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Big-bounce cosmology from quantum gravity: The case of a cyclical Bianchi I universe

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We analyze the classical and quantum dynamics of a Bianchi I model in the presence of a small negative cosmological constant characterizing its evolution in term of the dust-time dualism. We demonstrate that in a canonical metric approach, the cosmological singularity is removed in correspondence to a positive defined value of the dust energy density. Furthermore, the quantum big bounce is connected to the Universe's turning point via a well-defined semiclassical limit. Then we can reliably infer that the proposed scenario is compatible with a cyclical universe picture. We also show how, when the contribution of the dust energy density is sufficiently high, the proposed scenario can be extended to the Bianchi IX cosmology and therefore how it can be regarded as a paradigm for the generic cosmological model. Finally, we investigate the origin of the observed cutoff on the cosmological dynamics, demonstrating how the big-bounce evolution can be mimicked by the same semiclassical scenario, where the negative cosmological constant is replaced via a polymer discretization of the Universe's volume. A direct proportionality law between these two parameters is then established.

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I. INTRODUCTION

The Wheeler-DeWitt (WDW) approach [1–3] to quantum cosmology [4,5] has two main relevant shortcomings, i.e., the absence of a unique definition of time [6] and the difficulty in removing or properly interpreting the primordial singularity [7–9].

Such a problem, mainly characterizing all the canonical metric approaches, is essentially addressed by the loop quantum cosmology [10–12], where, adopting a scalar field as a relational time, it is shown that the existence of a big bounce removes the singularity.

However, this important result does not overcome some subtleties concerning its derivation that are relevant on a general ground too. First of all, it is not clear if the choice of any relation time and, in particular, the scalar field one, is suitable to describe the early Universe quantum dynamics [13,14]. Then attention is called to the question concerning whether or not the symmetry preservation, characterizing loop quantum cosmology, is the correct quantization procedure of a cosmological model [15].

The present paper analyzes a cosmological model that contains features of interest to the deep understanding of the two points mentioned above. In fact, we consider a canonical minisuperspace model using a dust fluid as external time, according to the time-dust dualism discussed in [16]. The very important feature of the obtained quantum cosmology is the emergence of a nonsingular cyclical universe, which is characterized by a quantum big bounce and a classical turning point, associated with the existence of a small negative cosmological constant, i.e., small enough to ensure that such a recollapsing feature is in the far future of the actual Universe.

An important aspect of such a cosmological scenario, which legitimizes the idea of a cyclical universe, is the possibility to link the quantum evolution to the standard isotropic behavior via a well-defined classical limit (see also [17–19] for this problem in alternative theories of gravity). In fact the presence of a negative cosmological constant induces a harmonic oscillator morphology in the system Hamiltonian (a part of a global minus sign), and this is responsible both for the existence of a classical limit and of the positive nature of the dust energy density. This latter fact solves, in our cosmological implementation, the basic problem of the approach discussed in [16].

In more detail, we consider the evolutionary quantum dynamics of a Bianchi I model in the presence of a negative cosmological constant, as represented in Misner-like variables [20,21]. Clearly, the classical limit corresponds to an increasingly isotropic universe, although we do not address

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here the role of the matter and then the reproduction of standard cosmology. This is because we aim to determine a cosmological behavior able to mimic a very general cosmological scenario near the singularity, according to the idea that the natural isotropization mechanism must be recognized in the inflationary scenario [22].

To this end, we investigate the implications of our dynamical model on the evolution of the Bianchi IX cosmology, which is, accordingly to the Belinski-Khalatnikov-Lifshitz (BKL) conjecture, the prototype for the evolution of a generic inhomogeneous universe on a sufficiently small spatial scale [23]. We demonstrate that, along the dynamics of the stable expectation values of the configurational variables, the presence of the Bianchi IX potential can be neglected, as soon as the value of the dust energy density is sufficiently large. Thus, for such a (nonsevere) restriction, the Bianchi I and Bianchi IX model quantum dynamics overlap near the primordial singularity and our result acquires a high degree of generality, i.e., our picture of a cyclical universe could have a very general implementation in the generic cosmological problem. Finally, we investigate which ingredient of our model is relevant in determining a cutoff physics and we show that there exists a direct relation between the negative cosmological constant presence and an effective semiclassical polymer dynamics [24,25], in which that constant is removed but the discrete nature of the universe volume is included.

In summary, the present paper discusses a cosmological scenario containing a number of very peculiar properties, suggesting that its features are physically meaningful and are not formal coincidences. In particular, we stress how, in the present canonical evolutionary quantum context, the emergence of a big bounce and of a cyclical universe is all natural and general in its structure, so much so as to encourage more general implementations.

This paper is organized as follows.

In Sec. II we describe the Bianchi I model in the presence of a negative cosmological constant from the classical and the quantum point of view. The first part of the section is devoted to analyzing the classical trajectories of the Misner-like variables near the singularities while in the second part we compare these classical behaviors with the related quantum expectation values.

In Sec. III we generalize, in a qualitative way, the properties found for the Bianchi I model to the more general Bianchi IX model, shedding light on the role played by the potential term.

Section IV is dedicated to the cosmological interpretation of the results obtained in the previous section, giving, in particular, a phenomenological explanation of how to extend the features of the Bianchi I and Bianchi IX model to the generic inhomogeneous Universe.

Then, in Sec. V, we see how the role of the negative cosmological constant is related to a cutoff physics, making use of a polymer quantization for the variable connected to the universe volume.

Brief concluding remarks complete the paper.

II. BIANCHI I QUANTUM DYNAMICS IN THE KUCHAŘ AND TORRE APPROACH

The cosmological scenario we are going to implement can be applied also to the isotropic universe [26], as soon as the role played here by the anisotropy variables is supplied by a massless (or even self-consistent) scalar field. Indeed, the kinetic term in the Hamiltonian of a scalar field on the isotropic universe dynamics is all isomorphic to that of an anisotropic variable in the Misner representation (i.e., β_+ or β_-) in the Hamiltonian of a Bianchi model, in particular, for types I and IX we address in this paper. The motivation to consider the present more general scheme rather than the isotropic universe must be individualized in the natural presence of the anisotropy terms near the cosmological singularity, in comparison to the necessity of postulating the presence of a kinetic scalar field contribution asymptotically to the singularity (a reasonable but not rigorously proved feature associated to the preinflationary inflaton dynamics [5]). Furthermore, the morphology of the Bianchi I and IX models outlines a high degree of generality with respect to the Robertson-Walker geometry since, as shown in [23], the generic cosmological solution, near the singularity, is an infinite series of Kasner epochs (periods of time in which the dynamics is Bianchi I-like), one for each space point (physically for each cosmological horizon). Such a basic result, known as the BKL conjecture, suggests that the analysis here addressed can be implemented to a very general picture and we can infer that the discussed scenario removes the cosmological singularity for a generic inhomogeneous universe, as far as its evolution admits the Bianchi IX oscillatory regime as a homogeneous prototype. In what follows, we prefer to deal with minisuperspace models, in order to avoid the nontrivial question of how the conjecture above can be rigorously implemented on a quantum level: the spatial decoupling of the space point in the asymptotic dynamics of an inhomogeneous universe towards the singularity is demonstrated in the classical sector, on the basis of statistical arguments [27], but it remains an open issue in a metric quantum dynamics. Let us consider a universe described by a Bianchi I model in the presence of a negative cosmological constant $-\Lambda$, with $\Lambda > 0$. It is useful to describe the model with respect to the Misner variables $\{\alpha, \beta_\pm\}$, where α expresses the isotropic volume of the Universe (the initial singularity is reached for $\alpha \to -\infty$) while $\beta_\pm$ accounts for the anisotropies of this model. In the Appendix we provide a brief derivation to show that the associated minisuperspace super-Hamiltonian takes the form[1]

[1]We use the $(-,+,+,+)$ signature of the metric and the geometric unit system $(c = G = \hbar = 1)$.

$$\mathcal{H} = \frac{e^{-3\alpha}}{24\pi}[-p_\alpha^2 + p_+^2 + p_-^2] - \pi e^{3\alpha}\Lambda, \tag{1}$$

where $\{p_\alpha, p_+, p_-\}$ are the conjugated momenta related to the Misner variables. In view of a later quantization of the model, it is convenient to introduce the auxiliary variable ρ such that

$$\rho = e^{\frac{3}{2}\alpha} \to p_\rho = \frac{2}{3}e^{-\frac{3}{2}\alpha}p_\alpha. \tag{2}$$

In terms of these new conjugated variables the super-Hamiltonian (1) takes the form

$$\mathcal{H} = -\frac{3}{32\pi}p_\rho^2 + \frac{p_+^2 + p_-^2}{24\pi\rho^2} - \pi\rho^2\Lambda. \tag{3}$$

We now perform a canonical quantization of the system, after the definition of a suitable Hilbert space, by replacing the space-phase variables with multiplicative operators for variables $\{\rho, \beta_+, \beta_-\}$ and differential operators for $\{p_\rho, p_+, p_-\}$, so that

$$p_i \to -i\frac{d}{dq_i}, \qquad q_i = \{\rho, \beta_+, \beta_-\}. \tag{4}$$

If now we introduce the wave function of the Universe $\psi(\rho, \beta_\pm)$ we can apply to it the quantum version of the super-Hamiltonian (3) in order to obtain the Wheeler-deWitt operator

$$\hat{\mathcal{H}}\psi(\rho,\beta_\pm) = \left[\frac{3}{32\pi}\partial_\rho^2 - \frac{\partial_+^2 + \partial_-^2}{24\pi\rho^2} - \pi\rho^2\Lambda\right]\psi(\rho,\beta_\pm). \tag{5}$$

A. Evolutionary quantum cosmology

Here we take into account the evolutionary quantum theory, as it is analyzed in [16,28]. In these works it is considered a system of normal Gaussian coordinates $X^\mu = (T, X^i)$, or in other words a synchronous reference system, for which the line element of the metric takes the form

$$ds^2 = -dT^2 + h_{ij}dX^i dX^j, \tag{6}$$

where the indices $\{i, j\}$ are summed over the spatial directions and h_{ij} is the spatial metric. In this way four components of the space-time metric $g_{\mu\nu}$ are fixed by the Gaussian conditions,

$$g_{00} + 1 = 0, \qquad g_{0i} = 0. \tag{7}$$

The physical meaning of the previous conditions is more clear in the context of the Arnowitt-Deser-Misner (ADM [29]) formalism, for which the space-time metric $g_{\mu\nu}$ is replaced by the lapse function N, the shift vector N^i, and the spatial metric h_{ij}. In the ADM procedure we perform a foliation of the space-time: the lapse function N represents the proper time separation between two neighboring leaves, while the shift vector N^i represents the displacement, with respect to a normal projection, of the local spatial coordinate system in the intersection with the successive leave. In the ADM formalism the space-time metric takes the form

$$ds^2 = N^2dt^2 - h_{ij}(N^i + dx^i)(N^j + dx^j). \tag{8}$$

If we make a comparison between the line elements (6) and (8) it is clear that the conditions (7) are equivalent to

$$N = 1, \qquad N^i = 0, \tag{9}$$

where the foliation of the space-time is such that $t = T$ and $x^i = X^i$. The relations (9) tell us that everywhere the proper time between two neighboring leaves is the same and that there is no displacement, with respect to the normal projection, between one leaf and another. If now we want to implement the Gaussian conditions in the action principles of general relativity, for example, in the vacuum case, we can follow two ways: in the first one we impose the conditions after the variation of the Einstein-Hilbert action, while in the other case we adjoin them to the action, making use of the Lagrangian multipliers technique, before the variation.

When we proceed in the first manner, we deal with the Einstein-Hilbert action in the vacuum

$$S^G = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}R, \tag{10}$$

and a variation of this action with respect to the space-time metric $g_{\mu\nu}$ leads to the Einstein equations in vacuum,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{11}$$

An equivalent form of the action (10) is obtained in the ADM formalism, for which we have

$$S^G[h_{ij}, N, N^i] = \int_{\mathbb{R}} dt \int_\Sigma d^3x[\dot{h}_{ij}P^{ij} - (N^i\mathcal{H}_i^G + N\mathcal{H}^G)], \tag{12}$$

where

$$\mathcal{H}^G = \mathcal{G}_{ijkl}P^{ij}P^{kl} - \frac{\sqrt{h}}{2k}\bar{R}, \tag{13}$$

$$\mathcal{H}_i^G = -2h_{ik}\nabla_j P^{kj}, \tag{14}$$

$$\mathcal{G}_{ijkl} = \frac{k}{\sqrt{h}}(h_{ik}h_{jl} + h_{jk}h_{il} - h_{ij}h_{kl}) \tag{15}$$

are respectively the super-Hamiltonian, the supermomentum, and the supermetric, and P^{ij} is the conjugated momenta to the spatial metric h_{ij}. The variation with respect to N and N^i gives the secondary constraints,

$$\mathcal{H}^G = \mathcal{H}_i^G = 0. \tag{16}$$

The Hamilton equations for h_{ij} and P_{ij}, once fixed $N = 1$ and $N^i = 0$, provide, together with the constraints (16), the Einstein equations in the synchronous reference frame.

The second way to proceed consists of adding the coordinate conditions (7) in the Einstein-Hilbert action by the multipliers M and M_i in such a way that an extra term S^F appears in the action,

$$S[g_{\mu\nu}, M, M_k] = S^G + S^F, \tag{17}$$

with

$$S^F[g_{\mu\nu}, M, M_k] = -\frac{1}{2\kappa}\int d^4x\left[-\frac{1}{2}M\sqrt{-g}(g^{00}+1) + M_i\sqrt{-g}g^{0i}\right] \tag{18}$$

and where we defined the quantity

$$\begin{cases} M := -\frac{H^G}{\sqrt{h}}, \\ M_i := \frac{H_i^G}{\sqrt{h}}. \end{cases} \tag{19}$$

Clearly the variation of the action (17) introduces a source term in the Einstein equations. The role of Lagrangian multipliers M, M_k is clear if we write the action (17) in the ADM formalism, in order to obtain

$$S[h_{ab}, N, N^i, M, M_k] = \int_{\mathbb{R}} dt \int_{\Sigma} d^3x[\dot{h}_{ij}P^{ij} - (N^i\mathcal{H}_i^G + N\mathcal{H}^G) + -\frac{1}{2}M\sqrt{h}(N - N^{-1}) + M_i\sqrt{h}NN^i]. \tag{20}$$

If we perform a variation by M and M_i we obtain the Gaussian conditions (9), while a variation with respect to N and N^i gives Eqs. (19) and fixes the multipliers M and M_i as functions of the canonical variables h_{ij}, P^{ij}. If we use Eqs. (9) and (19) to eliminate the presence of the mutipliers N, N^i and M, M_i, the action (20) clearly reduces to the canonical action (17).

Looking at the action (17), it is not invariant under arbitrary transformations of space-time coordinates and this is due to the fact that we have introduced a privileged coordinate system, i.e., the normal Gaussian coordinates. However, it is always possible to restore the diffeomorphism invariance making a parametrization of the coordinates. It means that if we take the Gaussian coordinates as functions of arbitrary coordinates x^α in such a way that $X^\mu = (T(x^\alpha), X^i(x^\alpha))$ the action (17) can be expressed as

$$S[g_{\alpha\beta}, M, M_k, X^\mu] = S^G + S^F = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}R + -\frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[-\frac{1}{2}M(g^{\alpha\beta}T_{,\alpha}T_{,\beta}+1) + M_i g^{\alpha\beta}T_{,\alpha}X^i_{,\beta}\right], \tag{21}$$

which is manifestly invariant under arbitrary transformations of x^α.

The form of the action (21) allows us to understand the nature of the source of the gravitational field, described by that part of the action appearing in the second row. In [16] this source term is defined as Gaussian reference fluid.

The variation of the action (21) by the metric $g_{\alpha\beta}$ gives the Einstein equations,

$$G_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{22}$$

where

$$T^{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta S^F}{\delta g_{\alpha\beta}} \tag{23}$$

is the energy-momentum tensor associated with the reference fluid. After the definition of the four-velocity vector

$$U^\alpha := -g^{\alpha\beta}T_{,\beta}, \tag{24}$$

it is possible to evaluate the energy-momentum tensor in order to give a clear physical interpretation of the presence model,

$$T^{\alpha\beta} = MU^\alpha U^\beta + M^{(\alpha}U^{\beta)}. \tag{25}$$

Equation (25) is equivalent to the Eckart energy-momentum tensor [30] that describes a heat-conducting fluid. The absence of a stress part in the energy-momentum tensor tells us that the Gaussian reference fluid behaves as a dust. In particular, if we impose only the time condition ($M^i = 0$) Eq. (25) becomes

$$T^{\alpha\beta} = MU^\alpha U^\beta, \tag{26}$$

which describes the behavior of an incoherent dust, where M is the rest mass density and U^α is the four-velocity.

If now we consider the canonical ADM form of the action (21) we have

$$S[h_{ij}, X^\mu, M, M_k] = \int_{\mathbb{R}} dt \int_\Sigma d^3x[\dot{h}_{ij}P^{ij} + \dot{X}^\mu P_\mu + - (N^i\mathcal{H}_i + N\mathcal{H}), \tag{27}$$

with

$$\mathcal{H} = \mathcal{H}^G + \mathcal{H}^D, \qquad \mathcal{H}_i = \mathcal{H}_i^G + \mathcal{H}_i^D, \tag{28}$$

where $P_\mu = (P, P_i)$ are the conjugated momenta to $X_\mu = (T, X_i)$. The quantities $\mathcal{H}^D$ and $\mathcal{H}_i^D$ are respectively the super-Hamiltonian and supermomentum contribution due to the reference fluid and, when we take into account the case of incoherent dust, they simply become

$$\mathcal{H}^D = P, \qquad \mathcal{H}_i^D = X^j_{,i}P_j = 0. \tag{29}$$

As before, the variation with respect to N and N^i gives us the constraints

$$\mathcal{H} = \mathcal{H}^G + \mathcal{H}^D = \mathcal{H}^G + P = 0, \tag{30}$$

$$\mathcal{H}_i = \mathcal{H}_i^G + \mathcal{H}_i^D = \mathcal{H}_i^G = 0. \tag{31}$$

The quantization procedure of the system composed by incoherent dust coupled with gravity [16] consists in associating with the canonical variables the following differential operators,

$$\hat{h}_{ij} = h_{ij}\times, \qquad \hat{P}^{ij} = -i\frac{\delta}{\delta h_{ij}}, \tag{32}$$

$$\hat{X}^\mu = X^\mu\times, \qquad \hat{P}_\mu = -i\frac{\delta}{\delta X^\mu}, \tag{33}$$

and evaluating the action of the quantum version of the constraints (30) and (31) on the physical states identified as the functional $\Psi[X^\mu, h_{ij}]$, i.e., the wave function of the Universe.

First of all, the condition $\mathcal{H}_i^D = X^j_{,i}P_j = 0$ tells us that

$$\frac{\delta}{\delta X^i}\Psi[X^\mu, h_{ij}] = 0, \tag{34}$$

so the wave function of the Universe does not depend on the spatial fluid variables X^i but only on the time fluid variable T. Furthermore, the quantum version of the constraint (31),

$$\hat{\mathcal{H}}_i\Psi[T, h_{ij}] = 0, \tag{35}$$

ensures us that $\Psi[T, h_{ij}]$ does not depend on the particular metric representation, but only on three geometries.

Remembering the definitions of the operators (32) and (33), the application of the constraint (30) on the physical states $\Psi[T, h_{ij}]$ leads us to the WDW equation that resembles a Schrodinger-like equation,

$$\hat{\mathcal{H}}\Psi[T, h_{ij}] = \left[\hat{\mathcal{H}}^G - i\frac{\delta}{\delta T}\right]\Psi[T, h_{ij}] = 0 \rightarrow i\frac{\delta}{\delta T}\Psi[T, h_{ij}] = \hat{\mathcal{H}}^G\Psi[T, h_{ij}], \tag{36}$$

which determines the evolution of the system with respect to the time variable T. It is easy to verify that a general solution for Eq. (36) is

$$\Psi(T, h_{ij}) = \int dE\psi(E, h_{ij})e^{-iET}, \tag{37}$$

leading to the time independent eigenvalue problem

$$\hat{\mathcal{H}}^G\psi = E\psi. \tag{38}$$

From Eq. (38) we can see that E is the eigenvalue of the super-Hamiltonian, and it is associated with the dust energy density via the relation $\rho_{\mathrm{dust}} = -\frac{E}{\sqrt{h}}$. For the Bianchi I model that we are taking into account the super-Hamiltonian $\mathcal{H}^G$ is of the form (3), which in the quantum version $\hat{\mathcal{H}}^G$ corresponds to Eq. (5), and the eigenvalue problem (38) takes the explicit form

$$\left[\frac{3}{32\pi}\partial_\rho^2 - \frac{\partial_+^2 + \partial_-^2}{24\pi\rho^2} - \pi\rho^2\Lambda\right]\psi(\rho, \beta_\pm) = E\psi(\rho, \beta_\pm). \tag{39}$$

The Kuchař and Torre approach is clearly a promising point of view for addressing the problem of time, viewed as a necessary weakening of the general relativity principle. Indeed, although the general covariance is preserved via a general reparametrization, the time evolution of the quantum gravitational field comes out from the privileged character of the Gaussian reference frame. But the real critical point of the formulation presented above is that the super-Hamiltonian spectrum is not positively defined and consequently the dust fluid has to possess a nonpositive energy density, a really unpleasant physical property, which is a serious shortcoming of the formulation. In [28], it has been demonstrated that real incoherent dust coupled to gravity plays the role of a physical clock and this issue constitutes a complementary approach to the present one.

Apart from the nontrivial question about how it is possible to make the Gaussian frame compatible with the energy conditions [16] (i.e., its energy-momentum tensor does not fulfil the condition to represent a physical fluid), we can see that a dualism exists between a physical clock for the gravitational field and a fluid of reference coupled to the gravitational field dynamics; see also [31–33]. From a more general point of view, we can infer that the coupling of the gravitational field to a given physical fluid is equivalent to inducing no longer vanishing super-Hamiltonian and/or supermomentum constraints. From a field theory point of view, we are arguing that the quantization of the gravitational field is affected by the choice of a specific gauge, i.e., of a real system of reference, by restoring a time evolution. In quantum gravity, the distinction between a real reference frame (a physical system) having a nonzero energy-momentum tensor and a simple system of coordinates (a mathematical reparametrization of the dynamics) is deep: while in general relativity the two concepts overlap, as soon as we take the real fluid as a test system, on the quantum level, the energy-momentum tensor of the reference frame participates in the gravitational field dynamics via the super-Hamiltonian spectrum.

The present study addresses the question concerning the positive character of the dust energy density, since we construct a quantum cosmology model for which such a property definitely holds. It is actually relevant that from such a regularization of the Kukhař and Torre model the relevant issues described below come out: the emergence of a cyclical universe, possessing a big-bounce feature and the proper classical limit. The basic ingredient for such a physical implementation of the clock-dust dualism is the presence of a small negative cosmological constant (also ensuring the Universe's turning point), while the evolutionary quantum dynamics is then crucial for the emergence of a cyclical picture. The physical meaning of our cosmological time consists of the possibility to restate the Bianchi I super-Hamiltonian eigenvalue as the energy density of a physical fluid, comoving with the synchronous reference system and, *de facto*, identified with the latter. In the classical limit, our Universe possesses a dust contribution (nonrelativistic matter) that is redshifted by the inflationary paradigm [5,34] up to such much small values that its present-day contribution to the Universe's critical parameter is negligible; see [26] and [35–37]. In other words, our physical dust is a valuable clock to describe the considered model evolution, but it is today so much diluted across the Universe that the difference with a formal system of coordinates is no longer appreciable and the general relativity principle is fully restored.

B. Semiclassical limit

Before dealing with the full quantum problem, it is interesting for our purposes to study the associated classical problem of Eq. (38), namely, the zeroth order of a WKB expansion of the evolutionary quantum system [38]. The constraint that we obtain is

$$\mathcal{H} = -\frac{3}{32\pi} p_\rho^2 + \frac{p_+^2 + p_-^2}{24\pi\rho^2} - \pi\rho^2\Lambda = E. \tag{40}$$

We can solve the classical dynamics by making use of the Hamiltonian equations and the constraint (40). We can find the solution for the isotropic variable ρ by taking into account the Hamiltonian equations,[2]

$$\begin{cases} \dot\rho = \frac{d\rho}{dt} = \frac{\partial\mathcal{H}}{\partial p_\rho} = -\frac{3}{16\pi} \\ p_\rho \dot p_\rho = \frac{dp_\rho}{dt} = -\frac{\partial\mathcal{H}}{\partial\rho} = \frac{p_+^2+p_-^2}{12\pi\rho^3} + 2\pi\rho\Lambda, \end{cases} \tag{41}$$

in order to obtain

$$\ddot\rho + \frac{p_+^2 + p_-^2}{64\pi^2\rho^3} + \frac{3}{8}\rho\Lambda = 0. \tag{42}$$

Recalling that $p_\rho = -\frac{16\pi}{3}\dot\rho$, the super-Hamiltonian constraint becomes

$$\dot\rho^2 - \frac{p_+^2 + p_-^2}{64\pi^2\rho^2} + \frac{3}{8}\rho^2\Lambda + \frac{3}{8\pi}E = 0. \tag{43}$$

It is possible to show that a solution to Eqs. (42) and (43) is given by

$$\rho(t) = \sqrt{\left(\frac{-E}{2\pi\Lambda}\right)\left[1 \pm \sqrt{1 + \frac{\Lambda(p_+^2+p_-^2)}{6E^2}}\sin\left(\sqrt{\frac{3\Lambda}{2}}t + \varphi\right)\right]}. \tag{44}$$

The solution (44) exhibits the usual initial singularity in the past for which $\rho = 0 \to \alpha = -\infty$ and furthermore a singularity in the future exists too, namely, a big crunch singularity. The value of the integration constant φ can be chosen in such a way that the initial singularity for the value $t = 0$ gives us

$$\varphi_0 = \arcsin\left(\mp\frac{1}{\sqrt{1 + \frac{\Lambda(p_+^2+p_-^2)}{6E^2}}}\right). \tag{45}$$

The classical behavior of the isotropic variable ρ is sketched in Fig. 1. Analogously, the classical dynamics of the anisotropies $\beta_\pm$ can be solved, including the solution (44) inside the Hamiltonian equations. This way, we have

[2] In the following we label the Gaussian time variable T as t.

$$\begin{cases} \dot{\beta_\pm} = \frac{\partial \mathcal{H}}{\partial p_\pm} = \frac{p_\pm}{12\pi\rho^2} == -\frac{\Lambda p_\pm}{6E}\left[1 \pm \sqrt{1+\frac{\Lambda(p_+^2+p_-^2)}{6E^2}}\sin\left(\sqrt{\frac{3\Lambda}{2}}t+\varphi_0\right)\right]^{-1} \\ \dot{p_\pm} = -\frac{\partial \mathcal{H}}{\partial \beta_\pm} = 0 \end{cases}. \tag{46}$$

The solution reads as

$$\beta_\pm(t) = \frac{p_\pm}{3\sqrt{p_+^2+p_-^2}}\ln\left|\frac{1+\frac{\sqrt{6}E}{\sqrt{\Lambda(p_+^2+p_-^2)}}\left(\sqrt{1+\frac{\Lambda(p_+^2+p_-^2)}{6E^2}}+\tan\left(\frac{1}{2}\sqrt{\frac{3\Lambda}{2}}t+\frac{\varphi_0}{2}\right)\right)}{1-\frac{\sqrt{6}E}{\sqrt{\Lambda(p_+^2+p_-^2)}}\left(\sqrt{1+\frac{\Lambda(p_+^2+p_-^2)}{6E^2}}+\tan\left(\frac{1}{2}\sqrt{\frac{3\Lambda}{2}}t+\frac{\varphi_0}{2}\right)\right)}\right| + \text{const.} \tag{47}$$

As we can see in Fig. 2, at the classical level the anisotropies of the model become important in magnitude towards the singularities in the past and in the future. So the presence of a negative cosmological constant in the semiclassical evolution case does not mine the divergence of the anisotropies towards the singularities, typical of the anisotropic models.

C. Dynamics of the quantum expectation values

Let us consider now the full quantum evolution case (39). The absence of a potential term for the anisotropies suggests considering for them a plane-waves solution, so that

$$\psi(\rho,\beta_\pm) = \frac{1}{2\pi}e^{ik_+\beta_+}e^{ik_-\beta_-}\chi(\rho), \tag{48}$$

where $\{k_+, k_-\}$ are the quantum numbers associated to $\{\beta_+, \beta_-\}$. Taking into account this shape of the wave function in Eq. (39) brings the following differential equation:

$$\left[\partial_\rho^2 + \frac{k_*^2}{\rho^2} - \Lambda_*\rho^2\right]\chi(\rho) = E_*\chi(\rho), \tag{49}$$

where

$$k_*^2 = \frac{4}{9}(k_+^2 + k_-^2), \qquad \Lambda_* = \frac{32\pi^2\Lambda}{3}, \qquad E_* = \frac{32\pi E}{3}. \tag{50}$$

Looking at Eq. (49) we can observe a formal analogy with the problem of the three-dimensional harmonic oscillator, where the angular momentum l is in correspondence with the continuous values $k_*^2 = -l(l+1)$. Following the analogy, we choose a solution for $\chi(\rho)$ of the form [39]

$$\chi(\rho) = e^{-\frac{\sqrt{\Lambda_*}\rho^2}{2}}\rho^{\frac{1}{2}+\frac{\sqrt{1-4k_*^2}}{2}}\xi(\rho). \tag{51}$$

The motivation of this choice is due to the fact that the terms $e^{\frac{\sqrt{\Lambda_*}\rho^2}{2}}$ and $\rho^{\frac{1}{2}+\frac{\sqrt{1-4k_*^2}}{2}}$ represent, respectively, the solutions of Eq. (49) in the limit $\rho \to \infty$ and $\rho \to 0$.

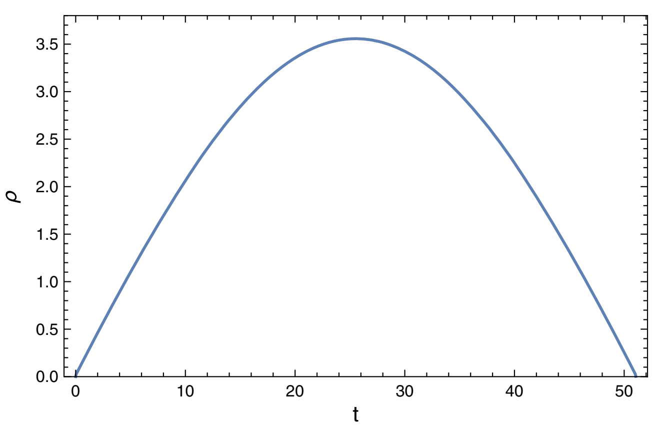

FIG. 1. The classical trajectory for the isotropic variable ρ exhibits a singularity in the past and another one in the future. The solution is sketched for the parameters: $\Lambda = 0.01$, $p_+ = p_- = 0.1$, $E = -0.397$.

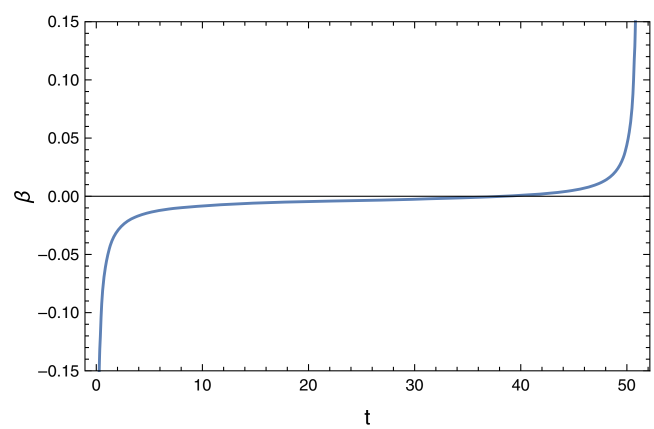

FIG. 2. The classical trajectory for the anisotropies $\beta_\pm$. Next to the singularities the anisotropies diverge. The solution is sketched for the parameters: $\Lambda = 0.01$, $p_+ = p_- = 0.1$, $E = -0.397$.

The solution (51) should take into account these two limit behaviors. We assume a finite power series expansion for the function $\xi(\rho)$ of the form

$$\xi(\rho) = \sum_{k=0}^{k'} c_{k,k'}\rho^k, \quad k, k' \in 2\mathbb{N}. \tag{52}$$

The reason is due to the fact that this is the only way to obtain physical acceptable solutions. Indeed, if we take into account a solution $\sum_{k=0}^{\infty} c_k\rho^k$ for the problem (49) we obtain a nonconverging series and then a diverging solution. To obtain a finite solution, as is done in Eq. (52), we must require the series to be truncated at a certain power associated to k'. Including expansion (52) in Eq. (49) we arrive at the following difference equation:

$$c_{k+2,k'}(k+2)\left[\sqrt{1-4k_*^2}+k+2\right] - c_{k,k'}\left[E_* + \sqrt{\Lambda_*}\left(\sqrt{1-4k_*^2}+2k+2\right)\right] = 0. \tag{53}$$

In order to obtain a finite solution we must set $c_{k+2,k'} = 0$. This restriction allows us to determine the behavior of the eigenvalue E, making use of the definitions (50),

$$E_* + \sqrt{\Lambda_*}(\sqrt{1-4k_*^2}+2k'+2) = 0$$
$$\Rightarrow E_{k',k_\pm} = -\frac{1}{4}\sqrt{\frac{3\Lambda}{2}}\left[\sqrt{1-\frac{16}{9}(k_+^2+k_-^2)}+2k'+2\right]. \tag{54}$$

In order to deal with a real eigenvalue, we consider a restriction for the values of $\{k_+, k_-\}$ of the form

$$(k_+^2+k_-^2) \le \frac{9}{16}. \tag{55}$$

This way we obtain a spectrum for the eigenvalues that assumes only negative real values and then the associated dust-energy density is always positive. Finally, always following the analogy with the three-dimensional harmonic oscillator, we can evaluate the coefficients $c_{k,k'}$ in terms of the Γ-function,

$$c^s_{k,k'} = \frac{(-1)^{\frac{k}{2}}((-1)^k+1)\Gamma\left[1+\frac{1}{2}\sqrt{1-\frac{16}{9}(k_+^2+k_-^2)}\right](\frac{32\pi^2\Lambda}{3})^{\frac{k}{4}}\frac{k'}{2}!}{\Gamma[1+\frac{k}{2}]\Gamma\left[1+\frac{n}{2}+\frac{1}{2}\sqrt{1-\frac{16}{9}(k_+^2+k_-^2)}\right](\frac{k'}{2}-\frac{k}{2})!}. \tag{56}$$

Now we can obtain the shape of the entire wave function, the solution to the problem of (39), that is

$$\psi(\rho,\beta_\pm) = Ae^{ik_+\beta_+}e^{ik_-\beta_-}e^{-\frac{\sqrt{\Lambda_*}\rho^2}{2}}\rho^{\frac{1}{2}+\frac{\sqrt{1-4k_*^2}}{2}}\sum_{k=0}^{k'} c^s_{k,k'}\rho^k, \tag{57}$$

where A is a normalization constant. Now we want to perform a comparison between the classical trajectories (44) and (47) and the expectation values of the associated operator $\hat{\rho}$ and $\hat{\beta}_\pm$. The states on which we evaluate them can be constructed taking into account the wave packets associated with the wave function (57) peaked around classical values $\{k'^*, k_+^*, k_-^*\}$, i.e.,

$$\Psi_{k'^*,k_\pm^*}(\rho,\beta_\pm) = A\int\int_R dk_\pm e^{-\frac{(k_+-k_+^*)^2}{2\sigma_+^2}}e^{-\frac{(k_--k_-^*)^2}{2\sigma_-^2}} \times \sum_{k'=1}^{\infty} e^{-\frac{(k'-k'^*)^2}{2\sigma^2}}e^{-iE_{k',k_\pm}t}\psi(\rho,\beta_\pm), \tag{58}$$

where the integrations on $\{k_+, k_-\}$ are restricted over the region $R = \{k_+, k_- \in \mathbb{R}|(k_+^2+k_-^2) \le \frac{9}{16}\}$ and we choose Gaussian weights to peak the wave packets. The evolution in time of the expectation value of the operator $\hat{\rho}$ is evaluated over such states,

$$\langle\hat{\rho}\rangle_t = \int_0^\infty d\rho\int_{-\infty}^{\infty} d\beta_\pm(\Psi_{k'^*,k_\pm^*})^*\rho\Psi_{k'^*,k_\pm^*}. \tag{59}$$

As we can see in Fig. 3 we have an overlap between the expectation value (black points) and classical trajectory (red continuous line) only for late time t. When we approach $t = 0$, the expectation value moves away from the classical trajectory and it does not exhibit a singular behavior. As a consequence, we can argue that in the evolutionary quantum model the singularity is avoided and

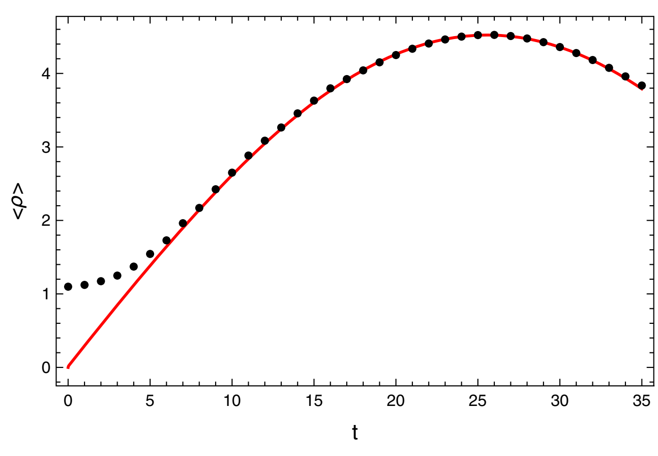

FIG. 3. The black points represent the expectation value $\langle\rho\rangle_t$ evaluated via numerical integration for the following choice of the integration parameters: $\Lambda = 0.01$, $k'^* = 5$, $k_+^* = k_-^* = 0.1$, $\sigma_+ = \sigma_- = 0.01$, $\sigma = 0.88$. The continuous red line represents the classical trajectory evaluated with the same classical parameters.

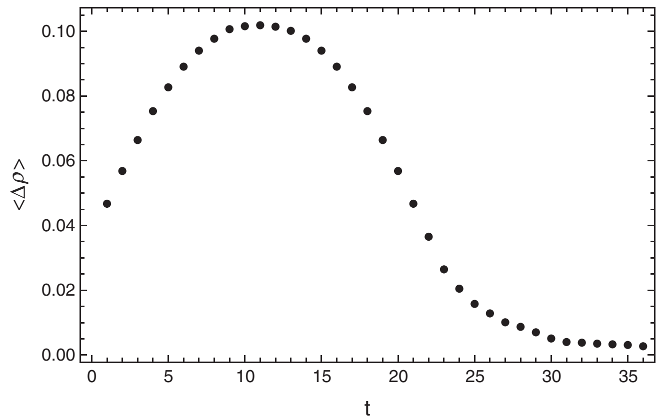

FIG. 4. The uncertainty of ρ as a function of time t that confirms that the expectation value $\langle\rho\rangle_t$ is a genuine quantity.

is replaced by a bounce. The validity of this argument is supported by the analysis of the uncertainty,

$$\langle\Delta\rho^2\rangle_t = \int_0^\infty d\rho \int_{-\infty}^{\infty} d\beta_\pm (\Psi_{k'^*,k^*_\pm})^* \rho^2 \Psi_{k'^*,k^*_\pm} - \langle\hat{\rho}\rangle_t^2, \quad (60)$$

essentially for two reasons. The first one is, as we can see in Fig. 4, that when we are near the singularity the uncertainty $\langle\Delta\rho^2\rangle$ has a maximum value but it remains always small compared to the expectation value and does not diverge in correspondence to the singularity. Thus, we can conclude that the expectation value (59) is a good indicator for the system next to the singularity. The second reason is the late time behavior. It is clear from Fig. 4 that as we get farther away from the singularity, the uncertainty becomes smaller and smaller and approaches 0 in the region of the overlap between expectation value and classical trajectory, guaranteeing that the Universe becomes more and more classical at late times.

The same approach can be used to compare expectation values related to the anisotropies with the corresponding classical trajectories. The evolution in time is

$$\langle\hat{\beta_\pm}\rangle_t = \int_0^\infty d\rho \int_{-\infty}^{\infty} d\beta_\pm (\Psi_{k'^*,k^*_\pm})^* \beta \Psi_{k'^*,k^*_\pm}. \quad (61)$$

As we can see in Fig. 5 again we have an overlap between the expectation value (black points) and the classical trajectory (red continuous line) only for late time t. At early times, the diverging behavior exhibited by the anisotropies at the classical level disappears in the quantum model. Indeed, when we approach the limit $t \to 0$ the anisotropies remain small and finite. As before, the validity of this argument is supported by the analysis of the uncertainty $\Delta\beta_\pm$, defined as

$$\langle\Delta\beta^2\rangle_t = \int_0^\infty d\rho \int_{-\infty}^{\infty} d\beta_\pm (\Psi_{k'^*,k^*_\pm})^* \beta^2 \Psi_{k'^*,k^*_\pm} - \langle\hat{\beta}\rangle_t^2. \quad (62)$$

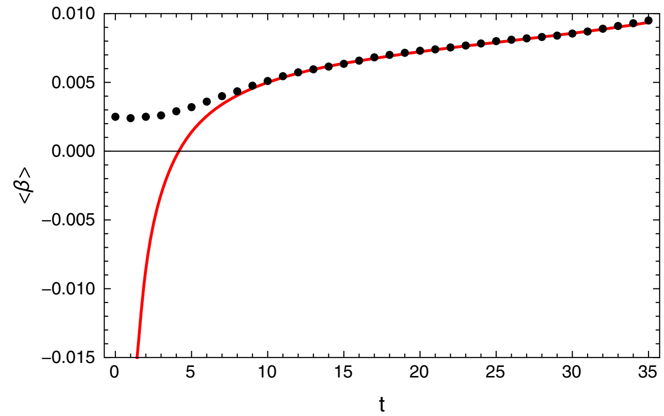

FIG. 5. The black points represent the expectation value $\langle\beta_\pm\rangle_t$ evaluated via numerical integration for the following choice of the integration parameters: $\Lambda = 0.01$, $k'^* = 5$, $k^*_+ = k^*_- = 0.1$, $\sigma_+ = \sigma_- = 0.01$, $\sigma = 0.88$. The continuous red line represents the classical trajectory evaluated with the same classical parameters.

As is shown in Fig. 6, the situation is exactly the same with respect to the case of the variable ρ, and this bring us to conclude in an analogous way that the (61) is a genuine quantity to describe the system next to the singularity and to recover the classical limit for late times. We conclude this section by noting how all the considerations discussed here for the initial singularity must remain valid when we consider the Bianchi I singularity in the future. In other words also the existing big crunch is removed in favor of a bounce and our model acquires a cyclical feature. The nondiverging character of the anisotropies in this scenario can have intriguing implications for the so-called big-bounce cosmologies [40] in view of the possibility to minimize the effect on anisotropic evolution.

III. IMPLICATION ON THE BIANCHI IX MODEL

Now, in order to implement the properties found above to a general one model, we analyze the Bianchi IX cosmology

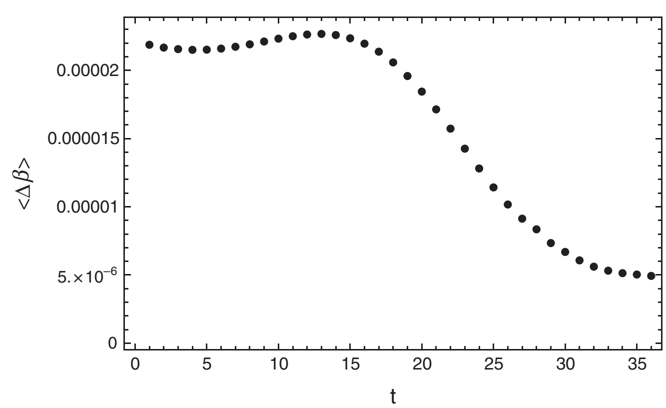

FIG. 6. The uncertainty of β as a function of time t that confirm that the expectation value $\langle\beta\rangle_t$ is a genuine quantity.

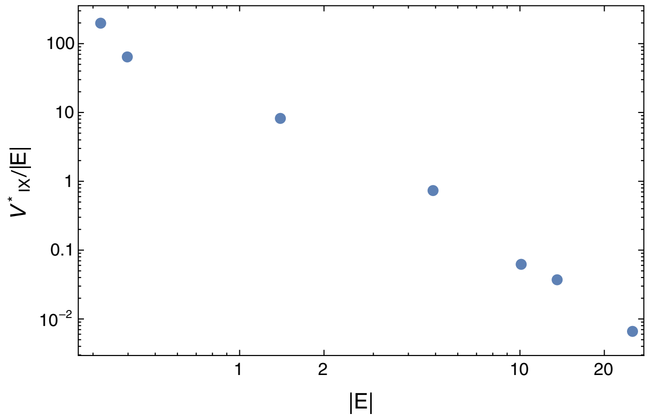

FIG. 7. The behavior of the quantity $V^*_{\mathrm{IX}}/|E|$ as a function of $|E|$ evaluated in correspondence to the bounce. The role of the Bianchi IX potential term becomes more and more marginal with the increase of the dust energy.

in the presence of a negative cosmological constant in the context of the evolutionary model. With respect to the configurational variables $\{\rho, \beta_+, \beta_-\}$ the super-Hamiltonian constraint takes the form

$$\mathcal{H} = -\frac{3}{32\pi} p_\rho^2 + \frac{p_+^2 + p_-^2}{24\pi\rho^2} + \frac{\pi}{2}\rho^{2/3} V_{\mathrm{IX}}(\beta_\pm) - \pi\rho^2\Lambda = E, \tag{63}$$

where the potential term, which accounts for the spatial curvature of the model, reads as

$$V_{\mathrm{IX}}(\beta_\pm) = e^{-8\beta_+} - 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+}[\cosh(4\sqrt{3}\beta_-) - 1]. \tag{64}$$

This potential is obtained by selecting the three constants of structure $(\lambda_l, \lambda_m, \lambda_n) = (1, 1, 1)$ in the general potential expression in Eq. (A20). As is well known, in the context of the Misner-like variables, it is clear that the difference between the Bianchi I model and the Bianchi IX model is the presence of the potential term $\frac{\pi}{2}\rho^{2/3} V_{\mathrm{IX}}(\beta_\pm)$. For this reason we want to see if a regime exists in which the potential term of the Bianchi IX model is negligible with respect to the kinetic plus the cosmological constant term. In other words, we want to see when it is possible to argue that the properties found in the previous section for the Bianchi I model are valid also for the Bianchi IX model. The importance to find a regime of this kind is due to the presence of the BKL conjecture, which allows one to extend such a conclusion to the generic cosmological solution. To this aim, we now assess the importance of the potential term $V^*_{\mathrm{IX}} = \frac{\pi}{2}\rho^{2/3} V_{\mathrm{IX}}(\beta_\pm)$ evaluated at the bounce as the dust energy E, estimated in (54), changes. As we can see in Fig. 7, the potential term of the Bianchi IX model becomes more and more negligible as the magnitude of the dust energy density increases. This means that, following the trajectory of a Bianchi IX cosmology, the relevant contribution comes from the kinetic plus cosmological constant term because the potential is more and more negligible as the parameter E becomes large. In this sense we can conclude, provided that the dust energy density is large enough to neglect the potential term, that the Bianchi IX model in the presence of a negative cosmological constant in the evolutionary context possesses the same qualitative features of the Bianchi I model previously found.

IV. PHENOMENOLOGICAL CONSIDERATIONS

Let us now provide a proper cosmological interpretation to the results we obtained in the previous sections and outline the main merits of the proposed scenario.

We considered a cosmological model that corresponds to type I Bianchi classification, i.e., having zero spatial curvature, and we included in the dynamics a small negative cosmological constant. The quantization of the model, to account for its behavior near the cosmological singularity, has been performed according to the Kuchař-Torre approach, relying on a basic dualism between an external clock and the presence of a real dust fluid in the model evolution. The weak point of such a viable perspective to construct a physical time in quantum gravity consists, in general, of the nonpositive definite nature of the dust energy density, emerging from the implementation of an external time (this fact reflects the nonpositive character of the super-Hamiltonian eigenvalue). However, in the considered model, this shortcoming of the dualism time dust is fully overcome, since the energy of the dust is always positive and this is a consequence of the negative value of the cosmological constant, which, from a purely formal point of view, allows one to compare the universe volume quantum dynamics to a harmonic oscillator, but having a global minus sign.

Then, studying the behavior of quantum expectation values and uncertainties, we get the very surprising and valuable issue of a big-bounce cosmology. What makes our model physically meaningful is the existence of a spontaneous classical limit, associated with the same harmonic structure removing the singularity. The quadratic potential associated with the negative cosmological constant is responsible for a localization of the physical quantum states near the classical trajectory, as the Universe has a sufficiently large volume.

These two important features of the model, i.e., the presence of a big bounce near the classical location of the singularity and the natural classical limit of the expanded Universe, together with the turning point in the Universe's late time evolution that the negative cosmological constant induces in the classical dynamics, suggest that our Bianchi I cosmology is an intriguing candidate for a cyclic universe.

This issue is in itself a remarkable scenario, but our interest in the constructed picture is actually for the potential degree of generality it could represent. In fact, in Sec. III, we inferred that the behavior of the Bianchi type I model can be extended, under suitable conditions (i.e., a sufficiently large value of the parameter E) to the most general Bianchi type IX cosmology, which is a good prototype for the generic cosmological Universe. In other words, it is a natural guess that the implementation of an evolutionary quantum gravity in the presence of a small negative cosmological constant can lead to a nonsingular cyclic universe even when we are referring to it as a generic inhomogeneous universe. According to the BKL conjecture [41] and to its quantum implementation (the so-called long-wavelength assumption), for each sufficiently small neighbor of a space point, physically corresponding to the cosmological horizon, the dynamical evolution is qualitatively that one of a Bianchi IX universe. Thus, we trace in the present analysis the basic dynamical features that could regularize the cosmological problem, without explicitly including an ultraviolet cutoff in the canonical Wheeler-DeWitt quantization of the system. Now we should shed light on the physical mechanism at the bottom of the dynamical picture traced above and, in this respect, we investigate which of our ingredients is related in the model to a cutoff physics.

V. PHYSICAL INTERPRETATION OF THE BIG BOUNCE

In this section we show how central the presence of the negative cosmological constant is for the appearance of the big bounce. To this aim we analyze here an evolutionary Bianchi I model without the negative cosmological constant and we consider a cutoff polymer dynamics that makes discrete the isotropic variable ρ in order to show how the behavior of the quantum expectation values of the previous section and the behavior of the polymer semiclassical dynamics are equivalent. This equivalence testifies the fact that the negative cosmological constant plays the role of a cutoff physics. The model is analyzed in the same configurational space variables $\{\rho,\beta_+,\beta_-\}$ and the physical choice is to define the isotropic variable ρ as a discrete variable and to leave unchanged the anisotropies $\{\beta_+,\beta_-\}$. We consider the polymer modification at a semiclassical level. This means that we are working with a modified super-Hamiltonian constraint obtained as the lowest order term of a WKB expansion for $\hbar\to 0$ of the full polymer quantum problem [24,25]. This procedure formally consists in the replacement

$$p_\rho^2 \to \frac{2}{\mu^2}[1-\cos(\mu p_\rho)], \tag{65}$$

where μ is the polymer scale, or equivalently the lattice spacing for the variable ρ. From the substitution (65) the super-Hamiltonian becomes

$$\mathcal{H}_p = -\frac{3}{16\pi\mu^2}[1-\cos(\mu p_\rho)] + \frac{p_+^2+p_-^2}{24\pi\rho^2}, \tag{66}$$

and again the super-Hamiltonian constraint is

$$\mathcal{H}_p = E. \tag{67}$$

As in the previous case, we can solve the semiclassical polymer dynamics by making use of the Hamiltonian equations

$$\begin{cases} \dot{\rho} = \frac{\partial\mathcal{H}_p}{\partial p_\rho} = -\frac{3}{16\pi\mu}\sin(\mu p_\rho) \\ \dot{p}_\rho = -\frac{\partial\mathcal{H}_p}{\partial\rho} = \frac{p_+^2+p_-^2}{12\pi\rho^3} \end{cases} \tag{68}$$

and the constraint (67). This way we obtain the following second order differential equation:

$$\ddot{\rho} + \frac{(p_+^2+p_-^2)(1-\frac{2\mu^2(p_+^2+p_-^2)}{9\rho^2}+\frac{16}{3}\pi\mu^2 E)}{64\pi^2\rho^3} = 0. \tag{69}$$

It is not possible to individuate an analytical solution for the differential equation above, and then we perform a numerical integration. In order to find a link between the presence of a negative cosmological constant and the polymer scale we make a comparison between the classical and quantum models analyzed in Sec. II and this new semiclassical polymer model. We impose that the initial condition for the numerical integration of the differential equation (69) is exactly equal to $\langle\rho\rangle_0$ adopted in Fig. 3, i.e., we are arguing that the initial condition for the semiclassical evolutionary polymer problem matches the expectation value of the quantum evolutionary model in the correspondence of the bounce determined in the previous section. In order to perform this comparison we obviously choose the same classical values for the parameters $\{p_+,p_-,E\}$ and the same corresponding parameters $\{k'^*,k_+^*,k_-^*\}$ around which we have built the wave packets that we have used in Sec. II. The only free parameter that we can fix is the polymer scale μ.

As we can see in Fig. 8 it is possible to individuate a special value for the parameter μ for which the behavior of $\rho(t)$ in the semiclassical polymer approach overlaps the expectation value $\langle\rho\rangle_t$ in the quantum evolutionary theory. Furthermore, as is expected for every kind of polymer approach, for late times the semiclassical polymer trajectory overlaps the classical one. In this way we show that near the singularity in the context of the evolutionary theory, a negative cosmological constant acts the same way as a polymer modification related to the isotropic variable, i.e., a cutoff physics.

It is possible to deepen the connection between the negative cosmological constant and the polymer scale making use of several numerical integrations related to

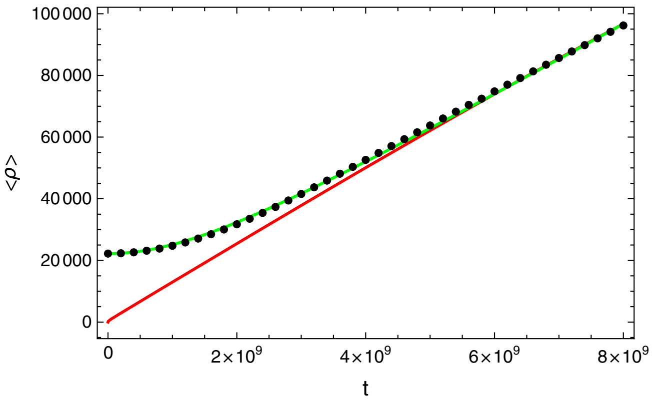

FIG. 8. The black points represent the expectation value $\langle \rho \rangle_t$ evaluated via numerical integration for the following choice of the integration parameters: $\Lambda = 10^{-20}$, $k'^{*} = 20$, $k_+^* = k_-^* = 0.1$, $\sigma_+ = \sigma_- = 0.01$, $\sigma = 0.88$. The continuous red line represents the classical trajectory while the green line represents the semiclassical polymer trajectory, where the polymer scale is fixed with the choice $\mu = 3.08 \times 10^5$.

different choices of the parameter values and seeing, time after time, if there is a general law. In Fig. 9 the behavior of $\log \mu$ as a function of $\log \sqrt{\Lambda}$ is shown, where the values of the numerical integration parameters $\{k', k_+, k_-\}$ used for evaluating the expectation value (59) (and obviously the correspondent polymer integration parameters $\{p_+, p_-, E\}$) are fixed for each line. As we can see, the slope of the lines is always the same, independently from the choice of the parameters, and it is equal to $-\frac{1}{2}$. It means that a universal law exists such that

$$\log \mu = \log \alpha_k - \frac{1}{2} \log \sqrt{\Lambda} \rightarrow \mu^2 \sqrt{\Lambda} = \alpha_k^2, \qquad (70)$$

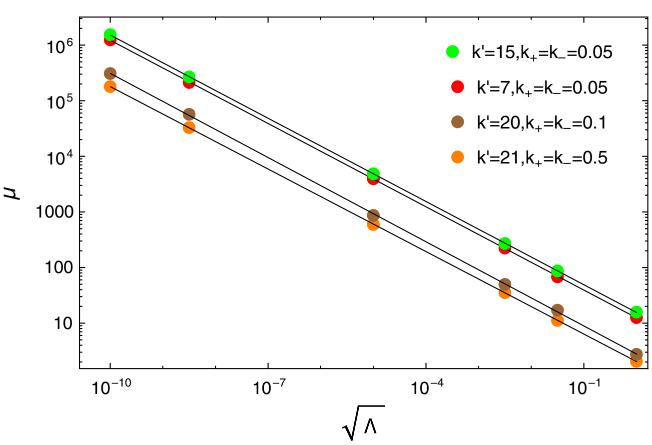

FIG. 9. The behavior of the polymer scale μ as a function of $\log \sqrt{\Lambda}$. It is evident that a law exists between the polymer scale and the negative cosmological constant, independently from the choice of the parameters.

where the constant $\alpha_k = \alpha_{k',k_+,k_-}$ depends on the values assigned to the parameters.

VI. CONCLUDING REMARKS

The main merit of the present work is in demonstrating how a rather general scenario for a cyclical universe can be recovered even within the metric canonical quantum approach, as far as a well-defined evolutionary theory is taken into account.

The basic ingredient of our approach is the small negative cosmological constant, which is responsible for the classical turning point, but, overall, it induces a harmonic oscillator morphology in the quantum universe volume dynamics. The Bianchi I cosmology we addressed here allows the simultaneous manifestation of significant properties, like the big bounce, the existence of a well-defined classical limit, and the positive character of the dust energy density, playing the role of a clock. However, what makes the present issues intriguingly cosmologically meaningful is the possibility to extend this picture to the Bianchi IX universe. In fact, this property suggests that the considered minisuperspace scheme can be generalized to the generic inhomogeneous cosmological problem. As far as we implement the long-wavelength approximation to the inhomogeneous quantum dynamics, we can factorize the Wheeler superspace into the local minisuperspaces, associated with space point neighbors. From a physical point of view, we can speak of causal regions evolving, independently of each other, according to the nonsingular cyclic dynamics we traced above. The implementation of the present ideas to a generic inhomogeneous universe, as well as the characterization of the role played by the matter, especially the radiation component, during the classical evolution, constitutes the natural perspective of the present analysis.

ACKNOWLEDGMENTS

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APPENDIX: DERIVATION OF THE SUPER-HAMILTONIAN FOR THE BIANCHI I MODEL IN THE PRESENCE OF A NEGATIVE COSMOLOGICAL CONSTANT

In this appendix we provide a brief derivation of the super-Hamiltonian (1) and we study the Bianchi I and Bianchi IX model, respectively the simplest and most general homogenous but anisotropic model. A generic homogeneous model with space-time metric g_{ij} has to preserve the invariance of the spatial line element under a suitable group of transformations. This means that the spatial line element

$$dl^2 = h_{\alpha\beta}(t,x)dx^\alpha dx^\beta, \tag{A1}$$

under the isometry $T: x \to x'$, has to be left invariant to the three-dimensional metric $h_{\alpha\beta}(t,x)$ so that in the transformed line element

$$dl^2 = h_{\alpha\beta}(t,x')dx'^\alpha dx'^\beta \tag{A2}$$

the spatial metric $h_{\alpha\beta}(t,x') = h_{\alpha\beta}(t,x)$. If we introduce three spatial vectors $\{l(x), m(x), n(x)\}$ that satisfy the homogeneity condition, the metric $h_{\alpha\beta}$ can be expressed in the form

$$h_{\alpha\beta} = a^2(t)l_\alpha l_\beta + b^2(t)m_\alpha m_\beta + c^2(t)n_\alpha n_\beta, \tag{A3}$$

where $a(t)$, $b(t)$, $c(t)$ are three different cosmic scale factors along the three spatial directions. Consequently, the vacuum Einstein equations in a synchronous reference and for a generic homogeneous cosmological model are

$$\begin{cases} -R_l^l = \frac{(\dot{a}bc)^\cdot}{abc} + \frac{1}{2(abc)^2}[\lambda_l^2 a^4 - (\lambda_m b^2 - \lambda_n c^2)^2] = 0 \\ -R_m^m = \frac{(a\dot{b}c)^\cdot}{abc} + \frac{1}{2(abc)^2}[\lambda_m^2 b^4 - (\lambda_l a^2 - \lambda_n c^2)^2] = 0 \\ -R_n^n = \frac{(ab\dot{c})^\cdot}{abc} + \frac{1}{2(abc)^2}[\lambda_n^2 c^4 - (\lambda_l a^2 - \lambda_m b^2)^2] = 0 \\ -R_0^0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0. \end{cases} \tag{A4}$$

The constants $(\lambda_l, \lambda_m, \lambda_n)$ are called constants of structure and they can only assume the values $(-1, 0, 1)$. The form of Eqs. (A4) takes into account the dynamics of the homogeneous models that are relevant near the singularity. In particular, we can only consider, in Eqs. (A4), the behavior of six models, called Bianchi I, II, VI, VII, VIII, and IX, that belong to the so-called Bianchi classification [42]. This classification contains all the nine possible models that respect the homogeneity constraint in the same way as $K = \{-1, 0, 1\}$ identifies the possible symmetry types for homogeneous and isotropic Friedmann-Robertson-Walker (FRW) three-spaces. In particular, three of them, the Bianchi I, the Bianchi V, and the Bianchi IX model, represent the anisotropic generalization of the flat, open, and closed FRW metrics, respectively.

Let us consider now a line element for a generic homogeneous space-time in the ADM [29] form,

$$ds^2 = N(t)^2dt^2 - h_{\alpha\beta}dx^\alpha dx^\beta, \tag{A5}$$

where $N(t)$ is the lapse function and where we redefined the three scale factors $\{a(t), b(t), c(t)\}$ in such a way as to have a spatial line element of the form

$$\begin{aligned} dl^2 &= h_{\alpha\beta}dx^\alpha dx^\beta \\ &= (e^{q_l}l_\alpha l_\beta + e^{q_m}m_\alpha m_\beta + e^{q_n}n_\alpha n_\beta)dx^\alpha dx^\beta = \eta_{ab}\omega^a\omega^b, \end{aligned} \tag{A6}$$

where we introduced the matrix $\eta_{ab} = \mathrm{diag}\{e^{q_l}, e^{q_m}, e^{q_n}\}$ and a set of three invariance forms $\omega^a = \omega^a_\alpha dx^\alpha$ with $\omega^a_\alpha = \{l_\alpha, m_\alpha, n_\alpha\}$.

In order to introduce the dynamical character of the gravitational field let us consider the Einstein-Hilbert action in the presence of a negative cosmological constant,

$$S = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}(R - 2\Lambda), \tag{A7}$$

where[3] $\kappa = 8\pi G$ and R is the Ricci scalar. Let us start by studying the variation of the previous action. It means that we have to evaluate the determinant of the space-time metric and the Ricci scalar for the particular case of the homogeneous space-time represented in the line element (A5). When we do this we obtain

$$\delta S_g = \delta\int_{t_1}^{t_2}\mathcal{L}(q_a, \dot{q}_a)dt = 0, \tag{A8}$$

where t_1 and t_2 are two fixed instants of time with $t_1 < t_2$ and the Lagrangian $\mathcal{L}$ is of the form

$$\mathcal{L} = -\frac{8\pi^2\sqrt{\eta}}{\kappa}\left[\frac{1}{2N}(\dot{q}_l\dot{q}_m + \dot{q}_l\dot{q}_n + \dot{q}_m\dot{q}_n) - N\bar{R} + N\Lambda\right]. \tag{A9}$$

In the Lagrangian (A9) we introduce the quantity $\eta = \det(\eta_{ab}) = e^{q_l+q_m+q_n} = e^{\sum_a q_a}$ while $\bar{R}$ represents the three-dimensional Ricci scalar and it is connected with the constants of structure in such a way that

$$\eta\bar{R} = -\frac{1}{2}\left[\sum_a \lambda_a^2 e^{2q_a} - \sum_{a\neq b}\lambda_a\lambda_b e^{q_a+q_b}\right], \tag{A10}$$

where the indices $\{a, b\}$ take values in $\{l, m, n\}$. The choice of the constants of structure that appear in Eq. (A10) determines the particular homogeneous model that we can select inside the Bianchi classification.

From the Lagrangian (A9) we can obtain the Hamiltonian of the system performing a Legendre transformation. The conjugated momenta to the generalized coordinate q_a are the following,

[3]For the calculation in this appendix we use the natural units $c = \hbar = 1$.

$$\begin{cases} p_l = \frac{\partial \mathcal{L}_g}{\partial \dot{q}_l} = -\frac{4\pi^2\sqrt{\eta}}{kN}(\dot{q}_m + \dot{q}_n) \\ p_m = \frac{\partial \mathcal{L}_g}{\partial \dot{q}_m} = -\frac{4\pi^2\sqrt{\eta}}{kN}(\dot{q}_n + \dot{q}_n) \\ p_n = \frac{\partial \mathcal{L}_g}{\partial \dot{q}_n} = -\frac{4\pi^2\sqrt{\eta}}{kN}(\dot{q}_l + \dot{q}_m), \end{cases} \tag{A11}$$

and taking into account the transformation

$$N\mathcal{H} = \sum_{a=l,m,n} p_a \dot{q}_a - \mathcal{L}_g, \tag{A12}$$

where $\mathcal{H}$ is the super-Hamiltonian of the system, we can put the action in the form

$$S_g = \int dt \left(\sum_a p_a \dot{q}^a - N\mathcal{H} \right) \tag{A13}$$

with

$$\mathcal{H} = \frac{k}{8\pi^2\sqrt{\eta}} \left[\sum_a (p_a)^2 - \frac{1}{2}\left(\sum\nolimits_b p_b\right)^2 - \frac{64\pi^4}{k^2}(\eta\bar{R} + \eta\Lambda) \right]. \tag{A14}$$

A very useful set of generalized coordinates is represented by the Misner variables $\{\alpha, \beta_+, \beta_-\}$ [4,20], i.e.,

$$\begin{cases} q_l = 2(\alpha + \beta_+ + \sqrt{3}\beta -) \\ q_m = 2(\alpha + \beta_+ - \sqrt{3}\beta -) \\ q_n = 2(\alpha - 2\beta_+). \end{cases} \tag{A15}$$

With respect to the Misner variables the metric η_{ab} assumes the form

$$\eta_{ab} = e^{2\alpha}(e^{2\beta})_{ab} \to \det(\eta_{ab}) = e^{6\alpha}. \tag{A16}$$

It is possible to show that the variable α represents the isotropic component of the Universe, being related to the volume, while the matrix $\beta_{ab} = \mathrm{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$ accounts for the anisotropy of the system. In terms of this new variable the action (A13) takes the form

$$S_g = \int (p_\alpha \dot{\alpha} + p_+ \dot{\beta}_+ + p_- \dot{\beta}_- - N\mathcal{H})dt, \tag{A17}$$

where

$$\mathcal{H} = \frac{k}{3(8\pi)^2} e^{-3\alpha}(-p_\alpha^2 + p_+^2 + p_-^2 + \mathcal{V}) - \frac{8\pi^2\Lambda}{\kappa} e^{3\alpha} \tag{A18}$$

and the scalar curvature term becomes

$$\mathcal{V} = -\frac{6(4\pi)^4}{k^2}\eta\bar{R} = \frac{3(4\pi)^4}{k^2} e^{4\alpha} V(\beta_\pm). \tag{A19}$$

The potential term $V(\beta_\pm)$ accounts for spatial curvature of the model and is given by the expression

$$V(\beta_\pm) = \lambda_l^2(e^{-8\beta_+ - 2e^{4\beta_+}}) + \lambda_m^2(e^{+4(\beta_+ + \sqrt{3}\beta_-)} - 2e^{-2(\beta_+ - \sqrt{3}\beta_-)}) + \lambda_n^2(e^{+4(\beta_+ - \sqrt{3}\beta_-)} - 2e^{-2(\beta_+ + \sqrt{3}\beta_-)}). \tag{A20}$$

When we choose the Bianchi I model we select a homogeneous model with the three constants of structure equal to 0, or in order words we are taking into account a model with zero spatial curvature. When we do this the Hamiltonian (A18) simply becomes

$$\mathcal{H}_I = \frac{k}{3(8\pi)^2} e^{-3\alpha}(-p_\alpha^2 + p_+^2 + p_-^2) - \frac{8\pi^2\Lambda}{\kappa} e^{3\alpha}. \tag{A21}$$

If now we make explicit $k = 8\pi G$ in the geometrical unit, so ($c = G = \hbar = 1$), the super-Hamiltonian (A21) reduces to the super-Hamiltonian in Eq. (1).

Finally, for $\lambda_m = \lambda_m = \lambda_m = 1$ we get the Bianchi IX model and the potential (A9).

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