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Wormholes, the weak energy condition, and scalar-tensor gravity

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We obtain a large class of Lorentzian wormhole spacetimes in scalar-tensor gravity, for which the matter stress energy does satisfy the weak energy condition. Our constructions have zero Ricci scalar and an everywhere finite, nonzero scalar field profile. Interpreting the scalar-tensor gravity as an effective on-brane theory resulting from a two-brane Randall-Sundrum model of warped extra dimensions, it is possible to link wormhole existence with that of extra dimensions. We study the geometry, matter content, gravitational redshift and circular orbits in such wormholes and argue that our examples are perhaps among those which may have some observational relevance in astrophysics in the future. We also study traversability and find that our wormholes are indeed traversable for values of the metric parameters satisfying the weak energy condition.

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I. INTRODUCTION

Curved spacetimes for which the Ricci scalar R is identically zero have been known in general relativity (GR) since the discovery of the Schwarzschild solution. While, in GR, the Schwarzschild is a vacuum spacetime for which both R and R_{ij} are zero, the Reissner-Nordström geometry has $R=0$ but $R_{ij}\neq 0$, thereby implying the presence of traceless matter. It has been further shown in Ref. [1] that a generalization of Schwarzschild spacetime with $|g_{00}|$ everywhere finite and nonzero and $R=0$ can be obtained. More recently [2], this $R=0$ wormhole spacetime has been found to be a solution in a scalar-tensor theory of gravity which is also the low-energy, effective, on-brane gravity theory for the warped two-brane Randall-Sundrum model. The $R=0$ spacetime in Ref. [1] when viewed as a solution in GR, requires matter that violates the weak energy condition (WEC). However, as shown in Ref. [2], in the context of the scalar-tensor theory, the required matter does not violate the WEC and the scalar field (radion) is also finite and nonzero everywhere. In our work here, we further generalize the spacetime studied in Refs. [1,2]. In particular, newer $R=0$ spacetimes with $|g_{00}|$ everywhere nonzero are constructed and shown to be solutions in scalar-tensor gravity. A subclass of such spacetimes are Lorentzian wormholes with the required matter satisfying the WEC.

In our analysis here, we begin with a general static, spherisymmetric line element for which the $R=0$ constraint is written as a differential equation for $\sqrt{|g_{00}|}=f(r)$. Obviously, this differential equation contains g_{rr} [or, $b(r)$ where $g_{rr}=\frac{1}{1-\frac{b(r)}{r}}$] and its derivatives in its coefficients. It can therefore be solved once we provide our choice for $b(r)$. Assuming $b(r)$ as one should for a Lorentzian wormhole, we find that for some specific choices (e.g. the Schwarzschild wormhole and the Ellis-Bronnikov wormhole [3,4]) the differential equation for $f(r)$ can indeed be solved. We obtain a wide class of geometries in this manner. Thereafter, we show how these geometries fare in terms of the WEC violation issue.

As mentioned above, in GR we require WEC violation for the matter that threads nonsingular wormhole spacetimes. However, it has been shown recently that, within GR, a wormhole (which is topologically different from the Morris-Thorne class) can be supported by a negative cosmological constant (which is not quite exotic, WEC-violating matter) [5]. Further, it is known that certain wormhole solutions which have been constructed in various modified theories of gravity—such as Brans-Dicke theory (scalar-tensor gravity with constant coupling ω) [6], $f(R)$ gravity [7], Gauss-Bonnet gravity [8], third-order Lovelock gravity [9], Eddington-inspired Born-Infeld gravity [10], mimetic gravity [11], and Dvali-Gabadadze-Porrati gravity [12]—do not require WEC-violating matter.

In our work here, we consider scalar-tensor gravity with a nonconstant coupling $\omega(\Phi)$ and in the presence of matter stress energy. Through our specific solutions, we show that there may not be WEC violation for the matter required to support a Lorentzian wormhole. It must be noted though that the timelike convergence condition is still violated, as it must be, for wormholes. The essential point is that the relation between the convergence condition and the energy condition is not as it is in GR, because of which there is extra freedom to avoid any violation in a modified theory of gravity (here, scalar-tensor gravity) [2].

Our paper is organized as follows. In Sec. II, we briefly recall scalar-tensor gravity both as an independent theory

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and in the context of braneworld gravity. We provide the details of our construction of $R = 0$ spacetimes in Sec. III. Sections IV and V deal with the status of the WEC. In Sec. VI, we study gravitational redshift in the wormhole spacetimes. The existence of stable and unstable circular orbits and the issue of traversability are addressed in Sec. VII. Finally, we conclude in Sec. VIII.

II. SCALAR-TENSOR GRAVITY

Scalar-tensor theories are well studied from various angles and perspectives. The first such theory was, of course, the Jordan-Brans-Dicke theory [13]. A scalar field Φ and an extra parameter ω characterizes such theories. Though originally considered a constant, a modified version incorporates a Φ-dependent ω [14]. In general, the Einstein field equations for such theories are given as

$$G_{\mu\nu} = \frac{\kappa}{\Phi} T^M_{\mu\nu} + \frac{1}{\Phi}(\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \nabla^\alpha \nabla_\alpha \Phi) + \frac{\omega(\Phi)}{\Phi^2}\left(\nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \Phi \nabla_\alpha \Phi\right) \tag{1}$$

where $T^M_{\mu\nu}$ is the matter energy-momentum tensor and the metric $g_{\mu\nu}$ is written in the Jordan frame where the action is not in the standard canonical (Einstein-Hilbert) form. Viewed as an independent competitor of GR, a scalar-tensor theory could be specified by providing a form for $\omega(\Phi)$. Note that Φ must satisfy the equation,

$$\nabla^\alpha \nabla_\alpha \Phi = \kappa \frac{T^M}{2\omega(\Phi) + 3} - \frac{1}{2\omega(\Phi) + 3} \frac{d\omega}{d\Phi} \nabla^\alpha \Phi \nabla_\alpha \Phi. \tag{2}$$

Various aspects of scalar-tensor theories have been extensively studied and we will not review them here. See Ref. [15] and references therein for detailed reviews. Taking the trace of Eq. (1) and using Eq. (2), one obtains

$$R = -\frac{2\kappa\omega T^M}{\Phi(2\omega + 3)} - \frac{3}{\Phi(2\omega + 3)} \times \left(\frac{d\omega}{d\Phi} - \frac{\omega(2\omega + 3)}{3\Phi}\right) \nabla^\alpha \Phi \nabla_\alpha \Phi.$$

Note that traceless matter implies $R = 0$ in GR. But, in scalar-tensor gravity, traceless matter does not imply $R = 0$ in general. But, for the specific form of $\omega(\Phi)$ given by

$$\frac{d\omega}{d\Phi} - \frac{\omega(2\omega + 3)}{3\Phi} = 0 \Rightarrow \omega(\Phi) = \frac{3c_0\Phi}{2(1 - c_0\Phi)}$$

$R = -\kappa c_0 T^M$ and hence, traceless matter implies $R = 0$. Here, c_0 is an integration constant. As we shall see, the case $c_0 = -1$ arises in the low-energy, effective on-brane quasi-scalar-tensor gravity theory developed by Kanno and Soda [16] in the context of the Randall-Sundrum two-brane model.

It is well known that scalar-tensor theories arise as effective theories of gravity in diverse contexts. For example, the low-energy effective gravity theory which emerges out of superstring theory contains a scalar known as the dilaton and the theory is a $\omega = -1$ Brans-Dicke theory [17]. In the braneworld scenario, our four-dimensional visible Universe is considered as a lower-dimensional hypersurface called a brane, which is embedded in a higher-dimensional bulk. In the low-energy limit, the field equation governing bulk gravity, leads to effective field equations for gravity on the brane. The presence of the extra-dimensional bulk leaves its imprint by modifying the Einstein field equations on the brane. The most popular among the on-brane effective Einstein equations was obtained by Shiromizu-Maeda-Sasaki [18] in the context of a single-brane model and contains a nonlocal term (bulk-Weyl-dependent $\mathcal{E}_{\mu\nu}$). The on-brane effective Einstein equations obtained by Kanno and Soda [16] in the context of the two-brane Randall-Sundrum model, however, do not contain any nonlocal contribution. Earlier, the nonlocal $\mathcal{E}_{\mu\nu}$ was used to obtain $R = 0$ spacetime solutions [19–22] (mostly wormhole solutions) in the single-brane effective theory of Shiromizu-Maeda-Sasaki. In the effective on-brane theory of Kanno and Soda, we have an effective energy-momentum tensor (other than the matter on the branes) constructed from the scalar radion field (which measures the inter-brane distance) and its derivatives. The radion field plays a crucial role in obtaining both static, spherically symmetric spacetimes and cosmological solutions [2,23,24]. It has been shown that the presence of the radion field can correctly reproduce the observed virial mass of galaxy clusters and observed galaxy rotation curves. Thus, the radion field can act as a possible dark matter candidate [25].

Let us now briefly review the low-energy effective on-brane quasi-scalar-tensor theory developed by Kanno and Soda [16]. The action is given as,

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g}\left(\mathcal{R} + \frac{12}{l^2}\right) - \sum_{i=A,B} \sigma_i \int d^4x \sqrt{-g^{i\,\text{brane}}} + \sum_{i=A,B} \int d^4x \sqrt{-g^{i\,\text{brane}}} \mathcal{L}^i_{\text{matter}}$$

where κ^2, $\mathcal{R}$ and $g^{i\,\text{brane}}_{\mu\nu}$ are the five-dimensional gravitational coupling constant, the five-dimensional curvature scalar and the induced metric on the branes, respectively. $\sigma_A = \frac{6}{\kappa^2 l}$ and $\sigma_B = -\frac{6}{\kappa^2 l}$ are the brane tensions. The positive-tension brane A (Planck brane) and the negative-tension brane B (visible brane) are respectively placed at fixed bulk locations ($y = 0$ and $y = l$). The bulk line element is,

$$ds^2 = e^{2\phi(x^\mu)} dy^2 + g_{\mu\nu}(y, x^\mu) dx^\mu dx^\nu$$

where the bulk curvature radius is l. Assuming the brane curvature radius L as large compared to the bulk curvature radius l, i.e., $(\frac{l}{L})^2 \ll 1$, Kanno and Soda used a low-energy expansion scheme (the gradient expansion method) wherein the bulk metric and the extrinsic curvature are expanded in powers of $(\frac{l}{L})^2$. They obtained the effective field equations,

$$G_{\mu\nu} = \frac{\kappa^2}{l\Phi}T^B_{\mu\nu} + \frac{\kappa^2(1+\Phi)}{l\Phi}T^A_{\mu\nu} + \frac{1}{\Phi}(\nabla_\mu\nabla_\nu\Phi - f_{\mu\nu}\nabla^\alpha\nabla_\alpha\Phi) - \frac{3}{2\Phi(1+\Phi)}\left(\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}f_{\mu\nu}\nabla^\alpha\Phi\nabla_\alpha\Phi\right) \tag{3}$$

on the visible brane. Here, $\Phi = e^{\frac{2d}{l}} - 1$, $T^A_{\mu\nu}$, $T^B_{\mu\nu}$ and $f_{\mu\nu}$ are respectively the radion field, the matter on the Planck brane, the matter on the visible brane and the metric tensor on the visible brane. The proper distance d between the branes is defined as

$$d(x) = \int_0^l e^{\phi(x)}dy.$$

The radion field Φ satisfies the following equation of motion on the visible brane:

$$\nabla^\alpha\nabla_\alpha\Phi = \frac{\kappa^2}{l}\frac{(T^A+T^B)}{2\omega+3} - \frac{1}{2\omega+3}\frac{d\omega}{d\Phi}\nabla^\alpha\Phi\nabla_\alpha\Phi \tag{4}$$

where both the traces T^A and T^B are taken with respect to the metric tensor $f_{\mu\nu}$ and the coupling function is given by

$$\omega(\Phi) = -\frac{3\Phi}{2(1+\Phi)}. \tag{5}$$

Using the radion equation of motion in the trace of the field equation (3), we obtain $R = \frac{\kappa^2}{l}T^B$, where R is the curvature scalar on the visible brane. Note that this is the $c_0 = -1$ case discussed earlier. Therefore, traceless matter implies $R = 0$ in this theory. Using the transformation $\xi = \sqrt{1+\Phi}$, we can express the radion equation of motion in the form

$$\nabla^\alpha\nabla_\alpha\xi - \frac{R}{6}\xi = \frac{\kappa^2}{6l}T^A\xi. \tag{6}$$

In the subsequent sections, we obtain our new solutions. It is important to note that the solutions could either be viewed as solutions in a scalar-tensor theory which has no link with extra dimensions or as solutions in the context of braneworld gravity. In obtaining the solutions, we take the Planck brane to be devoid of matter, i.e., $T^A_{\mu\nu} = 0$. Further, henceforth, we refer to this on-brane scalar-tensor gravity as the Kanno-Soda theory of gravity because of the specific form of the coupling function $\omega(\Phi)$ given in Eq. (5).

III. $R = 0$ SPACETIMES

We begin with the static, spherically symmetric line element given as

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $f(r)$ and $b(r)$ are unknown functions to be determined. We are interested in vanishing curvature scalar ($R = 0$) solutions. The $R = 0$ constraint yields the following second-order differential equation for $f(r)$:

$$\left(1-\frac{b}{r}\right)f''(r) + \frac{4r-3b-b'r}{2r^2}f'(r) - \frac{b'}{r^2}f(r) = 0. \tag{7}$$

The above equation can also be rewritten as a first-order differential equation for $b(r)$. Attempts have been made to find $R = 0$ spacetime solutions in the past [1,2,19–24,26,27]. In most of these articles, the authors have (a) considered known forms of $f(r)$ and obtained solutions by solving the first-order differential equation for $b(r)$ or (b) used some known $R = 0$ solutions present in the literature and analyzed them in the context of an effective theory of gravity. In our work, we take a different route. We specify the form of $b(r)$ and solve the second-order differential equation for $f(r)$. This way we hope to obtain a more general solution different from those obtained earlier. Another advantage of specifying the form of the shape function $b(r)$ is that we can choose to have wormhole solutions with a desired spatial shape. One can take different forms of $b(r)$. In our work here, we take $b(r) = 2m + \frac{\beta}{r}$. Putting $f(r) = \frac{F(r)}{r}$ in the $R = 0$ equation, we obtain

$$\left(1-\frac{2m}{r}-\frac{\beta}{r^2}\right)F''(r) + \left(\frac{m}{r^2}+\frac{\beta}{r^3}\right)F'(r) - \frac{m}{r^3}F(r) = 0.$$

After some manipulations, the above equation can be rewritten in the form

$$\frac{d}{dr}\left[\frac{1}{r}(r^2-2mr-\beta)^{3/2}\frac{d}{dr}\left(\frac{F(r)}{\sqrt{r^2-2mr-\beta}}\right)\right] = 0$$

which can be integrated to obtain

$$f(r) = C_1\left(m+\frac{\beta}{r}\right) + C_2\sqrt{1-\frac{2m}{r}-\frac{\beta}{r^2}} \tag{8}$$

where C_1 and C_2 are integration constants. Therefore, the line element becomes

$$ds^2 = -\left[C_1\left(m+\frac{\beta}{r}\right) + C_2\sqrt{1-\frac{2m}{r}-\frac{\beta}{r^2}}\right]^2 dt^2 + \frac{dr^2}{1-\frac{2m}{r}-\frac{\beta}{r^2}} + r^2 d\Omega^2. \quad (9)$$

It is useful to note that, when $m = 0$ or $\beta = 0$, the $|g_{00}|$ is of the general form, $(C_3 b(r) + C_4\sqrt{1-\frac{b(r)}{r}})^2$. However, when m, β are both nonzero, this generic form does not hold and we have $|g_{00}| = (C_3 b_1(r) + C_4\sqrt{1-\frac{b(r)}{r}})^2$. However, it is important to note that, in all cases, the redshift function is finite and nonzero everywhere.

We now look at special cases of our general solution. Some of these special cases reproduce the $R = 0$ spacetimes obtained earlier by other authors. For $\beta = 0$, $C_1 m = \kappa$ and $C_2 = \lambda$, the spacetime takes the form

$$ds^2 = -\left[\kappa + \lambda\sqrt{1-\frac{2m}{r}}\right]^2 dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2. \quad (10)$$

This spatially Schwarzschild spacetime has been obtained in Refs. [1,2,19,27]. This spacetime represents either a wormhole or a naked singularity for $\kappa \neq 0$. For $\kappa = 0$ and $\lambda = 1$, the spatially Schwarzschild solution reduces to the Schwarzschild spacetime. Taking $\kappa = 1$ and $\lambda = 0$ in this solution, we obtain the spatial Schwarzschild wormhole obtained in Ref. [20]. For $2m = \frac{r_0^2}{r_0 - M}$, $\beta = -\frac{M r_0^2}{r_0 - M}$, $C_1 m = 1$ and $C_2 = 0$, the general metric takes the form

$$ds^2 = -\left(1-\frac{2M}{r}\right)^2 dt^2 + \frac{dr^2}{(1-\frac{r_0}{r})(1-\frac{r_1}{r})} + r^2 d\Omega^2 \quad (11)$$

where $r_1 = \frac{M r_0}{r_0 - M}$. This metric has been obtained in Refs. [20,21]. This metric reduces to the extreme Reissner-Nordström form for $r_1 = r_0$, i.e., $r_0 = 2M$. The above metric fails to produce spatial Ellis geometry because $r_1 = -r_0$ implies $r_0 = r_1 = 0$. However, we can obtain spatial Ellis geometry from our general solution by taking $m = 0$ and $\beta = r_0^2$. The line element becomes

$$ds^2 = -\left(\frac{C_1 r_0^2}{r} + C_2\sqrt{1-\frac{r_0^2}{r^2}}\right)^2 dt^2 + \frac{dr^2}{1-\frac{r_0^2}{r^2}} + r^2 d\Omega^2. \quad (12)$$

This new metric has not been obtained before. Note that, due to the presence of the $\frac{1}{r}$ factor in $|g_{00}|$, the above metric is different from the spatial Ellis geometry obtained in Ref. [28] for a vanishing torsion scalar in the context of modified teleparallel gravity. For $\beta = -Q^2$, the general metric reduces to the spatial Reissner-Nordström metric. The spatial Reissner-Nordström metric further reduces to the Reissner-Nordström metric for $C_1 = 0$ and $C_2 = 1$.

Our focus is on the general metric (9). Depending on the signs and values of m, β, C_1 and C_2, the general spacetime may represent a black hole, a wormhole or a naked singularity. The nature of the solution depends on the presence of the zeros of $f(r)$ and $g(r)$. We will focus on solutions for which $m > 0$, $\beta > 0$ and $\eta = \frac{C_2}{C_1 m} > -1$. The solution for such choices does represent a Lorentzian wormhole.

For later use, let us now write down the solution in isotropic coordinates. The transformation from the Schwarzschild radial coordinate r to the isotropic coordinate R is given as

$$r = \left(1 + \frac{m}{R} + \frac{m^2+\beta}{4R^2}\right)R. \quad (13)$$

Using the transformation between r and R we obtain, in isotropic coordinates, the line element,

$$ds^2 = -\frac{h^2(R)}{U^2(R)} dt^2 + U^2(R)(dR^2 + R^2 d\Omega_2^2) \quad (14)$$

where

$$h(x) = (C_1 m - C_2)(q_1 + x)(q_2 + x), \quad (15)$$

$$U(x) = 1 + x^2 + 2\mu x \quad (16)$$

and

$$x = \frac{\sqrt{m^2+\beta}}{2R}, \qquad \mu = \frac{m}{\sqrt{m^2+\beta}}. \quad (17)$$

The constants q_1 and q_2 are given as

$$q_1 = \frac{\mu(1+\eta)}{1-\sqrt{\mu^2\eta^2+1-\mu^2}}, \quad q_2 = \frac{\mu(1+\eta)}{1+\sqrt{\mu^2\eta^2+1-\mu^2}} \quad (18)$$

and $\eta = \frac{C_2}{C_1 m}$. It is easy to show that η and μ are related to q_1 and q_2 as follows:

$$\mu = \frac{q_1 q_2 + 1}{q_1 + q_2}, \qquad \eta = \frac{q_1 q_2 - 1}{q_1 q_2 + 1}. \quad (19)$$

It can be shown that, for a spacetime without a horizon $\eta > -1$ is necessary. We will use the isotropic coordinate form of the line element while discussing the weak energy condition.

Defining $\xi = \sqrt{1+\phi}$ one can solve the radion field equation as shown in Ref. [2]. Obviously one ends up introducing a new constant which we denote as γ. We have

$$\xi = \frac{2\gamma}{q_1 - q_2} \log\left|\frac{q_1 + x}{q_2 + x}\right| \tag{20}$$

where we have omitted an overall additive constant. It is easy to see that $\xi^2 - 1$ is never zero as long as $q_1 > q_2 > 0$. Note that the domain of x is from 0 to 1, where $x = 1$ corresponds to the wormhole throat.

We will now turn towards analyzing the weak energy condition inequalities and demonstrate that it can indeed be satisfied for the wormhole we have constructed.

IV. WEAK ENERGY CONDITION

The weak energy condition comprises the inequalities

$$\rho \geq 0, \quad \rho + \tau \geq 0, \quad \rho + p \geq 0$$

for a diagonal energy-momentum tensor with energy density ρ, radial pressure τ and tangential pressure p defined in the static observer's frame. Physically, the WEC means that the matter energy density is always non-negative in any frame of reference.

For our case here, we have, for the lhs of the WEC inequalities,

$$\frac{\kappa^2}{l}\rho = \frac{16\mu^2(\mu^2-1)x^4}{m^2U^2}(\xi^2-1) + \frac{16\gamma\mu^2x^4}{m^2(q_1+x)^2(q_2+x)^2}\left[\gamma - \xi\frac{q_1+q_2-2q_1q_2\mu+2x(1-q_1q_2)+x^2(2\mu-q_1-q_2)}{U}\right], \tag{21}$$

$$\frac{\kappa^2}{l}(\rho+\tau) = \frac{8\mu^2x^3}{m^2(q_1+x)^2(q_2+x)^2} \times\left[8\gamma^2x + 4\gamma\xi(q_1q_2-x^2) - \frac{(q_1+q_2+2x+(q_1+q_2)x^2+2q_1q_2x)(q_1+x)(q_2+x)}{U}(\xi^2-1)\right], \tag{22}$$

$$\frac{\kappa^2}{l}(\rho+p) = \frac{\kappa^2}{l}\left[2\rho - \frac{1}{2}(\rho+\tau)\right] \tag{23}$$

where the last equation follows from the traceless requirement on the matter energy-momentum tensor.

For arbitrary values of the parameters it is not possible to satisfy these inequalities. However, one can isolate the negativity by looking at the requirements that emerge near $x = 0$. In the limit $x \to 0$, we have

$$\frac{\kappa^2}{l}\rho = \frac{16\mu^2}{m^2}\left[(\mu^2-1)(\xi_0^2-1) + \frac{\gamma^2}{q_1^2q_2^2} - \frac{\gamma\xi_0}{q_1^2q_2^2}(q_1+q_2-2\mu q_1q_2)\right]x^4 + \mathcal{O}(x^5), \tag{24}$$

$$\frac{\kappa^2}{l}(\rho+\tau) = \frac{8\mu^2}{m^2q_1q_2}[-(\xi_0^2-1)(q_1+q_2)+4\gamma\xi_0]x^3 + \mathcal{O}(x^4), \tag{25}$$

$$\frac{\kappa^2}{l}(\rho+p) = -\frac{\kappa^2}{2l}(\rho+\tau) + \mathcal{O}(x^4) \tag{26}$$

where $\xi_0 = \xi(x=0)$. Note that, at the leading order, [i.e., terms which are $\mathcal{O}(x^3)$], $(\rho+\tau)$ and $(\rho+p)$ are opposite in sign. This violates the WEC in the limit $x \to 0$. Therefore, we must set the coefficient of x^3 in $(\rho+\tau)$ to zero. This yields the following expression for γ:

$$\gamma^2 = \frac{(q_1-q_2)^2(q_1+q_2)}{4(q_1+q_2)(\log\frac{q_1}{q_2})^2 - 8(\log\frac{q_1}{q_2})(q_1-q_2)} \tag{27}$$

which reduces to the expression quoted in Ref. [2] for $q_1 = q$ and $q_2 = 1$. With the above condition satisfied, it can be shown that in $(\rho+\tau)$, the $\mathcal{O}(x^4)$ term vanishes as well and the $\mathcal{O}(x^5)$ term is always positive for any positive q_1 and q_2. Therefore, it is clear from the traceless condition that, at the $\mathcal{O}(x^4)$, positivity of ρ implies positivity of $(\rho+p)$. Note that, at this point, q_1 and q_2 can be chosen so that ρ and γ^2 are positive. However, we choose those values of q_1 and q_2 for which the coefficient of the $\mathcal{O}(x^4)$ term in ρ vanishes. This gives us the following requirement on γ^2:

$$\gamma^2 = \frac{(q_1-q_2)^2q_1^2q_2^2(1-\mu^2)}{4(1-\mu^2)q_1^2q_2^2(\log\frac{q_1}{q_2})^2 + 2(q_1-q_2)(q_1+q_2-2q_1q_2\mu)\log\frac{q_1}{q_2} - (q_1-q_2)^2}. \tag{28}$$

Equating the above two expressions for γ^2, one arrives at a relation between q_1 and q_2 which must be obeyed:

$$[8q_1^2q_2^2(1-\mu^2) + 2(q_1+q_2)(q_1+q_2-2q_1q_2\mu)] \times \log\frac{q_1}{q_2} = q_1^2 - q_2^2. \tag{29}$$

Thus, once we are able to choose a q_1 and a q_2 which satisfy the above relation, we can use their values to find γ. The set of values for q_1, q_2 and γ can therefore be used to write down the line element and the scalar field profile for a Lorentzian wormhole in scalar-tensor gravity which will satisfy all the WEC inequalities.

The central issue at this point is whether we can obtain an analytical handle on the relation between q_1 and q_2. To get rid of the logarithm, we can assume $q_1 = q_2e^y$. This will eventually lead us to a quartic equation in q_2 for which we can obtain the positive, real-valued roots in terms of y. Choosing y one can then obtain q_1, q_2 and then γ.

We will now illustrate the above statements with examples and also demonstrate our claim that the WEC is indeed satisfied for the wormholes we have constructed. Putting $q_1 = q_2e^y$ in Eq. (29), we obtain

$$Aq_2^4 + Bq_2^2 + C = 0 \Rightarrow q_2^2 = \frac{-B \pm \sqrt{B^2-4AC}}{2A} \tag{30}$$

where,

$$A = -4, \qquad B = 2(e^{-y}-1)^2, \tag{31}$$

$$C = (e^{-2y}-1)^2 + 2e^{-y}(e^{-2y}+1) - \frac{1}{2y}(e^{-y}+1)^2(1-e^{-2y}). \tag{32}$$

Therefore, we fix q_1 and q_2 and hence μ, η and γ by choosing a particular value of y. It can be shown that $-1 < \eta < 1$ for the whole range $-\infty < y < \infty$. Therefore, the definitions of q_1 and q_2 indicate that both are positive. Figure 1 shows the parametric plot for q_1 and q_2. Figure 2 shows the dependence of γ on q_1. Figures 3–5 show the lhs of the WEC inequalities for three sets of parameter values. It is to be noted that WEC inequalities are satisfied.

V. THE $m = 0$ LIMIT AND WEC

Let us now take the limit $m \to 0$. In this limit $\mu \to 0$ and η diverges. But, the product $\delta = \mu\eta$ is finite. In terms of δ, we have

$$q_1 = \frac{\delta}{1-\sqrt{1+\delta^2}}, \qquad q_2 = \frac{\delta}{1+\sqrt{1+\delta^2}},$$
$$q_1q_2 = -1. \tag{33}$$

Therefore, q_1 and q_2 have opposite signs. In this limit, the factor $\frac{\mu^2}{m^2}$ in the energy density and pressures must be

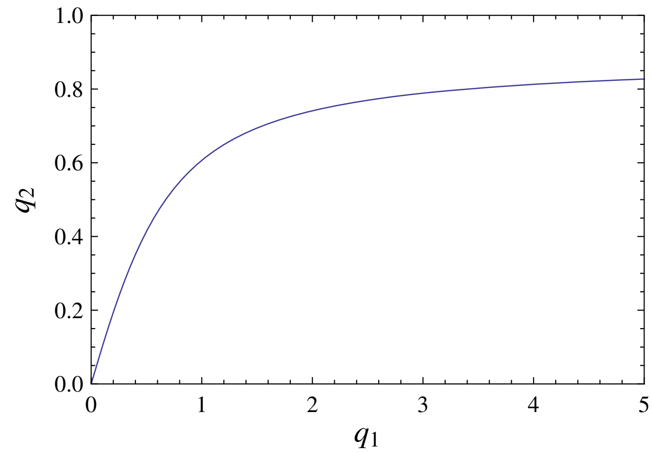

FIG. 1. Plot of q_2 as a function of q_1.

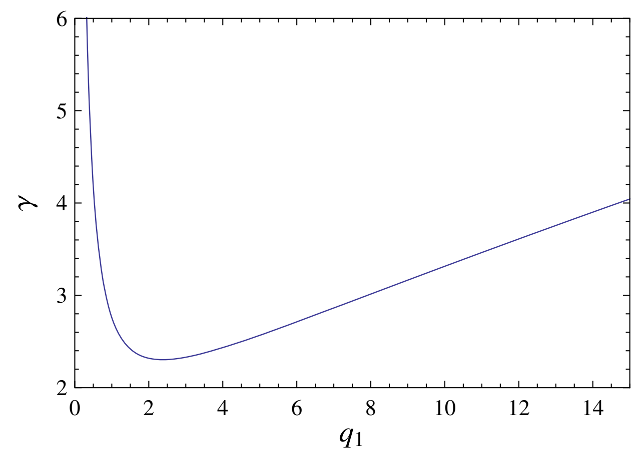

FIG. 2. Plot showing γ as a function of q_1.

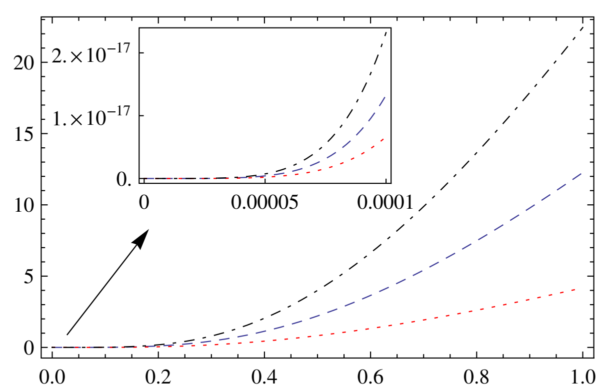

FIG. 3. Plots of $\frac{\kappa^2}{l}\rho$ (blue dashed curve), $\frac{\kappa^2}{l}(\rho+\tau)$ (red dotted curve) and $\frac{\kappa^2}{l}(\rho+p)$ (black dot-dashed curve) as a function of x for $y = 0.2948$ which corresponds to $\eta \simeq -0.5$, $q_1 \simeq 0.669$, $q_2 \simeq 0.498$, $\gamma \simeq 3.419$ and $\mu \simeq 1.142$.

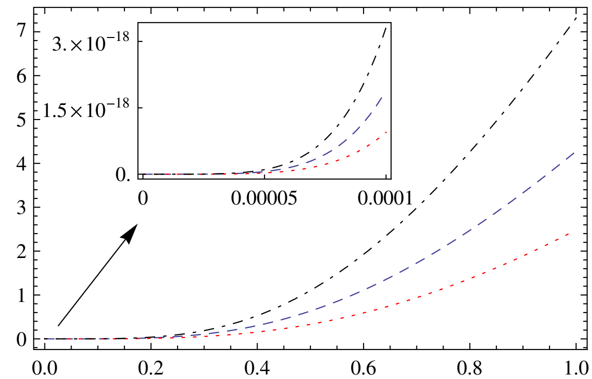

FIG. 4. Plots of $\frac{\kappa^2}{l}\rho$ (blue dashed curve), $\frac{\kappa^2}{l}(\rho+\tau)$ (red dotted curve) and $\frac{\kappa^2}{l}(\rho+p)$ (black dot-dashed curve) as a function of x for $y = 0.7475$ which corresponds to $\eta \simeq 0.0$, $q_1 \simeq 1.453$, $q_2 \simeq 0.688$, $\gamma \simeq 2.437$ and $\mu \simeq 0.934$.

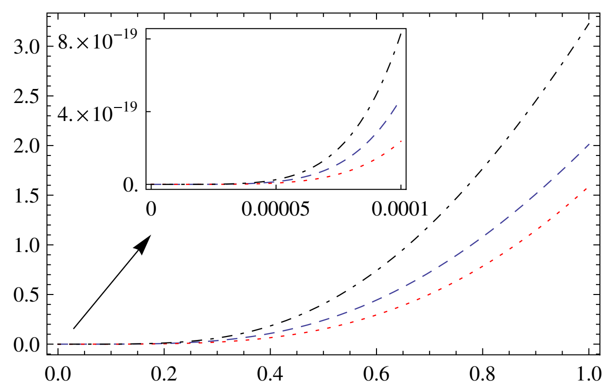

FIG. 5. Plots of $\frac{\kappa^2}{l}\rho$ (blue dashed curve), $\frac{\kappa^2}{l}(\rho+\tau)$ (red dotted curve) and $\frac{\kappa^2}{l}(\rho+p)$ (black dot-dashed curve) as a function of x for $y = 1.5267$ which corresponds to $\eta \simeq 0.5$, $q_1 \simeq 3.716$, $q_2 \simeq 0.807$, $\gamma \simeq 2.4$ and $\mu \simeq 0.884$.

replaced by $\frac{1}{\beta}$. It can be shown that $h(x)$ does not have zeros in the range $0 \leq x \leq 1$ for $\delta > 0$, thereby representing a wormhole. But, for $\delta < 0$, it does have a zero and represents a naked singularity. Let us first consider the case $\delta > 0$. Like the $m \neq 0$ case, here also, one must consider the expression (27). With this, it can be shown that, in the limit $x \to 0$, the coefficient of the $\mathcal{O}(x^4)$ term in ρ cannot be set to zero since it is negative for all $\delta > 0$, thereby violating the WEC. It also implies that the expression in Eq. (28) is not valid. However, we can make use of the freedom to add an additive constant to ξ and make this coefficient positive. If we do so, then we find that $\gamma^2 < 0$ in Eq. (27), i.e., γ is imaginary. On the other hand, it can be shown that if we force γ^2 to be positive, then either ρ or $(\rho+\tau)$ or both become negative in the limit $x \to 0$. Therefore, the wormholes without the mass term "m" violate the WEC. Figure 6 shows the WEC violation for $m = 0$. We have carried out a similar analysis for the naked

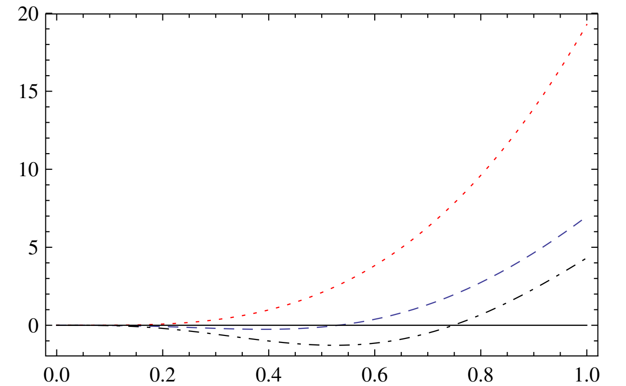

FIG. 6. Plots of $\frac{\kappa^2}{l}\rho$ (blue dashed curve), $\frac{\kappa^2}{l}(\rho+\tau)$ (red dotted curve) and $\frac{\kappa^2}{l}(\rho+p)$ (black dot-dashed curve) as a function of x for $\delta = 0.5$.

singularity case ($\delta < 0$) where we found that the WEC can be satisfied in the limit $x \to 0$, but the radion field ξ vanishes at a point between the singularity and spatial infinity. This gives an invalid solution. Therefore, WEC cannot be satisfied for the naked singularity.

VI. GRAVITATIONAL REDSHIFT

Let us now look at the gravitational redshift of a light signal propagating in the $R = 0$ wormhole spacetimes discussed above. Note that by virtue of having a g_{00} which is everywhere finite and nonzero, the gravitational redshift is always finite. By definition, wormhole spacetimes are horizon-free. Assume that a light signal is emitted at a frequency ω_0 from the wormhole throat ($x = 1$) and it travels to spatial infinity ($x = 0$) where a static observer receives it at a frequency ω_∞. Therefore, the fractional change in frequency due to the gravitational redshift is given by the standard formula.

$$\frac{\Delta\omega}{\omega_\infty} = \frac{|g_{00}(x=0)|^{1/2}}{|g_{00}(x=1)|^{1/2}} - 1 = \frac{2q_1 q_2(1+\mu)}{(q_1+1)(q_2+1)} - 1. \qquad (34)$$

We choose values of q_1 and q_2 which satisfy Eq. (29) and hence, the WEC. Figure 7 shows the fractional change in frequency as a function of q_1. It is worth noting that the light signal may have a negative gravitational redshift, i.e., a gravitational blueshift ($\Delta\omega < 0$) for certain values of q_1 and q_2. For the three sets of q_1 and q_2 given in Figs. 3–5, $\frac{\Delta\omega}{\omega_\infty}$ are respectively -0.429, -0.066 and 0.3265. For $q_1 \simeq 1.672$ and $q_2 \simeq 0.713$, $\frac{\Delta\omega}{\omega_\infty}$ vanishes. Therefore, the wormholes can exhibit both redshift and blueshift. It can be shown that, for those q_1 and q_2 satisfying the WEC, the maximum redshift is $\frac{\Delta\omega}{\omega_\infty}|_{\max} = (\sqrt{1+\sqrt{5}} - 1) \simeq 0.798$. This redshift does not exceed 80% for such values of q_1 and q_2. However, we can have more than 80% redshift if we choose q_1 and q_2 arbitrarily, which obviously violates the WEC. Thus, in a certain sense, as mentioned above, the

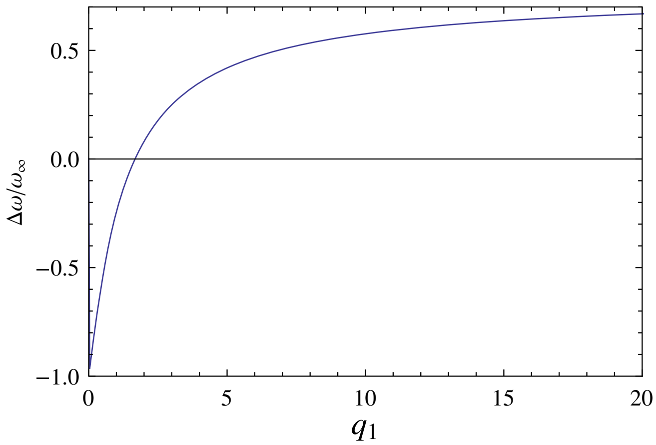

FIG. 7. Plot showing the fractional change in frequency suffered by a signal emitted from the wormhole throat and propagating to spatial infinity, as a function of q_1 with q_2 and q_1 satisfying Eq. (29).

WEC does restrict the gravitational redshift to an amount below 80%. It is possible to use this result (along with other signatures) to identify whether the cause of an observed gravitational frequency shift is indeed due to a wormhole.

VII. EFFECTIVE POTENTIALS, CIRCULAR ORBITS AND TRAVERSABILITY

A. Effective potentials and circular orbits

The geodesic equations for a test particle moving in the equatorial ($\theta = \frac{\pi}{2}$) plane of the spacetime (9) reduce to the following first integrals:

$$\dot{t} = \frac{E}{f^2(r)}, \qquad \dot{\phi} = \frac{L}{r^2}, \qquad \frac{f^2(r)\dot{r}^2}{1-\frac{b(r)}{r}} + V(r) = E^2 \tag{35}$$

where the energy E and the angular momentum L are constants of motion. An overdot represents differentiation with respect to the parameter λ (affine for timelike and arbitrary for null geodesics). The effective potential $V(r)$ is given by

$$V(r) = \left(-s + \frac{L^2}{r^2}\right) f^2(r) \tag{36}$$

where the normalization constant $s = \dot{x}_\mu \dot{x}^\mu$ is -1 for timelike geodesics and is 0 for null geodesics. We now look for circular orbits which correspond to $\dot{r} = 0$ (i.e. $V = E^2$) and $V'(r) = 0$ (i.e. extrema of the effective potential). Minima $[V''(r) > 0]$ and maxima $[V''(r) < 0]$ of the effective potential correspond to the stable and unstable circular orbits, respectively. For the line element in Eq. (9), the effective potential can be written as

$$V(z) = \left[-s + \frac{\mu^2 l^2}{(1+\mu)^2} z^2\right] f^2(z) \tag{37}$$

where

$$f(z) = \frac{1}{(1+\eta)}\left(1 + \frac{1-\mu}{\mu} z + \eta\sqrt{1 - \frac{2\mu}{1+\mu} z - \frac{1-\mu}{1+\mu} z^2}\right), \tag{38}$$

$l = \frac{L}{m}$ and $z = \frac{r_0}{r} = \frac{m+\sqrt{m^2+\beta}}{r} = \frac{m(1+\mu)}{\mu r}$, with r_0 being the throat radius. Here, we have normalized $f(r)$ by dividing it by $(C_1\alpha + C_2)$ such that $|g_{tt}| \to 1$ as $r \to \infty$. The effective potential is plotted in Figs. 8 and 9 for the parameter values satisfying the WEC. By choosing y, we find q_2 from Eq. (30), q_1 from $q_1 = q_2 e^y$ and hence μ and η from Eq. (19). It should be noted that both stable and unstable circular orbits exist for the timelike case. However, for the null case, no stable circular orbit exists for the parameter values satisfying the WEC. But, stable circular orbits exist if we choose the parameter values arbitrarily which, of course, violates the WEC. To check this, we first note that the extrema $z_{\rm ex}$ of the potential are given by

$$\frac{1}{f(z_{\rm ex})}\frac{df}{dz}\bigg|_{z_{\rm ex}} + \frac{1}{z_{\rm ex}} = 0. \tag{39}$$

Using Eqs. (7) and (39), it can be shown that

$$\frac{d^2V}{dz^2}\bigg|_{z_{\rm ex}} = -\frac{4l^2\mu^3}{(1+\mu)^3}\frac{f^2(z_{\rm ex})}{1-\frac{2\mu}{1+\mu}z_{\rm ex} - \frac{1-\mu}{1+\mu}z_{\rm ex}^2}\left(\frac{1+\mu}{\mu} - \frac{3}{2}z_{\rm ex}\right). \tag{40}$$

Therefore, minima ($\frac{d^2V}{dz^2} > 0$) exist if $z_{\rm ex}$ are real and $z_{\rm ex} > \frac{2(1+\mu)}{3\mu}$. Note that we must have $0 \le z_{\rm ex} \le 1$. This is possible when $\mu > 2$. Figure 10 shows the plots for the potential for parameter values chosen arbitrarily. We have checked that the WEC is violated for these parameter values. Therefore, stable null circular orbits exist if the WEC is violated. It should be noted that, for the parameter values used in Fig. 10, q_1 and q_2 become complex conjugates of each other because the square root terms in the expressions of q_1 and q_2 become imaginary. But, the metric function $h(x)$ still remains real since $(q_1 + q_2)$ and $q_1 q_2$ are real. It can also be shown that $\xi(x)$ is real. To verify this, let $q_1 = a + ib$ where a and b are real. Therefore, $q_2 = a - ib$. Writing $(q_1 + x) = \sqrt{(a+x)^2 + b^2}e^{i\theta(x)}$ and $(q_2 + x) = \sqrt{(a+x)^2 + b^2}e^{-i\theta(x)}$ where $\tan\theta(x) = \frac{b}{a+x}$, we obtain

$$\xi(x) = \frac{2\gamma}{2ib}\log\left[\frac{a+ib+x}{a-ib+x}\right] = \frac{2\gamma\theta(x)}{b} \tag{41}$$

which is real.

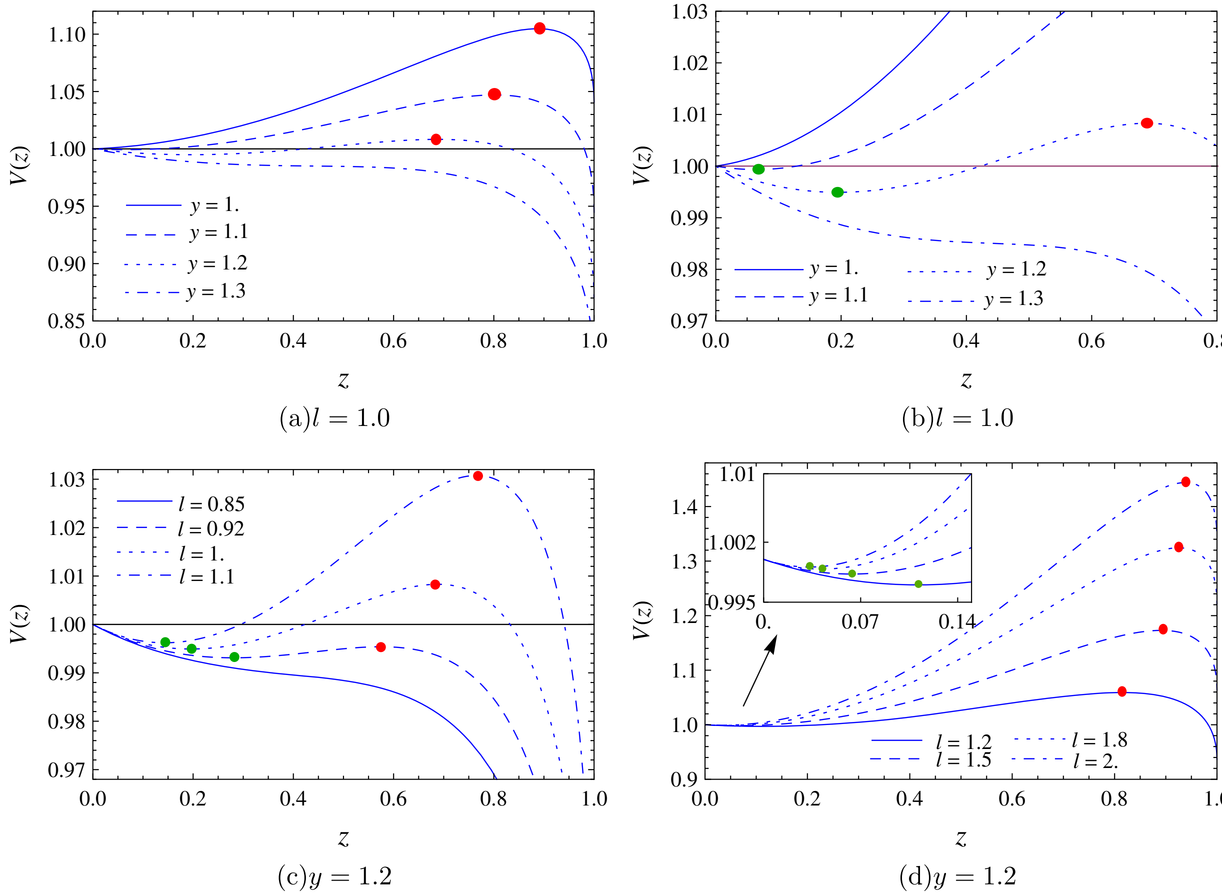

FIG. 8. Plots of the effective potential for timelike geodesics for (a), (b) different y and (c), (d) different l. (b) is the zoomed in version of (a). The green and red dots represent respectively the stable and unstable circular orbits.

B. Traversability

Traversability of a wormhole demands that the tidal force felt by a human traveler moving radially, must be within tolerable limits. In an orthonormal basis $\{e_{\hat{0}'}, e_{\hat{1}'}, e_{\hat{2}'}, e_{\hat{3}'}\}$ attached to the traveler frame, the tidal acceleration between two parts of his/her body, separated by the deviation vector $\xi^{\hat{i}'}$ is given by [29]

$$\Delta a^{\hat{j}'} = -c^2 R^{\hat{j}'}_{\hat{0}'\hat{k}'\hat{0}'} \xi^{\hat{k}'}, \tag{42}$$

where $R^{\hat{i}'}_{\hat{j}'\hat{k}'\hat{l}'}$ is the Riemann tensor. At the throat, the components of the tidal acceleration are given by

$$\Delta a^{\hat{1}'}|_{r_0} = \begin{cases} \frac{\beta c^2}{r_0^4} \xi^{\hat{1}'} & : C_1 \neq 0, \\ \left(\frac{2m}{r_0^3} + \frac{3\beta}{r_0^4}\right) c^2 \xi^{\hat{1}'} & : C_1 = 0, \end{cases} \tag{43}$$

$$\Delta a^{\hat{2}',\hat{3}'}|_{r_0} = \frac{\sqrt{m^2 + \beta}}{r_0^3} \bar{\gamma}_0^2 v_0^2 \xi^{\hat{2}',\hat{3}'} \tag{44}$$

where $\bar{\gamma} = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$ and $v = \pm \frac{\sqrt{g_{rr}} dr}{\sqrt{|g_{tt}|} dt}$ is the radial velocity of the traveler as measured by a static observer. v_0 denotes the radial velocity at the throat. For the wormhole solutions, we must have $C_1 \neq 0$. Note that, the radial component of the tidal acceleration vanishes for $\beta = 0$.

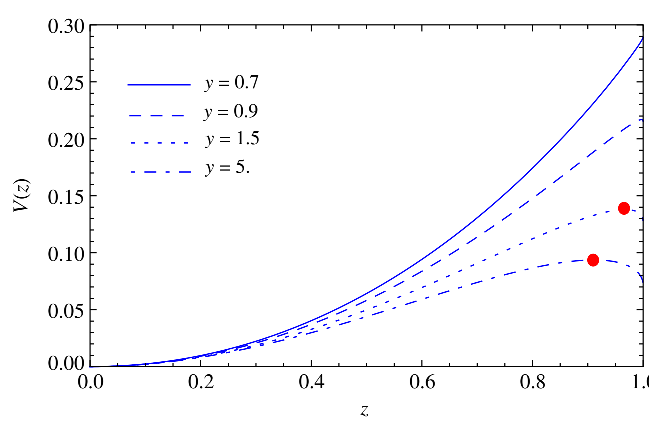

FIG. 9. Plots of the effective potential for null geodesics for different y. Red dots represent unstable circular orbits. Here, we have taken $l = 1.0$.

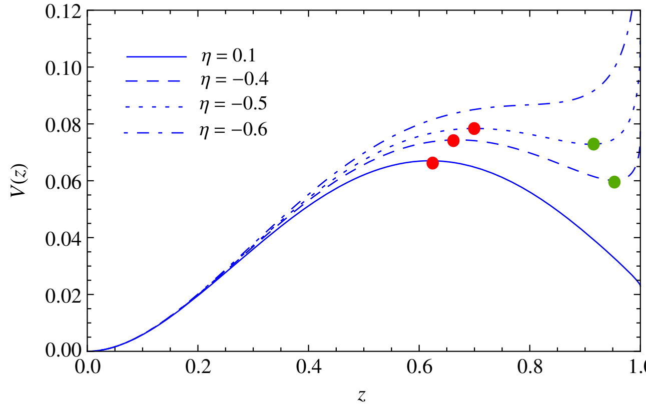

FIG. 10. Plots of the effective potential for null geodesics for different η. The green and red dots represent respectively the stable and unstable circular orbits. Here, we have taken $\mu = 5.0$ and $l = 1.0$.

Therefore, it can be made arbitrarily small by choosing an appropriate value of β. We now restrict the radial component below one Earth gravity, i.e., $|\Delta a^{\hat{1}'}| \leq g$. Thus, for a traveller of typical size $|\xi| \sim 2$ meters, we obtain

$$m \geq \sqrt{\frac{2c^2}{g}}\sqrt{\frac{\mu^2|(1-\mu)|}{(1+\mu)^3}}, \tag{45}$$

$$\beta \begin{cases} \geq \frac{2c^2}{g}\frac{|(1-\mu)|(1-\mu)}{(1+\mu)^2} & : \mu \leq 1, \\ \leq \frac{2c^2}{g}\frac{|(1-\mu)|(1-\mu)}{(1+\mu)^2} & : \mu \geq 1 \end{cases} \tag{46}$$

where we have used the expressions $\mu = \frac{m}{\sqrt{m^2+\beta}}$ and $r_0 = m + \sqrt{m^2+\beta} = \frac{m(1+\mu)}{\mu}$. Note that m and β are related through μ. For a given metric parameter μ, the constraint on m (and hence on β) puts a constraint on the throat radius. The constraint on r_0 is given by

$$r_0 \geq \sqrt{\frac{2c^2}{g}}\sqrt{\frac{|(1-\mu)|}{(1+\mu)}}. \tag{47}$$

For $\mu = 0.934$ which corresponds to Fig. 4, we obtain $r_0 \geq 3.9 R_E$, where $R_E = 6400$ km is the Earth's radius. However, we can reduce the lower limit on r_0 if we choose μ to be close to 1.

If the wormhole is to be traversable, not only the tidal acceleration but also the magnitude of acceleration felt by the traveler as he or she travels through it, must be within tolerable limits. In the orthonormal basis of the traveler frame, the magnitude of the acceleration is given by [29]

$$a = \mp \frac{1}{f(r)}\sqrt{1 - \frac{b(r)}{r}}(\bar{\gamma} f(r))' c^2. \tag{48}$$

Let us now consider both $\bar{\gamma}$ and $\bar{\gamma}'$ to be finite at the throat. Therefore, at the throat r_0, we have

$$|a|_{r_0} = \frac{|C_2|}{|C_1| r_0^2}\bar{\gamma}_0 c^2 = \frac{m|\eta|}{r_0^2}\bar{\gamma}_0 c^2. \tag{49}$$

For the wormhole solutions, we must have $\eta > -1$. Therefore, for a given r_0 and m which restrict the tidal acceleration below one Earth gravity, the magnitude of the acceleration can be made arbitrarily small by taking η to be very close to zero which, of course, satisfies the WEC (Fig. 4).

VIII. CONCLUSION

In this work, we have obtained a class of static, spherically symmetric $R = 0$ spacetimes which generalize certain known $R = 0$ line elements. We have shown that our spacetimes can arise in a scalar-tensor theory of gravity, an example of which is the low-energy, effective on-brane gravity theory developed by Kanno and Soda [16] in the context of the Randall-Sundrum two-brane model. Some special cases of the general solution reproduce the $R = 0$ spacetimes obtained earlier by other authors in the context of GR and in the on-brane effective gravity theory due to Shiromizu-Maeda-Sasaki [18] for a single-brane scenario. A subclass of the general spacetimes found by us represents $R = 0$ Lorentzian wormholes. We have studied the WEC for the matter that supports such wormholes and have shown that they can satisfy WEC. This is in contrast to GR, where such wormholes must violate WEC. We have also shown that the mass term in the metric functions is necessary to satisfy the WEC for these wormholes. Note that most of the wormhole solutions obtained in Ref. [6] are in vacuum-Brans-Dicke theory. However, the solutions we have obtained include both matter and a nonconstant, i.e., a Φ-dependent coupling $\omega(\Phi)$. As an additional curiosity, we also looked at the WEC for the naked singularity with the parameter $m = 0$, and have found that it is violated.

Apart from constructing solutions and checking the WEC, we have calculated the gravitational redshift of a light signal traveling in such wormholes and have found that the WEC restricts the amount of redshift to be below 80%. Furthermore, we obtained the effective potentials for particle motion and investigated the possibility of circular orbits in these wormhole spacetimes. It has been shown that both stable and unstable circular orbits exist for the timelike case. For the null case, only unstable circular orbits are possible if the WEC is satisfied. But, stable null circular orbits exist if the WEC is violated. Finally, we analyzed traversability in such wormholes and found that they are traversable for values of the metric parameters satisfying the WEC. Therefore, we now have wormholes which satisfy the WEC and are traversable. In addition, they may also be viewed as on-brane spacetimes if one considers the existence of warped extra dimensions of the Randall-Sundrum variety, in a two-brane context. In such a context the scalar field (radion) is related to the inter-brane distance and we have seen that this can always be kept finite and nonzero everywhere.

Recently, it has been argued that the universal ringdown waveforms from a binary coalescence indicate the presence of light rings, rather than of horizons [30]. Therefore, an object (e.g. a traversable wormhole) with a light ring will display a similar initial ringdown stage (waveform), even when its quasinormal mode spectrum is completely different from that of a black hole. However, their late-time ringdown stages are different [30,31]. It has further been shown that, if one does not use the thin-shell wormhole (the one used in Ref. [30]) obtained by special matching near the Schwarzschild radius but prefers a wormhole configuration constructed without thin shells instead, a wormhole may ring like a black hole or differently at all times, depending on the values of the wormhole and black hole parameters [32]. This leads to the hope that the Lorentzian wormholes we have obtained here, may be relevant in such studies and may have an observational consequence in astrophysics, in the future.

Further, it is to be noted that the $R = 0$ spacetime solution we have obtained can be used in the context of any other theory of gravity (different from the one considered here) to study the relation between the energy conditions and wormholes. We have noted that a subclass of our $R = 0$ spacetimes represents naked singularities. Therefore, as an observational test, it will be interesting to study gravitational lensing in the spacetimes we have obtained here and to see whether we can find ways to distinguish between wormholes and naked singularities [33] through observations.

ACKNOWLEDGMENTS

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