dp_2 \cdots dp_d \theta(k^2 - \Sigma_{s=2}^d p_s^2) \sqrt{k^2 - \Sigma_{s=2}^d p_s^2} |\varphi_{q_1 p_2 \cdots p_d}|^2 \right\} \\
& + \mathrm{sym}\{1, 2, \ldots, d\}.
\end{aligned}
$$

β-functions.— To find the β-functions of the coupling constants, we rely on the fact that the LHS of (33) is of the form

$$
\partial_t \Gamma_{\mathrm{kin}} = \int d\mathbf{p} |\varphi(\mathbf{p})|^2 \left(\beta(Z_k) \sum_s p_s^2 + \beta(\mu_k) \right).
$$

In fact, this allows us to identify the β-functions with the coefficients of an expansion in powers of the field momenta of the integrands in A and B, up to an $o(p^3)$. Respectively, the terms with momenta of order p_i^2 convoluted with the fields $\varphi_{\ldots, p_i, \ldots}$ will contribute to the flow of the wave function renormalization, while the zeroth order will be proportional to the scaling of the mass. All remaining terms, falling out of the truncation, must be discarded. Hence, we have, for $d \geq 3$,

$$A \simeq \frac{\lambda_k \partial_t Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l} \int dq_1 dp_2 \cdots dp_d \left[\frac{2}{3} k^3 + k(\Sigma_{s=2}^d p_s^2) \right] |\varphi_{q_1 p_2 \cdots p_d}|^2 \right. \\ \left. + \frac{1}{l^{d-1}} \int dp_1 dq_2 \cdots dq_d \Omega_{d-1} \left[\frac{k^{d+1}}{d+1} + \left(\frac{1}{d-1} - \frac{1}{2} \right) k^{d-1} p_1^2 \right] |\varphi_{p_1 q_2 \cdots q_d}|^2 \right\} \\ + \mathrm{sym}\{1, 2, \ldots, d\}.$$

For the B terms, one finds

$$B \simeq \lambda_k \frac{k^2 (2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^2} \left\{ \frac{1}{l^{d-1}} \int dp_1 dq_2 \cdots dq_d \Omega_{d-1} \left[\frac{k^{d-1}}{d-1} - \frac{k^{d-3}}{2} p_1^2 \right] |\varphi_{p_1 q_2 \cdots q_d}|^2 \right. \\ \left. + \frac{2}{l} \int dq_1 dp_2 \cdots dp_d \left[k - \frac{1}{2k} (\Sigma_{s=2}^d p_s^2) \right] |\varphi_{q_1 p_2 \cdots p_d}|^2 \right\} \\ + \mathrm{sym}\{1, 2, \ldots, d\}. \tag{A6}$$

Now, we concentrate on the colored symmetric terms. Note that the procedure and result of the above integrals will not change for each colored term in $\mathrm{sym}\{\cdot\}$, up to a simple relabeling. Thus, collecting all terms, we obtain an expression of the form

$$\partial_t \Gamma_{\mathrm{kin}} = \int dp_1 \cdots dp_d |\varphi_{p_1 \cdots p_d}|^2 \sum_{j=1}^d [f(k) + g(k) p_j^2 + h(k) (\Sigma_{i=1}^{j-1} p_i^2 + \Sigma_{i=j+1}^d p_i^2)] \\ = \int dp_1 \cdots dp_d |\varphi_{p_1 \cdots p_d}|^2 \left\{ d f(k) + [g(k) + (d-1) h(k)] \sum_{i=1}^d p_i^2 \right\}. \tag{A7}$$

This, by comparison between the two sides of the equation, leads to the following dimensionful β-functions for the parameters Z_k and μ_k:

$$\beta(Z_k) = \frac{\lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[2(d-1) \frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d+1}{2})} \frac{k^{d-1}}{l^{d-1}} \right] + 2 Z_k \left[(d-1) \frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d-1}{2})} \frac{k^{d-1}}{l^{d-1}} \right] \right\}, \\ \beta(\mu_k) = -\frac{d \lambda_k}{(Z_k k^2 + \mu_k)^2} \left\{ \partial_t Z_k \left[\frac{4}{3} \frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d+3}{2})} \frac{k^{d+1}}{l^{d-1}} \right] + 2 Z_k \left[2 \frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d+1}{2})} \frac{k^{d+1}}{l^{d-1}} \right] \right\}. \tag{A8}$$

Already at this level, one realizes that each β-function does not have homogeneous scaling in k and dimensions in l. This feature clearly comes from the pattern of the convolution of the interaction which is specific to TGFTs.

2. φ^4-terms

The second order $(II)_W$ of (33) will provide the β-function for λ_k, which completes the set of β-functions of the model. Defining R'_k and P'_k such that

$$R_k(\mathbf{p}, \mathbf{p}') = R'_k(\mathbf{p}) \Theta\left(k^2 - \sum_s p_s^2 \right) \delta(\mathbf{p} - \mathbf{p}'), \\ P_k(\mathbf{p}, \mathbf{p}') = P'_k(\mathbf{p}) \delta(\mathbf{p} - \mathbf{p}'), \tag{A9}$$

the terms of interest take the form

$$
\begin{aligned}
(II)_W &= \mathrm{Tr}[\partial_t R_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1} \cdot F_k \cdot (P_k)^{-1}] \\
&= \int_{D^{*\times 5}} d\mathbf{p} d\mathbf{p}' d\mathbf{p}'' d\mathbf{q} d\mathbf{q}' \partial_t R'_k(\mathbf{p}) \Theta\left(k^2 - \sum_s p_s^2\right) \delta(\mathbf{p}-\mathbf{p}')(P'_k)^{-1}(\mathbf{p}')\delta(\mathbf{p}'-\mathbf{p}'') \\
&\quad \times F_k(\mathbf{p}'',\mathbf{q})(P'_k)^{-1}(\mathbf{q})\delta(\mathbf{q}-\mathbf{q}')F_k(\mathbf{q}',\mathbf{p})(P'_k)^{-1}(\mathbf{p}) \\
&= \int_{D^*} d\mathbf{p} \partial_t R'_k(\mathbf{p}) \Theta\left(k^2 - \sum_s p_s^2\right)(P'_k)^{-1}(\mathbf{p}) \int_{D^*} d\mathbf{q} F_k(\mathbf{p},\mathbf{q})(P'_k)^{-1}(\mathbf{q}) F_k(\mathbf{q},\mathbf{p})(P'_k)^{-1}(\mathbf{p}).
\end{aligned} \tag{A10}
$$

We focus on the intermediate convolution $F_k \cdot P_k^{-1} \cdot F_k$ which expands as

$$
\begin{aligned}
(F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p},\mathbf{p}) &= \lambda_k^2 \int_{D^*} d\mathbf{q} F(\mathbf{p},\mathbf{q})(P'_k)^{-1}(\mathbf{q}) F_k(\mathbf{q},\mathbf{p}) \\
&= \lambda_k^2 \int_{D^*} dq_1 \cdots dq_d \left[\int_{\mathcal{D}} dm_1 \varphi_{m_1 p_2 \cdots p_d} \bar{\varphi}_{m_1 q_2 \cdots q_d} \delta(p_1 - q_1)\right. \\
&\quad \left. + \int_{\mathcal{D}^{\times d-1}} dm_2 \cdots dm_d \varphi_{p_1 m_2 \cdots m_d} \bar{\varphi}_{q_1 m_2 \cdots m_d} \prod_{i=2}^d \delta(p_i - q_i) + \mathrm{sym}\{1,2,\ldots,d\}\right] \\
&\quad \times (P'_k)^{-1}(\mathbf{q}) \left[\int_{\mathcal{D}} dm'_1 \varphi_{m'_1 q_2 \cdots q_d} \bar{\varphi}_{m'_1 p_2 \cdots p_d} \delta(p_1 - q_1)\right. \\
&\quad \left. + \int_{\mathcal{D}^{\times d-1}} dm'_2 \cdots dm'_d \varphi_{q_1 m'_2 \cdots m'_d} \bar{\varphi}_{p_1 m'_2 \cdots m'_d} \prod_{i=2}^d \delta(p_i - q_i) + \mathrm{sym}\{1,2,\ldots,d\}\right].
\end{aligned}
$$

At this level, the product of colored symmetric terms generates a list of terms (among which are cross terms) that we must all carefully analyze. First, we deal with the case when the product involves two terms of the same color, and then we will treat the cross-colored case. Below, we further specialize the study to the product of terms of color 1 and then on the cross terms 1–2 in the above expansion. We refer to the first type of term as $(F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p},\mathbf{p})|_{1,1}$ and to the overall contribution after tracing over remaining indices as $(II)_W|_{1,1}$ (respectively, the symbol $|_{1,2}$ will stand for the cross term product of the colors 1 and 2). This evaluation is, of course, without loss of generality because one can quickly infer the result for all remaining products. All these contributions, at the end, must be summed.

We have

$$
\begin{aligned}
(F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p},\mathbf{p})|_{1,1} &= \lambda_k^2 \int_{D^*} dq_1 \cdots dq_d \int_{\mathcal{D}} dm_1 \varphi_{m_1 p_2 \cdots p_d} \bar{\varphi}_{m_1 q_2 \cdots q_d} \delta(p_1 - q_1)(P'_k)^{-1}(\mathbf{q}) \\
&\quad \times \int_{\mathcal{D}^{\times d-1}} dm'_2 \cdots dm'_d \varphi_{q_1 m'_2 \cdots m'_d} \bar{\varphi}_{p_1 m'_2 \cdots m'_d} \prod_{i=2}^d \delta(p_i - q_i) \\
&\quad + \lambda_k^2 \int_{D^*} dq_1 \cdots dq_d \int_{\mathcal{D}^{\times d-1}} dm_2 \cdots dm_d \varphi_{p_1 m_2 \cdots m_d} \bar{\varphi}_{q_1 m_2 \cdots m_d} \delta(p_i - q_i)(P'_k)^{-1}(\mathbf{q}) \\
&\quad \times \int_{\mathcal{D}} dm'_1 \varphi_{m'_1 q_2 c \cdots q_d} \bar{\varphi}_{m'_1 p_2 \cdots p_d} \delta(p_1 - q_1) \\
&\quad + \lambda_k^2 \int_{D^*} dq_1 \cdots dq_d \int_{\mathcal{D}} dm_1 \varphi_{m_1 p_2 \cdots p_d} \bar{\varphi}_{m_1 q_2 \ldots q_d} \delta(p_1 - q_1)(P'_k)^{-1}(\mathbf{q}) \\
&\quad \times \int_{\mathcal{D}} dm'_1 \varphi_{m'_1 q_2 \cdots q_d} \bar{\varphi}_{m'_1 p_2 \cdots p_d} \delta(p_1 - q_1) \\
&\quad + \lambda_k^2 \int_{D^*} dq_1 \cdots dq_d \int_{\mathcal{D}^{\times d-1}} dm_2 \cdots dm_d \varphi_{p_1 m_2 \cdots m_d} \bar{\varphi}_{q_1 m_2 \cdots m_d} \prod_{i=2}^d \delta(p_i - q_i)(P'_k)^{-1}(\mathbf{q}) \\
&\quad \times \int_{\mathcal{D}^{\times d-1}} dm'_2 \cdots dm'_d \varphi_{q_1 m'_2 \cdots m'_d} \bar{\varphi}_{p_1 m'_2 \cdots m'_d} \prod_{i=2}^d \delta(p_i - q_i).
\end{aligned}
$$

The first two terms, after the δ's in $\mathbf{q}$ are integrated out, become proportional to the product of two square moduli of the fields, and thus they represent disconnected interactions. They can be discarded for the same reasons invoked above. As a remainder, we get

$$
\begin{aligned}
(F_k \cdot P_k^{-1} \cdot F_k)(\mathbf{p},\mathbf{p})|_{1,1} \simeq{}& \frac{\lambda_k^2}{l}\int_{\mathcal{D}^{\times d-1}} dq_2 \cdots dq_d \int_{\mathcal{D}^{\times 2}} dm_1 dm_1' \varphi_{m_1 p_2\cdots p_d}\bar{\varphi}_{m_1 q_2\cdots q_d}\varphi_{m_1' q_2\cdots q_d}\bar{\varphi}_{m_1' p_2\cdots p_d}(P_k')^{-1}(p_1, q_2, \ldots, q_d)\\
&+\frac{\lambda_k^2}{l^{d-1}}\int_{\mathcal{D}} dq_1 \int_{\mathcal{D}^{\times 2d-2}} dm_2 \cdots dm_d dm_2' \cdots dm_d' \varphi_{p_1 m_2\cdots m_d}\bar{\varphi}_{q_1 m_2\cdots m_d}\varphi_{q_1 m_2'\cdots m_d'}\bar{\varphi}_{p_1 m_2'\cdots m_d'}\\
&\times (P_k')^{-1}(q_1, p_2, \ldots, p_d). \qquad (A11)
\end{aligned}
$$

Then, plugging back (A11) into $(II)_W$ and concentrating on the contribution of this term, one finds

$$
\begin{aligned}
(II)_W|_{1,1} ={}& \lambda_k^2 \int_{D^*} d\mathbf{p}\Theta\left(k^2 - \sum_s p_s^2\right)\frac{[\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2}\\
&\times\left\{\frac{1}{l}\int_{\mathcal{D}^{\times d-1}} dq_2\cdots dq_d \int_{\mathcal{D}^{\times 2}} dm_1 dm_1' \varphi_{m_1 p_2\cdots p_d}\bar{\varphi}_{m_1 q_2\cdots q_d}\varphi_{m_1' q_2\cdots q_d}\bar{\varphi}_{m_1' p_2\cdots p_d}\right.\\
&\times\left[Z_k(k^2 - p_1^2 - \Sigma_{i=2}^d q_i^2)\Theta(k^2 - p_1^2 - \Sigma_{i=2}^d q_i^2) + Z_k(p_1^2 + \Sigma_{i=2}^d q_i^2) + \mu_k\right]^{-1}\\
&+\frac{1}{l^{d-1}}\int_{\mathcal{D}} dq_1 \int_{\mathcal{D}^{\times 2d-2}} dm_2\cdots dm_d dm_2'\cdots dm_d' \varphi_{p_1 m_2\cdots m_d}\bar{\varphi}_{q_1 m_2\cdots m_d}\varphi_{q_1 m_2'\cdots m_d'}\bar{\varphi}_{p_1 m_2'\cdots m_d'}\\
&\left.\times\left[Z_k(k^2 - q_1^2 - \Sigma_{i=2}^d p_i^2)\Theta(k^2 - q_1^2 - \Sigma_{i=2}^d p_i^2) + Z_k(q_1^2 + \Sigma_{i=2}^d p_i^2) + \mu_k\right]^{-1}\right\}.
\end{aligned}
$$

With the same principle used for the evaluation of the β-functions of Z_k and μ_k, any explicit dependence on the $2d$ momenta involved in the four fields in the spectral sums of (A10) must be discarded. In other words, any term of the form $p_i^\alpha \varphi_{\cdots p_i \cdots}\bar{\varphi}_{\cdots p_i \cdots}\cdot(\varphi\bar{\varphi})$ falls out of the truncation. After taking the limit (again we drop the symbol $\lim_{l\to 0}$), we expand the expression at zeroth order and get

$$
\begin{aligned}
(II)_W|_{1,1} \simeq{}& \frac{\lambda_k^2}{l}\int_{\mathbb{R}^{2d}} dm_1 dm_1' dp_2\cdots dp_d dq_2\cdots dq_d \varphi_{m_1 p_2\cdots p_d}\bar{\varphi}_{m_1 q_2\cdots q_d}\varphi_{m_1' q_2\cdots q_d}\bar{\varphi}_{m_1' p_2\cdots p_d}\\
&\times\int_{\mathbb{R}} dp_1 \frac{[\partial_t Z_k(k^2 - p_1^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2}\frac{\theta(k^2 - p_1^2)}{Z_k(k^2 - p_1^2)\theta(k^2 - p_1^2) + Z_k p_1^2 + \mu_k}\\
&+\frac{\lambda_k^2}{l^{d-1}}\int_{\mathbb{R}^{2d}} dp_1 dq_1 dm_2\cdots dm_d dm_2'\cdots dm_d' \varphi_{p_1 m_2\cdots m_d}\bar{\varphi}_{q_1 m_2\cdots m_d}\varphi_{q_1 m_2'\cdots m_d'}\bar{\varphi}_{p_1 m_2'\cdots m_d'}\\
&\times\int_{\mathbb{R}^{d-1}} dp_2\cdots dp_d \frac{[\partial_t Z_k(k^2 - \Sigma_{i=2}^d p_i^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2}\frac{\theta(k^2 - \Sigma_{i=2}^d p_i^2)}{Z_k(k^2 - \Sigma_{i=2}^d p_i^2)\theta(k^2 - \Sigma_{i=2}^d p_i^2) + Z_k(\Sigma_{i=2}^d p_i^2) + \mu_k}.
\end{aligned}
$$

The θ's turn out to be redundant in both the terms, and we can simplify their contributions. Call $\mathcal{V}_i$ the vertex of color i of the effective interaction. Rather than using the explicit form of that vertex, we will simply use $\mathcal{V}_i$ in the following, when no confusion might arise.

We split the previous terms into two pieces,

$$
\begin{aligned}
&(II)'_W|_{1,1} \\
&= \frac{1}{l}\frac{\lambda_k^2 k^2(2+\partial_t)Z_k}{(Z_k k^2+\mu_k)^3}\int dq_2\cdots dq_d dp_2\cdots dp_d dm_1 dm'_1 \varphi_{m_1 p_2\cdots p_d}\bar{\varphi}_{m_1 q_2\cdots q_d}\varphi_{m'_1 q_2\cdots q_d}\bar{\varphi}_{m'_1 p_2\cdots p_d}\int dp_1\theta(k^2-p_1^2) \\
&\quad -\frac{1}{l}\frac{\lambda_k^2\partial_t Z_k}{(Z_k k^2+\mu_k)^3}\int dq_2\cdots dq_d dp_2\cdots dp_d dm_1 dm'_1 \varphi_{m_1 p_2\cdots p_d}\bar{\varphi}_{m_1 q_2\cdots q_d}\varphi_{m'_1 q_2\cdots q_d}\bar{\varphi}_{m'_1 p_2\cdots p_d}\int dp_1 p_1^2\theta(k^2-p_1^2) \\
&= 2\frac{\lambda_k^2 k^3}{l}\left[\frac{(2+\partial_t)Z_k}{(Z_k k^2+\mu_k)^3}-\frac{1}{3}\frac{\partial_t Z_k}{(Z_k k^2+\mu_k)^3}\right]\mathcal{V}_1 = \frac{2\lambda_k^2 k^3}{l(Z_k k^2+\mu_k)^3}\left[2Z_k+\frac{2}{3}\partial_t Z_k\right]\mathcal{V}_1.
\end{aligned}
$$

The second integral can be computed as

$$
\begin{aligned}
(II)''_W|_{1,1} &= \frac{1}{l^{d-1}}\frac{\lambda_k^2 k^2(2+\partial_t)Z_k}{(Z_k k^2+\mu_k)^3}\int dp_1 dq_1 dm_2\cdots dm_d dm'_2\cdots dm'_d \varphi_{p_1 m_2\cdots m_d}\bar{\varphi}_{q_1 m_2\cdots m_d}\varphi_{q_1 m'_2\cdots m'_d}\bar{\varphi}_{p_1 m'_2\cdots m'_d} \\
&\quad\times\int dp_2\cdots dp_d\theta(k^2-\Sigma_{i=2}^d p_i^2) \\
&\quad -\frac{1}{l^{d-1}}\frac{\lambda_k^2\partial_t Z_k}{(Z_k k^2+\mu_k)^3}\int dp_1 dq_1 dm_2\cdots dm_d dm'_2\cdots dm'_d \varphi_{p_1 m_2\cdots m_d}\bar{\varphi}_{q_1 m_2\cdots m_d}\varphi_{q_1 m'_2\cdots m'_d}\bar{\varphi}_{p_1 m'_2\cdots m'_d} \\
&\quad\times\int dp_2\cdots dp_d(\Sigma_{i=2}^d p_i^2)\theta(k^2-\Sigma_{i=2}^d p_i^2) \\
&= \int dp_1 dq_1 dm_2\cdots dm_d dm'_2\cdots dm'_d \varphi_{p_1 m_2\cdots m_d}\bar{\varphi}_{q_1 m_2\cdots m_d}\varphi_{q_1 m'_2\cdots m'_d}\bar{\varphi}_{p_1 m'_2\cdots m'_d} \\
&\quad\times\left[\frac{1}{l^{d-1}}\frac{\lambda_k^2 k^2(2+\partial_t)Z_k}{(Z_k k^2+\mu_k)^3}\int d\Omega_{d-1}\int_0^k dr r^{d-2}-\frac{1}{l^{d-1}}\frac{\lambda_k^2\partial_t Z_k}{(Z_k k^2+\mu_k)^3}\int d\Omega_{d-1}\int_0^k dr r^d\right] \\
&= \frac{\lambda_k^2}{l^{d-1}(Z_k k^2+\mu_k)^3}\left[\frac{2k^{d+1}(2+\partial_t)Z_k\pi^{\frac{d-1}{2}}}{(d-1)\Gamma_E(\frac{d-1}{2})}-\frac{2\pi^{\frac{d-1}{2}}k^{d+1}\partial_t Z_k}{(d+1)\Gamma_E(\frac{d-1}{2})}\right]\mathcal{V}_1 \\
&= \frac{\lambda_k^2 k^{d+1}\pi^{\frac{d-1}{2}}}{l^{d-1}(Z_k k^2+\mu_k)^3}\left[\frac{\partial_t Z_k}{\Gamma_E(\frac{d+3}{2})}+\frac{2Z_k}{\Gamma_E(\frac{d+1}{2})}\right]\mathcal{V}_1.
\end{aligned}
$$

A simple check of the dimensions of these terms and the dimension of the interaction term of the effective action can be given as

$$
[(II)'_W] = [(II)''_W] = 2[\lambda]-4+2d+4[\varphi],
$$

which, considering that $[\varphi] = -\frac{d+2}{2}$, fixes $[\lambda] = 4$ as expected.

Let us now focus on the cross term given by the product of the contribution of color 1 and 2,

$$(II)_W|_{1,2} = \lambda_k^2 \int_{D^{*\times 2}} d\mathbf{p}d\mathbf{j} \frac{\Theta(k^2 - \sum_s p_s^2)}{(Z_k k^2 + \mu_k)^2} \frac{\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k}{\Theta(k^2 - \sum_s j_s^2) Z_k(k^2 - \sum_s j_s^2) + Z_k \sum_s j_s^2 + \mu_k}$$
$$\times \Bigg[\int_{\mathcal{D}^{\times 2}} dm_1 dn_2 \varphi_{m_1 j_2 \cdots j_d} \bar{\varphi}_{m_1 p_2 \cdots p_d} \varphi_{p_1 n_2 p_3 \cdots p_d} \bar{\varphi}_{j_1 n_2 j_3 \cdots j_d} \delta(p_1 - j_1)\delta(p_2 - j_2)$$
$$+ \int_{\mathcal{D}^{2d-2}} dm_2 \cdots dm_d dn_1 dn_3 \cdots dn_d \varphi_{j_1 m_2 \cdots m_d} \bar{\varphi}_{p_1 m_2 \cdots m_d} \varphi_{n_1 p_2 n_3 \cdots n_d} \bar{\varphi}_{n_1 j_2 n_3 \cdots n_d}$$
$$\times \delta(p_1 - j_1)\delta(p_2 - j_2) \prod_{i=3}^{d} \delta^2(p_i - j_i)$$
$$+ \int_{D^*} dm_1 dn_1 dn_3 \cdots dn_d \varphi_{m_1 j_2 \cdots j_d} \bar{\varphi}_{m_1 p_2 \cdots p_d} \varphi_{n_1 p_2 n_3 \cdots n_d} \bar{\varphi}_{n_1 j_2 n_3 \cdots n_d} \delta^2(p_1 - j_1) \prod_{i=3}^{d} \delta(p_i - j_i)$$
$$+ \int_{D^*} dm_2 \cdots dm_d dn_2 \varphi_{j_1 m_2 \cdots m_d} \bar{\varphi}_{p_1 m_2 \cdots m_d} \varphi_{p_1 n_2 p_3 \cdots p_d} \bar{\varphi}_{j_1 n_2 j_3 \cdots j_d} \delta^2(p_2 - j_2) \prod_{i=3}^{d} \delta(p_i - j_i) \Bigg].$$

If we integrate the deltas over the j variables, the second term is again a disconnected 4-point function that we neglect. In rank $d > 3$, the first term falls out of the truncation: it generates a "matrixlike" convolution with two momenta distinguished from the other $d - 2$ labels. However, at the boundary value $d = 3$, it will contribute to the flow. We find

$$(II)_W|_{1,2} = \delta_{d,3} \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^2} \int dp_1 dp_2 dp_3 dm_1 dn_2 dj_3 \varphi_{m_1 p_2 j_3} \bar{\varphi}_{m_1 p_2 p_3} \varphi_{p_1 n_2 p_3} \bar{\varphi}_{p_1 n_2 j_3}$$
$$\times \frac{\Theta(k^2 - \sum_s p_s^2)[\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k]}{\Theta(k^2 - p_1^2 - p_2^2 - j_3^2) Z_k(k^2 - p_1^2 - p_2^2 - j_3^2) + Z_k(p_1^2 + p_2^2 + j_3^2) + \mu_k}$$
$$+ \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^2} \frac{1}{l} \int_{D^* \times \mathcal{D}} dp_1 \cdots dp_d dj_2 dm_1 dn_1 dn_3 \cdots dn_d \varphi_{m_1 j_2 p_3 \cdots p_d} \bar{\varphi}_{m_1 p_2 \cdots p_d} \varphi_{n_1 p_2 n_3 \cdots n_d} \bar{\varphi}_{n_1 j_2 n_3 \cdots n_d}$$
$$\times \frac{\Theta(k^2 - \sum_s p_s^2)[\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k]}{\Theta(k^2 - p_1^2 - j_2^2 - \Sigma_{i=3}^d p_i^2) Z_k(k^2 - p_1^2 - j_2^2 - \Sigma_{i=3}^d p_i^2) + Z_k(p_1^2 + j_2^2 + \Sigma_{i=3}^d p_i^2) + \mu_k}$$
$$+ \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^2} \frac{1}{l} \int_{D^* \times \mathcal{D}} dp_1 \cdots dp_d dj_1 dm_2 \cdots dm_d dn_2 \varphi_{j_1 m_2 \cdots m_d} \bar{\varphi}_{p_1 m_2 \cdots m_d} \varphi_{p_1 n_2 p_3 \cdots p_d} \bar{\varphi}_{j_1 n_2 p_3 \cdots p_d}$$
$$\times \frac{\Theta(k^2 - \sum_s p_s^2)[\partial_t Z_k(k^2 - \sum_s p_s^2) + 2k^2 Z_k]}{\Theta(k^2 - j_1^2 - \Sigma_{i=2}^d p_i^2) Z_k(k^2 - j_1^2 - \Sigma_{i=2}^d p_i^2) + Z_k(j_1^2 + \Sigma_{i=2}^d p_i^2) + \mu_k}.$$

In the continuum limit, the previous integrals can be evaluated at 0-momentum truncation and the Θ in the denominator put to 1. One realizes that the first term is proportional to $\delta_{d,3}\mathcal{V}_3$, while the second and third terms are (1,2)-colored symmetric contributions and are proportional to $\mathcal{V}_2$ and $\mathcal{V}_1$, respectively. Casting away the $p_i^2 \varphi_{p_i}^4$-terms, one infers

$$(II)_W|_{1,2} \simeq \delta_{d,3} \frac{\lambda_k^2 k^2 (2 + \partial_t) Z_k}{(Z_k k^2 + \mu_k)^3} \mathcal{V}_3$$
$$+ \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \frac{1}{l} \mathcal{V}_2 \int dp_1 \theta(k^2 - p_1^2)[\partial_t Z_k(k^2 - p_1^2) + 2k^2 Z_k] + \mathrm{sym}\{1 \to 2\}. \tag{A12}$$

Performing the integrals over the external momenta,

$$(II)_W|_{1,2} = \frac{\lambda_k^2 k^2}{(Z_k k^2 + \mu_k)^3} \left\{ \delta_{d,3}(2 + \partial_t) Z_k \mathcal{V}_3 + \frac{k}{l} \left[-\frac{2}{3} \partial_t Z_k + 2(2 + \partial_t) Z_k \right] (\mathcal{V}_2 + \mathcal{V}_1) \right\}. \tag{A13}$$

We are in position to sum all contributions. Taking into account the color symmetry of the vertices, the coefficients obtained from $(II)_W|_{i,i}$ contribute once for each color i, while the terms coming from the cross terms, i.e., $(II)_W|_{i,j\neq i}$, will appear once for each couple of colors (i,j), $j\neq i$. Thus the later terms gain a factor $2(d-1)$. Especially, the term $\delta_{d,3}\mathcal{V}_3$ in (A13) and the like, at $d=3$, acquires a factor of 2. Performing these operations, the β-function for λ_k reads

$$\beta(\lambda_k) = \frac{2\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left\{\partial_t Z_k\left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d+3}{2})}\frac{k^{d+1}}{l^{d-1}} + \frac{4(2d-1)}{3}\frac{k^3}{l} + 2\delta_{d,3}k^2\right]\right.$$
$$\left. + 2Z_k\left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d+1}{2})}\frac{k^{d+1}}{l^{d-1}} + 2(2d-1)\frac{k^3}{l} + 2\delta_{d,3}k^2\right]\right\}. \tag{A14}$$

Dimensionful β-functions.— We write the full set of dimensionful β-functions for the model as

$$\begin{cases} \beta(Z_k) = \frac{\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[2(d-1)\frac{k}{l} - \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d+1}{2})}\frac{k^{d-1}}{l^{d-1}}\right] + 2Z_k\left[(d-1)\frac{k}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d-1}{2})}\frac{k^{d-1}}{l^{d-1}}\right]\right\} \\ \beta(\mu_k) = -\frac{d\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[\frac{4}{3}\frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d+3}{2})}\frac{k^{d+1}}{l^{d+1}}\right] + 2Z_k\left[2\frac{k^3}{l} + \frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d+1}{2})}\frac{k^{d+1}}{l^{d-1}}\right]\right\} \\ \beta(\lambda_k) = \frac{2\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left\{\partial_t Z_k\left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d+3}{2})}\frac{k^{d+1}}{l^{d-1}} + \frac{4(2d-1)}{3}\frac{k^3}{l} + 2\delta_{d,3}k^2\right]\right. \\ \quad\left. + 2Z_k\left[\frac{\pi^{\frac{d-1}{2}}}{\Gamma_E(\frac{d+1}{2})}\frac{k^{d+1}}{l^{d-1}} + 2(2d-1)\frac{k^3}{l} + 2\delta_{d,3}k^2\right]\right\} \end{cases}, \tag{A15}$$

which is reported in Sec. III E, Eq. (49).

APPENDIX B: EVALUATION OF β-FUNCTIONS IN THE GAUGE INVARIANT CASE

The computation of the dimensionful β-functions for the gauge projected model follows roughly the same steps of the calculations of the model without constraints. However, because of the presence of the extra deltas of the gauge projection, the analysis requires, at some point, a different technique. In this appendix, we provide details of the procedure for obtaining the system of the dimensionful RG equations, namely, Eq. (98) of Sec. IV C, and underline the differences with the previous calculus.

We start by expanding Eq. (97) of Sec. IV B and focus, first on the φ^2-terms and then calculate higher order terms.

1. φ^2-terms

Referring to the conventions introduced at the beginning of Sec. IV B, say (93)–(96), for the scaling of the kinetic term, we have

$$\begin{aligned}(I^g)_W &= -\mathrm{Tr}[\partial_t R_k\cdot(P_k)^{-1}\cdot F_k\cdot(P_k)^{-1}] \\ &= -\lambda_k\int_{D^*} d\mathbf{p}\Theta(k^2-\Sigma_s p_s^2)\frac{[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{(Z_k k^2+\mu_k)^2}\frac{\delta(\Sigma p)}{\delta^2(\Sigma p)} \\ &\quad\times\left[\frac{1}{l^{d-1}}\int_{\mathcal{D}^{\times d-1}} dm_2\cdots dm_d|\varphi_{p_1 m_2\cdots m_d}|^2\delta^2(\Sigma p)\delta^2(p_1+m_2+\cdots+m_d)\right. \\ &\quad\left.+\frac{1}{l}\int_{\mathcal{D}} dm_1|\varphi_{m_1 p_2\cdots p_d}|^2\delta^2(\Sigma p)\delta^2(m_1+p_2+\cdots+p_d)+\mathrm{sym}\{1,2,\ldots,d\}\right].\end{aligned} \tag{B1}$$

In the same perspective, the square deltas can be reduced as $\delta^2(p)=\delta(p)\delta(0)=\frac{1}{l}\delta(p)$. The second integral in the above expression can be directly computed by integrating over p_1 the $\delta(\sum p)$ as

$$
\begin{aligned}
(I^g)'_W = & -\frac{\lambda_k}{l^2}\int_{D^*} d\mathbf{p}\Theta(k^2-\Sigma_s p_s^2)\frac{[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{(Z_k k^2+\mu_k)^2}\delta(\Sigma p) \\
& \times \int_{\mathcal{D}} dm_1 |\varphi_{m_1 p_2\cdots p_d}|^2 \delta(m_1+p_2+\cdots+p_d) + \mathrm{sym}\{1,2,\ldots,d\} \\
= & -\frac{\lambda_k}{l^2(Z_k k^2+\mu_k)^2}\int_{D^*} dm_1 dp_2\cdots dp_d |\varphi_{m_1 p_2\cdots p_d}|^2 \delta(m_1+p_2+\cdots+p_d) \\
& \times \int_{\mathcal{D}} dp_1 \Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]\delta(\Sigma p) + \mathrm{sym}\{1,2,\ldots,d\} \\
= & -\frac{\lambda_k}{l^2(Z_k k^2+\mu_k)^2}\int_{D^*} d\mathbf{p}|\varphi_{p_1 p_2\cdots p_d}|^2\delta(p_1+p_2+\cdots+p_d)[dk^2(2+\partial_t)Z_k - d\partial_t Z_k \Sigma_{s=1}^d p_s^2] \\
= & -\frac{d\lambda_k}{l^2(Z_k k^2+\mu_k)^2}\int_{D^*} d\mathbf{p}|\varphi_{p_1 p_2\cdots p_d}|^2\delta(p_1+p_2+\cdots+p_d)[2k^2 Z_k + \partial_t Z_k[k^2-\Sigma_{s=1}^d p_s^2]]. \qquad \text{(B2)}
\end{aligned}
$$

We discuss now the first term in the brackets in (B1) that we denote

$$
\begin{aligned}
(I^g)''_W = & -\frac{\lambda_k}{l^d(Z_k k^2+\mu_k)^2}\int_{D^*} dp_1 dm_2\cdots dm_d |\varphi_{p_1 m_2\cdots m_d}|^2\delta(p_1+m_2+\cdots+m_d) \\
& \times \int_{\mathcal{D}^{\times d-1}} dp_2\cdots dp_d \Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]\delta\left(\sum p\right) + \mathrm{sym}\{1,2,\ldots,d\}. \qquad \text{(B3)}
\end{aligned}
$$

Because of the combinatorial pattern chosen for the interaction, the case $d=3$ represents again a special situation that we deal with by direct evaluation. We integrate over the third variable, imposing the constraint $p_3 = -(p_1+p_2)$. The resulting domain of integration of p_2 is known, in the continuous limit, as the θ distribution and is nonzero when $-2p_2^2 - 2p_2 p_1 + (k^2 - 2p_1^2) \geq 0$. The boundary of this inequality, solved in p_2, is given by the roots

$$
p_2^{\pm} = \frac{1}{2}\left(-p_1 \pm \sqrt{2k^2 - 3p_1^2}\right). \qquad \text{(B4)}
$$

The nonzero values of the Heaviside distribution hold when $p_2 \in [p_2^-, p_2^+]$. There is still a residual constraint over p_1 which has to be imposed in order to keep real the square root appearing in (B4), that is, $3p_1^2 \leq 2k^2$. Thus, Eq. (B3) becomes

$$
\begin{aligned}
(I^g)''_{W;d=3} = & -\frac{\lambda_k}{l^3(Z_k k^2+\mu_k)^2}\int dp_1 dm_2 dm_3 |\varphi_{p_1 m_2 m_3}|^2\delta(p_1+m_2+m_3) \\
& \times \theta(2k^2-3p_1^2)\int_{\frac{1}{2}(-p_1-\sqrt{2k^2-3p_1^2})}^{\frac{1}{2}(-p_1+\sqrt{2k^2-3p_1^2})} dp_2\{\partial_t Z_k[k^2-2(p_2^2+p_1^2+p_2 p_1)]+2k^2 Z_k\} \\
& + \mathrm{sym}\{1,2,\ldots,d\} \\
= & -\frac{\lambda_k}{l^3(Z_k k^2+\mu_k)^2}\int dp_1 dm_2 dm_3 |\varphi_{p_1 m_2 m_3}|^2\delta(p_1+m_2+m_3)\theta(2k^2-3p_1^2) \\
& \times \left\{k^2\sqrt{2k^2-3p_1^2}(2+\partial_t)Z_k - \frac{3}{2}\sqrt{2k^2-3p_1^2}\partial_t Z_k p_1^2 - \frac{1}{6}(2k^2-3p_1^2)^{3/2}\partial_t Z_k\right\} + \mathrm{sym}\{1,2,\ldots,d\}. \qquad \text{(B5)}
\end{aligned}
$$

Expanding the last result up to the third order in momenta, one obtains

$$
\begin{aligned}
(I^g)''_{W;d=3} \simeq & -\frac{\lambda_k}{l^3(Z_k k^2+\mu_k)^2}\int dp_1 dm_2 dm_3 |\varphi_{p_1 m_2 m_3}|^2\delta(p_1+m_2+m_3) \\
& \times \left[k^3\left(\sqrt{2}-\frac{\sqrt{8}}{6}\right)\partial_t Z_k + 2\sqrt{2}k^3 Z_k - \frac{3}{\sqrt{2}}k(1+\partial_t)Z_k p_1^2\right] + \mathrm{sym}\{1,2,\ldots,d\} \\
\simeq & -\frac{\lambda_k}{l^3(Z_k k^2+\mu_k)^2}\int_{D^*} d\mathbf{p}|\varphi_{p_1 p_2 p_3}|^2\delta(p_1+p_2+p_3)\left[2\sqrt{2}dk^3\left(\frac{1}{3}\partial_t+1\right)Z_k - \frac{3}{\sqrt{2}}k(1+\partial_t)Z_k\left(\sum_{s=1}^d p_s^2\right)\right], \qquad \text{(B6)}
\end{aligned}
$$

where in the last line we include the symmetry factors. From this point, and combining it with (B2) restricted at $d = 3$, we write the β-functions for the couplings μ_k and Z_k as

$$\beta_{d=3}(Z_k) = \frac{\lambda_k}{(Z_k k^2 + \mu_k)^2}\left[\frac{3}{\sqrt{2}}\frac{k}{l^3}(1+\partial_t)Z_k + \frac{3}{l^2}\partial_t Z_k\right];$$
$$\beta_{d=3}(\mu_k) = -\frac{3\lambda_k}{(Z_k k^2 + \mu_k)^2}\left[2\sqrt{2}\frac{k^3}{l^3}\left(1+\frac{1}{3}\partial_t\right)Z_k + \frac{k^2}{l^2}(2+\partial_t)Z_k\right]. \tag{B7}$$

At rank $d \geq 4$, the term (B3) has more integrations to perform and becomes simpler if expressed in spherical coordinates. Considering that the coordinate p_1 is convoluted with the field, we will change basis from $(p_2, \ldots, p_d)$ to (r, Ω_{d-1}). The $\delta(\sum p)$ defines the hyperplane orthogonal to a vector $\mathbf{N}$ of norm $\|\mathbf{N}\| = \sqrt{d}$ and components (in Cartesian coordinates) $\mathbf{N} = (1, 1, \ldots, 1)$. We will call $\mathbf{n}$ the projection of this vector on the subspace orthogonal to p_1 and $\mathbf{P}$ the generic vector on this subspace. In this setting the Dirac delta function becomes

$$\delta(p_1 + \langle \mathbf{P}, \mathbf{n}\rangle) = \delta(p_1 + r\sqrt{d-1}\cos\vartheta) = \frac{\delta(\frac{p_1}{r\sqrt{d-1}} + \cos\vartheta)}{r\sqrt{d-1}}, \tag{B8}$$

where ϑ represents the angle between $\mathbf{P}$ and $\mathbf{n}$. Considering that the scalar product, as the rest of the integrand, is rotational invariant on the $(d-1)$-dimensional space, we can set ϑ to be one of the angles appearing in the spherical measure. After the change of coordinates, Eq. (B3) reads

$$(I^g)''_{W;d>3} = -\frac{\lambda_k}{l^d(Z_k k^2 + \mu_k)^2}\int dp_1 dm_2 \cdots dm_d \int dr d\Omega_{d-2}\int_0^\pi d\vartheta r^{d-2}\sin^{d-3}\vartheta \frac{\delta(\frac{p_1}{r\sqrt{d-1}} + \cos\vartheta)}{r\sqrt{d-1}}$$
$$\times |\varphi_{p_1 m_2 \cdots m_d}|^2 \delta(p_1 + m_2 + \cdots + m_d)\theta(k^2 - p_1^2 - r^2)[\partial_t Z_k(k^2 - p_1^2 - r^2) + 2k^2 Z_k]$$
$$+ \mathrm{sym}\{1, 2, \ldots, d\}. \tag{B9}$$

We focus on the integral over ϑ and change the variable from ϑ to $X = \cos\vartheta$ and get, for $d > 3$,

$$\int_0^\pi d\vartheta \sin^{d-3}\vartheta\, \delta\left(\frac{p_1}{r\sqrt{d-1}} + \cos\vartheta\right) = \int_{-1}^1 dX(1-X^2)^{\frac{d-4}{2}}\delta\left(\frac{p_1}{r\sqrt{d-1}} + X\right) = \left[1 - \frac{p_1^2}{r^2(d-1)}\right]^{\frac{d-4}{2}}. \tag{B10}$$

Substituting (B10) in (B9), we get

$$(I^g)''_{W;d>3} = -\frac{\lambda_k}{l^d(Z_k k^2 + \mu_k)^2}\frac{\Omega_{d-2}}{\sqrt{d-1}}\int dp_1 dm_2 \cdots dm_d |\varphi_{p_1 m_2 \cdots m_d}|^2 \delta(p_1 + m_2 + \cdots + m_d)$$
$$\times \theta(k^2 - p_1^2)\left[\theta(d-5)\int_0^{\sqrt{k^2 - p_1^2}} dr r^{d-3}\left[1 - \frac{p_1^2}{(d-1)r^2}\right]^{\frac{d}{2}-2}[\partial_t Z_k(k^2 - p_1^2 - r^2) + 2k^2 Z_k]\right.$$
$$\left. + \delta_{d,4}\int_0^{\sqrt{k^2 - p_1^2}} dr r[\partial_t Z_k(k^2 - p_1^2 - r^2) + 2k^2 Z_k]\right] + \mathrm{sym}\{1, 2, \ldots, d\}. \tag{B11}$$

Expanding the result of the integral over ϑ at the second order in p_1, we obtain an integral over r of the form

$$(I^g)''_{W;d>3} \simeq -\frac{\lambda_k}{l^d(Z_k k^2 + \mu_k)^2}\frac{\Omega_{d-2}}{\sqrt{d-1}}\int dp_1 dm_2 \cdots dm_d |\varphi_{p_1 m_2 \cdots m_d}|^2 \delta(p_1 + m_2 + \cdots + m_d)$$
$$\times \theta(k^2 - p_1^2)\left[\theta(d-5)\int_0^{\sqrt{k^2 - p_1^2}} dr r^{d-3}\left[1 - \frac{d-4}{2(d-1)r^2}p_1^2\right][\partial_t Z_k(k^2 - p_1^2 - r^2) + 2k^2 Z_k]\right.$$
$$\left. + \delta_{d,4}\int_0^{\sqrt{k^2 - p_1^2}} dr r[\partial_t Z_k(k^2 - p_1^2 - r^2) + 2k^2 Z_k]\right] + \mathrm{sym}\{1, 2, \ldots, d\}. \tag{B12}$$

Computing the last integral and expanding the result, we expand the RHS of (B12) to the second order in the momenta convoluted with the fields, and this yields

$$(I^g)''_{W;d>3} \simeq -\frac{\lambda_k}{l^d(Z_kk^2+\mu_k)^2}\frac{\Omega_{d-2}}{\sqrt{d-1}}\int dp_1dm_2\cdots dm_d|\varphi_{p_1m_2\cdots m_d}|^2\delta(p_1+m_2+\cdots+m_d)$$
$$\times\theta(k^2-p_1^2)\left\{\theta(d-5)\left[\frac{2k^d}{d-2}Z_k+\frac{2k^d}{d(d-2)}\partial_tZ_k-p_1^2k^{d-2}\left[\frac{d}{(d-1)}Z_k+\frac{d\partial_tZ_k}{(d-1)(d-2)}\right]\right]\right.$$
$$\left.+\delta_{d,4}\left[\frac{1}{2}\left[\frac{1}{2}\partial_tZ_k+2Z_k\right]k^4-p_1^2k^2\left[\frac{1}{2}\partial_tZ_k+Z_k\right]\right]\right\}+\mathrm{sym}\{1,2,\ldots,d\}. \tag{B13}$$

We sum (B2) and (B13) and write at rank $d=4$,

$$(I^g)_{W;d=4}\simeq-\frac{\lambda_k}{(Z_kk^2+\mu_k)^2}\int d\mathbf{p}|\varphi_{p_1p_2\cdots p_4}|^2\delta\left(\sum_s p_s\right)$$
$$\times\left\{\frac{2\pi}{l^4\sqrt{3}}\left[2\left[\frac{1}{2}\partial_tZ_k+2Z_k\right]k^4-k^2\left[\frac{1}{2}\partial_tZ_k+Z_k\right]\left(\sum_s p_s^2\right)\right]+\frac{4}{l^2}\left[k^2(2+\partial_t)Z_k-\partial_tZ_k\left(\sum_{s=1}^d p_s^2\right)\right]\right\}, \tag{B14}$$

and at rank $d>4$, summing (B2) and (B13) gives

$$(I^g)_W\simeq\frac{\lambda_k}{(Z_kk^2+\mu_k)^2}\int_{D^*}d\mathbf{p}|\varphi_{p_1p_2\ldots p_d}|^2\delta\left(\sum_s p_s\right)$$
$$\times\left\{-\frac{1}{l^d}\frac{\Omega_{d-2}}{\sqrt{d-1}}\left\{dk^d\left[\frac{(2+\partial_t)Z_k}{d-2}-\frac{\partial_tZ_k}{d}\right]-k^{d-2}\left[\frac{d\partial_tZ_k}{(d-1)(d-2)}+\frac{dZ_k}{d-1}\right]\left(\sum_s p_s^2\right)\right\}\right.$$
$$\left.-\frac{d}{l^2}\left[k^2(2+\partial_t)Z_k-\partial_tZ_k\left(\sum_{s=1}^d p_s^2\right)\right]\right\}. \tag{B15}$$

Hence, we write the β-functions for the couplings μ_k and Z_k at rank $d=4$ as

$$\beta_{d=4}(Z_k)=\frac{\lambda_k}{(Z_kk^2+\mu_k)^2}\left\{\frac{2\pi}{\sqrt{3}}\left[\frac{1}{2}\partial_tZ_k+Z_k\right]\frac{k^2}{l^4}+\partial_tZ_k\frac{4}{l^2}\right\};$$
$$\beta_{d=4}(\mu_k)=-\frac{4\lambda_k}{(Z_kk^2+\mu_k)^2}\left\{\frac{\pi}{\sqrt{3}}\left[\frac{1}{2}\partial_tZ_k+2Z_k\right]\frac{k^4}{l^4}+(2+\partial_t)Z_k\frac{k^2}{l^2}\right\}, \tag{B16}$$

and for $d>4$, as

$$\beta_{d>4}(Z_k)=\frac{d\lambda_k}{(Z_kk^2+\mu_k)^2}\left\{\partial_tZ_k\left[\frac{\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}}\Gamma_E(\frac{d}{2})}\frac{k^{d-2}}{l^d}+\frac{1}{l^2}\right]+\frac{2\pi^{\frac{d-2}{2}}Z_k}{(d-1)^{\frac{3}{2}}\Gamma_E(\frac{d-2}{2})}\frac{k^{d-2}}{l^d}\right\};$$
$$\beta_{d>4}(\mu_k)=-\frac{d\lambda_k}{(Z_kk^2+\mu_k)^2}\left\{\partial_tZ_k\left[\frac{k^d}{l^d}\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma_E(\frac{d+2}{2})}+\frac{k^2}{l^2}\right]+2Z_k\left[\frac{k^d}{l^d}\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma_E(\frac{d}{2})}+\frac{k^2}{l^2}\right]\right\}. \tag{B17}$$

We note that setting $d=3$ in (B17), we recover (B7). We can therefore extend the last formulas to $d=3$ and will denote them $\beta_{d\neq4}(Z_k)$ and $\beta_{d\neq4}(\mu_k)$. The case $d=4$ must be distinguished from the rest of the ranks, because we observe that $\beta_{d=4}(Z_k)$ is not the evaluation of $\beta_{d\neq4}(Z_k)$ at $d=4$. Note that the mass equation can, however, be recovered from $\beta_{d\neq4}(\mu_k)$ at $d=4$.

2. φ^4-terms

The next order of the truncation made on the Wetterich equation, i.e., $(II^g)_W=\mathrm{Tr}[\partial_tR_k\cdot(P_k)^{-1}\cdot F_k\cdot(P_k)^{-1}\cdot F_k\cdot(P_k)^{-1}]$, provides the β-function for the coupling λ_k. Introducing the notation $\hat{\varphi}_{\mathbf{p}}=\varphi_{\mathbf{p}}\delta(\sum p)$ for the gauge invariant field, we write

$$(II^g)_W = \lambda_k^2 \int_{D^{*\times 2}} d\mathbf{p}d\mathbf{r} \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]\delta(\Sigma p)}{(Z_k k^2+\mu_k)^2[Z_k\Sigma_s r_s^2+\mu_k+\Theta(k^2-\Sigma_s r_s^2)Z_k(k^2-\Sigma_s r_s^2)]\delta(\Sigma r)\delta^2(\Sigma p)}$$
$$\times\left[\int_{\mathcal{D}^{d-1}} dm_2\cdots dm_d \hat{\varphi}_{r_1 m_2\cdots m_d}\hat{\bar{\varphi}}_{p_1 m_2\cdots m_d}\delta(\Sigma p)\delta(r_1+p_2+\cdots+p_d)\prod_{i=2}^d \delta(p_i-r_i)\right.$$
$$\left.+\int_{\mathcal{D}} dm_1 \hat{\varphi}_{m_1 r_2\cdots r_d}\hat{\bar{\varphi}}_{m_1 p_2\cdots p_d}\delta(\Sigma p)\delta(p_1+r_2+\cdots+r_d)\delta(p_1-r_1)+\mathrm{sym}\{1,2,\ldots,d\}\right]$$
$$\times\left[\int_{\mathcal{D}^{\times d-1}} dn_2\cdots dn_d \hat{\varphi}_{p_1 n_2\cdots n_d}\hat{\bar{\varphi}}_{r_1 n_2\cdots n_d}\delta(\Sigma r)\delta(p_1+r_2+\cdots+r_d)\prod_{i=2}^d \delta(r_i-p_i)\right.$$
$$\left.+\int_{\mathcal{D}} dn_1 \hat{\varphi}_{n_1 p_2\cdots p_d}\hat{\bar{\varphi}}_{n_1 r_2 c\ldots r_d}\delta(\Sigma r)\delta(r_1+p_2+\ldots p_d)\delta(r_1-p_1)+\mathrm{sym}\{1,2,\ldots,d\}\right], \quad \text{(B18)}$$

where the redundant Θ-functions are set to 1. The combinatorics of the present model is the same as studied in the previous appendix, and we therefore proceed in the same way by collecting different types of colored contributions. We first discuss the contribution obtained by the product of color 1–1,

$$(II^g)_W|_{1,1} = \lambda_k^2 \int_{D^{*\times 2}} d\mathbf{p}d\mathbf{r} \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{(Z_k k^2+\mu_k)^2[Z_k\Sigma_s r_s^2+\mu_k+\Theta(k^2-\Sigma_s r_s^2)Z_k(k^2-\Sigma_s r_s^2)]}$$
$$\times\left[\int_{\mathcal{D}^{\times 2}} dm_1 dn_1 \hat{\varphi}_{m_1 r_2\cdots r_d}\hat{\bar{\varphi}}_{m_1 p_2\cdots p_d}\hat{\varphi}_{n_1 p_2\cdots p_d}\hat{\bar{\varphi}}_{n_1 r_2\cdots r_d}\delta(p_1+r_2+\cdots+r_d)\delta(r_1+p_2+\cdots+p_d)\delta^2(r_1-p_1)\right.$$
$$+\int_{\mathcal{D}^{\times 2d-2}} dm_2\cdots dm_d dn_2\cdots dn_d \hat{\varphi}_{r_1 m_2\cdots m_d}\hat{\bar{\varphi}}_{p_1 m_2\cdots m_d}\hat{\varphi}_{p_1 n_2\cdots n_d}\hat{\bar{\varphi}}_{r_1 n_2\cdots n_d}$$
$$\left.\times\,\delta(r_1+p_2+\cdots+p_d)\delta(p_1+r_2+\cdots+r_d)\prod_{i=2}^d \delta^2(r_i-p_i)+\text{disconnected}\right], \quad \text{(B19)}$$

where the terms denoted by "disconnected" describe disconnected interactions which we discard. Integrating over r_i, in the delta functions which are not convoluted with the fields, and replacing the redundant δ by $1/l$, one gets

$$(II^g)_W|_{1,1} \simeq \lambda_k^2 \int_{D^{*\times 2}} dm_1 dp_2\cdots dp_d dn_1 dr_2\cdots dr_d \frac{\hat{\varphi}_{m_1 r_2\cdots r_d}\hat{\bar{\varphi}}_{m_1 p_2\cdots p_d}\hat{\varphi}_{n_1 p_2\cdots p_d}\hat{\bar{\varphi}}_{n_1 r_2\cdots r_d}}{(Z_k k^2+\mu_k)^2}$$
$$\times\frac{1}{l}\int_{\mathcal{D}} dp_1 \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{Z_k(p_1^2+\Sigma_{i=2}^d r_i^2)+\mu_k+\Theta[k^2-p_1^2-\Sigma_{i=2}^d r_i^2]Z_k[k^2-p_1^2-\Sigma_{i=2}^d r_i^2]}\delta(\Sigma p)\delta(p_1+\Sigma_{i=2}^d r_i)$$
$$+\lambda_k^2 \int_{D^{*\times 2}} dp_1 dm_2\cdots dm_d dr_1 dn_2\cdots dn_d \frac{\hat{\varphi}_{r_1 m_2\ldots m_d}\hat{\bar{\varphi}}_{p_1 m_2\cdots m_d}\hat{\varphi}_{p_1 n_2\cdots n_d}\hat{\bar{\varphi}}_{r_1 n_2\cdots n_d}}{(Z_k k^2+\mu_k)^2}$$
$$\times\frac{1}{l^{d-1}}\int_{\mathcal{D}^{\times(d-1)}} dp_2\cdots dp_d \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{Z_k(r_1^2+\Sigma_{i=2}^d p_i^2)+\mu_k+\Theta[k^2-r_1^2-\Sigma_{i=2}^d p_i^2]Z_k[k^2-r_1^2-\Sigma_{i=2}^d p_i^2]}$$
$$\times\,\delta(\Sigma p)\delta(r_1+p_2+\cdots+p_d). \quad \text{(B20)}$$

Once again, the case $d=3$ requires special care during the evaluation of the above integrals. For $d=3$, we have by direct evaluation

$$(II^g)_{W;d=3}|_{1,1} \simeq \lambda_k^2 \int_{D^{*\times 2}} dm_1 dp_2 dp_3 dn_1 dr_2 dr_3 \frac{\hat{\varphi}_{m_1 r_2 r_3}\hat{\bar{\varphi}}_{m_1 p_2 p_3}\hat{\varphi}_{n_1 p_2 p_3}\hat{\bar{\varphi}}_{n_1 r_2 r_3}}{(Z_k k^2+\mu_k)^2}$$
$$\times\frac{1}{l}\int_{\mathcal{D}} dp_1 \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{Z_k(p_1^2+\Sigma_{i=2}^3 r_i^2)+\mu_k+\Theta[k^2-p_1^2-\Sigma_{i=2}^3 r_i^2]Z_k[k^2-p_1^2-\Sigma_{i=2}^3 r_i^2]}\delta(\Sigma p)\delta(p_1+r_2+r_3)$$
$$+\lambda_k^2 \int_{D^{*\times 2}} dp_1 dm_2 dm_3 dr_1 dn_2 dn_3 \frac{\hat{\varphi}_{r_1 m_2 m_3}\hat{\bar{\varphi}}_{p_1 m_2 m_3}\hat{\varphi}_{p_1 n_2 n_3}\hat{\bar{\varphi}}_{r_1 n_2 n_3}}{(Z_k k^2+\mu_k)^2}$$
$$\times\frac{1}{l^2}\int_{\mathcal{D}^{\times 2}} dp_2 dp_3 \frac{\Theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2 Z_k]}{Z_k(r_1^2+\Sigma_{i=2}^3 p_i^2)+\mu_k+\Theta[k^2-r_1^2-\Sigma_{i=2}^3 p_i^2]Z_k[k^2-r_1^2-\Sigma_{i=2}^3 p_i^2]}\delta(\Sigma p)\delta(r_1+p_2+p_3). \quad \text{(B21)}$$

We integrate over p_1 the first term and over p_3 the second term, replace redundant deltas by appropriate factors $1/l$, and then put to 0 all momentum variables involved in the field convolutions, to get

$$\begin{aligned} (II^g)_{W;d=3}|_{1,1} &\simeq \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\frac{k^2(2+\partial_t)Z_k}{l^2}\mathcal{V}_1 + \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\frac{1}{l^3}\int_{-\sqrt{k^2/2}}^{\sqrt{k^2/2}} dr[\partial_t Z_k(k^2-2r^2)+2k^2 Z_k]\mathcal{V}_1 \\ &\simeq \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left[\frac{k^2(2+\partial_t)Z_k}{l^2}+\frac{k^3}{l^3}\left[\sqrt{2}(\partial_t+2)Z_k-\frac{\sqrt{2}}{3}\partial_t Z_k\right]\right]\mathcal{V}_1 \\ &\simeq \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left[\left[\frac{k^2}{l^2}+\frac{2\sqrt{2}}{3}\frac{k^3}{l^3}\right]\partial_t Z_k + 2\left[\frac{k^2}{l^2}+\sqrt{2}\frac{k^3}{l^3}\right]Z_k\right]\mathcal{V}_1. \end{aligned} \tag{B22}$$

At rank $d>3$, using again the spherical coordinates (R,Ω_{d-1}), and taking the continuum limit, we write

$$\begin{aligned} (II^g)_{W;d>3}|_{1,1} &\simeq \lambda_k^2\int dm_1dp_2\cdots dp_d dn_1 dr_2\cdots dr_d \frac{\hat{\varphi}_{m_1r_2\cdots r_d}\hat{\bar{\varphi}}_{m_1p_2\cdots p_d}\hat{\varphi}_{n_1p_2\cdots p_d}\hat{\bar{\varphi}}_{n_1r_2\cdots r_d}}{l(Z_k k^2+\mu_k)^2} \\ &\quad\times\int dp_1 \frac{\theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2Z_k]}{Z_k(p_1^2+\Sigma_{i=2}^d r_i^2)+\mu_k+\theta[k^2-p_1^2-\Sigma_{i=2}^d r_i^2]Z_k[k^2-p_1^2-\Sigma_{i=2}^d r_i^2]}\delta(\Sigma p)\delta(p_1+\Sigma_{i=2}^d r_i) \\ &\quad+\lambda_k^2\int dp_1dm_2\cdots dm_d dr_1dn_2\cdots dn_d \frac{\hat{\varphi}_{r_1m_2\cdots m_d}\hat{\bar{\varphi}}_{p_1m_2\cdots m_d}\hat{\varphi}_{p_1n_2\cdots n_d}\hat{\bar{\varphi}}_{r_1n_2\cdots n_d}}{l^{d-1}(Z_k k^2+\mu_k)^2} \\ &\quad\times\int dR\int d\Omega_{d-1}\frac{R^{d-2}\theta(k^2-p_1^2-R^2)[\partial_t Z_k(k^2-p_1^2-R^2)+2k^2Z_k]}{Z_k(r_1^2+R^2)+\mu_k+\theta(k^2-r_1^2-R^2)Z_k[k^2-r_1^2-R^2]} \\ &\quad\times\delta(p_1+R\sqrt{d-1}\cos\vartheta)\delta(r_1+R\sqrt{d-1}\cos\vartheta) \\ &=\lambda_k^2\int dm_1dp_2\cdots dp_d dn_1 dr_2\cdots dr_d \frac{\hat{\varphi}_{m_1r_2\cdots r_d}\hat{\bar{\varphi}}_{m_1p_2\cdots p_d}\hat{\varphi}_{n_1p_2\cdots p_d}\hat{\bar{\varphi}}_{n_1r_2\cdots r_d}}{l(Z_k k^2+\mu_k)^2} \\ &\quad\times\int dp_1 \frac{\theta(k^2-\Sigma_s p_s^2)[\partial_t Z_k(k^2-\Sigma_s p_s^2)+2k^2Z_k]}{Z_k(p_1^2+\Sigma_{i=2}^d r_i^2)+\mu_k+\theta[k^2-p_1^2-\Sigma_{i=2}^d r_i^2]Z_k[k^2-p_1^2-\Sigma_{i=2}^d r_i^2]}\delta(\Sigma p)\delta(p_1+\Sigma_{i=2}^d r_i) \\ &\quad+\lambda_k^2\int dp_1dm_2\cdots dm_d dr_1dn_2\cdots dn_d \frac{\hat{\varphi}_{r_1m_2\cdots m_d}\hat{\bar{\varphi}}_{p_1m_2\cdots m_d}\hat{\varphi}_{p_1n_2\cdots n_d}\hat{\bar{\varphi}}_{r_1n_2\cdots n_d}}{l^{d-1}(Z_k k^2+\mu_k)^2} \\ &\quad\times\int dR\int d\Omega_{d-2}\int_0^\pi \frac{R^{d-2}}{R^2(d-1)}d\vartheta\sin^{d-3}\vartheta\,\delta\left(\frac{p_1}{R\sqrt{d-1}}+\cos\vartheta\right)\delta\left(\frac{r_1}{R\sqrt{d-1}}+\cos\vartheta\right) \\ &\quad\times\frac{\theta(k^2-p_1^2-R^2)[\partial_t Z_k(k^2-p_1^2-R^2)+2k^2Z_k]}{Z_k(r_1^2+R^2)+\mu_k+\theta(k^2-r_1^2-R^2)Z_k(k^2-r_1^2-R^2)} \\ &=\lambda_k^2\int dm_1dp_2\cdots dp_d dn_1 dr_2\cdots dr_d \frac{\hat{\varphi}_{m_1r_2\cdots r_d}\hat{\bar{\varphi}}_{m_1p_2\cdots p_d}\hat{\varphi}_{n_1p_2\cdots p_d}\hat{\bar{\varphi}}_{n_1r_2\cdots r_d}}{l(Z_k k^2+\mu_k)^2} \\ &\quad\times\frac{\theta[k^2-2(\Sigma_{i=2}^d p_i^2+\Sigma_{1<i<j}p_ip_j)]\{\partial_t Z_k[k^2-2(\Sigma_{i=2}^d p_i^2+\Sigma_{1<i<j}p_ip_j)]+2k^2Z_k\}\delta(\Sigma_{i=2}^d(r_i-p_i))}{Z_k[\Sigma_{i=2}^d r_i^2+(\Sigma_{i=2}^d p_i)^2]+\mu_k+\theta[k^2-\Sigma_{i=2}^d r_i^2-(\Sigma_{i=2}^d p_i)^2]Z_k[k^2-\Sigma_{i=2}^d r_i^2-(\Sigma_{i=2}^d p_i)^2]} \\ &\quad+\lambda_k^2\Omega_{d-2}\int dp_1dm_2\cdots dm_d dr_1dn_2\cdots dn_d \frac{\hat{\varphi}_{r_1m_2\cdots m_d}\hat{\bar{\varphi}}_{p_1m_2\cdots m_d}\hat{\varphi}_{p_1n_2\cdots n_d}\hat{\bar{\varphi}}_{r_1n_2\cdots n_d}}{l^{d-1}(Z_k k^2+\mu_k)^2} \\ &\quad\times\int dR\left[1-\frac{r_1^2}{R^2(d-1)}\right]^{\frac{d-4}{2}}\frac{R^{d-3}}{\sqrt{d-1}}\delta(p_1-r_1)\frac{\theta(k^2-p_1^2-R^2)[\partial_t Z_k(k^2-p_1^2-R^2)+2k^2Z_k]}{Z_k(r_1^2+R^2)+\mu_k+\theta(k^2-r_1^2-R^2)Z_k(k^2-r_1^2-R^2)}. \end{aligned} \tag{B23}$$

Considering that all interaction terms which explicitly depend on the momenta involved in their fields fall out of our truncation, considering also that the deltas $\delta(p_1-r_1)$ and $\delta(\Sigma_{i=2}^d(r_i-p_i))$ turn out to be redundant with the gauge invariance conditions, we can then set to zero the labels p_i and r_i appearing in the integrals, coefficients of the gauge projected fields, and get

$$
\begin{aligned}
(II^g)_{W;d>3}|_{1,1} \simeq{}& \frac{\lambda_k^2 k^2 (2+\partial_t) Z_k}{l^2 (Z_k k^2 + \mu_k)^3} \int_{D^{*\times 2}} dm_1 dp_2 \cdots dp_d dn_1 dr_2 \cdots dr_d \hat{\varphi}_{m_1 r_2 \cdots r_d} \hat{\bar{\varphi}}_{m_1 p_2 \cdots p_d} \hat{\varphi}_{n_1 p_2 \cdots p_d} \hat{\bar{\varphi}}_{n_1 r_2 \cdots r_d} \\
&+ \frac{2\pi^{\frac{d-2}{2}} \lambda_k^2}{(Z_k k^2 + \mu_k)^3 \Gamma_E(\frac{d-2}{2}) \sqrt{d-1}} \frac{k^d}{l^d} \left[\frac{(2+\partial_t) Z_k}{d-2} - \frac{\partial_t Z_k}{d} \right] \\
&\times \int_{D^{*\times 2}} dp_1 dm_2 \cdots dm_d dr_1 dn_2 \cdots dn_d \hat{\varphi}_{r_1 m_2 \cdots m_d} \hat{\bar{\varphi}}_{p_1 m_2 \cdots m_d} \hat{\varphi}_{p_1 n_2 \cdots n_d} \hat{\bar{\varphi}}_{r_1 n_2 \cdots n_d} \\
={}& \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^3} \left\{ (2+\partial_t) Z_k \left[\frac{\pi^{\frac{d-2}{2}}}{\Gamma_E(\frac{d}{2}) \sqrt{d-1}} \frac{k^d}{l^d} + \frac{k^2}{l^2} \right] - \frac{2\pi^{\frac{d-2}{2}} \partial_t Z_k}{d\sqrt{d-1} \Gamma_E(\frac{d-2}{2})} \frac{k^d}{l^d} \right\} \mathcal{V}_1,
\end{aligned}
\tag{B24}
$$

where we used for the colored vertex the same notation introduced in Sec. A 2. We note that setting $d = 3$ in the last result leads us to (B22). Then, we prolong $(II^g)_W$ to $d \geq 3$.

Inspecting the 2-color cross terms, we focus on the product of terms 1–2. Discarding the disconnected interactions and the terms which fall out of the chosen truncation, while paying special care to the case $d = 3$, one has

$$
\begin{aligned}
(II^g)_W|_{1,2} \simeq{}& \lambda_k^2 \int_{D^{*\times 2}} d\mathbf{p} d\mathbf{r} \frac{\Theta(k^2 - \Sigma_s p_s^2)[\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{(Z_k k^2 + \mu_k)^2 [Z_k \Sigma_s r_s^2 + \mu_k + \Theta(k^2 - \Sigma_s r_s^2) Z_k (k^2 - \Sigma_s r_s^2)]} \\
&\times \left[\delta_{d,3} \int_{D^2} dm_1 dn_2 \hat{\varphi}_{m_1 r_2 r_3} \hat{\bar{\varphi}}_{m_1 p_2 p_3} \hat{\varphi}_{p_1 n_2 p_3} \hat{\bar{\varphi}}_{r_1 n_2 r_3} \delta(p_1 + r_2 + p_3) \delta(p_1 + r_2 + r_3) \delta(r_1 - p_1) \delta(p_2 - r_2) \right. \\
&+ \int_{D^*} dm_1 dn_1 dn_3 \cdots dn_d \hat{\varphi}_{m_1 r_2 \cdots r_d} \hat{\bar{\varphi}}_{m_1 p_2 \cdots p_d} \hat{\varphi}_{n_1 p_2 n_3 \cdots n_d} \hat{\bar{\varphi}}_{n_1 r_2 n_3 \cdots n_d} \\
&\times \delta(p_1 + r_2 + \cdots + r_d) \delta(r_1 + p_2 + r_3 + \cdots + r_d) \delta^2(r_1 - p_1) \prod_{i=3}^{d} \delta(p_i - r_i) \\
&+ \int_{\mathcal{D}^*} dn_2 dm_2 \cdots dm_d \hat{\varphi}_{r_1 m_2 \cdots m_d} \hat{\bar{\varphi}}_{p_1 m_2 \cdots m_d} \hat{\varphi}_{p_1 n_2 p_3 \cdots p_d} \hat{\bar{\varphi}}_{r_1 n_2 r_3 \cdots p_d} \\
&\left. \times \delta(r_1 + p_2 + \cdots + p_d) \delta(p_1 + r_2 + p_3 + \cdots + p_d) \delta^2(p_2 - r_2) \prod_{i=3}^{d} \delta(p_i - r_i) \right] \\
\simeq{}& \frac{\lambda_k^2}{(Z_k k^2 + \mu_k)^2} \left\{ \delta_{d,3} \int_{D^2} dm_1 dn_2 dr_3 dp_1 dp_2 dp_3 \hat{\varphi}_{m_1 p_2 r_3} \hat{\bar{\varphi}}_{m_1 p_2 p_3} \hat{\varphi}_{p_1 n_2 p_3} \hat{\bar{\varphi}}_{p_1 n_2 r_3} \delta(p_1 + p_2 + p_3) \delta(p_1 + p_2 + r_3) \right. \\
&\times \frac{\Theta(k^2 - \Sigma_s p_s^2)[\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (p_1^2 + p_2^2 + r_3^2) + \mu_k + \Theta[k^2 - (p_1^2 + p_2^2 + r_3^2)] Z_k [k^2 - (p_1^2 + p_2^2 + r_3^2)]} \\
&+ \frac{1}{l} \int_{D^{*\times 2}} dm_1 dp_2 \cdots dp_d dn_1 dr_2 dn_3 \cdots dn_d \hat{\varphi}_{m_1 r_2 p_3 \cdots p_d} \hat{\bar{\varphi}}_{m_1 p_2 \cdots p_d} \hat{\varphi}_{n_1 p_2 n_3 \cdots n_d} \hat{\bar{\varphi}}_{n_1 r_2 n_3 \cdots n_d} \\
&\times \int_{\mathcal{D}} dp_1 \delta(p_1 + r_2 + p_3 + \cdots + p_d) \delta(\Sigma p) \\
&\times \frac{\Theta(k^2 - \Sigma_s p_s^2)[\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (p_1^2 + r_2^2 + \Sigma_{i=3}^d p_i^2) + \mu_k + \Theta[k^2 - (p_1^2 + r_2^2 + \Sigma_{i=3}^d p_i^2)] Z_k [k^2 - (p_1^2 + r_2^2 + \Sigma_{i=3}^d p_i^2)]} \\
&+ \frac{1}{l} \int_{D^{*\times 2}} dr_1 dm_2 \cdots dm_d dp_1 dn_2 dp_3 \cdots dp_d \hat{\varphi}_{r_1 m_2 \cdots m_d} \hat{\bar{\varphi}}_{p_1 m_2 \cdots m_d} \hat{\varphi}_{p_1 n_2 p_3 \cdots p_d} \hat{\bar{\varphi}}_{r_1 n_2 p_3 \cdots p_d} \\
&\times \int_{\mathcal{D}} dp_2 \delta(r_1 + p_2 + \cdots + p_d) \delta(\Sigma p) \\
&\left. \times \frac{\Theta(k^2 - \Sigma_s p_s^2)[\partial_t Z_k (k^2 - \Sigma_s p_s^2) + 2k^2 Z_k]}{Z_k (r_1^2 + p_2^2 + \cdots + p_d^2) + \mu_k + \Theta[k^2 - (r_1^2 + p_2^2 + \cdots + p_d^2)] Z_k [k^2 - (r_1^2 + p_2^2 + \cdots + p_d^2)]} \right\}.
\end{aligned}
\tag{B25}
$$

Performing the integral over p_1 and p_2 in the last two terms and evaluating at the 0-momentum we find

$$\begin{aligned}(II^g)_W|_{1,2} &\simeq \delta_{d,3}\frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\frac{k^2}{l^2}(2+\partial_t)Z_k\mathcal{V}_3 + \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\frac{k^2}{l^2}(2+\partial_t)Z_k[\mathcal{V}_2+\mathcal{V}_1] \\ &\simeq \frac{\lambda_k^2}{(Z_k k^2+\mu_k)^3}\frac{k^2}{l^2}(2+\partial_t)Z_k[\delta_{d,3}\mathcal{V}_3+\mathcal{V}_2+\mathcal{V}_1]. \end{aligned}\tag{B26}$$

The combinatorics of the φ^4 is the same with or without the presence of (gauge) constraints; the contribution to the coefficients coming from the color symmetry is the same as for the previous model. Collecting all contributions, $(II^g)_W|_{i,i}$ (B24), $i=1,\ldots,d$, and $(II^g)_W|_{i,j}$ (B26), $i<j$, $i,j=1,\ldots,d$, the β-function for λ_k, in any rank d, expresses as

$$\beta(\lambda_k) = \frac{2\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left\{\partial_t Z_k\left[\frac{2\pi^{\frac{d-2}{2}}}{d\sqrt{d-1}\Gamma(\frac{d}{2})}\frac{k^d}{l^d}+(2d-1)\frac{k^2}{l^2}\right]+2Z_k\left[\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma(\frac{d}{2})}\frac{k^d}{l^d}+(2d-1)\frac{k^2}{l^2}\right]\right\}. \tag{B27}$$

Dimensionful β-functions.— Let us collect all β-functions. At rank $d\neq 4$, we gather (B17) and (B27) for the complete system of β-functions for the gauge invariant TGFT model which is expressed as

$$\begin{cases}\beta_{d>4}(Z_k) &= \frac{d\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[\frac{\pi^{\frac{d-2}{2}}}{(d-1)^{\frac{3}{2}}\Gamma_E(\frac{d}{2})}\frac{k^{d-2}}{l^d}+\frac{1}{l^2}\right]+\frac{2\pi^{\frac{d-2}{2}}Z_k}{(d-1)^{\frac{3}{2}}\Gamma_E(\frac{d-2}{2})}\frac{k^{d-2}}{l^d}\right\} \\ \beta_{d\neq 4}(\mu_k) &= -\frac{d\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\partial_t Z_k\left[\frac{k^d}{l^d}\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma_E(\frac{d+2}{2})}+\frac{k^2}{l^2}\right]+2Z_k\left[\frac{k^d}{l^d}\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma_E(\frac{d}{2})}+\frac{k^2}{l^2}\right]\right\} \\ \beta_{d\neq 4}(\lambda_k) &= \frac{2\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left\{\partial_t Z_k\left[\frac{2\pi^{\frac{d-2}{2}}}{d\sqrt{d-1}\Gamma(\frac{d}{2})}\frac{k^d}{l^d}+(2d-1+2\delta_{d,3})\frac{k^2}{l^2}\right]+2Z_k\left[\frac{\pi^{\frac{d-2}{2}}}{\sqrt{d-1}\Gamma(\frac{d}{2})}\frac{k^d}{l^d}+(2d-1+2\delta_{d,3})\frac{k^2}{l^2}\right]\right\}\end{cases}, \tag{B28}$$

which is reported in (98) in Sec. IV C, and at $d=4$, we obtain the expression

$$\begin{cases}\beta_{d=4}(Z_k) = \frac{\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\frac{2\pi}{\sqrt{3}}\left[\frac{1}{2}\partial_t Z_k + Z_k\right]\frac{k^2}{l^4}+\partial_t Z_k\frac{4}{l^2}\right\} \\ \beta_{d=4}(\mu_k) = -\frac{4\lambda_k}{(Z_k k^2+\mu_k)^2}\left\{\frac{\pi}{\sqrt{3}}\left[\frac{1}{2}\partial_t Z_k + 2Z_k\right]\frac{k^4}{l^4}+(2+\partial_t)Z_k\frac{k^2}{l^2}\right\} \\ \beta_{d=4}(\lambda_k) = \frac{2\lambda_k^2}{(Z_k k^2+\mu_k)^3}\left\{\partial_t Z_k\left[\frac{2\pi}{4\sqrt{3}}\frac{k^4}{l^4}+7\frac{k^2}{l^2}\right]+2Z_k\left[\frac{\pi}{\sqrt{3}}\frac{k^4}{l^4}+7\frac{k^2}{l^2}\right]\right\}\end{cases} \tag{B29}$$

as reported in (99).

APPENDIX C: SCALING AND CANONICAL DIMENSIONS IN SCALAR/TENSOR GROUP FIELD THEORY

We provide here the scaling dimensional analysis for a scalar and tensor field theory on $\mathbb{R}^d$. This is useful in order to explain the notion of scaling dimension in the tensor case as discussed in Sec. III E.

1. Scalar field theory

Consider a $\sum_{l=2}^{k_{\max}}\lambda_l\phi^l$ real scalar field theory in dimension d (l should be even and $k_{\max}$ a fixed even integer). We use the usual propagator $(-\Delta+m^2)^{-1}$ introduced by the usual Gaussian field measure. Introducing a momentum cutoff Λ, it is known that the power counting of any amplitude $A_{\mathcal{G}}$ of graph $\mathcal{G}$ (with set $\mathcal{V}$ of vertices and set $\mathcal{E}$ of propagator lines) can be written as

$$|A_{\mathcal{G}}| \propto |\prod_{v\in\mathcal{V}}\lambda_v|\Lambda^{\omega_{\rm div}(\mathcal{G})},$$
$$\omega_{\rm div}(\mathcal{G}) = -2E + d(E - V + 1), \tag{C1}$$

where $|\mathcal{V}| = V$, the quantity $E = |\mathcal{E}|$ gives the number of internal propagators (introducing a suppression of Λ^{-2E}), and $E - V + 1$ counts the number of loops producing divergences and where internal momenta should be cut off. Using the combinatorial property $2E = l \cdot V - N_{\rm ext}$, where $N_{\rm ext}$ is the number of external legs of $\mathcal{G}$, and $l \cdot V := \sum_{l=2}^{k_{\max}} lV_l$ is the number of half-lines exiting from all vertices in the graph, we recast the degree of divergence $\omega_{\rm div}(\mathcal{G})$ as

$$\omega_{\rm div}(\mathcal{G}) = \left(1 - \frac{d}{2}\right)N_{\rm ext} + d + \left(\left(-1 + \frac{d}{2}\right)l \cdot -d\right)V. \tag{C2}$$

Let $n_0 \in [\![2, k_{\max}]\!]$. We introduce the scaling dimension $\{\lambda_{n_0}\}$ in units of the momentum of a coupling $\lambda_{n_0;k} = \bar{\lambda}_{n_0;k}k^{\{\lambda_{n_0}\}}$ to be the quantity such that, for all graphs $\mathcal{G}$ with $N_{\rm ext} = n_0$, the scaling dimension of the amplitude associated with $\mathcal{G}$ must coincide with $\{\lambda_{n_0}\}$. This definition is natural in the sense that evaluating the RG flow of the coupling $\lambda_{n_0;k}$, all amplitudes with external data coinciding with ϕ^{n_0} must be summed and equated to $k\partial_k\lambda_{n_0;k}$ and hence must be of the same scaling dimension. In other terms, we have an equality

$$\{A_{\mathcal{G}}\} = \sum_l V_l\{\lambda_l\} + \omega_{\rm div}(\mathcal{G}) = \{\lambda_{n_0}\}. \tag{C3}$$

We can check the consistency of this equation at the zeroth order of perturbation. Consider $\mathcal{G}$ being the graph made by 1 vertex ϕ^{n_0}, with 4 external propagators attached to the external legs, with no loops. The graph does not have any internal line or loops, so then we get $\omega_{\rm div}(\mathcal{G}) = 0$; on the other hand, $\sum_l V_l\{\lambda_l\} = V_{n_0}\{\lambda_{n_0}\} = \{\lambda_{n_0}\}$, so that (C3) holds.

Introducing the notations, $\hat{V} = \sum_{l\neq n_0}V_l$, $\hat{l} \cdot \hat{V} = \sum_{l\neq n_0}lV_l$, $\{\hat{\lambda}\} \cdot \hat{V} = \sum_{l\neq n_0}V_l\{\lambda_l\}$, from (C3) and (C2), we infer for a graph $\mathcal{G}$ with $N_{\rm ext} = n_0$,

$$\begin{aligned}0 &= \{\lambda_{n_0}\}(V_{n_0} - 1) + \left(\left(-1 + \frac{d}{2}\right)n_0 - d\right)V_{n_0}\\ &\quad + \left(1 - \frac{d}{2}\right)n_0 + \{\hat{\lambda}\} \cdot \hat{V} + d + \left(\left(-1 + \frac{d}{2}\right)\hat{l} \cdot -d\right)\hat{V},\\ 0 &= ((-2 + d)n_0 + 2\{\lambda_{n_0}\} - 2d)(V_{n_0} - 1)\\ &\quad + ((-2 + d)\hat{l} \cdot + 2\{\hat{\lambda}\} \cdot -2d)\hat{V}.\end{aligned} \tag{C4}$$

The graph being arbitrary (with arbitrary V_{n_0} and $\hat{V}$), for this equation to hold, we must set for all $n \in [\![2, k_{\max}]\!]$,

$$\{\lambda_n\} = \frac{1}{2}(2 - d)n + d. \tag{C5}$$

We note that the scaling dimension for a coupling λ_k is precisely (-1) times the coefficient of V_k in (C2). It can be seen that at $d = 2$, something special happens, the scaling dimension of any λ_n becomes fixed to 2. This is, of course, the superrenormalizable $P(\phi)_2$-model.

The above notion of scaling dimension coincides with the notion of canonical dimension set up at the level of the action yielding the usual dimension (in momentum unit)

$$[\phi] = -\frac{d+2}{2},$$
$$[\lambda_n] = -dn + d - n[\phi] = d + \frac{n}{2}(2 - d); \tag{C6}$$

the last equation comes from the interaction term $\lambda_n \int[\prod_{i=1}^n {\rm d}^d p_i\phi(p_i)]\delta(\sum_i p_i)$ setting its canonical dimension to be that of the action which is 0. Let us finally comment that the notion of relevant and marginal couplings are defined by $\{\lambda_n\} = [\lambda_{n_0}] \geq 0$, and restricting the initial action to couplings with such scaling dimensions, one obtains a perturbatively renormalizable theory. As a consequence, the set of β-functions is stable and can be made autonomous by simple rescaling of the couplings.

2. Tensorial group field theory

As expected in the case of tensor fields, observables and amplitudes are more involved. Fortunately, the previous treatment can be extended to achieve a notion of scaling dimension. The expression of the amplitude $|A_{\mathcal{G}}| \propto |\prod_{v\in\mathcal{V}}\lambda_v|\Lambda^{\omega_{\rm div}(\mathcal{G})}$ remains valid in this case for graphs built with tensorial interactions. The degree of divergence is of the form [25], in similar notations as previously introduced,

$$\begin{aligned}\omega_{\rm div}(\mathcal{G}) &= -2E + F_{\rm int}\\ &= -\Omega_{\mathcal{G}} - \frac{1}{2}[(d-3)N_{\rm ext} - 2(d-1)]\\ &\quad + \frac{1}{2}[(d-3)l \cdot -2(d-1)]V.\end{aligned} \tag{C7}$$

We have restricted the result of [25] to $G_D = U(1)$, $D = 1$, and focused on connected diagrams such that $C_\partial = 1$. The case $C_\partial \geq 1$ can be treated in a similar way. E is the number of propagator lines, $F_{\rm int}$ is the number of internal faces or closed loops associated with momenta integrated. To be useful, the divergence degree must be further expanded in terms of $\Omega_{\mathcal{G}} = 2(\deg(\mathcal{G}) - \deg(\partial\mathcal{G}))/(d-1)!$ which is related to the difference between the degree $\deg(\mathcal{G})$ of the graph $\mathcal{G}$ [19] and the degree of the boundary graph $\partial\mathcal{G}$. Roughly speaking, the boundary graph $\partial\mathcal{G}$ collects the external data of $\mathcal{G}$: it defined by the set of external legs of $\mathcal{G}$ but also the particular pattern of convolution of external momenta. For the following developments, we mainly rely on the fact that $\Omega_{\mathcal{G}}$ is bounded [24]. Once again, $N_{\rm ext}$ is the number of external legs of the diagram, V_l is the number of vertices having degree l, $l\cdot V \coloneqq \sum_{l=2}^{k_{\max}} lV_l$ is the number of exiting half-lines of all vertices, and $V=\sum_l V_l$ is the total number of vertices.

Let us fix $\partial\mathcal{G}_0$ the boundary graph of $\mathcal{G}_0$ that we will identify with a particular pattern b, namely a colored graph, characterizing an interaction $\lambda_{n_0;b}{\rm Tr}_{n_0;b}((\phi\cdot\bar\phi)^{n_0/2})$. We write $\partial\mathcal{G}_0=b$. The scaling dimension $\{\lambda_{n_0;b}\}$ of $\lambda_{n_0;b;k}=\tilde\lambda_{n_0;b;k}k^{\{\lambda_{n_0;b}\}}$ will be evaluated at large k, at the dominant order in k in the power counting. In that situation, for a given boundary $\partial\mathcal{G}_0$ data we will target all graphs $\mathcal{G}$ such that $\partial\mathcal{G}=\partial\mathcal{G}_0=b$ and such that the divergence degree $\omega_{\rm div}(\mathcal{G})$ is maximal. One realizes that the procedure of evaluating a scaling dimension is generally difficult to handle in practice simply because listing all graphs with given boundary data is not obvious in the nonlocal case. However, in some particular cases, we have enough information and can quickly reach a result.

In $\mathcal{G}$, among the V_{n_0} vertices with n_0 exiting half-lines, there are vertices which do not reproduce the pattern b, so we further split V_{n_0} in the number $V_{n_0;b}$ of vertices with contraction pattern b and the rest: $V_{n_0}=V_{n_0;b}+V'_{n_0}$. We then denote $\hat V=\sum_{l\neq n_0}V_l+V'_{n_0}$ and $\hat l\cdot\hat V\coloneqq\sum_{l=2;l\neq n_0}^{k_{\max}} lV_l+n_0V'_{n_0}$. The scaling dimension of the amplitude of a graph $\mathcal{G}$ with boundary $\partial\mathcal{G}=b$ should be the same as $\{\lambda_{n_0;b}\}$, and we have

$$\{\lambda_{n_0;b}\}=(V_{n_0;b}\{\lambda_{n_0;b}\}+\{\hat\lambda\}\cdot\hat V)|_{\mathcal{G}/\omega_{\rm div}(\mathcal{G})\ \text{is maximal}} + \max_{\mathcal{G}/\partial\mathcal{G}=\partial\mathcal{G}_0=b}\omega_{\rm div}(\mathcal{G}), \tag{C8}$$

where $\{\hat\lambda\}\cdot\hat V\coloneqq\sum_{b'\neq b}V_{n_0;b'}{}'\{\lambda_{n_0;b'}\}+\sum_{l\neq n_0;b''}V_{l;b''}\{\lambda_{l;b''}\}$. This set of equations might be difficult to solve in general. Note that the scaling dimension might even depend on the quantity $\min_{\mathcal{G}/\partial\mathcal{G}=\partial\mathcal{G}_0=b}\Omega_{\mathcal{G}}$. This is a feature proper to tensor models in rank $d\geq 3$, not to scalar theories, not even to matrix models. Indeed, dealing with a matrix model, after a large N expansion, the scaling dimension of a coupling λ_{n_0} does not depend on the genus of the ribbon graphs but only on the data of external legs and the number of multitraces building the invariant [41,42]. This can be further justified: any boundary graph of any ribbon graph can be recast as the boundary graph of a planar ribbon graph, with null genus. Another crucial piece of information is that $\Omega_{\mathcal{G}}$ is always positive, and it is either 0 or, whenever $\deg(\partial\mathcal{G})>0$, then $\Omega_{\mathcal{G}}\geq d-2$ (a theorem in [24]; see also Proposition 1 of the second reference therein). Hence to maximize the divergence degree, $\Omega_{\mathcal{G}}$ should be chosen minimal.

There are cases where we can find $\{\lambda_{n_0;b}\}$. This occurs when the pattern b is a melonic boundary, i.e., $\deg(\partial\mathcal{G})\coloneqq 0$. Then, we know that we can construct a tensor melonic graph $\mathcal{G}$, namely defined by $\deg(\mathcal{G})\coloneqq 0$, such that $\partial\mathcal{G}=b$. Indeed, this can easily be achieved by adding to any of the external legs of b a two-point function like a sunshine graph. In that case, $\Omega_{\mathcal{G}}$ reaches its smallest value of 0. Hence, we restrict to the class of graphs $\deg(\partial\mathcal{G})=0$ since they make the $\omega_{\rm div}(\mathcal{G})$ the largest possible.

Inspecting the rest of the terms of the divergence degree, we must note that

$$\forall l\geq 2,\quad (d-3)l-2(d-1)\geq 2d-6-2d+1=-5; \tag{C9}$$

hence, in (C7), the term $\frac{1}{2}[(d-3)l-2(d-1)]V$ can be negative or positive, since all $V_k\geq 0$. At fixed d, let us assume that $k'_{\max}$ is the integer such that

$$(d-3)k'_{\max}-2(d-1)=0, \tag{C10}$$

such that $V_{k_{\max}}$ becomes an arbitrary number in (C7).

Now let us assume $k'_{\max}=k_{\max}$ and $n_0=k_{\max}$. To make the degree of divergence maximal, we consider a graph without vertices of valence $l<k_{\max}=n_0$ such that $V_{l<k_{\max}}=0$ [otherwise, one can prove that the graph amplitude is convergent $\omega_{\rm div}(\mathcal{G})<0$ and it is certainly not maximal; see the proof of this claim on page 141 in [25]]. From (20), (C7), (C8), and $\Omega_{\mathcal{G}}=0$, we obtain

$$\{\lambda_{k_{\max};b}\}=V_{k_{\max};b}\{\lambda_{k_{\max};b}\}, \tag{C11}$$

which, from the arbitrariness of $V_{k_{\max};b}$ (infinitely many graphs), imposes $\{\lambda_{k_{\max};b}\}=0$. We inspect the case $n_0\leq k_{\max}$. By a similar argument which optimizes the divergence degree, we use graphs such that $V_{l<k_{\max}}=0$ [otherwise, the degree of divergence may be not maximal since $(d-3)l-2(d-1)<0$]. We use (20), (C7), $V_{l<k_{\max}}=0$, and $\Omega_{\mathcal{G}}=0$, to write (C8) in the form

$$\{\lambda_{n_0;b}\}=(\{\lambda_{k_{\max};b}\}\cdot V_{k_{\max};b})|_{\mathcal{G}/\omega_{\rm div}(\mathcal{G})\ \text{is maximal}} + \max_{\mathcal{G}/\partial\mathcal{G}=\partial\mathcal{G}_0=b}\omega_{\rm div}(\mathcal{G})=\max_{\mathcal{G}/\partial\mathcal{G}=\partial\mathcal{G}_0=b}\omega_{\rm div}(\mathcal{G}),$$
$$\{\lambda_{n_0;b}\}=-\frac{1}{2}[(d-3)n_0-2(d-1)]. \tag{C12}$$

Notice that for this type of graphs, the scaling dimension does not depend anymore on b.

The second case $k'_{\max}<k_{\max}$ leads to difficulties since the divergence degree can, in fact, be arbitrarily large. We can build graphs using only vertices $V_{k'_{\max}<k\leq k_{\max}}$ with all

positive coefficients in $\omega_{\rm div}(\mathcal{G})$. This is precisely the signal of nonrenormalizability.

To deal only with perturbatively renormalizable theories (with a finite number of marginal and relevant directions in the IR) we must restrict the parameter space $(d, k_{\max})$ to cases when, for all $l \in [\![2, k_{\max}]\!]$, we have $(d-3)l - 2(d-1) \leq 0$. That analysis has been carried out in [25], and it results that the constraints $d \leq 5$ and $k_{\max}(d) \leq 6$ must be imposed. In this case, perturbative renormalization at all orders, fixing the scaling dimensions to be positive, and as given in (C12) are equivalent statements. Thus, in nonlocal theories the notions of scaling dimension and canonical dimension in the sense of Sec. III A can very well differ. Scaling dimensions, as expected, govern renormalization analysis.

Finally, one might be interested in the restriction at rank $d = 2$ or the matrix case. We obtain at that rank

$$\{\lambda_n\}|_{d=2} = 1 + \frac{n}{2}, \tag{C13}$$

which differs from the scaling dimension of the coupling λ_n of $\mathrm{Tr}(M^n)$ of matrix models in a single trace formalism [41,42]. In that case, one finds $-\frac{n}{2} + 1$. This can easily be explained from the fact that in the above reference the authors did not include a kinetic term in the form of a Laplacian. Thus the factor $(d-3)n$ must be replaced by $(d-1)n$ in (C7) to obtain the corresponding scaling. All the above reasoning applies to matrices, and even in a simpler fashion, because in that case there is always a planar graph with vanishing genus $\Omega_{\mathcal{G}} = 2g_{\mathcal{G}} = 0$, the boundary of which is precisely of the form of $\mathrm{Tr}(M^n)$, for all n. Moreover, we mention that the extension of the above formula (C12) at $C_\partial > 1$, $d = 2$ (and without the Laplacian) will lead to the multitrace scaling dimension as worked out in the above reference as well.

The discussion for the tensorial group field theory with the gauge projection is straightforward: we must again maximize the degree of divergence of a graph which is given by [26–28]

$$\omega_{\rm div}(\mathcal{G}) = -2E + F_{\rm int} - R, \tag{C14}$$

where R is the rank of the so-called incidence matrix between line and faces. The divergence degree for $G = U(1)$ can be recast in a way similar to (C7) by just adding $-R$ to that expression. There is a useful relation proving that $F_{\rm int} - R \leq (d-2)(E - V + 1)$ which allows one to find an upper bound on the divergence degree. The rest of the reasoning applies and, for couplings such that $\deg(\partial\mathcal{G}) = 0$ and $\Omega_{\mathcal{G}} = 0$, leads to

$$\{\lambda_{n;b}\} = -\frac{1}{2}((d-4)n - 2(d-2)). \tag{C15}$$

This shows that there is a shift in d in the scaling dimension $d \to d-1$. Restricted to $N_{\rm ext} = 4, 2$, we obtain the scaling dimensions $\{\lambda_{4;b}\}|_{n=4} = -\frac{1}{2}(4d - 16 - 2d + 4) = 6 - d$ as found in Sec. IV C.

[1] D. Oriti, The microscopic dynamics of quantum space as a group field theory, in *Foundations of Space and Time*, edited by G. Ellis *et al.* (Cambridge University Press, Cambridge, UK, 2012); D. Oriti, The group field theory approach to quantum gravity, in *Approaches to Quantum Gravity*, edited by D. Oriti (Cambridge University Press, Cambridge, UK, 2009); L. Freidel, Group field theory: An overview, *Int. J. Theor. Phys.* ***44**, 1769 (2005)*; D. Oriti, Quantum gravity as a quantum field theory of simplicial geometry, in *Mathematical and Physical Aspects of Quantum Gravity*, edited by B. Fauser *et al.* (Birkhaeuser, Basel, 2006); A. Baratin and D. Oriti, Ten questions on group field theory (and their tentative answers), J. Phys. Conf. Ser. **360**, 012002 (2012); T. Krajewski, Group field theories, *Proc. Sci.*, QGQGS2011 (2011) 005 [arXiv: 1210.6257]; D. Oriti, The group field theory approach to quantum gravity: Some recent results, in *The Planck Scale*, edited by J. Kowalski-Glikman *et al.* (AIP, New York, 2009).

[2] D. V. Boulatov, A model of three-dimensional lattice gravity, Mod. Phys. Lett. A **07**, 1629 (1992); H. Ooguri, Topological lattice models in four-dimensions, *Mod. Phys. Lett. A* ***07**, 2799 (1992)*.

[3] J. Ambjorn, B. Durhuus, and T. Jonsson, Three-dimensional simplicial quantum gravity and generalized matrix models, Mod. Phys. Lett. A **06**, 1133 (1991); N. Sasakura, Tensor model for gravity and orientability of manifold, *Mod. Phys. Lett. A* ***06**, 2613 (1991)*; M. Gross, Tensor models and simplicial quantum gravity in $> 2 - D$, *Nucl. Phys. B, Proc. Suppl.* ***25**, 144 (1992)*.

[4] P. Di Francesco, P. H. Ginsparg, and J. Zinn-Justin, 2-D Gravity and random matrices, Phys. Rep. **254**, 1 (1995).

[5] T. Thiemann, *Modern Canonical Quantum General Relativity* (Cambridge University Press, Cambridge, UK, 2007); A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A status report, Classical Quantum Gravity **21**, R53 (2004); C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, UK, 2006).

[6] C. Rovelli, Basis of the Ponzano-Regge-Turaev-Viro-Ooguri quantum gravity model in the loop representation basis, Phys. Rev. D **48**, 2702 (1993).

[7] A. Perez, The spin foam approach to quantum gravity, Living Rev. Relativ. **16**, 3 (2013); C. Rovelli, Zakopane lectures on loop gravity, *Proc. Sci.*, QGQGS2011 (2011) 003 [arXiv:1102.3660].

[8] M. P. Reisenberger and C. Rovelli, Space-time as a Feynman diagram: The connection formulation, Classical Quantum Gravity **18**, 121 (2001).

[9] M. X. Han and M. Zhang, Asymptotics of spinfoam amplitude on simplicial manifold: Euclidean theory, Classical Quantum Gravity **29**, 165004 (2012).

[10] A. Baratin and D. Oriti, Group Field Theory with Noncommutative Metric Variables, Phys. Rev. Lett. **105**, 221302 (2010); Group field theory and simplicial gravity path integrals: A model for Holst-Plebanski gravity, Phys. Rev. D **85**, 044003 (2012).

[11] D. Oriti, Group field theory as the 2nd quantization of loop quantum gravity, arXiv:1310.7786; D. Oriti, Group field theory and loop quantum gravity, arXiv:1408.7112; D. Oriti, J. P. Ryan, and J. Thürigen, Group field theories for all loop quantum gravity, New J. Phys. **17**, 023042 (2015).

[12] D. Oriti, Disappearance and emergence of space and time in quantum gravity, Stud. Hist. Phil. Mod. Phys. **46**, 186 (2014); B. L. Hu, Can spacetime be a condensate? Int. J. Theor. Phys. **44**, 1785 (2005); T. Konopka, F. Markopoulou, and L. Smolin, Quantum graphity, arXiv:hep-th/0611197.

[13] T. A. Koslowski, Dynamical quantum geometry (DQG programme), arXiv:0709.3465.

[14] T. Koslowski and H. Sahlmann, Loop quantum gravity vacuum with nondegenerate geometry, SIGMA **8**, 026 (2012).

[15] J. Ambjorn, A. Goerlich, J. Jurkiewicz, and R. Loll, Nonperturbative quantum gravity, Phys. Rep. **519**, 127 (2012).

[16] D. Benedetti and J. Henson, Spacetime condensation in $(2+1)$-dimensional CDT from a Horava-Lifshitz minisuperspace model, Classical Quantum Gravity **32**, 215007 (2015).

[17] J. Mielczarek, Big bang as a critical point, arXiv:1404.0228.

[18] B. Dittrich, F. C. Eckert, and M. Martin-Benito, Coarse graining methods for spin net and spin foam models, New J. Phys. **14**, 035008 (2012); B. Bahr, B. Dittrich, F. Hellmann, and W. Kaminski, Holonomy spin foam models: Definition and coarse graining, Phys. Rev. D **87**, 044048 (2013); B. Dittrich, M. Martìn-Benito, and E. Schnetter, Coarse graining of spin net models: Dynamics of intertwiners, New J. Phys. **15**, 103004 (2013).

[19] R. Gurau, The $1/N$ expansion of colored tensor models, Ann. Henri Poincare **12**, 829 (2011); R. Gurau and V. Rivasseau, The $1/N$ expansion of colored tensor models in arbitrary dimension, Europhys. Lett. **95**, 50004 (2011); R. Gurau, The complete $1/N$ expansion of colored tensor models in arbitrary dimension, Ann. Henri are **13**, 399 (2012).

[20] R. Gurau, Double scaling limit in arbitrary dimensions: A toy model, Phys. Rev. D **84**, 124051 (2011); W. Kamiński, D. Oriti, and J. P. Ryan, Towards a double-scaling limit for tensor models: Probing sub-dominant orders, New J. Phys. **16**, 063048 (2014); S. Dartois, R. Gurau, and V. Rivasseau, Double scaling in tensor models with a quartic interaction, J. High Energy Phys. 09 (2013) 088; V. Bonzom, R. Gurau, J. P. Ryan, and A. Tanasa, The double scaling limit of random tensor models, J. High Energy Phys. 09 (2014) 051.

[21] R. Gurau, Universality for random tensors, Ann. Inst. Henri Poincaré Probab. Statist. **50**, 1474 (2014).

[22] V. Rivasseau, The tensor theory space, Fortschr. Phys. **62**, 835 (2014).

[23] J. Ben Geloun, J. Magnen, and V. Rivasseau, Bosonic colored group field theory, Eur. Phys. J. C **70**, 1119 (2010); J. Ben Geloun, T. Krajewski, J. Magnen, and V. Rivasseau, Linearized group field theory and power counting theorems, Classical Quantum Gravity **27**, 155012 (2010); J. Ben Geloun, R. Gurau, and V. Rivasseau, EPRL/FK group field theory, Europhys. Lett. **92**, 60008 (2010); J. Ben Geloun and V. Bonzom, Radiative corrections in the Boulatov-Ooguri tensor model: The 2-point function, Int. J. Theor. Phys. **50**, 2819 (2011); J. Ben Geloun and E. R. Livine, Some classes of renormalizable tensor models, J. Math. Phys. (N.Y.) **54**, 082303 (2013); S. Carrozza, D. Oriti, and V. Rivasseau, Renormalization of tensorial group field theories: Abelian U(1) models in four dimensions, Commun. Math. Phys. **327**, 603 (2014); T. Krajewski, Schwinger-Dyson equations in group field theories of quantum gravity, arXiv:1211.1244; M. Raasakka and A. Tanasa, Combinatorial Hopf algebra for the Ben Geloun-Rivasseau tensor field theory, Seminaire Lotharingien de Combinatoire **70**, B70d (2014); D. O. Samary, Closed equations of the two-point functions for tensorial group field theory, Classical Quantum Gravity **31**, 185005 (2014); J. Ben Geloun, On the finite amplitudes for open graphs in Abelian dynamical colored Boulatov-Ooguri models, J. Phys. A **46**, 402002 (2013); T. Krajewski and R. Toriumi, Polchinski's equation for group field theory, Fortschr. Phys. **62**, 855 (2014).

[24] J. Ben Geloun and V. Rivasseau, A renormalizable 4-dimensional tensor field theory, Commun. Math. Phys. **318**, 69 (2013); J. Ben Geloun and V. Rivasseau, Addendum to "A renormalizable 4-dimensional tensor field theory, Commun. Math. Phys. **322**, 957 (2013).

[25] J. Ben Geloun, Renormalizable models in rank $d \geq 2$ tensorial group field theory, Commun. Math. Phys. **332**, 117 (2014).

[26] S. Carrozza, D. Oriti, and V. Rivasseau, Renormalization of a SU(2) tensorial group field theory in three dimensions, Commun. Math. Phys. **330**, 581 (2014).

[27] D. O. Samary and F. Vignes-Tourneret, Just renormalizable TGFT's on $U(1)^d$ with gauge invariance, Commun. Math. Phys. **329**, 545 (2014).

[28] S. Carrozza, *Tensorial Methods and Renormalization in Group Field Theories, Springer Theses 2014* (Springer, New York, 2014).

[29] J. Ben Geloun and D. O. Samary, 3D Tensor field theory: Renormalization and one-loop β-functions, Ann. Henri Poincare **14**, 1599 (2013).

[30] J. Ben Geloun, Two and four-loop β-functions of rank 4 renormalizable tensor field theories, Classical Quantum Gravity **29**, 235011 (2012).

[31] D. O. Samary, Beta functions of $U(1)^d$ gauge invariant just renormalizable tensor models, Phys. Rev. D **88**, 105003 (2013).

[32] S. Carrozza, Discrete renormalization group for SU(2) tensorial group field theory, Ann. Inst. Henri Poincare Comb. Phys. Interact **2**, 49 (2015).
[33] V. Rivasseau, Why are tensor field theories asymptotically free?, Europhys. Lett. **111**, 60011 (2015).
[34] V. Lahoche and D. Oriti, Renormalization of a tensorial field theory on the homogeneous space SU(2)/U(1), arXiv: 1506.08393.
[35] T. Krajewski and R. Toriumi, Polchinski's equation for group field theory, Fortschr. Phys. **62**, 855 (2014).
[36] T. Krajewski and R. Toriumi, Polchinski's exact renormalisation group for tensorial theories: Gaussian universality and power counting, arXiv:1511.09084.
[37] V. Lahoche, D. Oriti, and V. Rivasseau, Renormalization of an Abelian tensor group field theory: Solution at leading order, J. High Energy Phys. 04 (2015) 095.
[38] D. Benedetti, J. Ben Geloun, and D. Oriti, Functional renormalization group approach for tensorial group field theory: A rank-3 model, J. High Energy Phys. 03 (2015) 084.
[39] E. Brezin and J. Zinn-Justin, Renormalization group approach to matrix models, Phys. Lett. B **288**, 54 (1992).
[40] D. Benedetti and F. Caravelli, The local potential approximation in quantum gravity, J. High Energy Phys. 06 (2012) 017; 10 (2012) 157(E).
[41] A. Eichhorn and T. Koslowski, Continuum limit in matrix models for quantum gravity from the functional renormalization group, Phys. Rev. D **88**, 084016 (2013).
[42] A. Eichhorn and T. Koslowski, Towards phase transitions between discrete and continuum quantum spacetime from the renormalization group, Phys. Rev. D **90**, 104039 (2014).
[43] J. Ben Geloun, R. Martini, and D. Oriti, Functional renormalization group analysis of a tensorial group field theory on $\mathbb{R}^3$, Europhys. Lett. **112**, 31001 (2015).
[44] D. Benedetti and V. Lahoche, Functional renormalization group approach for tensorial group field theory: A rank-6 model with closure constraint, Classical Quantum Gravity **33**, 095003 (2016).
[45] T. Delepouve and R. Gurau, Phase transition in tensor models, J. High Energy Phys. 06 (2015) 178.
[46] D. Benedetti and R. Gurau, Symmetry breaking in tensor models, Phys. Rev. D **92**, 104041 (2015).
[47] A. Baratin, S. Carrozza, D. Oriti, J. Ryan, and M. Smerlak, Melonic phase transition in group field theory, Lett. Math. Phys. **104**, 1003 (2014).
[48] S. Gielen, D. Oriti, and L. Sindoni, Cosmology from Group Field Theory Formalism for Quantum Gravity, Phys. Rev. Lett. **111**, 031301 (2013); S. Gielen, D. Oriti, and L. Sindoni, Homogeneous cosmologies as group field theory condensates, J. High Energy Phys. 06 (2014) 013; L. Sindoni, Effective equations for GFT condensates from fidelity, arXiv:1408.3095; S. Gielen and D. Oriti, Quantum cosmology from quantum gravity condensates: Cosmological variables and lattice-refined dynamics, New J. Phys. **16**, 123004 (2014); S. Gielen, Perturbing a quantum gravity condensate, Phys. Rev. D **91**, 043526 (2015).
[49] R. Gurau, A generalization of the Virasoro algebra to arbitrary dimensions, Nucl. Phys. **B852**, 592 (2011).
[50] R. Gurau, The Schwinger Dyson equations and the algebra of constraints of random tensor models at all orders, Nucl. Phys. **B865**, 133 (2012).
[51] V. Bonzom, R. Gurau, and V. Rivasseau, Random tensor models in the large N limit: Uncoloring the colored tensor models, Phys. Rev. D **85**, 084037 (2012).
[52] A. Abdesselam, On the volume conjecture for classical spin networks, J. Knot Theory Ramifications **21**, 3 (2012).
[53] S. Carrozza and A. Tanasa, $O(N)$ random tensor models, arXiv:1512.06718.
[54] J. Polchinski, Renormalization and effective Lagrangians, Nucl. Phys. **B231**, 269 (1984).
[55] C. Wetterich, Exact evolution equation for the effective potential, Phys. Lett. B **301**, 90 (1993).
[56] B. Delamotte, An introduction to the nonperturbative renormalization group, Lect. Notes Phys. **852**, 49 (2012).
[57] T. R. Morris, The exact renormalization group and approximate solutions, Int. J. Mod. Phys. A **09**, 2411 (1994).
[58] J. Berges, N. Tetradis, and C. Wetterich, Nonperturbative renormalization flow in quantum field theory and statistical physics, Phys. Rep. **363**, 223 (2002).
[59] R. Gurau, Colored group field theory, Commun. Math. Phys. **304**, 69 (2011); R. Gurau, Lost in translation: Topological singularities in group field theory, Classical Quantum Gravity **27**, 235023 (2010).
[60] R. Gurau and J. P. Ryan, Colored tensor models—A review, SIGMA **8**, 020 (2012).
[61] V. Bonzom, R. Gurau, A. Riello, and V. Rivasseau, Critical behavior of colored tensor models in the large N limit, Nucl. Phys. **B853**, 174 (2011); V. Bonzom, R. Gurau, and V. Rivasseau, The Ising model on random lattices in arbitrary dimensions, Phys. Lett. B **711**, 88 (2012); D. Benedetti and R. Gurau, Phase transition in dually weighted colored tensor models, Nucl. Phys. **B855**, 420 (2012); R. Gurau and J. P. Ryan, Melons are branched polymers, Ann. Henri Poincare **15**, 2085 (2014).
[62] D. F. Litim, Renormalisation group and the Planck scale, Phil. Trans. R. Soc. A **369**, 2759 (2011).
[63] D. F. Litim, Optimized renormalization group flows, Phys. Rev. D **64**, 105007 (2001).
[64] M. Salmhofer, *Renormalization: An introduction* (Springer, Berlin, 1999).
[65] D. Benedetti, Critical behavior in spherical and hyperbolic spaces, J. Stat. Mech. (2015) P01002.
[66] R. Gurau and O. J. Rosten, Wilsonian renormalization of noncommutative scalar field theory, J. High Energy Phys. 07 (2009) 064.
[67] D. F. Litim and F. Sannino, Asymptotic safety guaranteed, J. High Energy Phys. 12 (2014) 178.
[68] M. Niedermaier and M. Reuter, The asymptotic safety scenario in quantum gravity, Living Rev. Relativ. **9**, 5 (2006); R. Percacci, *Asymptotic safety, in Approaches to Quantum Gravity*, edited by D. Oriti (Cambridge University Press, Cambridge, UK, 2009); M. Reuter and F. Saueressig, Quantum Einstein gravity, New J. Phys. **14**, 055022 (2012).
[69] V. Rivasseau, Quantum gravity and renormalization: The tensor track, AIP Conf. Proc. **1444**, 18 (2011); V. Rivasseau, The tensor track, III, Fortschr. Phys. **62**, 81 (2014).