Trapping effects in inflation: Blue spectrum at small scales

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We consider the inflationary model in which the inflaton ϕ couples to another scalar field χ via the interaction $g^2(\phi - \phi_0)^2\chi^2$ with a small coupling constant g ($g^2 \sim 10^{-7}$). We assume that there is a sequence of "trapping points" ϕ_{0i} along the inflationary trajectory where particles of the χ field become massless and are rather effectively produced. We calculate the power spectrum of inflaton field fluctuations originating from a backreaction of χ particles produced using the Schwinger's "in-in" formalism. We show that the primary curvature power spectrum produced by these backreaction effects is blue, which leads to a strong overproduction of primordial black holes (PBHs) in the subsequent radiation era.

I. INTRODUCTION

In many recent works a large class of inflationary models has been considered in which a generation of inflaton field fluctuations in a course of inflation is implemented (partly, at least) through an interaction between inflaton and other quantum fields. In particular, in [1–9] the interaction of the type

$$\mathcal{L}_{\text{int}} = -\frac{g^2}{2}(\phi - \phi_0)^2\chi^2 \qquad (1)$$

has been thoroughly exploited. Here, ϕ is the inflaton field, and χ is some other scalar field. It has been shown that the interaction of this type leads to a particle creation during inflation because when ϕ, in a process of slow roll, comes nearer to ϕ_0, the particles of the χ field become massless and are effectively produced. During the short time interval of χ-particle production, a feature in the power spectrum of scalar curvature fluctuations is generated. In early works [2] (and, also, in [4]) the process has been studied in the mean-field approximation (i.e., the variance $\langle\chi^2\rangle$, solely, has been used to quantify the backreaction of χ particles, produced during inflation, on the inflaton field). In subsequent works the calculations were refined, going beyond the mean-field treatment. Methods used for the calculation include (i) the analytic description of particle creation with the coupling (1) developed in the theory of preheating after inflation [10,11], (ii) cosmological perturbation theory for the field equations (see, e.g., [12]) (the method is used in [6,7]) and (iii) Schwinger's "in-in" formalism generalized to compute cosmological perturbations (this method is used in [8,9]).

In the present work we calculate the power spectrum of inflaton field fluctuations originating from a backreaction of χ particles produced during inflation, via the coupling (1), on

the inflaton field. We studied, in contrast with [5–7], only the case of weak coupling, $g^2 \sim 10^{-7}$, and, exclusively, the region of small scales, $k/H \gg 1$ (k is the comoving wave number, H is the Hubble parameter during inflation, and the scale factor a is equal to 1 at the initial time of the inflation era). If g^2 is so small, the "trapped inflation" scenario [3,5] is ineffective [6], but just for this region of g^2 values, the interesting phenomenon was predicted in [8]. Namely, authors of [8] (see also [13]) argued that in a case of closely spaced trapping points, i.e., if we have a sequence of points $\phi_{0,i}$ ($i = 1, \ldots, n$) where particles χ_i become massless, the total (accumulative) power spectrum of inflaton fluctuations at small scales would be blue. This conclusion is important because such small scale fluctuations might effect primordial black hole (PBH) formation when the fluctuations cross the horizon inside after the end of inflation.

The approach used in the present work is, technically, rather similar to those of Refs. [14–16]. In these works the original scenario of stochastic inflation [17] had been reformulated using functional methods. In [14–16], as in [17], the Fourier components of the inflaton field, as a whole, are split into long-wavelength modes (with wavelengths a/k longer than the horizon scale H^{-1}) and short-wavelength ones. This splitting is performed now at the action level, and short wavelengths are integrated out via a path integral over the subhorizon part of the whole field. The long-wavelength modes of the field are assumed to be classical, but long-short couplings (due to a time dependence of the window function, in particular) yield the semiclassical correction to their equation of motion. In our case, the key difference with this approach is that, instead of the short-wavelength part of the inflaton field, we have the independent field χ interacting with the inflaton field. Furthermore, in our case the inflaton field is not coarse-grained by the splitting of its own modes; the semiclassical corrections to the equation of motion for modes of the inflaton field arise due to loops of the χ field only. This approach, in application to calculations of the

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power spectrum of inflaton fluctuations, is justified if the correction to this spectrum from these loops is formed at around a time when the corresponding mode exits the horizon.

For concrete calculations, in this paper we use the closed-time-path (CTP) functional formalism of Schwinger and Keldysh [18]. The application of this formalism to cosmological problems had been suggested in pioneering works by Calzetta and Hu [19] and Jordan [20]. It is well known (see, e.g., [21,22]) that this formalism is especially useful for studying cosmological backreaction problems. Most importantly, this approach, which operates with the in-in effective action, yields the real and causal equations of motion describing a time evolution of the system (inflaton scalar field, in our case). Furthermore, the CTP functional formalism is closely related to the influence functional formalism of Feynman and Vernon [23] because in both methods the full quantum system can be divided into two parts: the distinguished subsystem (the "open system" or, simply, the "system") and the remaining degrees of freedom (the "environment"; in our case, this is the χ field). We are interested in the state of the system as influenced by the overall effect of the environment, so the environmental degrees of freedom must be integrated out. It is easy to verify that integrating out these variables in a CTP path integral, the generating CTP functional can be expressed in terms of the influence functional of Feynman and Vernon. Correspondingly, in the semiclassical approximation, the effective CTP action is expressed through the influence action.

The influence of the environment on the (open) system is, by definition, the backreaction effect. The influence action is, in general, complex; its real part contains the dissipational kernel, which yields the dissipative terms in the effective equations of motion. The imaginary part contains the noise kernel accounting for the fluctuations induced on the system through its coupling to the environment (here we use the terminology of [21,22]).

As is well known, the dynamical evolution of the (open) system is not deterministic; even in the semiclassical approximation, it is, in general, stochastic [21,22]. This is well illustrated by the quantum Brownian model [24,25], in which the Feynman-Vernon idea of a stochastic force from the environment acting on the system was exploited. In this approach, the time evolution of the system's degrees of freedom is described by the Langevin equation.

The kernels of the influence action of two interacting quantum fields in de Sitter space was found in a pioneering work by Hu, Paz and Zhang [26]. Later, similar influence actions and functionals were considered in [27,28] (in Minkowski space), in [29], in warm inflation models (see [30] and references therein), and in works on effective field theory [31].

Naturally, the backreaction of the environment on the open system cannot be too strong, so as to make the separation scheme meaningless. In our case the open system is the inflaton field; i.e., we consider the scenario of effective single field inflation. The backreaction is significant during the time Δt when the χ field is light, so the condition for a reasonable separation is $H\Delta t \lesssim 1$ [7], where H is the Hubble parameter during inflation.

If the environmental scalar field is minimally coupled to gravity and nearly massless (and this is just the case considered in the present work), an additional problem connected with infrared divergences arises. The kernels of the influence action are expressed through the momentum integrals. In particular, a one-loop contribution from integrating out the environment field is given by momentum integrals over the product of four mode functions of this field. These integrals are divergent in the massless limit (see, e.g., [32]).

The main feature of massless, minimally coupled (MMC) scalar fields is the absence of normalizable de Sitter invariant states; in other words, there is no de Sitter–invariant Fock vacuum state [33,34]. In particular, the Bunch-Davies vacuum breaks the de Sitter invariance, when $m = \xi = 0$. The consequences of this breaking that are important for us include the following: (i) the mean squared fluctuations of the MMC field grow linearly with time during inflation [35], and (ii) the de Sitter–invariant two-point function (propagator) becomes infrared divergent in the limit $m = \xi = 0$ [36].

For a regularization of the infrared divergence we use, in the mode expansion of the χ field, the comoving infrared cutoff Λ [37,38]. We put $\Lambda = H$; this cutoff value is very natural [39–41] if we want to set initial conditions for all modes at the beginning of inflation because the physics inside of the initial Hubble radius H^{-1} cannot determine the initial conditions for super-Hubble modes (clearly, there is no causal process for preparing the initial state in a space box having a size larger than the horizon).

As a result of our work, we obtain a qualitative confirmation of the main conclusion of [8] about the blueness of the power spectrum at small scales. Our calculation, however, differs from those of [8] in some important ways. In particular, we were not able to derive and use their basic formula for the power spectrum. Furthermore, our results are quite sensitive to a value of the infrared cutoff parameter Λ (see below), while we do not see anything similar in the formulas of [8]; in particular, their final expression for the noise-driven power spectrum does not contain the infrared cutoff explicitly.

The plan of the paper is as follows. In the second section we formulate our theoretical approach and obtain the equation for the amplitude of inflaton field fluctuations needed for a power spectrum calculation. The third section contains the derivation of the power spectrum formula. In the last section we present the results of our power spectrum calculations and the main conclusions about constraints on the parameters of the model following from PBH overproduction predictions.

II. EFFECTIVE ACTION AND LANGEVIN EQUATION

As stated in the Introduction, we use the CTP formalism of Schwinger and Keldysh [18] and the influence functional method of Feynman and Vernon [23]. Our Lagrangian is

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - V(\phi) - \frac{g^2}{2} \phi^2 \chi^2,$$
$$\phi = \Phi - \Phi_0. \tag{2}$$

The splitting between the system and the environment in our case is as follows: the system sector contains all the modes of the inflaton field ϕ, and the environment contains the modes of the massless scalar χ field with physical wavelengths shorter than the critical length $\lambda_c = 2\pi/\Lambda$ at the initial conformal time η_i. We set $a(\eta_i) = 1$, so a physical length $\lambda_{\text{phys}} = a(\eta)\lambda$ coincides with the comoving length λ at $\eta = \eta_i$. We approximate the space-time during inflation by a de Sitter metric,

$$ds^2 = a^2(\eta)(d\eta^2 - d\vec{x}^2), \qquad a(\eta) = \frac{-1}{H\eta}. \tag{3}$$

The generating CTP functional is defined by introducing sources for the ϕ field modes only:

$$e^{iW_{\text{CTP}}[J^+, J^-]} = \int d\phi_f d\phi_i^+ d\phi_i^-$$
$$\times \int_{\phi_i^+}^{\phi_f} \mathcal{D}\phi^+ \int_{\phi_i^-}^{\phi_f} \mathcal{D}\phi^-$$
$$\times e^{i(S_0[\phi^+] - S_0[\phi^-] + J^+\phi^+ - J^-\phi^- + S_{\text{IF}}[\phi^+, \phi^-; \infty])}$$
$$\cdot \rho_\phi(\phi_i^+, \phi_i^-; t_i). \tag{4}$$

Here, ρ_ϕ is the initial density matrix for the ϕ field.

The expressions for free field actions and for the action describing the interaction of the fields are given by

$$S_{\text{int}}[\phi, \chi] = -\frac{g^2}{2} \int d^4 x\, a^4(\eta) \phi^2 \chi^2, \tag{5}$$

$$S_0[\phi] = \int d^4 x\, a^2(\eta) \left[\frac{\phi'^2}{2} - \frac{(\nabla\phi)^2}{2} - a^2(\eta) V(\phi) \right], \tag{6}$$

$$S_0[\chi] = \int d^4 x\, a^2(\eta) \left[\frac{\chi'^2}{2} - \frac{(\nabla\chi)^2}{2} \right]. \tag{7}$$

The influence action is expressed by the formula

$$e^{iS_{\text{IF}}[\phi^+, \phi^-; t_f]} = \int d\chi_f d\chi_i^+ d\chi_i^-$$
$$\times \int_{\chi_i^+}^{\chi_f} \mathcal{D}\chi^+ \int_{\chi_i^-}^{\chi_f} \mathcal{D}\chi^-$$
$$\times e^{i(S_0[\chi^+] + S_{\text{int}}[\phi^+, \chi^+] - S_0[\chi^-] - S_{\text{int}}[\phi^-, \chi^-])}$$
$$\cdot \rho_\chi(\chi_i^+, \chi_i^-; t_i), \tag{8}$$

where ρ_χ is the initial density matrix for the χ field. The CTP or in-in effective action containing all quantal corrections to the field expectation value is given by

$$\Gamma(\overline{\phi}^+, \overline{\phi}^-) = W_{\text{CTP}}[J^+, J^-] - J^+\overline{\phi}^+ + J^-\overline{\phi}^-, \tag{9}$$

$$\overline{\phi}^\pm = \frac{\delta W_{\text{CTP}}[J^+, J^-]}{\delta J^\pm}. \tag{10}$$

In the semiclassical approximation, when loops of the ϕ field can be neglected and the density matrix ρ_ϕ is diagonal, the expectation values of the ϕ field modes are described, as follows from Eqs. (4) and (9), by the CTP effective action

$$\Gamma_{\text{CTP}}[\phi^+, \phi^-] \sim S_0[\phi^+] - S_0[\phi^-] + S_{\text{IF}}[\phi^+, \phi^-]. \tag{11}$$

Here we suppose that an evolution of the system field becomes semiclassical (for long-wavelength modes) due to the interaction with the environment (see [42–44] and [45] and references therein). The imaginary part of the influence action drives the system to this semiclassical behavior, in the course of fast inflationary expansion. The time of decoherence is proportional to $a^4 g^4$ [26].

In the CTP formalism, the time integration in the expressions for actions S is carried out along the closed path going from the initial time to $+\infty$ and back. Field values generally are not considered to be the same on the forward and backward parts of the contour, which is equivalent to doubling of degrees of freedom, i.e., considering two fields, ψ^+ and ψ^-. In formulas below we will use the linear transformation (Keldysh rotation), which leads to two new fields ϕ_c and ϕ_Δ, defined by

$$\phi_c = \frac{1}{2}(\phi^+ + \phi^-), \qquad \phi_\Delta = \phi^+ - \phi^-. \tag{12}$$

The influence action S_{IF} is calculated perturbatively (see, e.g., [27,46–48]); we keep the terms proportional to g^2 and g^4. The inclusion of the terms of order g^4 is crucial because this is the lowest order at which an imaginary part of the action appears. Just this part determines the stochastic forces in the equation of motion for the system field. The influence action $S_{\text{IF}}[\phi_c, \phi_\Delta]$ is given by the following formulas, separately for its real and imaginary parts:

$$\mathrm{Im} S_{\mathrm{IF}} = g^4 \int d^4x \int d^4x' \phi_\Delta(x)\phi_c(x)\phi_\Delta(x')\phi_c(x')$$
$$\times \mathrm{Re} G_{++}^{\Lambda 2}(x,x'), \tag{13}$$

$$\mathrm{Re} G_{++}^2(x,x') = -\frac{1}{2}\{\langle \chi(x)\chi(x')\rangle^2 + \langle \chi(x')\chi(x)\rangle^2\}, \tag{14}$$

$$\mathrm{Re} S_{\mathrm{IF}} = g^2 \int d^4x \phi_c(x)\phi_\Delta(x) i G_{++}^\Lambda(x,x) a^4(\eta)$$
$$-\frac{g^4}{2}\int d^4x \int d^4x' \phi_\Delta(x)\phi_c(x)[\phi_\Delta^2(x') + 4\phi_c^2(x')]$$
$$\cdot \mathrm{Im} G_{++}^{\Lambda 2}(x,x') \cdot \theta(\eta - \eta'), \tag{15}$$

$$\mathrm{Im} G_{++}^2(x,x') = -(\theta(\eta - \eta') - \theta(\eta' - \eta))$$
$$\times \frac{1}{2i}\{\langle \chi(x)\chi(x')\rangle^2 - \langle \chi(x')\chi(x)\rangle^2\}. \tag{16}$$

In these relations, the function $G_{++}(x,x')$ is the real time propagator [49,50] of the χ particles on the contour,

$$G_{++}(x,x') = i\langle T\chi(x)\chi(x')\rangle. \tag{17}$$

The upper index Λ in the propagator symbols in Eqs. (13) and (15) signifies a necessity of the cutoff in integration over inner momenta.

The real part of the square of the propagator needed for a calculation of the imaginary part of S_{IF} can be obtained if mode functions of the χ field, $\chi_{\vec{q}}(\eta, \vec{x})$, are known:

$$\mathrm{Re} G_{++}^{\Lambda 2}(\eta, \eta', \vec{k}) = -(2\pi)^3 \int_{q>\Lambda} d^3q \int_{q'>\Lambda} d^3q' \delta(\vec{q} + \vec{q}' - \vec{k})$$
$$\cdot \mathrm{Re}\{\chi_q(\eta)\chi_q^*(\eta')\chi_{q'}(\eta)\chi_{q'}^*(\eta')\}, \tag{18}$$

$$\chi_{\vec{q}}(\eta, \vec{x}) = \chi_q(\eta) e^{i\vec{q}\vec{x}}. \tag{19}$$

For the environment χ field we assume the Bunch-Davies vacuum, i.e.,

$$\chi_q(\eta) = \frac{1}{(2\pi)^{3/2}} \frac{e^{-ik\eta}}{a(\eta)\sqrt{2q}} \left(1 - \frac{i}{q\eta}\right). \tag{20}$$

The integration in (18), using (20), reduces to calculations of integrals of the form [51]

$$\int_{q>\Lambda} d^3q \int_{q'>\Lambda} d^3q' \delta(\vec{q} + \vec{q}' - 2\vec{k}_0) \cdot \frac{\cos[(q + q')(\eta - \eta')]}{q^n q^m}$$
$$= \frac{\pi}{k_0^{m+n-3}} \left[\int_{\frac{\Lambda}{k_0}+2}^\infty \frac{du}{u^{n-1}} \int_{u-2}^{u+2} \frac{dz}{z^{m-1}} \cos(k_0(\eta - \eta')(u + z))\right.$$
$$\left. + \int_{\frac{\Lambda}{k_0}}^{\frac{\Lambda}{k_0}+2} \frac{du}{u^{n-1}} \int_{\frac{\Lambda}{k_0}}^{u+2} \frac{dz}{z^{m-1}} \cos(k_0(\eta - \eta')(u + z))\right]. \tag{21}$$

The semiclassical equation of motion for the system field is obtained by extremizing the CTP effective action Γ_{CTP},

$$\left.\frac{\delta\Gamma_{\mathrm{CTP}}}{\delta\phi_\Delta}\right|_{\phi_\Delta=0} = 0. \tag{22}$$

This is the average equation of motion; it has to be interpreted as an average over random (stochastic) forces. Such an interpretation was suggested many years ago, in studies of quantum Brownian motion [24] by the Feynman and Vernon method. To take these stochastic forces into account, one must keep in mind the imaginary part of the effective action [Eq. (13)] which is, to lowest order, quadratic in ϕ_Δ and, for this reason, does not contribute to the average equation. A standard trick for this aim is the use of the Hubbard-Stratonovich transformation [52], which introduces an auxiliary random field $\xi(x)$. Namely, the imaginary part of the effective action is rewritten in the form

$$e^{-\mathrm{Im} S_{\mathrm{IF}}} = \int \mathcal{D}\xi P[\xi] e^{-i\int d^4x \phi_c(x)\phi_\Delta(x)\xi(x)}$$
$$\equiv \langle e^{-i\int d^4x \phi_c(x)\phi_\Delta(x)\xi(x)}\rangle_\xi. \tag{23}$$

Here, $P[\xi]$ is a normalized probability distribution on a space of functions $\xi(x)$,

$$P[\xi] = N e^{-\frac{1}{2}\int d^4x \int d^4x' \xi(x)\nu^{-1}(x,x')\xi(x')}. \tag{24}$$

The kernel $\nu^{-1}(x,x')$ is defined by the relation

$$\nu(x,x') = g^4 a^4(\eta) a^4(\eta') \mathrm{Re} G_{++}^{\Lambda 2}(x,x') = \langle \xi(x)\xi(x')\rangle_\xi. \tag{25}$$

After this transformation one obtains the stochastic effective action

$$\Gamma_{\mathrm{CTP}}[\phi_c, \phi_\Delta] = \mathrm{Re}\Gamma_{\mathrm{CTP}}[\phi_c, \phi_\Delta] - \int d^4x \phi_c(x)\phi_\Delta(x)\xi(x). \tag{26}$$

Statistical averages are defined as functional integrals over the $\xi(x)$ field, i.e., the averages over all realizations of $\xi(x)$,

$$\langle (\ldots) \rangle_\xi = \int \mathcal{D}[\xi] \mathcal{P}(\xi)(\ldots). \qquad (27)$$

It follows from Eqs. (13) and (25) that a correlation of the random forces is determined by an imaginary part of the influence action.

The functional variation of the stochastic effective action leads to the stochastic Langevin equation for the system (inflaton) field [8]. Decomposing ϕ on the mean field and the classical perturbation

$$\phi(\eta, \vec{x}) = \phi_0(\eta) + \delta\phi(\eta, \vec{x}), \qquad (28)$$

one obtains, finally, the linearized Langevin equation for the inflaton perturbation

$$\delta\phi''(\vec{k}, \eta) + 2aH\delta\phi'(\vec{k}, \eta) + k^2\delta\phi(\vec{k}, \eta) + a^2 m_\phi^2 \delta\phi(\vec{k}, \eta)$$
$$= \frac{g^2}{a^2}\phi_0\xi, \qquad (29)$$

$$m_\phi^2 = \frac{d^2 V}{d\phi_0^2} + g^2\langle\chi^2\rangle. \qquad (30)$$

Here, the $\langle\chi^2\rangle$ factor arises from the relation

$$G_{++}^\Lambda(x, x') = i\langle T\chi^+(x)\chi^+(x')\rangle \overset{x \to x'}{\to} i\langle\chi^2\rangle. \qquad (31)$$

Equation (29) contains, in its right-hand side, the term proportional to $\xi(x)$ [this term is absent in the average Eq. (22)], which is the fluctuation induced by the (colored) stochastic noise.

Note that deriving the stochastic Langevin equation (29), we neglected the dissipative term, proportional to $g^4 \mathrm{Im} G_{++}^{\Lambda 2}$, hoping that it does not lead to a large error due to a small value of g^2.

III. NOISE-DRIVEN POWER SPECTRUM OF INFLATON FLUCTUATIONS

The power spectrum of the quantum field fluctuations $\delta\phi$ is the function $P_\phi(k, \eta)$, which is given by the relations

$$\langle\delta\phi(\vec{x}, \eta)\delta\phi(\vec{x} + \vec{r}, \eta)\rangle = \int \frac{d^3 k}{(2\pi)^3} P_\phi(k, \eta)e^{-i\vec{k}\vec{r}}, \quad (32)$$

$$P_\phi(k, \eta) \equiv 2\pi^2 k^{-3}\Delta_\phi^2(k, \eta). \qquad (33)$$

The particular solution of Eq. (29) in a case when $m_\phi \approx 0$ is given by the formulas (see, e.g., [51])

$$\delta\phi^{(p)}(k, \eta) = -\int_{\eta_i}^\eta d\eta' g(k, \eta, \eta')\xi(k, \eta')\phi_0(\eta'), \quad (34)$$

$$g(k, \eta, \eta') = \frac{1}{a(\eta)a(\eta')}\left[\frac{\sin k(\eta - \eta')}{k} \cdot \left(1 + \frac{1}{k^2\eta\eta'}\right)\right.$$
$$\left. - \frac{\cos k(\eta - \eta')}{k^2\eta\eta'}(\eta - \eta')\right]. \qquad (35)$$

Using Eq. (34), the spectrum quantity $\Delta_\phi^2(k, \eta)$ is obtained from the integral

$$\frac{2\pi^2}{k^3}\Delta_\phi^2(k, \eta)\delta(\vec{k} - \vec{k}')$$
$$= \int_{\eta_i}^\eta d\eta' \int_{\eta_i}^\eta d\eta''\phi_0(\eta')\phi_0(\eta'')\langle\xi(k, \eta')\xi(k', \eta'')\rangle_\xi$$
$$\times g(k, \eta, \eta')g(k', \eta, \eta''). \qquad (36)$$

The ξ correlator in rhs of Eq. (36) is expressed through the Fourier transform of the ξ correlator in $(\vec{x}, \eta)$ space [Eq. (25)],

$$\langle\xi(k, \eta')\xi(k', \eta'')\rangle_\xi = (2\pi)^3\delta(\vec{k} - \vec{k}')g^4 a^4(\eta')a^4(\eta'')$$
$$\times \mathrm{Re}G_{++}^{\Lambda 2}(\eta', \eta'', k). \qquad (37)$$

Using this equation, the noise-driven power spectrum can be expressed as

$$\Delta_\phi^2(k) = -\frac{g^4 k^3}{\pi^2}\int_{\eta_i}^\eta d\eta' \int_{\eta_i}^\eta d\eta'' a^4(\eta')a^4(\eta'')\phi_0(\eta')$$
$$\times \phi_0(\eta'')g(k, \eta, \eta')g(k, \eta, \eta'')\mathrm{Re}G_{++}^{\Lambda 2}(\eta', \eta'', \vec{k}). \quad (38)$$

The spectrum in Eq. (38) is, in general, not scale invariant due to a finite duration of the inflation stage and, also, due to existence of the (infrared) cutoff Λ. Really, one can rewrite Eq. (38) in a form [51,53]

$$\Delta_\phi^2(k) = -\frac{g^4}{\pi^2}\int_{k\eta_i}^{k\eta}\frac{dz'}{(z')^4}\int_{k\eta_i}^{k\eta}\frac{dz''}{(z'')^4}\phi_0(\eta')\phi_0(\eta'')f(k\eta, z')$$
$$\times f(k\eta, z'')F\left(z', z'', \frac{\Lambda}{k}\right), \qquad (39)$$

$$f(k\eta, z') \equiv k^3 H^{-2}g(k, \eta, \eta')$$
$$= (z'k\eta + 1)\sin(k\eta - z')$$
$$- (k\eta - z')\cos(k\eta - z'), \qquad (40)$$

$$F\left(z', z'', \frac{\Lambda}{k}\right) \equiv k^3 H^{-4}\mathrm{Re}G_{++}^{\Lambda 2}(\eta', \eta'', \vec{k}). \qquad (41)$$

Note, once more, that this relatively simple spectrum formula is obtained in a massless limit: it is assumed that

both m_ϕ and m_χ are close to zero ($m_{\phi,\chi}^2 \ll H^2$). In this case, the corresponding mode functions are given by formula (20). It was shown in [8] that the approximation of massless fields is justified if $H\Delta t \cong 1$, where Δt is a time scale on which the particle production happens, $\Delta t \sim (g\dot\phi)^{-1/2}$ [3,5,11]. The number density of the χ particles produced is estimated as $m_\chi\langle\chi^2\rangle$, and, for a massless field, $\langle\chi^2\rangle \cong H^3 t/(4\pi^2)$ [35]. The effective mass of χ is equal to $g^2\phi_0^2$, as follows from the coupling term in Lagrangian (2),

$$m_\chi^2 = g^2\phi_0^2 \approx g^2\dot\phi^2(t_0 - t)^2. \tag{42}$$

Here we assume, for simplicity, that $\dot\phi$ is approximately constant during the slow-roll period of inflation. Near the moment $t = t_0$ (t_0 is the time when the inflaton field reaches the trapping point) the χ particles are almost massless and the particle production process is effective. It follows from Eq. (42) that, if $H\Delta t \cong 1$, $m_\chi^2 \ll H^2$ inside the time interval Δt, and the number density of the χ particles is $n_\chi \sim m_\chi\langle\chi^2\rangle \sim H^3 \sim (\Delta t)^{-3}$. The effective mass of ϕ is about $g^2\langle\chi^2\rangle$, which is also rather small due to the smallness of the coupling constant.

It follows from these arguments that for a calculation of the noise-driven power spectrum using Eq. (38), we must limit ourselves (for each trapping point) by the integration over those η's which correspond to cosmic times t close to t_0. Following [8], we approximate $\phi_0(\eta)$ by the relation

$$\phi_0(\eta) = \frac{v}{H}\ln\frac{\eta}{\eta_0}, \tag{43}$$

where η_0 is the conformal time corresponding to t_0, i.e., $\eta_0 = -1/a(t_0)H$, and v is the slow-roll velocity of ϕ_0, $v = |\dot\phi|$. Using this approximation, the integration region in (38) for each trapping point reduces to the interval

$$\eta_0 e^{H\Delta t/2} < \eta', \eta'' < \eta_0 e^{-H\Delta t/2}. \tag{44}$$

IV. RESULTS AND DISCUSSIONS

Calculating the power spectrum in which we are interested, mostly, in the region of rather large comoving wave numbers, $k \gg H$. It appears that the spectrum expression is rather sensitive to a value of Λ, if $k \gg \Lambda$. The cutoff value enters the spectrum expression through the factor $\mathrm{Re}G_{++}^{\Lambda2}$, in which the leading term at large ratio k/Λ is proportional to $(k/\Lambda)^2$ [51]:

$$\mathrm{Re}G_{++}^{\Lambda2}(\eta',\eta'',\vec{k}) \sim \frac{H^4}{k^3}\left(\frac{k}{2\Lambda}\right)^2 \cos\left[\frac{2\Lambda}{k}(k\eta' - k\eta'')\right]. \tag{45}$$

As discussed in the Introduction (see also [54]), an infrared cutoff is necessary to avoid an infrared singularity in the free propagator associated with a minimally coupled massless scalar field in de Sitter geometry. We assume, following [36], that the reasonable cutoff value Λ is close to H.

The results of the power spectrum calculation (for one trap and different values of g^2 and Λ) are shown in Figs. 1 and 2. Note that the peak value of the power spectrum shifts with a change of the trap position η_0 in such a way that $k_{\mathrm{peak}}\eta_0 \sim 1$. This means that the power spectrum is formed in the near-horizon region, where the inflaton field can be considered as classical, and, therefore, our semiclassical approach is justified.

The accumulative power spectra for a series of equally spaced traps (with interval ΔN, in e-foldings, between them) are shown in Figs. 3 and 4. It is seen that the accumulative power spectrum is blue. As seen from Fig. 4, it is more or less wiggly, depending on the relation between ΔN and $H\Delta t$ (we always assume $\Delta N \geq H\Delta t$).

The resulting power spectrum obtained from Figs. 3 and 4 (approximating the curves by its envelope) can be described by the following formula:
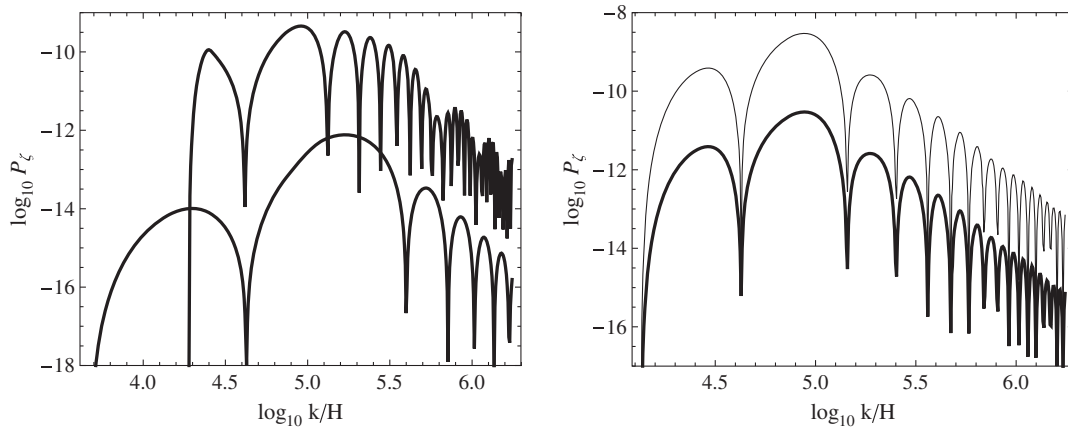

FIG. 1. Curvature perturbation power spectra generated by a single trap at $N = 10$ (N is the e-fold number, counting from the beginning of inflation). Left panel: $g^2 = 10^{-9}$ ($H\Delta t = 0.3$, upper curve); $g^2 = 10^{-5}$ ($H\Delta t = 3$, lower curve). For both curves, $\Lambda = H$. Right panel: $g^2 = 10^{-7}$ ($H\Delta t = 1$). Upper curve: $\Lambda = 0.1H$. Lower curve: $\Lambda = H$.
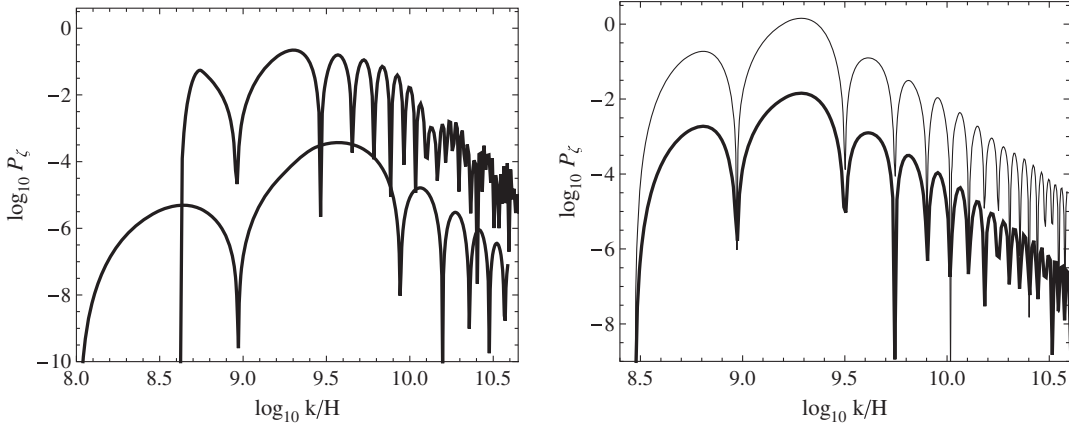
FIG. 2. Curvature perturbation power spectra generated by a single trap at $N = 20$. Left panel: $g^2 = 10^{-9}$ ($H\Delta t = 0.3$, upper curve); $g^2 = 10^{-5}$ ($H\Delta t = 3$, lower curve). For both curves, $\Lambda = H$. Right panel: $g^2 = 10^{-7}$ ($H\Delta t = 1$). Upper curve: $\Lambda = 0.1H$. Lower curve: $\Lambda = H$.

$$\mathcal{P}_\zeta(k) = \left(\frac{H}{v}\right)^2 \Delta_\phi^2 \approx \alpha g^2 \left(\frac{k}{H}\right)^2, \qquad \alpha \approx 5 \times 10^{-14}. \quad (46)$$

This form of the k dependence of $\mathcal{P}_\zeta(k)$ is similar to the analogous dependence predicted in [8] (see [55] where the numerical results of [8] were parametrized).

Now it is convenient to rescale the scale factor and, correspondingly, the comoving wave number, setting $a = 1$ at the present time (rather than $a = 1$ at the initial time of the inflation era, as was set before). After this rescaling Eq. (46) is rewritten as

$$\mathcal{P}_\zeta(k) \approx \alpha g^2 \left(\frac{k}{Ha_{\text{start}}}\right)^2, \qquad \alpha \approx 5 \times 10^{-14}, \quad (47)$$
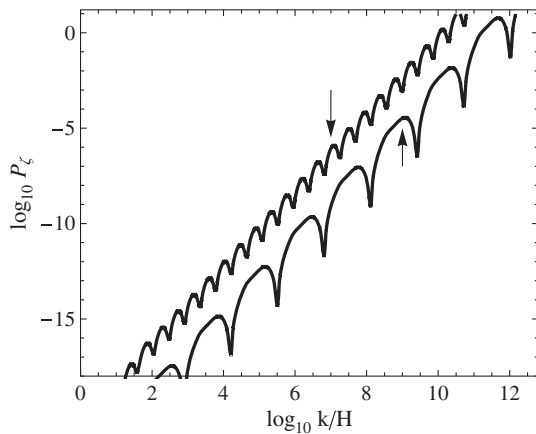
where a_{start} is a scale factor at the beginning of inflation. It is related to a scale factor at the end of inflation, a_{end}, by the formula

$$\frac{a_{\text{end}}}{a_{\text{start}}} = e^{N_{\text{inf}}}, \quad (48)$$

where N_{inf} is the total number of e-folds during inflation. The value of a_{end} can be easily estimated by the approximate equation

$$a_{\text{end}} = a_{\text{eq}} \frac{T_{\text{eq}}}{\sqrt{M_{\text{Pl}} H}}, \quad (49)$$

where a_{eq} and T_{eq} are the scale factor and temperature of the Universe at the time of the matter-radiation equality ($T_{\text{eq}} \sim 3$ eV). As a result, the spectrum amplitude contains the factor $\exp(2N_{\text{infl}})/H$. Observations at cosmologically
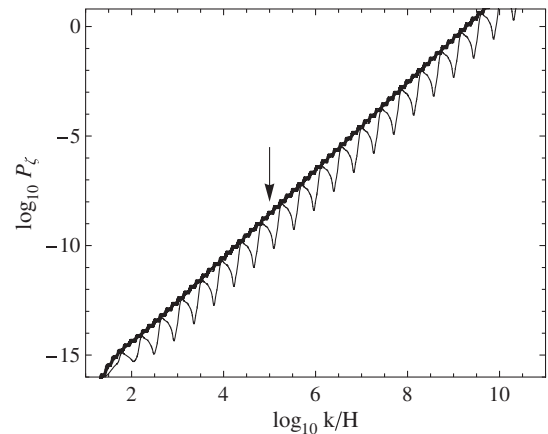

FIG. 3. Curvature perturbation power spectra generated by a series of traps. Upper curve: $g^2 = 10^{-7}$ ($H\Delta t = 1$), $\Delta N = 1$. Lower curve: $g^2 = 10^{-9}$ ($H\Delta t = 3$), $\Delta N = 3$. For both cases, $\Lambda = H$. Arrows show, for each curve, the position of k at which $g^2(k/\Lambda) = 1$.

FIG. 4. Curvature perturbation power spectra generated by a series of traps, for $g^2 = 10^{-5}$ ($H\Delta t = 0.3$), $\Lambda = H$. The thin curve is for $\Delta N = 1$; the thick one is for $\Delta N = 0.3$. Arrows show the position of k at which $g^2(k/\Lambda) = 1$.
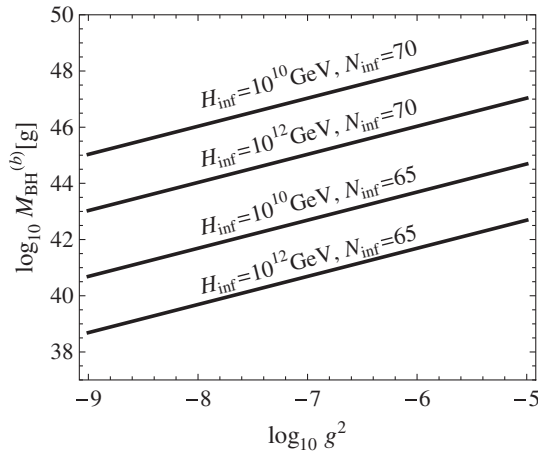
FIG. 5. The values of border masses of PBHs for different values of model parameters.

large scales ($k \lesssim 10$ Mpc^{-1}) show (see, e.g., [56]) that the amplitude of the primary power spectrum does not exceed $\sim 10^{-9}$. To match this condition, the duration of inflation in our model must not be too long ($N_{\text{inf}} \lesssim 70 \div 75$) and the energy scale of inflation must be rather high ($H \gtrsim 10^8 \div 10^{10}$ GeV).

It is important to check when (at which values of k and/or PBH mass) the resulting power spectrum (47) reaches values that are prohibited by the PBH overproduction. In this work we assume that, roughly, PBHs are overproduced when, for some value of k, $\mathcal{P}_\zeta(k) > \mathcal{P}_\zeta^{\text{PBH}}$ where $\mathcal{P}_\zeta^{\text{PBH}} \sim 10^{-2}$ is the PBH production threshold (see, e.g., [57,58]). Then, having the relation between k and horizon mass [58],

$$k \approx 2 \times 10^{23} (M_h[\text{g}])^{-1/2} \text{ Mpc}^{-1}, \qquad (50)$$

and, assuming that, approximately, the PBH mass is of order of the horizon mass, $M_{\text{BH}} \approx M_h(k)$, we obtain, from Eq. (47), the corresponding border value of M_{BH},

$$M_{\text{BH}}^{(b)} \approx \frac{3 \times 10^{14} \text{ GeV}}{H} \cdot e^{2N_{\text{inf}}} \cdot \frac{\alpha g^2}{\mathcal{P}_\zeta^{\text{PBH}}} \text{ g}. \qquad (51)$$

Clearly, the production of PBHs with masses $M_{\text{BH}} < M_{\text{BH}}^{(b)}$ is prohibited by the present constraints [57–60] if the power spectrum grows with k as strongly as Eq. (47) predicts.

The resulting dependence of $M_{\text{BH}}^{(b)}$ on g^2, N_{inf} and H is shown in Fig. 5. It is seen that the inadmissibly large PBH overproduction in the inflation model considered in the present paper is predicted for all reasonable values of parameters N_{inf}, H and for some interval of values of the coupling constant g^2. This is the main conclusion of the paper.

The model of inflation with trapping points (in a particular case of the weak coupling constant g^2) can survive if, for some reason, the growth of the power spectrum with an increase of k becomes gradually slower and, finally, stops at some value $\mathcal{P}_{\zeta,\text{max}} < 10^{-2}$. One can imagine two reasons, at least, for such a behavior: (i) an account of the dissipation term in the influence action S_{IF} will, in general, lead to a damping of the power of high-k modes [8] and (ii) an account of higher order terms in the perturbative expansion (these terms correspond to diagrams with two or more χ loops) also can change a form of the spectrum. Our calculations based on perturbation theory are reliable if $g^2(k/\Lambda) \lesssim 1$ (the corresponding value of k is denoted by arrows in Figs. 3 and 4). So, parts of the spectrum curves in these figures which are to the right of the arrows have to be considered as extrapolations (hoping that the accounting of higher order terms in g^2 does not slow down the growth of the spectrum).

Note, in the end, that our numerical results for the power spectrum depend rather strongly on a choice of the infrared cutoff value Λ (see Figs. 1 and 2). Another thing that is worth mentioning is that power spectra from individual trapping points are overlapped weakly, so the envelope curve of the accumulative spectrum almost does not depend on the spacing (see Figs. 3 and 4).

ACKNOWLEDGMENTS

[1] L. A. Kofman and A. D. Linde, Nucl. Phys. **B282**, 555 (1987).

[2] D. J. H. Chung, E. W. Kolb, A. Riotto, and I. I. Tkachev, Phys. Rev. D **62**, 043508 (2000).

[3] L. Kofman, A. D. Linde, X. Liu, A. Maloney, L. McAllister, and E. Silverstein, J. High Energy Phys. 05 (2004) 030.

[4] A. E. Romano and M. Sasaki, Phys. Rev. D **78**, 103522 (2008).

[5] D. Green, B. Horn, L. Senatore, and E. Silverstein, Phys. Rev. D **80**, 063533 (2009).

[6] N. Barnaby, Z. Huang, L. Kofman, and D. Pogosyan, Phys. Rev. D **80**, 043501 (2009).

[7] N. Barnaby, Phys. Rev. D **82**, 106009 (2010).

[8] W. Lee, K. W. Ng, I. C. Wang, and C. H. Wu, Phys. Rev. D **84**, 063527 (2011).

[9] C. H. Wu, K. W. Ng, W. Lee, D. S. Lee, and Y. Y. Charng, J. Cosmol. Astropart. Phys. 02 (2007) 006.

[10] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994).

[11] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Rev. D **56**, 3258 (1997).

[12] D. Seery, K. A. Malik, and D. H. Lyth, J. Cosmol. Astropart. Phys. 03 (2008) 014.

[13] G. C. Liu, K. W. Ng, and I. C. Wang, Phys. Rev. D **90**, 103531 (2014).

[14] M. Morikawa, Phys. Rev. D **42**, 1027 (1990).

[15] B. L. Hu and Y. Zhang, University of Maryland, Report No. 90-186; B. L. Hu, in *Proceedings of SILARG VII, Cocoyoc, Mexico, 1990*, edited by J. C. D'Olivo *et al.* (World Scientific, Singapore, 1991).

[16] S. Matarrese, M. A. Musso, and A. Riotto, J. Cosmol. Astropart. Phys. 05 (2004) 008.

[17] A. A. Starobinsky, in *Fundamental Interactions*, edited by V. N. Ponomarev (MGPI Press, Moscow, 1984), p. 54; A. A. Starobinsky, in *Lecture Notes in Physics*, edited by H. J. de Vega and N. Sanchez (Springer, New York, 1986), Vol. 246, p. 107.

[18] J. Schwinger, J. Math. Phys. (N.Y.) **2**, 407 (1961); P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. (N.Y.) **4**, 1 (1963); **4**, 12 (1963); L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964).

[19] E. Calzetta and B. L. Hu, Phys. Rev. D **35**, 495 (1987); **37**, 2878 (1988); **40**, 656 (1989).

[20] R. D. Jordan, Phys. Rev. D **33**, 444 (1986).

[21] E. Calzetta and B. L. Hu, Phys. Rev. D **49**, 6636 (1994).

[22] B. L. Hu, arXiv:gr-qc/9403061.

[23] R. Feynman and F. Vernon, Ann. Phys. (N.Y.) **24**, 118 (1963); R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965); H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics* (World Scientific, Singapore, 1990).

[24] A. O. Caldeira and A. J. Leggett, Physica (Amsterdam) **121A**, 587 (1983); Ann. Phys. (N.Y.) **149**, 374 (1983).

[25] H. Grabert, P. Schramm, and G. L. Ingold, Phys. Rep. **168**, 115 (1988).

[26] B. L. Hu, J. P. Paz, and Y. Zhang, arXiv:gr-qc/9512049.

[27] M. Gleiser and R. O. Ramos, Phys. Rev. D **50**, 2441 (1994).

[28] A. Berera, M. Gleiser, and R. O. Ramos, Phys. Rev. D **58**, 123508 (1998).

[29] J. Yokoyama, Phys. Rev. D **70**, 103511 (2004).

[30] R. O. Ramos and L. A. da Silva, J. Cosmol. Astropart. Phys. 03 (2013) 032.

[31] D. Boyanovsky, New J. Phys. **17**, 063017 (2015); Phys. Rev. D **92**, 023527 (2015); **93**, 043501 (2016).

[32] D. Boyanovsky, Phys. Rev. D **85**, 123525 (2012); **86**, 023509 (2012).

[33] B. Allen, Phys. Rev. D **32**, 3136 (1985).

[34] B. Allen and A. Folacci, Phys. Rev. D **35**, 3771 (1987).

[35] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. A **360**, 117 (1978); A. Vilenkin and L. H. Ford, Phys. Rev. D **26**, 1231 (1982); A. Linde, Phys. Lett. **116B**, 335 (1982); A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).

[36] L. H. Ford and L. Parker, Phys. Rev. D **16**, 245 (1977).

[37] D. H. Lyth, J. Cosmol. Astropart. Phys. 12 (2007) 016.

[38] N. Bartolo, S. Matarrese, M. Pietroni, A. Riotto, and D. Seery, J. Cosmol. Astropart. Phys. 01 (2008) 015.

[39] T. Prokopec, N. C. Tsamis, and R. P. Woodard, Ann. Phys. (Amsterdam) **323**, 1324 (2008).

[40] W. Xue, K. Dasgupta, and R. Brandenberger, Phys. Rev. D **83**, 083520 (2011).

[41] K. Larjo and D. A. Lowe, Phys. Rev. D **85**, 043528 (2012).

[42] D. Polarski and A. A. Starobinsky, Classical Quantum Gravity **13**, 377 (1996).

[43] J. Lesgourgues, D. Polarski, and A. A. Starobinsky, Nucl. Phys. **B497**, 479 (1997); Classical Quantum Gravity **14**, 881 (1997).

[44] C. Kiefer, D. Polarski, and A. A. Starobinsky, Int. J. Mod. Phys. D **07**, 455 (1998).

[45] C. Kiefer and D. Polarski, Adv. Sci. Lett. **2**, 164 (2009).

[46] F. Lombardo and F. D. Mazzitelli, Phys. Rev. D **53**, 2001 (1996).

[47] C. Greiner and B. Muller, Phys. Rev. D **55**, 1026 (1997).

[48] E. A. Calzetta, B. L. Hu, and F. D. Mazzitelli, Phys. Rep. **352**, 459 (2001).

[49] K. c. Chou, Z. b. Su, B. l. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).

[50] N. P. Landsman and C. G. van Weert, Phys. Rep. **145**, 141 (1987).

[51] F. C. Lombardo and D. Lopez Nacir, Phys. Rev. D **72**, 063506 (2005).

[52] R. L. Stratonovich, Sov. Phys. Dokl. **2**, 416 (1958); J. Hubbard, Phys. Rev. Lett. **3**, 77 (1959).

[53] E. A. Calzetta and S. Gonorazky, Phys. Rev. D **55**, 1812 (1997).

[54] A. Vilenkin, Nucl. Phys. **B226**, 527 (1983).

[55] C. M. Lin and K. W. Ng, Phys. Lett. B **718**, 1181 (2013).

[56] P. A. R. Ade *et al.* (Planck Collaboration), arXiv:1502.01589.

[57] B. J. Carr, K. Kohri, Y. Sendouda, and J. Yokoyama, Phys. Rev. D **81**, 104019 (2010).

[58] E. Bugaev and P. Klimai, Phys. Rev. D **83**, 083521 (2011).

[59] E. Bugaev and P. Klimai, Phys. Rev. D **79**, 103511 (2009).

[60] A. S. Josan, A. M. Green, and K. A. Malik, Phys. Rev. D **79**, 103520 (2009).