Cosmological solutions of $f(T)$ gravity

Andronikos Paliathanasis,[1,*] John D. Barrow,[2,†] and P. G. L. Leach[3,4,5,‡]

[1]*Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia, Chile*
[2]*DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom*
[3]*Department of Mathematics and Institute of Systems Science, Research and Postgraduate Support, Durban University of Technology, P.O. Box 1334, Durban 4000, South Africa*
[4]*School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa*
[5]*Department of Mathematics and Statistics, University of Cyprus, Lefkosia 1678, Cyprus*

(Received 2 June 2016; published 28 July 2016)

In the cosmological scenario in $f(T)$ gravity, we find analytical solutions for an isotropic and homogeneous universe containing a dust fluid and radiation and for an empty anisotropic Bianchi I universe. The method that we apply is that of movable singularities of differential equations. For the isotropic universe, the solutions are expressed in terms of a Laurent expansion, while for the anisotropic universe we find a family of exact Kasner-like solutions in vacuum. Finally, we discuss when a nonlinear $f(T)$-gravity theory provides solutions for the teleparallel equivalence of general relativity and derive conditions for exact solutions of general relativity to solve the field equations of an $f(T)$ theory.

DOI: 10.1103/PhysRevD.94.023525

I. INTRODUCTION

One of the most important unsolved problems of modern astronomy and particle physics is the identity of the "dark energy" that is evidently responsible for the observed acceleration of the universal expansion. The dynamics can be quite accurately described by the inclusion of a simple cosmological constant term to Einstein's equations but its required magnitude is mysterious and unmotivated by fundamental physics. Some new fundamental theory might eventually be able to provide a natural explanation (see, for an example, [1]) or there may be more complicated explanations in which the dark energy is not a constant stress but some time-dependent scalar (or effective scalar) field. Effective scalar fields are available in the many deviant theories of gravity that have been proposed as generalizations of Einstein's general theory of relativity. In the past it had been expected that deviations from Einstein's theory would only arise in situations of high, or formally infinite, spatial curvature—so-called curvature "singularities"—where the entire theory breaks down. However, the unusual challenge posed by the acceleration of the Universe is that it may require modifications to Einstein's theory in the late Universe when spatial curvature is very low. So-called "modified theories of gravity" provide one of these scenarios to explain the acceleration of the Universe. In contrast to the explicit dark-energy models, such as quintessence, phantom fields, Chaplygin gas or many others (see [2–5], and references therein), in which an energy-momentum tensor which violates the strong energy condition is added to the field equations of general relativity (GR), in modified gravity theories the dark energy often has a geometric origin and is related to new dynamical terms which follow from the modification of the Einstein-Hilbert action.

A particular modified theory of gravity which has attracted the interests of cosmologists is so-called $f(T)$ teleparallel gravity[1] [7–9]. Inspired by the formulation of $f(R)$ gravity, in which the Lagrangian of the gravitational field equations is a function, f, of the Ricci scalar R of the underlying geometry [10], $f(T)$ gravity is a similar generalization. Now, instead of using the torsionless Levi-Civita connection of GR, the curvatureless Weitzenböck connection is used in which the corresponding dynamical fields are the four linearly independent vierbeins, and T is related to the antisymmetric connection which follows from the nonholonomic basis [11–13].

A linear $f(T)$ theory leads to the teleparallel equivalent of GR (TEGR) [14]. However, $f(T)$ gravity does not coincide with $f(R)$ gravity. One of the main differences is that for a nonlinear $f(R)$ function, gravity is a fourth-order theory, whereas $f(T)$ gravity is always a second-order theory. This follows because T includes only first derivatives of the vierbeins. Moreover, while $f(T)$ gravity is a second-order theory and in the limit of a linear function, $f = R$, GR is recovered, in general $f(T)$ gravity provides different structural properties from those of GR. However, from the analysis of the cosmological data and the Solar System tests of GR we know that deviations from GR must

*anpaliat@phys.uoa.gr
†J.D.Barrow@damtp.cam.ac.uk
‡leach.peter@ucy.ac.cy

[1]For a recent review on $f(T)$ gravity see [6].

be small, and so $f(T)$ must be close to a linear form [15–18].

Even though $f(T)$ gravity is a second-order theory, very few exact analytical solutions of the field equations are known. Some power-law solutions in a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime can be found in [19,20], while some power-law solutions in anisotropic spacetimes are given in [21]. Finally, some analytical solutions in the case of static spherically symmetric spacetimes can be found in [22,23], and references therein.

In this work we are interested to determine exact solutions of the field equations in $f(T)$ gravity in the cosmological scenario of an isotropic and homogeneous universe and for the Bianchi I spacetime. Specifically, the method that we use is that of the singularity analysis of differential equations. Singularity analysis is complementary to symmetry analysis (for a discussion between the two methods see [24]). The application of Noether point symmetries for $f(T)$ gravity can be found in [19,20,22]. Recently, singularity analysis was applied in the cosmological scenario of $R+\alpha R^n$ gravity [25] and it was proved that, if n is a rational number and $n>1$, then the gravitational field equations pass the singularity test and the analytical solution of the field equations can be written as a Laurent expansion around the movable singularity of the field equations. The application of singularity analysis in gravitational studies is not new and has provided interesting results [26–28].

The plan of the paper is as follows. In Sec. II we define our model with $f(T)$ gravity in a spatially flat FLRW spacetime and the resulting gravitational field equations are presented. The singularity analysis of the field equations for some functions proposed in the literature for $f(T)$ is performed in Secs. III and IV. Specifically, we consider the power-law model $f(T)=T+\alpha(-T)^n$, which has been proposed in [7] as was the same model with the cosmological constant term: $f(T)=T+\alpha(-T)^n-\Lambda$. For these two models we find that the solution of the field equations for the FLRW universe can be written analytically in a Laurent expansion. However, the singularity analysis fails in the Bianchi I spacetime, but in the latter model we find that there exists an exact vacuum solution of the field equation which leads to a Kasner-like universe. In Sec. V, we construct conditions which allow solutions of GR to be recovered in $f(T)$ gravity. Finally, in Sec. VI, we discuss our results and draw our conclusions.

II. $f(T)$ GRAVITY

We briefly discuss the basic assumptions of $f(T)$ teleparallel gravity. The vierbein fields $\mathbf{e}_i(x^\mu)$, as nonholonomic frames in spacetime, are the dynamical variables of teleparallel gravity and consequently of the $f(T)$ gravity. The vierbein fields form an orthonormal basis for the tangent space at each point x^μ of the manifold, that is, $g(e_i,e_j)=\mathbf{e}_i\cdot\mathbf{e}_i=\eta_{ij}$, where η_{ij} is the line element of four-dimensional Minkowski spacetime. In a coordinate basis the vierbeins can be written as $e_i=h_i^\mu(x)\partial_i$, for which the metric tensor is defined as follows:

$$g_{\mu\nu}(x)=\eta_{ij}h_\mu^i(x)h_\nu^j(x). \tag{1}$$

The curvatureless Weitzenböck connection, which is considered in teleparallel gravity, has the non-null torsion tensor [29,30]

$$T^\beta_{\mu\nu}=\hat{\Gamma}^\beta_{\nu\mu}-\hat{\Gamma}^\beta_{\mu\nu}=h_i^\beta(\partial_\mu h_\nu^i-\partial_\nu h_\mu^i), \tag{2}$$

while the Lagrangian density of the teleparallel gravity, from which the gravitational field equations are derived, is the scalar

$$T=S_\beta{}^{\mu\nu}T^\beta{}_{\mu\nu}, \tag{3}$$

where

$$S_\beta{}^{\mu\nu}=\frac{1}{2}(K^{\mu\nu}{}_\beta+\delta^\mu_\beta T^{\theta\nu}{}_\theta-\delta^\nu_\beta T^{\theta\mu}{}_\theta) \tag{4}$$

and $K^{\mu\nu}{}_\beta$ is the contorsion tensor that is defined by

$$K^{\mu\nu}{}_\beta=-\frac{1}{2}(T^{\mu\nu}{}_\beta-T^{\nu\mu}{}_\beta-T_\beta{}^{\mu\nu}). \tag{5}$$

It equals the difference between the Levi-Civita connections in the holonomic and the nonholonomic frame.

The action for $f(T)$ gravity is

$$S_{f(T)}=\frac{1}{16\pi G}\int d^4xe(f(T))+S_m, \tag{6}$$

in which $e=\det(e^i_\mu)=\sqrt{-g}$. Variation with respect to the vierbein gives the gravitational field equations:

$$\begin{aligned}&e^{-1}\partial_\mu(ee_i^\rho S_\rho{}^{\mu\nu})f_T-e_i^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}f_T\\&\quad+e_i^\rho S_\rho{}^{\mu\nu}\partial_\mu(T)f_{TT}+\frac{1}{4}e_i^\nu f(T)\\&\quad=4\pi Ge_i^\rho\mathcal{T}_\rho{}^\nu,\end{aligned} \tag{7}$$

where f_T and f_{TT} denote the first and second derivatives, respectively, of the function $f(T)$ with respect to T and the tensor $\mathcal{T}_\rho{}^\nu$ denotes the energy-momentum tensor of the matter source S_m. Furthermore, from (7) we recover GR when $f_{TT}=0$.

A. Modified Friedmann equations

In order to recover the cosmological scenario of a spatially flat FLRW spacetime, we consider the diagonal frame for the vierbein:

$$h^i_\mu(t)=\mathrm{diag}(1,a(t),a(t),a(t)). \tag{8}$$

In the holonomic frame the spacetime has the line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),$$

where $a(t)$ is the cosmological scale factor. For this frame we calculate the Lagrangian density

$$T = -6\left(\frac{\dot{a}}{a}\right)^2 = -6H^2, \tag{9}$$

where $H = \dot{a}/a$ is the Hubble parameter, while the gravitational field equations (7) become

$$12H^2 f_T(T) + f(T) = 16\pi G\rho \tag{10}$$

and

$$48H^2\dot{H}f_{TT}(T) - 4(\dot{H} + 3H^2)f_T(T) - f(T) = 16\pi Gp, \tag{11}$$

in which ρ and p denote the energy density and pressure, respectively, of the energy-momentum tensor $\mathcal{T}_\rho{}^\nu$, from which we have the conservation equation

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{12}$$

However, Eqs. (10) and (11) can be rewritten as

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_T) \tag{13}$$

and

$$2\dot{H} + 3H^2 = -8\pi G(p + p_T), \tag{14}$$

where ρ_T and p_T are the effective energy density and pressure, respectively, of the geometric fluid which follow from the modification of the gravitational action integral. Specifically, ρ_T and p_T depend upon T and f_T and are

$$\rho_T = \frac{1}{16\pi G}[2Tf_T(T) - f(T) - T] \tag{15}$$

and

$$p_T = \frac{1}{16\pi G}[4\dot{H}(2Tf_{TT}(T) + f_T(T) - 1)] - \rho_T. \tag{16}$$

An effective equation of state parameter for the geometric fluid can be defined as usual by

$$w_T \equiv \frac{p_T}{\rho_T} = -1 + \frac{4\dot{H}[2Tf_{TT}(T) + f_T(T) - 1]}{2Tf_T(T) - f(T) - T}. \tag{17}$$

From this, if we consider that $f(T) = T + F(T)$, then (17) takes the simpler form

$$w_T = -\frac{F - TF_T + 2T^2F_{TT}}{(1 + F_T + 2TF_{TT})(F - 2TF_T)}. \tag{18}$$

In [16], the model $f_I \equiv f(T) = T + \alpha(-T)^n$ has been proposed as an alternative to the dark-energy models and fits some of the cosmological data quite well. Furthermore, the parameters of that model have been derived from cosmography in [31], while in [17] it has been constrained within the Solar System and it has been found that the perturbation to GR solution is given in terms of powers r^{2-2n} of distance r from a central point mass. Furthermore, in [17], they performed the analysis by including the cosmological constant term, i.e.,$f_{II} \equiv f(T) = T + \alpha(-T)^n - \Lambda$. These two models, f_I and f_{II}, are the models we study here. In what follows we will consider the two models $f_I(T)$ and $f_{II}(T)$, with $n \neq 0$, 1 (as we are in the teleparallel equivalence of GR) and $n \neq \frac{1}{2}$ (so we are close to GR in the limit). In order to check the latter condition, consider $f(T) = F(T) + \beta\sqrt{-T}$, in (6), where $F(T)$ is an arbitrary function. Then,

$$S = \frac{1}{16\pi G}\int d^4xeF(T) + \frac{\beta}{16\pi G}\int d^4xe\sqrt{-T}, \tag{19}$$

where, using (9), the second term becomes a total derivative, i.e., $e\sqrt{-T} = a^2\dot{a} = \frac{1}{3}\frac{d}{dt}(a^3)$, which does not affect the field equations. Furthermore, if we consider that $F(T) = -\beta\Lambda$, then the action (6) with the use of (9) becomes

$$S = \frac{\beta}{16\pi G}\int d^4x(a^2\dot{a} - a^3\Lambda) = -\frac{\beta}{16\pi G}\int d^4x(a^3\Lambda). \tag{20}$$

Hence. the field equations (9)–(11) cannot be recovered. Consider the diagonal frame

$$h^i_\mu(t) = \mathrm{diag}(a^{-3}(\tau), a(\tau), a(\tau), a(\tau)), \tag{21}$$

where the line element is that of FLRW spacetime with a lapse function $N(\tau) = a^{-3}(\tau)$, i.e., $dt = N(\tau)d\tau$. Again, $\sqrt{-T}$ is a linear function of $\dot{a}$, and the gravitational Lagrangian is a total derivative, which is something that has not been observed recently in [32].

However, in the case of vacuum, Eq. (10) can be written as

$$f - 2Tf_T = 0, \tag{22}$$

which indeed admits as a solution the case $f(T) = \sqrt{-T}$ but also has a special solution $f(T)|_{T->0} = 0$, which means that $a(t) = \text{const}$, and we have the solution of GR in empty spacetime.

For the fluid components of the field equations, we take a dust fluid, with $p_m = 0$, and a radiation fluid, $p_r = \frac{1}{3}\rho_r$. We assume that the two fluids are not interacting and are minimally coupled to gravity; hence, (12) for each fluid gives $\rho_m = \rho_{m0}a^{-3}$ and $\rho_r = \rho_{r0}a^{-4}$. At this point we should mention that Eq. (11), which is a second-order equation with respect to the scale factor, still has to be solved and the solution is constrained by the first modified Friedmann equation (10).

B. Anisotropic Bianchi I spacetime

The second scenario that we consider in this work is the determination of an analytical solution in a Bianchi I spacetime. To do that we consider the diagonal frame

$$h^i_\mu(t) = \mathrm{diag}(1, a(t), b(t), c(t)), \tag{23}$$

where the line element is that of Bianchi I spacetime with unknown scale factors $a(t)$, $b(t)$ and $c(t)$:

$$ds^2 = -dt^2 + a^2(t)dx^2 + b(t)dy^2 + c(t)dz^2. \tag{24}$$

The Lagrangian density for (23) is

$$T = -\frac{2}{abc}(c\dot{a}\dot{b} + b\dot{a}\dot{c} + a\dot{b}\dot{c}), \tag{25}$$

from which we can see that (9) is recovered in the isotropic scenario, $a(t) = b(t) = c(t)$, which is the spatially flat FLRW universe.

With the use of a Lagrange multiplier in (6) the Lagrangian of the field equations can be constructed:

$$L(a,b,c,\dot{a},\dot{b},\dot{c},T) = 2f_{,T}(c\dot{a}\dot{b} + b\dot{a}\dot{c} + a\dot{b}\dot{c}) + abc(f_{,T}T - f), \tag{26}$$

where we have assumed that there is no other matter source.

The gravitational field equations are the Euler-Lagrange equations with respect to the variables a, b and c, Eq. (25), which follow from $\frac{\partial L}{\partial T} = 0$, and the constraint equation

$$2f_{,T}(c\dot{a}\dot{b} + b\dot{a}\dot{c} + a\dot{b}\dot{c}) - abc(f_{,T}T - f) = 0. \tag{27}$$

This can be derived from the variation of the lapse function N, when $dt = N(\tau)d\tau$, where we have assumed that $N(t) = 1$. For the spacetime (23), we perform our analysis for the same models $f_I(T)$ and $f_{II}(T)$ introduced explicitly in the last section.

III. ANALYTICAL SOLUTIONS IN FLRW SPACETIMES

In order to determine the analytic solution of the field equations we apply the method of singularity analysis and we follow the Ablowitz-Ramani-Segur algorithm [33–35], which is based upon the existence of movable singularities for the differential equations and is in the spirit of the approach of Kowalevski [36]. We refer the reader to the following works for the basic properties of the singularity analysis: Refs. [37–39].

We perform our analysis for the two different models, f_I and f_{II}, that we discussed above for the two cases for the fluid terms: (a) dust and (b) dust plus radiation.

A. Dust fluid

The analyses for the two different models with only a dust fluid present are as follows.

1. Model $f_I(T)$

We substitute $a(\tau) = a_0\tau^\sigma$ in (11) and we search for the dominant terms in order to determine the power σ. Note that $\tau = (t - t_0)$ and t_0 is the position of the singularity. We have two different possibilities, $n < 1$ and $n > 1$ with $n \neq \frac{1}{2}$. Note that $n = 1$ is the special case of teleparallel GR.

(a) Case $n < 1$.—For values of n smaller than one we find the dominant behavior $\sigma = \frac{2}{3}$ for a_0 an arbitrary value. That means that the singularity of the differential equation is that when $a(t_0) \to 0$, while in the same time $\dot{a}(t_0) \to \infty$. In order to determine the position of the resonances we substitute $a(\tau) = a_0\tau^{\frac{2}{3}} + m\tau^{\frac{2}{3}+s}$ in (11), linearize around the $m \simeq 0$ and solve the remaining polynomial which follows from the dominant terms determining s. The polynomial is $s(s+1) = 0$, which gives the two solutions $s_1 = -1$ and $s_2 = 0$. The value of s_1 is essential for the existence of the singularity and gives a check that our analysis is correct. The second resonance gives us the position of the second constant of integration which is at the dominant term. Recall that one constant of integration is the position of the singularity t_0. Furthermore, as the dominant term is not a solution of (11) because there are remainder terms, the solution is expressed in a right Painlevé series with a step $\frac{1}{3}$, so

$$a(\tau) = a_0\tau^{\frac{2}{3}} + \sum_{N=1}^{+\infty} a_N \tau^{\frac{2+N}{3}}. \tag{28}$$

In the solution (28), the only arbitrary constants are the position of the singularity, t_0, and the coefficient a_0. The coefficients a_N have to be determined from (11) and (10).

First consider the case $n = -1$. We substitute the solution (28) into (11) and (10) which gives $16\pi G\rho_{m0} = \frac{8}{3}a_0^3$. The nonzero coefficients a_N are the a_M with $M = 12\lambda$, $\lambda \in \mathbb{N}$ and $\frac{a_{12}}{a_0} = -\frac{9}{320}\alpha$, $a_{24} = \frac{33}{160}\alpha a_{12}$, $a_{36} = \frac{23373}{45760}\alpha a_{24}$, etc., occur every 12 terms.

Since there are so many zero coefficients of the a_N very close to the singularity at $a(t_0)$, the solution of the field equation is well approximated by the power-law solution $a(\tau) = a_0\tau^{\frac{2}{3}}$, which is that of the dust fluid. That means that close to the singularity the dominant term in the gravitational field equations is the linear term T, while the dynamical parts contributed by T^n only change the dynamics far from the movable singularity.

(b) Case $n > 1$.—For $n > 1$, the dominant term is $a(\tau) = a_0\tau^{\frac{2}{3}n}$. We assume that $\frac{2}{3}n \notin \mathbb{N}^*$ and we calculate that the dominant terms are $\tau^{-2+\frac{2}{3}n+s}$, which gives the resonances $s = -1$, $s = 0$, so as before the solution is

expressed in a right Painlevé series. In contrast to the $n < 1$ case, n now has to be a rational number in order for the singularity analysis to work. The step of the right Painlevé series depends on n and is determined from the denominator of the dominant term with $\sigma = \frac{2}{3}n$.

On the other hand, when $n = \frac{3}{2}\mu$, $\mu \in \mathbb{N}^*$, in order to perform the singularity analysis we substitute $a \to b^{-1}(\tau)$, from which we see that the dominant behavior is $b(\tau) = b_0\tau^{-\mu}$. The resonances are again at $s = -1$ and $s = 0$ but, as the dominant behavior is not a solution of the field equations, the solution is expressed again as a right Painlevé series with step one.

Now consider the case $n = 2$. The analytical solution is

$$\frac{a(\tau)}{a_0} = \tau^{\frac{4}{3}} + \sum_{N=1}^{+\infty} a_N \tau^{\frac{4+N}{3}}, \tag{29}$$

where the only nonzero coefficients are the a_Σ with $\Sigma = 6\lambda$, $\lambda \in \mathbb{N}$. The constant of integration is a_0. For the leading coefficients we have $a_6 = (288\alpha)^{-1}$, $a_{12} = 17(2880\alpha)^{-1}a_6$, $a_{18} = 835(205632\alpha)^{-1}a_{12}$, etc., and $16\pi G\rho_{m0} = -\frac{1024}{3}a_0^3\alpha$, which means that $\alpha < 0$ for $\rho_{m0} > 0$. We can see that the solution (29) passes the consistency test. Before we proceed to our analysis for the second model $f_{II}(T)$, we note that the dominant term follows from the $(-T)^n$ term of the action and it is the power solution of the power-law model $f(T) = (-T)^n$ [20]; that is, the universe is dominated by the geometric effective fluid ρ_T, p_T. The fluid has a constant equation of state parameter $w_T = \frac{n-1}{n}$ which is always positive for $n > 1$.

On the other hand, for $n = \frac{3}{2}$, which means $\mu = 1$, the solution for the scale factor is

$$(a(\tau))^{-1} = b_0\tau^{-1} + \sum_{N=1}^{+\infty} b_N \tau^{-1+N}. \tag{30}$$

For the coefficients b_N, we have the relations $\frac{b_1}{b_0} = (12\sqrt{6}\alpha)^{-1}$, $\frac{b_2}{b_0} = -(12\sqrt{6}\alpha)^{-1}\frac{b_1}{b_0}$, $\frac{b_3}{b_0} = (9(12\sqrt{6})^2\alpha^2)^{-1}\frac{b_1}{b_0}$, $\frac{b_4}{b_0} = (\frac{15}{19}(12\sqrt{6})^4\alpha^4)^{-1}\frac{b_1}{b_0}$, etc., while (10) gives $16\pi G\rho_{m0} = \frac{12\sqrt{6}}{b_0^3}\alpha > 0$. From (30), we observe that near the singularity the effective fluid is that of radiation. We continue our analysis with the model $f_{II}(T)$ in which the cosmological constant is considered.

2. Model $f_{II}(T)$

The singularity analysis for $f_{II}(T)$ provides the same results as that of $f_I(T)$. This means that the cosmological constant term does not effect the dominant behavior near the singularity or the resonances. The only differences which arise are that the coefficient terms of the Laurent expansion now also depend upon Λ. We demonstrate this by deriving the coefficients for the cases $n = -1$, $n = 2$ and $n = \frac{3}{2}$.

For $n = -1$, the solution of the field equations for $f_{II}(T)$ is again given by (28), where the nonzero coefficients are now $a_{\bar{M}}$ with $\bar{M} = 6\lambda$, $\lambda \in \mathbb{N}$. In the analysis above the nonzero coefficients occurred every 12 steps. The values of the first coefficients are now

$$\left(\frac{a_6}{a_0}\right) = \frac{\Lambda}{24}, \qquad \left(\frac{a_{12}}{a_0}\right) = \left(\frac{\Lambda^2 - 81\alpha}{2880}\right) \quad \text{and}$$
$$\left(\frac{a_{18}}{a_0}\right) = \frac{\Lambda(\Lambda^2 - 1994\alpha)}{362880}.$$

Thus, we can see, for $\Lambda = 0$, that the coefficients have the values of the model $f_I(T)_{|n\to -1}$. Note that we have $16\pi G\rho_{m0} = \frac{8}{3}a_0^3$.

In the case when $n = 2$, the solution of field equation is the right Painlevé series, (29). The nonzero coefficients are a_Σ, with $\Sigma = 6\lambda$, $\lambda \in \mathbb{N}$, where the first coefficients are

$$\left(\frac{a_6}{a_0}\right) = (288\alpha)^{-1},$$
$$\left(\frac{a_{12}}{a_0}\right) = 10\alpha(17 - 162\alpha\Lambda)\left(\frac{\alpha_6}{a_0}\right)^2 \quad \text{and}$$
$$\left(\frac{a_{18}}{a_0}\right) = 84(167 - 1944\alpha\Lambda)\left(\frac{\alpha_6}{a_0}\right)^3, \quad \text{etc.}$$

Hence, we can see that the cosmological constant affects the dynamics from the twelfth term of the Laurent expansion and for $\Lambda = 0$ we have the same coefficients as before. Furthermore, the first Friedmann equation gives $16\pi G\rho_{m0} = -\frac{1024}{3}a_0^3\alpha$.

Finally, for the case of $n = \frac{3}{2}$, the solution of the field equations is (30), where from (10) we have $16\pi G\rho_{m0} = \frac{12\sqrt{6}}{b_0^3}\alpha$ and from (11) that

$$\left(\frac{b_1}{b_0}\right) = (12\sqrt{6}\alpha)^{-1},$$
$$\left(\frac{b_2}{b_0}\right) = -(12\sqrt{6}\alpha)^{-1}\left(\frac{b_1}{b_0}\right) \quad \text{and}$$
$$\frac{b_3}{b_0} = \frac{1 - 54\alpha^2\Lambda}{7776\sqrt{6}\alpha^3}.$$

From these coefficients we can see that, when $\Lambda = 0$, the solution reduces to that of the model $f_I(T)_{|n\to\frac{3}{2}}$.

B. Dust and radiation fluids

In a more general scenario we assume that the matter source of the field equations includes a part from the cold dark matter (dust), a radiation component. We use the model $f_{II}(T)$, because the cosmological constant does not affect the dominant term or the resonances. Again, we consider two possible cases, $n < 1$ and $n > 1$.

(a) Case $n < 1$.—We follow the same steps as before and we find that the dominant term of Eq. (11) is $a(\tau) = a_0\tau^{\frac{1}{2}}$. Now a_0 is not arbitrary as above, but $\bar{\rho}_{r0} = \frac{1}{2}(a_0)^4$, where $\bar{\rho}_{r0} = \frac{16\pi G}{3}\rho_{r0}$. This means that the radiation fluid dominates in the early universe as expected. For the resonances, we find that they are $s_1 = -1$ and $s_2 = \frac{1}{2}$ and now the position of the second constant of integration in the Laurent expansion is in the coefficient a_1. The Laurent expansion is a right Painlevé series and is

$$a(\tau) = a_0\tau^{\frac{1}{2}} + a_1\tau + \sum_{N=2}^{+\infty} a_N\tau^{\frac{1+N}{2}}. \tag{31}$$

In this case it is important to prove the consistency of the solution. We do that by replacing (31) in (11). We assume that $n = -1$. We find that

$$\bar{\rho}_{r0} = \frac{1}{2}(a_0)^4, \qquad a_2 = -\frac{7}{8}\frac{a_1^2}{a_0},$$
$$a_3 = \frac{5}{4}\frac{a_1^3}{a_0}, \qquad a_4 = -\frac{273}{128}\frac{a_1^4}{a_0} + \frac{a_0}{18}\Lambda, \quad \text{etc.},$$

where again $\bar{\rho}_{r0} = \frac{1}{2}(a_0)^4$ and from (10) we have $16\pi G\rho_{m0} = 9a_0^2 a_1$.

(b) Case $n > 1$.—When $n > 1$ the dominant term in the movable singularity of the field equation (11) follows from the term $(-T)^n$ in the action and does not correspond to a radiation fluid as occurred in the previous case with $n < 1$. We find that the dominant behavior is $a(\tau) = a_0\tau^{\frac{2}{3}n}$, for $\frac{2}{3}n \notin \mathbb{N}^*$. Straightforwardly, we calculate the resonances and they are $s_1 = -1$ and $s_2 = 0$; that is, the solution is expressed in a right Painlevé series where the coefficient a_0 is the second constant of integration. This is possible for $n \in \mathbb{Q}$.

Again, when $\frac{2}{3}n = \mu \in \mathbb{N}^*$, we change variable via $a(\tau) \to (b(\tau))^{-1}$. We find that the field equations pass the singularity analysis when μ is an even number, $\mu = 2\zeta$, where the dominant behavior is $b(\tau) = b_0\tau^{-\frac{3}{2}\zeta}$, with $\bar{\rho}_{r0} = -2^{-3\zeta}3^{-1+9\zeta}(6\zeta - 1)a_0^{-4}$ with resonances $s_1 = -1$ and $s_2 = \frac{3}{2}\zeta$. Hence the solution is expressed in a right Painlevé series in which the step is $\frac{1}{2}$ for ζ an odd number and 1 when ζ is an even number. The position of the second constant of integration depends upon the value of the resonance s_2.

IV. ANALYTICAL SOLUTIONS IN BIANCHI I SPACETIME

The exact solution of the vacuum field equations which follow from the Lagrangian function (26) in GR, i.e., $f(T) = T$, is the Kasner spacetime where the coefficient functions of the spacetime (24) are power law, that is, $\chi(t) = t^{p_i}$, and the $p_i = (p_i, p_2, p_3)$ are solutions of the following system:

$$\sum_{i=1}^{3} p_i = 1, \qquad \sum_{i=1}^{3} p_i^2 = 1. \tag{32}$$

These are called Kasner's relations.

However, in modified theories of gravity it is possible for Kasner-like solutions to exist but Kasner's relations may have to be modified because the components of the geometric fluids exist. This has been considered first for the higher-order theories of gravity by Barrow and Clifton in [40–42]. Kasner-like solutions have been studied for the $f(X) = R^n$, $f(X) = (R_{\mu\nu}R^{\mu\nu})^n$ and $f(X) = (R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda})^n$ theories of gravity. Specifically Kasner's relations (32) have been modified such that the right-hand sides of Eq. (32) do not equal one but depend upon the power n defining the Lagrangian of the theory, but Kasner's metric or that of Minkowski spacetime can still be recovered.

Before we study the existence of analytical solutions in the models $f_I(T)$ and $f_{II}(T)$ we consider the power-law theory $f(T) = (-T)^n$ for which we study the existence of a Kasner-like solution.

A. Kasner-like solution

Consider $f(T) = (-T)^n$, and assume that

$$a(t) = a_0 t^{p_1}, \qquad b(t) = b_0 t^{p_2}, \qquad c(t) = c_0 t^{p_3}.$$

We find that the field equations which follow from the Lagrangian (24) are satisfied either when

$$p_1 = p_2 = p_3, \quad \text{where } n = \frac{1}{2}, \tag{33}$$

or when p_i satisfies the two conditions

$$\sum_{i=1}^{3} p_i = 2n - 1, \qquad \sum_{i=1}^{3} p_i^2 = (2n-1)^2, \quad \text{for } n > 0, \tag{34}$$

or

$$\sum_{i=1}^{3} p_i = \rho_0, \qquad \sum_{i=1}^{3} p_i^2 = \rho_0^2, \quad \text{for } n > 1. \tag{35}$$

Solution (33) has been derived in [43], but it is that of an isotropic universe for the theory $f(T) = \sqrt{-T}$, but (as discussed above) this Lagrangian cannot recover the field equations of GR in an appropriate limit. Furthermore, from (34), Kasner's spacetime is recovered only when $n = 1$, while from (35), and for $n > 1$, Kasner's solution is recovered always for $\rho_0 = 1$. Moreover, there exists consistency of (34) for every value of n, while solution (34) has the universe expanding when $n > \frac{1}{2}$ in (34), or $\rho_0 > 0$ in

(35), and $t=0$ describes the position of the spacetime Weyl curvature singularity. Finally, from (34), we observe that for positive values of n (or positive ρ_0) one of the resonances always has a different sign from the others, i.e., if p_2, p_3 are positive, then $p_1<0$. This means that the chaotic dynamical behavior on approach to the singularity in the Mixmaster universe, via an infinite sequence of Kasner eras, can occur as in GR,

We see that by rescaling via $\bar{p}_i=\frac{1}{2n-1}p_i$, or $\bar{p}_i=\frac{1}{\rho_0}p_i$, conditions (34) and (35) simply become the GR Kasner relations

$$\sum_{i=1}^{3}\bar{p}_i=1, \qquad \sum_{i=1}^{3}\bar{p}_i^2=1. \tag{36}$$

The existence of solutions (34) and (35) means that the field equations in $f(T)$ gravity, for the diagonal frame (23), admit an anisotropic exact solution. This is contrary to the claim in a recent review of $f(T)$ gravity [6], which is based on results in Ref. [43]. To see this more clearly, note that the constraint equation $\tilde{G}^0_0=0$, where $\tilde{G}^\mu_\nu$ is the modified Einstein tensor, is again Eq. (22) for the case of vacuum. This admits the general solution $f(T)=\sqrt{-T}$ and also the particular solution $T=0$, with $f(T)|_{T\to 0}=0$, for the power-law case. It is easy to see that the solution (34) allows (25) to take a zero value.

B. Singularity analysis

We perform our singularity analyses for the models $f_I(T)$ and $f_{II}(T)$. As in the case of the isotropic universe, we will study the two different cases for which $n<1$ and $n>1$.

1. Model $f_I(T)$

(a) Case $n<1$.—For values of $n<1$, we find that the dominant term in the field equations is the linear term in the action; that is, we are in the limit of GR as for the FLRW universe in the previous section. Hence, the dominant terms are $(a(t),b(t),c(t))=(a_0t^{p_1},b_0t^{p_2},c_0t^{p})$, where a_0, b_0, c_0 are arbitrary constants and the p_i satisfy the Kasner relations (32). However, since the p_i satisfy the Kasner relations we have that $T(t)=0$. Hence the singularity analysis fails.

(b) Case $n>1$.—In the second case, when $n>1$, we find that the dominant terms are $(a(t),b(t),c(t))=(a_0t^{p_1},b_0t^{p_2},c_0t^{p})$, where again a_0, b_0, c_0 are arbitrary constants and the p_i satisfy the modified Kasner relations (34). This solution also gives $T(t)=0$, which means that the singularity analysis fails. However in this case we observe that Kasner's solution (32) solves the field equations.

2. Model $f_{II}(T)$

For the second model, namely $f_{II}(T)$, the singularity analysis fails to provide us with a solution because the dominant terms ensure $T(t)=0$. In contrast to the model $f_I(T)$, we now find $f_{II}(T)\neq 0$, which means that the field equations are not satisfied.

V. TEGR IN NONLINEAR $f(T)$ GRAVITY

We rewrite the gravitational field equations (7) as follows:

$$e_i^\rho\mathbf{G}f_T+\frac{1}{4}e_i^\rho[(f-Tf_T)]+e_i^\rho S_\rho{}^{\mu\nu}\partial_\mu(T)f_{TT}=4\pi Ge_i^\rho\mathcal{T}_\rho{}^\nu, \tag{37}$$

where $\mathbf{G}$ is the Einstein tensor in the teleparallel equivalence

$$e_i^\rho\mathbf{G}=\left(e^{-1}\partial_\mu(ee_i^\rho S_\rho{}^{\mu\nu})-e_i^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu}+\frac{1}{4}e_i^\nu T\right). \tag{38}$$

Recall that the Lagrangian density T is related to the Ricci scalar by

$$T=-R+2e^{-1}\partial_\nu(eT_\rho{}^{\rho\nu}). \tag{39}$$

If $(f-Tf_T)=0$, that is $f(T)=T$ or $f(T)_{|T\to 0}=0$ and $T=0$, then Eq. (37) becomes

$$e_i^\rho\mathbf{G}f_T+e_i^\rho S_\rho{}^{\mu\nu}\partial_\mu(T)f_{TT}=4\pi Ge_i^\rho\mathcal{T}_\rho{}^\nu. \tag{40}$$

A vacuum solution of $f(T)$ gravity is therefore also a vacuum solution of GR if and only if

$$R=2e_\nu^{-1}\partial_\nu(eT_\rho{}^{\rho\nu})=0. \tag{41}$$

However, if we assume a nonzero energy-momentum tensor $e_i^\rho\mathcal{T}_\rho{}^\nu$, then solution (40) is again one of GR if $f_T\neq 0$, $e_i^\rho S_\rho{}^{\mu\nu}\partial_\mu(T)f_{TT}=0$ and condition $R=2e_\nu^{-1}\partial_\nu(eT_\rho{}^{\rho\nu})$ holds. The latter conditions have been derived in [44]. In the case of vacuum it is not necessary that f_T be a nonzero constant. It can be also zero when GR is recovered as we saw in Sec. IV A with the case of power-law $f(T)$.

We conclude that vacuum solutions in GR can be recovered in $f(T)$ gravity as in the case of the fourth-order $f(R)$ gravity [45]. However, it is necessary to select the correct frame in which $R=2e_\nu^{-1}\partial_\nu(eT_\rho{}^{\rho\nu})$. Note also that a vacuum solution of GR may correspond only to a special solution of $f(T)$ gravity and may not be stable in initial data space [46].

For the Bianchi I model, a power-law solution, $a(t)=a_0t^{p_1}$, $b(t)=b_0t^{p_2}$, $c(t)=c_0t^{p_3}$, solves the vacuum field equations if $T=0$, $f(T)|_{T\to 0}=0$, which provides the constraint equation

$$(p_1p_2+p_1p_3+p_2p_3)=0, \tag{42}$$

if and only if the left-hand side of (40) is well defined. In the case of $f(T) = T + \alpha(-T)^n$, where $f(0) = 0$, $f_T(0) = 1$, we have that $e_i^\rho S_\rho{}^{\mu\nu}\partial_\mu(T) f_{TT} = 0$ only when $n > 1$ and then GR is recovered.

On the other hand, in $f(T) = (-T)^n$ gravity we have that $f(0) = 0$, $f_T(0) = 0$ and $e_i^\rho S_\rho{}^{\mu\nu}\partial_\mu(T) f_{TT} = 0$ for $n > 1$, where condition (43) provides us with (35), where the Kasner solution is recovered again for $\rho_0 = 1$ without necessarily having $f_T(0) \neq 0$. However, for values of n where $n \in (0, 1)$, the quantities $f_T(0)$, $f_{TT}(0)$ are infinite but if the constant ρ_0 has the value $\rho_0 = 2n - 1$, then the right-hand side part of (40) is well defined.

A. Cosmological constant

If we include the cosmological constant, then Eq. (37) becomes

$$e_i^\rho(\mathbf{G} + \Lambda) f_T + \frac{1}{4} e_i^\rho[(f - (T + \Lambda) f_T)] + e_i^\rho S_\rho{}^{\mu\nu}\partial_\mu(T) f_{TT} = 4\pi G e_i^\rho \mathcal{T}_\rho{}^\nu. \tag{43}$$

The above analysis holds and we reduce to the solutions of GR with the cosmological constant when $f(T)_{|T\to\Lambda} = 0$ and $T = -\Lambda$ [44]. Again, in the vacuum scenario, $f_{,T}(-\Lambda)$ can be zero and GR can be recovered with the proper frame for the cosmological constant Λ.

In order to demonstrate this, note that in (10) and (11) and for the diagonal frame we considered in Sec. II, that $f(T) = (-T - \Lambda)^n$; this means that $f(-\Lambda) = 0$, and $f_{,T}(-\Lambda) \to 0$ for $n > 1$ or $f_{,T}(-\Lambda) \to \infty$ for $n < 1$.

From the field equations, (10) and (11), we find the de Sitter solutions

$$a(t) = a_0 \exp\left(\pm\sqrt{\frac{\Lambda}{6(1-2n)}}t\right), \qquad n \neq 0, \tag{44}$$

$$a(t) = a_0 \exp\left(\pm\sqrt{\frac{\Lambda}{6}(1+\Lambda)t}\right), \qquad n = \frac{1}{2}, \qquad \Lambda \neq -1, \tag{45}$$

and

$$a(t) = a_0 \exp\left(\pm\sqrt{\frac{\Lambda}{6}}t\right), \qquad n > 1. \tag{46}$$

The latter is that in which $T = -\Lambda$. This is the solution through which we recover GR. We observe that (44) and (45) provide us with GR solutions but for a cosmological constant $\tilde{\Lambda} = \frac{\Lambda}{1-2n}$, $\bar{\Lambda} = \Lambda(1 + \Lambda)$. This means that in $f(T) = (-T - \Lambda)^n$ gravity there exists a solution in which the geometric fluid with components ρ_T, p_T has a constant equation of state parameter $w_T = -1$. That follows from the results of [44] because $f(T) \neq 0$ for $T = \frac{\Lambda}{1-2n}$. Then, a new cosmological constant has to be considered. Recall that for $n = \frac{1}{2}$, the function $f(T) = (-T - \Lambda)^{\frac{1}{2}}$ is well defined for $\Lambda \neq 0$, in contrast to the situation when $\Lambda = 0$.

Including a matter source in (10) and (11), like that of a dust fluid, in order to recover ΛCDM cosmology we can see that the use of the condition $T = -\Lambda$ gives the scale factor (46), which means that GR cannot be recovered by that condition—at least for the frame that we have considered. We know that $f(T)$ gravity is not invariant under Lorentz transformations which is one of the main issues with the theory; see [47,48]. Therefore, in order for GR to be recovered, the frame should be that such condition (41) is satisfied.

Consider again the field equations (10) and (11) without a matter source ρ, p, for a function f such as $T = -\Lambda$, $f(-\Lambda) = 0$, with $e_i^\rho S_\rho{}^{\mu\nu}\partial_\mu(T) f_{TT} = 0$. The field equations become

$$(-T + \Lambda) f_T + (f - (T + \Lambda) f_T) = 16\pi G\rho \tag{47}$$

and

$$-(4\dot{H} - T + \Lambda) f_T - (f - (T + \Lambda) f_T) + 48H^2\dot{H} f_{TT} = 16\pi G p. \tag{48}$$

The de Sitter solution (46) solves (47) and (48) when $p = -\rho$ and $\rho = \frac{f_{,T}}{8\pi G}\Lambda$, with $f_{,T} \neq 0$, or when $f_T = 0$. In the latter case we can say directly that $f(T)$ provides us with the solution of the teleparallel equivalence of general relativity with a cosmological constant in the vacuum, while for $f_{,T} \neq 0$ a new fluid term has to be introduced in order to eliminate the remaining terms of $f(T)$ gravity. This is something that is not necessary when $\Lambda = 0$.

Before we close this section we should remark that when $f(-\Lambda)$ and $f_T(-\Lambda)$ are nonzero constants then the gravitational field equations become those of GR with a cosmological constant $\hat{\Lambda}$ which is different to Λ. Indeed, their solution will be that of TEGR while we cannot say that GR is always recovered because of the constraint equation

$$R = 2e_\nu^{-1}\partial_\nu(eT_\rho{}^{\rho\nu}) + \Lambda. \tag{49}$$

VI. CONCLUSIONS

In this paper the method of movable singularities of differential equations was applied in order to determine analytical solutions of the field equations in $f(T)$ gravity in a cosmological scenario. The models that we considered are $f_1(T) = T + \alpha(-T)^n$ and $f_2(T) = T + \alpha(-T)^n - \Lambda$, where GR is recovered for $\alpha \to 0$. For the right-hand side of the field equations, i.e., the energy-momentum tensor, we have considered two perfect fluids: a dust fluid which corresponds to the cold dark matter and a blackbody radiation term. We prove that the solution of these models is given as a right Painlevé series and the cosmological constant does not play any significant role in the existence of the movable singularity or on the resonances. The cosmological constant modifies only the coefficients of the Painlevé series.

We studied two different cases in which the total fluid is (a) dust and (b) dust plus radiation. For the case (a) we found that the field equations always pass the singularity test. When $n < 1$, the dominant term gives with dust term, as in GR, while far from the movable singularity, which corresponds to $a(t_0) \to 0$, $\dot{a}(t_0) \to \infty$, the term $\alpha(-T)^n - \Lambda$ plays a dominant role. On the other hand, when $n > 1$, the dominant term corresponds to the $(-T)^n$ term of the action, which provides an effective perfect fluid with a constant equation of state parameter, $w_T = \frac{n-1}{n}$.

However, the situation is different when we add a radiation fluid. In this case we showed that, when $n < 1$, the dominant behavior is that of a radiation fluid in GR. For $n > 1$ we have two possible cases. For n such that $\frac{2}{3}n \notin \mathbb{N}^*$ the dominant term is that of $(-T)^n$ and, when $\frac{2}{3}n \in \mathbb{N}^*$, we found that the field equations pass the singularity test only if $\frac{2}{3}n$ is an even number. The dominant term is then $a(\tau) = a_0\tau^{\frac{1}{2}n}$. Furthermore, for both cases (a) and (b), the field equations pass the singularity analysis for $n > 1$ only if n is a rational number.

We compare our results with the fourth-order gravity defined by the Lagrangian $f_I(R) = R + \alpha R^n$ that has been studied from the point of view of the singularity analysis in [25] without a radiation fluid. There, it was found that the field equations pass the singularity test when n is a rational number greater than one and the dominant term is that of the term R^n in the Lagrangian for $n > 1$ with $n \neq \frac{5}{4}, 2$. Of course, the two different theories $f(T) = T^n$ and $f(R) = R^m$ provide power-law solutions. That means that at a level close to the movable singularity the two different theories, $f_I(T)$ and $f_I(R)$, provide a similar behavior for $n, m > 1$.

Another issue that deserves comment is that the movable singularity in the modified Friedmann equation (11) for the models studied corresponds to a spacetime singularity because either [when $a(t_0) \to 0$] the Hubble function, the deceleration parameter, or one of their higher derivatives of the scale factor becomes singular. Of course, that does not mean that the method of movable singularities of differential equations cannot be applied in cosmological models with no singularities. A movable singularity at $t \to t_0$, when it exists, can provide a solution such as $a(t_0) \to \infty$. That is possible when the dominant behavior is negative. This is clear from the analysis we perform in the Bianchi I spacetime.

When considering the Bianchi I spacetime we found that the vacuum field equations admit an anisotropic Kasner-like solution which is contrary to the existing results in the literature [6,43]. We did that by studying the field equations for the power-law model $f(T) = (-T)^n$. The modified Kasner relations depend upon the power n and the sum of the Kasner indices, and their squares are $(2n-1)$ and $(2n-1)^2$, or ρ_0 and ρ_0^2, respectively, where for $n = 1$ or $\rho_0 = 1$ we are in the limit of teleparallel equivalence of GR. As far as the two models $f_I(T)$ and $f_{II}(T)$ are concerned, we found that the singularity analysis failed to provide us with the analytical solution of the field equations. However, the dominant terms are also solutions of the field equations for the $f_I(T)$ model, where for $n < 1$ the Kasner solution is recovered, while for $n > 1$ the Kasner-like solution follows. Furthermore, we note that the results are different from that of $f(R) = R^m$ gravity, where two families of Kasner-like solutions exist while the power m of the theory cannot be arbitrary.

In $f(T)$ gravity for the two spacetimes that we considered we show that the vacuum field equations are satisfied when the solution guarantees $T = 0$ and $f(T)|_{T\to 0} = 0$. In the case of the FLRW spacetime the solution is that of the four-dimensional Minkowski spacetime. For the Bianchi I spacetime if we consider a power-law solution, then condition (42) should be satisfied and the Kasner metric solves (42). We expect that an $f(T)$ Mixmaster universe to have similar chaotic behavior to that displayed in GR on approach to a spacetime singularity.

We also studied when solutions of the teleparallel equivalence of GR can be recovered in $f(T)$ gravity. We found that when $T = T_0$ and $f(T_0) = 0$, the field equations do not admit terms which diverge at infinity. The solution of GR is recovered for the proper frame for an arbitrary value of $f_T(T_0)$ for the vacuum case with or without a cosmological constant and also when $f_T(T_0) \neq 0$ when a fluid is included in the field equations.

The knowledge that the field equations form an integrable system is important for the existence of real solutions. Symmetries and singularity analyses are two independent methods which they provide us with information if the system is integrable. In a forthcoming work we would like to extend that approach and in other gravitational theories.

ACKNOWLEDGMENTS

A. P. acknowledges financial support of FONDECYT Grant No. 3160121. J. D. B. acknowledges support from the STFC. P. G. L. L. thanks the Instituto de Ciencias Físicas y Matemáticas of the UACh for the hospitality provided while this work carried out and acknowledges the National Research Foundation of South Africa and the University of KwaZulu-Natal for financial support. The views expressed in this paper should not be attributed to either institution.

[1] J. D. Barrow and D. J. Shaw, Phys. Rev. Lett. **106**, 101302 (2011); Phys. Rev. D **83**, 043518 (2011).
[2] B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988).
[3] J. M. Overduin and F. I. Cooperstock, Phys. Rev. D **58**, 043506 (1998).
[4] A. W. Brookfield, C. van de Bruck, D. F. Mota, and D. Tocchini-Valentini, Phys. Rev. Lett. **96**, 061301 (2006).
[5] M. C. Bento, O. Bertolami, and A. A. Sen, Phys. Rev. D **66**, 043507 (2002).
[6] Y. F. Cai, S. Capozziello, M. De Laurentis, and E. N. Saridakis, arXiv:1511.0786.
[7] G. R. Bengochea and R. Ferraro, Phys. Rev. D **79**, 124019 (2009).
[8] R. Ferraro and F. Fiorini, Phys. Rev. D **75**, 084031 (2007).
[9] E. V. Linder, Phys. Rev. D **81**, 127301 (2010).
[10] H. A. Buchdahl, Mon. Not. R. Astron. Soc. **150**, 1 (1970).
[11] K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979).
[12] M. Tsamparlis, Phys. Lett. A **75**, 27 (1979).
[13] H. I. Arcos and J. G. Pereira, Int. J. Mod. Phys. D **13**, 2193 (2004).
[14] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl., 217 (1928); , 224 (1928) [translated by, arXiv:physics/0503046].
[15] P. Wu and H. W. Yu, Phys. Lett. B **693**, 415 (2010).
[16] G. R. Bengochea, Phys. Lett. B **695**, 405 (2011).
[17] L. Iorio, N. Radicella, and M. L. Ruggiero, J. Cosmol. Astropart. Phys. 08 (2015) 021.
[18] S. Basilakos, Phys. Rev. D **93**, 083007 (2016).
[19] K. Atazadeh and F. Darabi, Eur. Phys. J. C **72**, 2016 (2012).
[20] S. Basilakos, S. Capozziello, M. De Laurentis, A. Paliathanasis, and M. Tsamparlis, Phys. Rev. D **88**, 103526 (2013).
[21] M. E. Rodrigues, A. V. Kpadonou, F. Rahaman, P. J. Oliveira, and M. J. S. Houndjo, Astrophys. Space Sci. **357**, 129 (2015).
[22] A. Paliathanasis, S. Basilakos, E. N. Saridakis, S. Capozziello, K. Atazadeh, F. Darabi, and M. Tsamparlis, Phys. Rev. D **89**, 104042 (2014).
[23] S. Capozziello, P. A. Gonzalez, E. N. Saridakis, and Y. Vasquez, J. High Energy Phys. 02 (2013) 039.
[24] A. Paliathanasis and P. G. L. Leach, arXiv:1605.04164.
[25] A. Paliathanasis and P. G. L. Leach, arXiv:1605.04204.
[26] S. Cotsakis and P. G. L. Leach, J. Phys. A **27**, 1625 (1994).
[27] J. Demaret and C. Scheen, J. Phys. A **29**, 59 (1996).
[28] S. Cotsakis, J. Demaret, Y. De Rop, and L. Querella, Phys. Rev. D **48**, 4595 (1993).
[29] K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979).
[30] J. W. Maluf, J. Math. Phys. (N.Y.) **35**, 335 (1994).
[31] S. Capozziello, V. F. Cardone, H. Farajollahi, and A. Ravanpak, Phys. Rev. D **84**, 043527 (2011).
[32] K. Bamba, S. D. Odintsov, and E. N. Saridakis, arXiv:1605.02461.
[33] M. J. Ablowitz, A. Ramani, and H. Segur, Lett. Nuovo Cimento **23**, 333 (1978).
[34] M. J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys. (N.Y.) **21**, 715 (1980).
[35] M. J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys. (N.Y.) **21**, 1006 (1980).
[36] S. Kowalevski, Acta Math. **12**, 177 (1889).
[37] A. Ramani, B. Grammaticos, and T. Bountis, Phys. Rep. **180**, 159 (1989).
[38] M. R. Feix, C. Géronimi, L. Cairó, P. G. L. Leach, R. L. Lemmer, and S. É. Bouquet, J. Phys. A **30**, 7437 (1997).
[39] K. Andriopoulos and P. G. L. Leach, Phys. Lett. A **359**, 199 (2006).
[40] J. D. Barrow and T. Clifton, Classical Quantum Gravity **23**, L1 (2006).
[41] T. Clifton and J. D. Barrow, Classical Quantum Gravity **23**, 2951 (2006).
[42] J. Middleton, Classical Quantum Gravity **27**, 225013 (2010).
[43] M. E. Rodrigues, M. J. S. Houndjo, D. Saez-Gomez, and F. Rahaman, Phys. Rev. D **86**, 104059 (2012).
[44] R. Ferraro and F. Fiorini, Phys. Rev. D **84**, 083518 (2011).
[45] J. D. Barrow and A. C. Ottewill, J. Phys. A **16**, 2757 (1983).
[46] J. D. Barrow, arXiv:1503.05723.
[47] B. Li, T. P. Sotiriou, and J. D. Barrow, Phys. Rev. D **83**, 064035 (2011).
[48] N. Tamanini and C. G. Bohmer, Phys. Rev. D **86**, 044009 (2012).