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Separating the Universe into real and fake energy densities

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The separate universe technique provides a means of establishing consistency relations between short-wavelength observables and the long-wavelength matter density fluctuations within which they evolve by absorbing the latter into the cosmological background. We extend it to cases where nongravitational forces introduce a Jeans scale in other species like dynamical dark energy or massive neutrinos. The technique matches the synchronous gauge matter density fluctuations to the local expansion using the acceleration equation and accounts for the temporal nonlocality and scale dependence of the long-wavelength response of small scale matter observables, e.g., the nonlinear power spectrum, halo abundance and the implied halo bias, and N-point correlation functions. Above the Jeans scale, the local Friedmann equation relates the expansion to real energy densities and a curvature that is constant in comoving coordinates. Below the Jeans scale, the curvature evolves and acts like a fake density component. In all cases, the matter evolution on small scales is correctly modeled as we illustrate using scalar field dark energy with adiabatic or isocurvature initial conditions across the Jeans scale set by its finite sound speed.

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I. INTRODUCTION

Describing the impact of a long-wavelength cosmological perturbation on small scale observables as a change in the background cosmology, or separate universe, has proven very useful both conceptually and as a tool for making precise and consistent predictions between observables. In the inflationary context, separate universe arguments provide insights into consistency relations between the N-point functions [1,2], the evolution of isocurvature fluctuations in multifield models [3], and the observable impact of compensated isocurvature fluctuations [4] within its domain of validity [5].

In the late universe context, they have enabled studies of baryon acoustic oscillations [6], supersample power spectrum covariance [7,8], position dependent power spectra [9–11], cosmic microwave background lensing covariance [12], and dark matter halo bias [13–16]. Moreover, to the extent that the separate universe construction holds, these observable effects can be modeled deep into the nonlinear regime with cosmological simulations [9,17–19], in principle complete with state-of-the-art treatments of astrophysical processes. In particular, long-wavelength modes have an impact on small scale observables that is nonlocal in time, complicating, for example, the modeling of the nonlinear power spectrum [20] and halo bias [21–23]. In the separate universe approach, these can simply be modeled as a change in cosmological parameters.

The separate universe construction has traditionally been limited to large scales where only gravitational forces act [24]. The effects of pressure or anisotropic stress gradients in stabilizing fluctuations at the Jeans scale would seem to prevent replacing fluctuations with a separate homogeneous and isotropic universe. While the nonrelativistic matter is effectively pressureless on cosmological scales, the real universe contains components that have relativistic stresses today or in the past, e.g., dark energy and massive neutrinos. In these cases, the response of short-wavelength observables to long-wavelength fluctuations can depend on their scale and redshift in addition to their amplitudes [23,25].

In this work, we extend separate universe techniques to the multicomponent, relativistic case by introducing fictitious components below the Jeans scale that preserve the illusion of a separate homogeneous and isotropic background from the perspective of the small-scale matter distribution. We begin in Sec. II with the construction of a separate expansion history that absorbs a long-wavelength matter density fluctuation even in this general case where nongravitational forces act on other components. Above the Jeans scale set by these forces, the construction yields a separate universe with the real energy densities and curvature of the long-wavelength fluctuations. Below the Jeans scale, nonconservation of curvature prevents these associations but retains the correct evolution of the small scale matter distribution.

In Sec. III, we discuss the relationship between the entropy, nonadiabatic stress, and curvature of the perturbations and the assignment of cosmological parameters in the separate universe above and below the Jeans scale. The separate universe construction determines how short-wavelength observables respond to long-wavelength modes including scale dependent and temporally nonlocal effects as illustrated in Sec. IV. We provide a concrete example of scalar field dark energy with adiabatic and isocurvature initial conditions in Sec. V. We discuss these results in Sec. VI.

II. SEPARATE UNIVERSE CONSTRUCTION

In this section, we develop the formal aspects of the separate universe construction in a fully relativistic and multicomponent context. We begin in Sec. II A by defining the local expansion history that absorbs a matter density fluctuation and its entire growth history into the background. This construction corresponds to exactly matching the separate universe acceleration equation with the synchronous gauge density perturbation and is always possible as shown in Sec. II B. In Sec. II C, we find this construction also matches the Friedmann and energy conservation equations of the other components if the separate universe curvature implied by the curvature perturbation is constant (see also Ref. [24]). We relate this condition to the Jeans scales of these components in Sec. II D.

A. Local expansion and density

In the separate universe approach, we seek to absorb a long-wavelength matter density fluctuation $\delta = \delta\rho_m/\bar{\rho}_m$, including its entire growth history, into the background of a separate universe [8,9,13,17,26],

$$\bar{\rho}_m(a)[1+\delta(a)] = \bar{\rho}_{mW}(a). \tag{1}$$

"W" here and throughout denotes locally averaged or "windowed" quantities on scales much smaller than the wavelength. In terms of defining the local cosmology, we can introduce the separate universe scale factor through $\bar{\rho}_{mW} \propto a_W^{-3}$, which then defines the matter density parameters

$$\frac{\Omega_m h^2}{a^3}(1+\delta) = \frac{\Omega_{mW} h_W^2}{a_W^3}. \tag{2}$$

Here, the Hubble constant $H_0 = 100h\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, and similarly H_{0W} is parametrized by the dimensionless h_W. Our convention is to set the scale factor of the separate universe a_W to agree with the global one a at high redshift,

$$\lim_{a\to 0} a_W(a) = a, \tag{3}$$

where

$$\lim_{a\to 0} \delta(a) = 0. \tag{4}$$

With this convention, the background energy densities in the matter at the same numerical values for a and a_W are always equal, and hence

$$\Omega_{mW} h_W^2 = \Omega_m h^2, \tag{5}$$

but the scale factors at the same time differ,

$$a_W = \frac{a}{(1+\delta)^{1/3}} \approx a\left(1-\frac{\delta}{3}\right). \tag{6}$$

In this construction, we assume that the two universes share a common universal time. We shall see that common clocks of the two universes require δ to be specified in synchronous gauge, a distinction that becomes important for scales near the horizon. Equivalently, in a gauge-invariant separate universe construction, the quantity of interest is the change in the e-folds of the expansion [3] which we can equate to the synchronous gauge matter density perturbation since it evolves only via metric perturbations (see Sec. II C),

$$\delta N = \ln a_W - \ln a \approx -\frac{\delta}{3}. \tag{7}$$

The difference in scale factors also implies that the separate universe has a different expansion rate. Using the definition $H = \dot{a}/a$ and Eq. (6), we obtain

$$\delta H^2 = H_W^2 - H^2 \approx -\frac{2}{3}H\dot{\delta} = -\frac{2}{3}H^2\delta'. \tag{8}$$

Here, overdots denote d/dt in both the global and separate universe, whereas $' = d/d\ln a$ will denote derivatives in the global universe only.

For the purpose of setting up the separate universe, it is sufficient to modify the expansion rate directly without making further distinctions about its purported sources in the local Friedmann equation. Notice that this construction in fact makes no direct use of components in the universe besides the matter. Other components affect the construction only through changing the matter growth history $\delta(a)$. On the other hand, this expansion history may or may not be generated by the local Friedmann equation with the physical components of the local universe. To understand this issue, we consider the impact of the other components in the following sections.

B. Newtonian cosmology and acceleration

In the Newtonian interpretation of the acceleration equation, the nonrelativistic matter can be considered as test particles tracking the evolution of some region of physical radius R in the global universe. It accommodates a perturbation δ as long as R is much smaller than its wavelength.

Let us suppose that in addition to the matter there are additional density and pressure components:

$$\rho_Q = \sum_{J\neq m} \rho_J, \qquad p_Q = \sum_{J\neq m} p_J. \tag{9}$$

Newtonian cosmology, the relativistic justification of which relies on the Birkhoff theorem [27,28], relates the acceleration $\ddot{R}$ to the enclosed active gravitational mass

$$\ddot{R} = -\frac{4\pi G}{3}[\rho_m + \rho_Q + 3p_Q]R \tag{10}$$

and so with the acceleration equation for the global universe,

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{\Omega_m H_0^2}{2a^3}\delta - \frac{4\pi G}{3}(\delta\rho_Q + 3\delta p_Q). \tag{11}$$

The evolution of this radius can be absorbed into a separate universe scale factor if $a_W \propto R$ so that

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}_W}{a_W}. \tag{12}$$

Using Eq. (6), we also have

$$\frac{\ddot{a}_W}{a_W} = \frac{\ddot{a}}{a} - \frac{2}{3}H\dot{\delta} - \frac{1}{3}\ddot{\delta}. \tag{13}$$

Thus, the separate universe condition is satisfied if

$$\begin{aligned}\ddot{\delta} + 2H\dot{\delta} &= \frac{3\Omega_m H_0^2}{2a^3}\delta + 4\pi G(\delta\rho_Q + 3\delta p_Q) \\ &= 4\pi G\sum_J(\delta\rho_J + 3\delta p_J) = 0. \end{aligned} \tag{14}$$

We shall see in the next section that this is exactly the equation of motion for the synchronous gauge matter density perturbation. In the separate universe approach, we are really just going to Lagrangian coordinates defined by the cold dark matter particles. Only their relationship to Eulerian coordinates, quantified by δ, is influenced by other species in the universe. On scales that are well below the horizon, the distinction between the synchronous gauge and other common gauges such as the conformal Newtonian or comoving gauge becomes irrelevant, and a Newtonian analysis for the density perturbation also applies.

C. Synchronous gauge and Friedmann equation

We can formalize the separate universe associations in the fully relativistic context of the Friedmann and acceleration equations. The assumption of a universal time implies that the δ we absorb into the local background is the synchronous gauge fluctuation. Its evolution must be compatible with the local Friedmann and acceleration equations for a real separate universe construction.

In the synchronous gauge, the metric is given by $g_{00} = -1$, $g_{0i} = 0$, and a perturbed spatial metric

$$g_{ij} = a^2(\gamma_{ij} + h_{ij}), \tag{15}$$

where γ_{ij} is the 3-metric of constant comoving curvature K and the scalar metric perturbations for a mode of Laplacian wave number k can be further decomposed into trace and trace-free pieces:

$$h_{ij} = \frac{h_L}{3}\gamma_{ij} - \left[\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2\right]\frac{h_L + 6\eta_T}{k^2}. \tag{16}$$

Covariant differentiation and raising and lowering of spatial indices is performed with respect to γ_{ij}. In the separate universe, h_L performs the role of the perturbation to the scale factor a and η_T to the spatial curvature K. Specifically, the perturbation to the 3D Ricci scalar ${}^{(3)}R = 6K/a^2$ on constant synchronous time slices (e.g., Ref. [29]),

$$\delta K = -\frac{2}{3}(k^2 - 3K)\eta_T, \tag{17}$$

whereas the effect of h_L is to change volumes and hence the effective Hubble rate by

$$\frac{\delta H}{H} = \frac{\dot{h}_L}{6H} = \frac{h_L'}{6}. \tag{18}$$

Note that the effect of a single k-mode perturbation is an anisotropic change in the expansion rate. For example, if $K = 0$, the normal modes are plane waves. For a k-mode directed in the x direction e^{ikx}, the scale factor and its time derivative only change in the same direction $\delta\ln a_x = h_L/2$, $\delta\ln a_y = \delta\ln a_z = 0$. Consequently, the separate universe construction only strictly applies to the angle averaged response of local observables to the long-wavelength mode, for example, number densities of dark matter halos or angle averaged power spectra.

The 00 and trace ii Einstein equations relate the metric to the energy density $\delta\rho_J$ and pressure δp_J perturbations of the various components (see, e.g., Refs. [29] and [30,31] for a similar notation),

$$-\frac{k^2 - 3K}{(aH)^2}\eta_T + \frac{1}{2}h_L' = \frac{4\pi G}{H^2}\sum_J \delta\rho_J, \tag{19}$$

$$h_L'' + \left(2 + \frac{H'}{H}\right)h_L' = -\frac{8\pi G}{H^2}\sum_J(\delta\rho_J + 3\delta p_J), \tag{20}$$

which are themselves governed by the continuity and Navier-Stokes equation

$$\delta\rho_J' + 3(\delta\rho_J + \delta p_J) = -\frac{k\bar{\rho}_J}{aH}u_J - \frac{\bar{\rho}_J + \bar{p}_J}{2}h_L',$$
$$\bar{\rho}_J u_J' + (\bar{\rho}_J - 3\bar{p}_J)u_J = \frac{k}{aH}\left[\delta p_J - \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)p_J\pi_J\right], \tag{21}$$

where $\bar{\rho}_J u_J$ is the momentum density.

If the separate universe construction holds exactly, then these synchronous gauge equations can be reabsorbed into the Friedmann, acceleration, and energy conservation equations. We can already see from the lack of a background Navier-Stokes equation that this can only be true if the momentum density generated by nongravitational gradients in the isotropic stress δp_J and anisotropic stress π_J can be ignored. We examine each of these equations in turn.

First, let us check the matter continuity equation (21) and its relation to the perturbation to the Hubble rate. For the matter, $p_m = 0$, and we further use the remaining gauge freedom of the synchronous gauge to choose the freely falling observers to be on a grid of the pressureless matter particles themselves which sets $u_m = 0$.

With the shorthand convention $\delta = \delta_m$, Eq. (21) becomes

$$\delta' = -\frac{1}{2}h_L' = -3\frac{\delta H}{H}. \tag{22}$$

This relation matches the separate universe construction in Eq. (18). Equation (20) then gives the evolution equation for δ as Eq. (14).

For the other components, if we take energy conservation in the background

$$\bar{\rho}_J' + 3(\bar{\rho}_J + \bar{p}_J) = 0 \tag{23}$$

and perturb the expansion rate, we obtain the purely gravitational pieces of their continuity equations (21). The perturbation to the Hubble rate means that derivatives with respect to the scale factor are perturbed as

$$\frac{d}{d\ln a_W} = \frac{1}{H_W}\frac{d}{dt} = \frac{H}{H_W}\frac{1}{H}\frac{d}{dt} \approx \left(1 - \frac{\delta H}{H}\right)\frac{d}{d\ln a}, \tag{24}$$

and so Eq. (23) becomes for the perturbations

$$\delta\rho_J' + 3(\delta\rho_J + \delta p_J) + 3\frac{\delta H}{H}(\bar{\rho}_J + \bar{p}_J) = 0. \tag{25}$$

This matches the continuity equation (21) when the effect of the divergence of the nongravitational peculiar velocities u_J can be ignored. These are generated through the Navier-Stokes equation (21) from pressure and anisotropic stress gradients which give the condition

$$\left(\frac{k}{aH}\right)^2 \mathcal{O}\left(\frac{\delta p_J}{\delta\rho_J}, \frac{\bar{p}_J\pi_J}{\delta\rho_J}\right) \ll 1 \tag{26}$$

for them to change the energy density fluctuation negligibly.

Next, through the change in the expansion rate (22), the ii Einstein equation (20) is related to the perturbation to the acceleration equation

$$H^2 + \frac{1}{2}\frac{dH^2}{d\ln a} = -\frac{4\pi G}{3}\sum_J(\bar{\rho}_J + 3\bar{p}_J). \tag{27}$$

Converting derivatives of the scale factor with Eq. (24), we obtain the perturbed acceleration equation in terms of the global a as

$$\left(\frac{\delta H}{H}\right)' + \left(2 + \frac{H'}{H}\right)\frac{\delta H}{H} = -\frac{4\pi G}{3H^2}\sum_J(\delta\rho_J + 3\delta p_J), \tag{28}$$

which matches Eq. (20) given (22). Notice that, unlike the continuity equations, the acceleration equations for the background and perturbations take exactly the same form with no further restrictions on scales, in agreement with the discussion of the Newtonian cosmology in the previous section.

Finally, the 00 Einstein equation (19) is the perturbation to the Friedmann equation

$$H^2 + \frac{K}{a^2} = \sum_J \frac{8\pi G}{3}\bar{\rho}_J, \tag{29}$$

or

$$\delta H^2 + \frac{\delta K}{a^2} = \sum_J \frac{8\pi G}{3}\delta\rho_J, \tag{30}$$

with the associations of Eq. (17) and (18). Conversely, these perturbations can be reabsorbed into a Friedmann equation of a local, separate universe if the curvature fluctuation $\delta K(a)$ can be replaced by a new constant curvature K_W in coordinates that comove with a_W,

$$K_W \equiv \frac{a_W^2}{a^2}(K + \delta K) \approx K + \delta K - \frac{2}{3}K\delta. \tag{31}$$

If the global universe is flat $K = 0$, the curvature perturbation δK itself must be constant. Thus, the existence of a real separate universe is intimately related to the conservation of curvature perturbations outside the horizon [32]. If $K \neq 0$, the curvature perturbation must evolve to account for the different local scale factor of the perturbed universe.

Thus, from the perspective of matching the synchronous gauge perturbation equations to background equations in the separate universe, the continuity equation requires nongravitational flows u_J to be negligible, and the Friedmann equation requires the curvature K_W to be constant. We shall now see that these are essentially the same criteria.

D. Curvature conservation and Jeans scale

To better understand why the constancy of curvature is related to having negligible nongravitational flows, it is useful to examine the redundant $0i$ Einstein equation which directly gives the evolution equation for the curvature fluctuation,

$$\left(1-\frac{3K}{k^2}\right)\eta_T' - \frac{K}{2k^2}h_L' = \frac{4\pi G}{H^2}\frac{aH}{k}\sum_J \bar{\rho}_J u_J. \quad (32)$$

This equation has no equivalent in the background given homogeneity and isotropy, and so in the separate universe construction should produce a tautology $0=0$. Using Eqs. (17) and (31), we can rewrite Eq. (32) as

$$K_W' = -\frac{8\pi G}{3}\frac{k^2}{H^2}\frac{aH}{k}\sum_J \bar{\rho}_J u_J. \quad (33)$$

The curvature is effectively constant when we can ignore the nongravitational velocities u_J. Given the Navier-Stokes equation (21)

$$\bar{\rho}_J u_J = \frac{k}{aH}\mathcal{O}(\delta p_J, p_J\pi_J), \quad (34)$$

we obtain the estimate

$$K_W' = 8\pi G\frac{k^2}{H^2}\mathcal{O}(\delta p_T, p_T\pi_T), \quad (35)$$

where "T" denotes the total of all components. We call the scale k_T at which this change in curvature per e-fold becomes comparable to the curvature fluctuation itself,

$$\frac{K_W'}{\delta K} = \mathcal{O}(1), \quad (36)$$

the total Jeans scale.

While this defines the total Jeans scale and relates it to nongravitational flows, it is useful to estimate its value in particular cases to relate it to more conventional definitions. For metric perturbations sourced by growing total density fluctuations $\delta\rho_T$, where $|\delta p_T/\delta\rho_T| \lesssim 1$, $h' = \mathcal{O}(4\pi G\delta\rho_T/H^2)$. Then, Eq. (19) gives the order of magnitude of the curvature fluctuation itself,

$$\delta K = 4\pi G a^2 \mathcal{O}(\delta\rho_T), \quad (37)$$

and so

$$\frac{K_W'}{\delta K} = \left(\frac{k}{aH}\right)^2 \mathcal{O}\left(\frac{\delta p_T}{\delta\rho_T}, \frac{\bar{p}_T\pi_T}{\delta\rho_T}\right). \quad (38)$$

For a single component with only isotropic, adiabatic stresses,

$$\delta p_T/\delta\rho_T = p_T'/\rho_T' \equiv c_{Ta}^2, \quad (39)$$

defining the adiabatic sound speed. This corresponds to the usual Jeans condition that pressure prevents further growth below the sound horizon or Jeans scale $c_{Ta}k_T/aH \approx 1$. If the sound speed is subluminal $c_{Ta} < 1$, the Jeans scale is always below the horizon scale. Note that in this case there is no difference between the constant curvature condition (38) and the negligible nongravitational flows condition (26).

More generally, the total pressure is composed of the adiabatic $(p_J'/\rho_J')\delta\rho_J = c_{Ja}^2\delta\rho_J$; internal nonadiabatic stress Γ_J of the various components,

$$\delta p_J = c_{Ja}^2\delta\rho_J + p_J\Gamma_J; \quad (40)$$

and their relative entropy fluctuations with the matter

$$S_{Jm} = \frac{\delta\rho_J}{\bar{\rho}_J + \bar{p}_J} - \frac{\delta\rho_m}{\bar{\rho}_m + \bar{p}_m}, \quad (41)$$

such that

$$\delta p_T = c_{Ta}^2\delta\rho_T + p_T\Gamma_T, \quad (42)$$

with the total nonadiabatic stress

$$p_T\Gamma_T = \sum_J [p_J\Gamma_J + S_{Jm}(\rho_J + p_J)(c_{Ja}^2 - c_{Ta}^2)]. \quad (43)$$

In the general case, the adiabatic sound speed c_{Ta} no longer bounds the total pressure.

Entropy fluctuations allow initial isocurvature conditions where the total Jeans scale can be made arbitrarily large compared with the horizon. In this case, we need to slightly generalize the estimate (38) since the total pressure fluctuation can be larger than the total density perturbation and we need to separate their contribution through h_L' to η_T in Eq. (19). This division can be readily identified by using the final synchronous Einstein equation, the redundant ij trace-free equation

$$-\left(\frac{k}{aH}\right)^2\eta_T + \frac{h_L'' + 6\eta_T''}{2} + \left(3 + \frac{H'}{H}\right)\frac{h_L' + 6\eta_T'}{2} = -\frac{8\pi G}{H^2}\sum_J p_J\pi_J, \quad (44)$$

and combining Eqs. (19) and (32) into

$$\left(1-\frac{3K}{k^2}\right)\left[\frac{h_L'+6\eta_T'}{2}-\frac{k^2}{(aH)^2}\eta_T\right]=\frac{4\pi G}{H^2}\delta\rho_{Tc}, \quad (45)$$

where

$$\delta\rho_{Tc}=\sum_J\left[\delta\rho_J+3\left(\frac{aH}{k}\right)\rho_J u_J\right] \quad (46)$$

defines the density perturbation in the comoving gauge [32]. With the assumption that the total anisotropic stress is negligible outside the horizon,

$$\delta K=4\pi Ga^2\mathcal{O}(\delta\rho_{Tc}), \quad (47)$$

which generalizes Eq. (37) for total density fluctuations that grow from isocurvature initial conditions.

Thus, isocurvature conditions result when the contributions to the comoving gauge density perturbations cancel between species of different sound speeds, leaving finite pressure perturbations. In this case, the curvature fluctuation is initially small but evolves significantly so that its small impact on the local curvature cannot be captured as a real separate universe. On the other hand, since our universe possesses adiabatic or initial curvature fluctuations, even if isocurvature modes S_{Jm} are comparable to the curvature fluctuations, their impact on the separate universe curvature above the horizon is negligible (see Sec. V). We shall also see there that an entropy fluctuation S_{Jm} forms dynamically from initial curvature fluctuations if J has an intrinsic nonadiabatic stress Γ_J as it must for a dynamical dark energy component (see Sec. V and Ref. [33]).

Finally, for collisionless particles, free streaming generates anisotropic stress, and their gradients generate higher moments. Anisotropic stress gradients also act as an effective viscosity in the Navier-Stokes equation (21) generating u_J and setting an effective Jeans scale in Eq. (34) called the free streaming scale.

The distinction between the constant curvature condition (38) and the nongravitational flows condition Eq. (26) is that the latter sets a Jeans scale for each component. If a component contributes negligibly to the total, then it has a negligible effect on the curvature even below its Jeans scale. In Sec. V, we shall see an example where, in the matter dominated regime, the dark energy has its own Jeans scale that is much larger than the total Jeans scale but does not impact the matter evolution. On the other hand, this distinction becomes irrelevant if, as in this case, the component eventually does dominate the expansion.

In summary, if the wavelength of δ is larger than the Jeans scales of all components, we call its absorption into a local background as a "real" separate universe construction since all of the perturbation equations can be absorbed into the background with real energy density and curvature in the Friedmann equation. If the wavelength is shorter than the total Jeans scale, we call this a "fake" separate universe construction. In this case, from the perspective of the matter, the local universe obeys an effective Friedmann equation. Here, the curvature in comoving coordinates evolves, but as we shall see in the next section, it is considered as an effective energy density component for the matter dynamics.

This correspondence enables and justifies a separate universe treatment of the response of small scale cosmological observables to a long-wavelength density perturbation even if that wavelength is below the total Jeans scale of the system.

III. SEPARATE UNIVERSE COMPONENTS

In the previous section, we have shown that a long-wavelength matter density fluctuation can always be reabsorbed into the background with an appropriate adjustment of the expansion rate to its local or separate universe value. By construction, this approach satisfies the acceleration equation or the Newtonian cosmology exactly. For the separate universe Friedmann equation to be truly satisfied in terms of real energy densities and curvature, the wavelength must be much longer than the total Jeans scale in order for the curvature to be constant in comoving coordinates.

Below the Jeans scale, if the nonmatter components only influence the small scale matter evolution through the expansion rate, they can be described by an effective energy density component. In this fake separate universe, the dynamical impact of a changing curvature is assigned to this fictitious energy density component.

In this section, by matching parameters in the global and separate universe Friedmann equation, we establish this correspondence explicitly. We begin in Sec. III A with the comparison of the two Friedmann equations. We relate parameters in the real separate universe in Sec. III B and in the fake separate universe in Sec. III C.

A. Friedmann matching

The Friedmann equation in the global background universe can, without loss of generality, be written as

$$\frac{H^2}{H_0^2}=\frac{\Omega_m}{a^3}+\Omega_Q F_Q(a)+\frac{\Omega_K}{a^2}, \quad (48)$$

where Q represents the sum over all components aside from the pressureless matter. Here, $F_Q(a=1)=1$, and so defines $\bar\rho_Q(a)$ relative to its value today. Its derivative gives the equation of state parameter

$$\frac{d\ln F_Q}{d\ln a}=-3(1+w_Q). \quad (49)$$

Let us first try a naive method for absorbing the energy density fluctuations δ_m and δ_Q into a local background. Taking an equation of the same form as the Friedmann equation,

$$H_W^2 = H_{0W}^2\left[\frac{\Omega_{mW}}{a_W^3} + \Omega_{QW}F_Q(a_W) + \frac{\Omega_{KW}}{a_W^2}\right], \quad (50)$$

we can attempt to set the local parameters using the Friedmann equation with perturbed energy densities

$$H_W^2 = H_0^2\left[(1+\delta)\frac{\Omega_m}{a^3} + (1+\delta_Q)\Omega_Q F_Q(a)\right] + H_{0W}^2\frac{\Omega_{KW}}{a_W^2}. \quad (51)$$

Using the $a_W(a)$ relationship (6), the Hubble rates in Eqs. (50) and (51) coincide if

$$\Omega_{mW}H_{0W}^2 = \Omega_m H_0^2,$$
$$\Omega_{QW}H_{0W}^2 = \Omega_Q H_0^2,$$
$$\delta_Q = -\frac{1}{3}\frac{d\ln F_Q}{d\ln a}\delta, \quad (52)$$

i.e., if the energy densities agree when a and a_W have the same numerical value and the entropy perturbation (41)

$$S_{Qm} = \frac{\delta_Q}{1+w_Q} - \delta \quad (53)$$

between the components vanishes. In this case, the same shift in the scale factor of Eq. (6) that absorbs the matter fluctuation would absorb the Q fluctuation as well for all time. However, Eq. (52) is not a necessary condition and in fact cannot be stably satisfied if Q contains dynamical dark energy components (see Sec. V and Ref. [33]).

More generally, in the separate universe construction, the synchronous gauge matter density fluctuation δ defines the Hubble rate in the separate universe H_W through Eq. (8) and the relationship between the scale factors at constant time through Eq. (6). We can relate H_W^2 to the energy densities and curvature to a more general form of the Friedmann equation,

$$\frac{H_W^2}{H_{0W}^2} = \frac{\Omega_{mW}}{a_W^3} + \Omega_{QW}F_Q(a_W) + \frac{\Omega_{KW}}{a_W^2} + \Omega_{SW}F_S(a_W), \quad (54)$$

where $F_S(a_W = 1) = 1$. The introduction of the S component allows us to match any expansion history, not just those defined by perturbations to the Ω_Q of the global universe. We shall see next that its presence indicates an entropy perturbation or nonadiabatic stress for a real separate universe and a fictitious energy density that accounts for the evolution of the curvature in a fake separate universe.

We can obtain these correspondences by equating the difference in the Friedmann equation Hubble rates defined by Eqs. (54) and (48) to that required by Eq. (8) to match the acceleration equation. Keeping terms linear in δ,

$$\frac{\delta H^2}{H_0^2} \equiv -\frac{2}{3}\frac{H^2}{H_0^2}\delta' = \frac{\Omega_m}{a^3}\delta - \frac{\Omega_Q}{3}F_Q'\delta + \frac{2}{3}\frac{\Omega_K}{a^2}\delta + \frac{2\delta h}{h}\frac{1}{a^2} + \Omega_{SW}\left[F_S - \frac{1}{a^2}\right]. \quad (55)$$

Here, $2\delta h/h \approx (H_{0W}^2 - H_0^2)/H_0^2$ is a constant associated with the expansion rates at two different times but the same numerical value of the scale factor, namely $a_W = 1$ and $a = 1$. Here and below, we use the notation $\delta X = X_W - X$ for a parameter X. Note that $\Omega_{SW} \propto \delta$ since this component is absent in the global universe.

In particular, $\delta h/h$ is defined by evaluating Eq. (55) at $a = 1$ using $\delta'(a = 1) \equiv \delta_0'$,

$$2\frac{\delta h}{h} = -\frac{2}{3}\delta_0' - \Omega_m\delta 0 + \frac{F_Q'}{3}\Omega_Q\delta 0 - \frac{2}{3}\Omega_K\delta 0. \quad (56)$$

Equality of the physical energy densities at the same scale factor sets

$$\frac{\delta\Omega_m}{\Omega_m} = \frac{\delta\Omega_Q}{\Omega_Q} = -2\frac{\delta h}{h}, \quad (57)$$

and

$$\Omega_{SW} + \Omega_{KW} = 1 - \Omega_{mW} - \Omega_{QW}, \quad (58)$$

by definition of H_{0W}^2. While this assumption for $\delta\Omega_Q$ is not fully general, we can absorb any remaining difference into the S component. In other words, we take $\delta\Omega_Q$ to define the division into Q and S components in the separate universe. We shall make explicit use of this fact in the dark energy isocurvature example in Sec. V.

These relations set the separate universe parameters of the energy density components that exist in the global universe. If we take $\Omega_{SW} = 0$, then the curvature is also determined, and there is no additional freedom that can be used to satisfy Eq. (55) at $a < 1$. Thus, for a general evolution of δ, $\Omega_{SW} \neq 0$ is required, and we can define F_S so as to satisfy Eq. (55). Conversely, for any desired evolution of δ, we can always construct a well-defined expansion history using S to satisfy both the acceleration and Friedmann equations.

B. Real separate universe

The distinction between a real and fake separate universe construction depends on whether the Friedmann components Ω_{SW} and Ω_{KW} truly represent an energy density and curvature in the local universe. In the construction of

Eq. (58), only their sum and not their individual values is specified. This ambiguity is related to the fact that in the Friedmann and acceleration equations it is not possible to distinguish between an energy density that scales as $1/a^2$ and a curvature component.

However, curvature has geometric effects which distinguish it, and moreover we can relate the curvature perturbation and the separate universe curvature using Eq. (31) at $a_W = 1$,

$$\Omega_{KW} \equiv -\frac{K_W}{H_{0W}^2} = -\frac{K}{H_0^2} - \frac{\delta K}{H_0^2} + \frac{K}{H_0^2}\left(\frac{2}{3}\delta + 2\frac{\delta h}{h}\right). \quad (59)$$

$$\frac{\delta K}{H_0^2} = \frac{8\pi G}{3H_0^2}\sum_J \delta\rho_J + \frac{2}{3}\delta', \quad (60)$$

where we have used Eqs. (17) and (19). Combining these equations with Eq. (56), we obtain

$$\delta\Omega_K = -\Omega_Q\delta_Q - \frac{F_Q'}{3}\Omega_Q\delta + 2(1-\Omega_K)\frac{\delta h}{h}$$
$$= -\Omega_Q(1+w_Q)S_{Qm} + 2(1-\Omega_K)\frac{\delta h}{h}. \quad (61)$$

Finally using Eq. (58), we obtain

$$\Omega_{SW} = \Omega_Q(1+w_Q)S_{Qm} \quad (62)$$

so that this component is associated with the entropy perturbation. All of the above relationships for separate universe cosmological parameters Ω_{JW} in terms of global universe perturbations are assumed to be evaluated at $a_W = 1$, and note that $\delta(a_W = 1) \approx \delta(a = 1) \equiv \delta_0$.

We obtain the same criteria from the standpoint of absorbing the energy density associated with δ_Q into the background at an arbitrary $a_W(a)$,

$$H_0^2\Omega_Q F_Q(a)(1+\delta_Q)$$
$$= H_{0W}^2\Omega_{QW}F_Q(a_W) + H_{0W}^2\Omega_{SW}F_S(a_W)$$
$$\approx H_0^2\Omega_Q\left[F_Q(a) - \frac{F_Q'(a)}{3}\delta\right] + H_0^2\Omega_{SW}F_S(a). \quad (63)$$

Employing the definition of the entropy

$$F_Q\delta_Q = -\frac{F_Q'}{3}(S_{Qm} + \delta), \quad (64)$$

we infer

$$\Omega_{SW}F_S(a) = -\Omega_Q\frac{F_Q'(a)}{3}S_{Qm}(a), \quad (65)$$

which gives Eq. (62) at $a = 1$ and defines the energy density scaling in terms of the evolution of the entropy. Note that the presence of an evolving entropy fluctuation does not prevent a real separate universe matching; each component represents a real energy density that exists in the global universe. It simply means that in the separate universe the background energy density components do not obey the same equations of state as in the global universe. On the other hand, we shall see next that nonconservation of the separate universe curvature below the Jeans scale does indicate that the separate universe construction involves fake components.

C. Fake separate universe

While the remapping of perturbations onto separate universe cosmological parameters in the previous section may seem fully general, it implicitly assumes that the separate universe curvature $K_W = \text{const.}$ whereas it is actually constructed in Eq. (31) out of the dynamical curvature and scale factor fluctuations in the global universe,

$$K_W' = \delta K' - \frac{2}{3}K\delta'. \quad (66)$$

Setting $K_W' = 0$ and combining the synchronous gauge metric equations, we obtain the condition

$$(\delta\rho_Q)' + 3(\delta\rho_Q + \delta p_Q) = (\bar\rho_Q + \bar p_Q)\delta'. \quad (67)$$

Not surprisingly, this is exactly the same condition in Eq. (25) for which the continuity equation Eq. (21) can be written as a perturbation to the background energy conservation equation. This condition holds to good approximation for scales above the Jeans scale for Q including any evolution in S_{Qm}. In this case, δ_Q evolves as in a separate universe.

Below the Jeans scale, the matter fluctuations still behave in a way that can be absorbed into a separate universe expansion rate but one that does not obey a true Friedmann equation. If we use the curvature and entropy decomposition in Eqs. (61) and (62), we must allow the curvature contribution to the Friedmann equation to have a general evolution,

$$\frac{H_W^2}{H_{0W}^2} = \frac{\Omega_{mW}}{a_W^3} + \Omega_{QW}F_Q(a_W) + \Omega_{KW}F_K(a_W)$$
$$+ \Omega_{SW}F_S(a_W), \quad (68)$$

where

$$\Omega_{KW}F_K(a_W) \equiv -\frac{K_W(a_W)}{H_{0W}^2 a_W^2}. \quad (69)$$

In this case, K_W is a nonconstant background curvature which has a fake equation of state $w_{KW} \neq -1/3$.

Alternatively, we can combine the curvature fluctuation into the effective energy density S,

$$\delta\Omega_K = 0,$$
$$\Omega_{SW} = -\delta\Omega_m - \delta\Omega_Q, \tag{70}$$

which evolves according to

$$\Omega_{SW} F_S(a) = \frac{\Omega_{SW}}{a^2} - \frac{2}{3}\frac{H^2}{H_0^2}\delta' - \frac{\Omega_m}{a^3}\delta + \frac{F_Q'}{3}\Omega_Q\delta - \frac{2}{3}\frac{\Omega_K}{a^2}\delta - \frac{2\delta h}{h}\frac{1}{a^2}. \tag{71}$$

In this case, S is a fake energy density component that accounts for both the curvature and entropy fluctuations. Of course, in practice, given that either alternative involves a fake component to the Friedmann equation, one can also simply set an expansion rate H_W without dividing its sources into separate energy density and curvature components.

In the intermediate regime, where the growth of δ depends on scale, we can either analyze the separate universes for each k-mode in turn or construct the real space δ that corresponds to the sum over the modes that contribute to the local mean averaged over a given physical scale.

IV. OBSERVABLE RESPONSE

In both the real and fake or super- and sub-Jeans scale separate universe constructions, the matter distribution on small scales responds to a long-wavelength fluctuation as if it were in a separate universe, i.e., through the change in cosmological parameters associated with the separate universe. In general, this means that these observables respond not just to the change in the local mean density at the epoch in which they are observed but also the whole history of its evolution. If this growth history depends on the scale of the long-wavelength mode, then the observable response will as well.

For nonlinear observables such as the bias and abundance of dark matter halos or the nonlinear matter or halo power spectra, we can employ cosmological simulations in the separate universe [9,17–19] to calibrate this response [8,9,14,15,15,16]. In future work, we will present results from cosmological simulations in the real and fake separate universe for dynamical dark energy and massive neutrino models.

Here, we illustrate the ideas with the observable being the growth of structure or power spectrum in the linear regime. If the short-wavelength mode is much smaller than the Jeans length of the other components Q, the evolution of $\delta_W(a)$ obeys the usual growth equation obtained by setting the $J = Q$ components to zero in Eq. (20), but with the scale factor and expansion rate of the separate universe

$$\frac{d^2\delta_W}{d\ln a_W^2} + \left(2 + \frac{d\ln H_W}{d\ln a_W}\right)\frac{d\delta_W}{d\ln a_W} = \frac{3}{2}\frac{H_{0W}^2}{H_W^2}\frac{\Omega_{mW}}{a_W^3}\delta_W. \tag{72}$$

Note that here the density fluctuation and growth are relative to the separate universe mean $\rho_{mW} = \rho_m(1+\delta)$. Using the relationship between the separate and global universes (6), (8), and (24), we can rewrite this in the global coordinates as

$$\delta_W'' + \left(2 - \frac{2}{3}\delta' + \frac{d\ln H}{d\ln a}\right)\delta_W' = \frac{3}{2}\frac{H_0^2}{H^2}\frac{\Omega_m}{a^3}(1+\delta)\delta_W. \tag{73}$$

Since the change in the growth due to δ is itself small, we can expand

$$\delta_W = \delta_- + \epsilon, \tag{74}$$

where δ_- is given by the unperturbed sub Jeans scale growth, i.e., by setting $\delta = 0$ in Eq. (73). Here, $\epsilon = \mathcal{O}(\delta)\delta_-$ is the second order correction from the long-wavelength mode that obeys

$$\epsilon'' + \left(2 + \frac{H'}{H}\right)\epsilon' - \frac{3}{2}\frac{H_0^2}{H^2}\frac{\Omega_m}{a^3}\epsilon = \frac{2}{3}\delta'\delta_-' + \frac{3}{2}\frac{H_0^2}{H^2}\frac{\Omega_m}{a^3}\delta\delta_-. \tag{75}$$

Notice that both the long-wavelength perturbation to the scale factor δ and the perturbation to the Hubble rate δ' enter as sources.

Solving this system, we can define the growth response function

$$\frac{d\ln D_W}{d\delta} = \frac{\epsilon}{\delta\delta_-}. \tag{76}$$

For growing modes in the matter dominated limit,

$$\frac{d\ln D_W}{d\delta} = \frac{13}{21}, \tag{77}$$

which is the usual second order result [34], accounting for the difference due to fluctuations being measured with respect to ρ_{mW}, i.e., in the global universe,

$$\frac{d\ln D}{d\delta} = \frac{d\ln D_W}{d\delta} + 1. \tag{78}$$

This implies that the power spectrum response to δ in the linear regime [8]

$$\frac{\partial\ln P}{\partial\delta} = 2\frac{d\ln D}{d\delta} - \frac{1}{3}\frac{d\ln k^3 P}{d\ln k} \tag{79}$$

depends on the growth history of the long-wavelength mode δ which can itself depend on scale. The second term on the rhs comes from the dilation of scales due to the separate universe scale factor [8]. One consequence of this is that the squeezed bispectrum and trispectrum becomes dependent on the scale of the long-wavelength mode. The latter also causes the supersample covariance of the power spectrum [8] to also depend on which modes contribute to the local mean within the sample.

These relations are also useful for setting up cosmological simulations of the separate universe with the same initial conditions as the global universe given a power spectrum normalization at $a_W = 1$ commonly used in codes.

V. SCALAR FIELD DARK ENERGY

As an illustration of the concepts in the previous sections, let us consider the concrete example of scalar field dark energy with a Lagrangian [35],

$$\mathcal{L} = P(X, Q), \qquad X = -\frac{1}{2}\nabla^\mu Q \nabla_\mu Q. \tag{80}$$

In this case, Q represents a single component, the scalar field itself, and takes the form of a perfect fluid with no anisotropic stress in the fluid rest frame.

In Sec. V A, we review how the scalar field equations of motion set the rest frame sound speed and require nonadiabatic stress. The separate universe construction for long-wavelength modes with initial adiabatic or curvature fluctuations differs above and below the sound horizon as shown in Sec. V B. In Sec. V C, we examine how this construction changes if the dark energy also has initial isocurvature fluctuations.

A. Sound horizon and equations of motion

To close the equations of motion (21) of dynamical dark energy in general, we need to specify the relationship between its pressure and energy density fluctuations, i.e., the sound speed. In the scalar field model, this is provided by the field equation given a specific Lagrangian. More generally, for a dark energy component that accelerates the expansion, $w_Q < -1/3$, and so typically the adiabatic sound speed

$$c_{Qa}^2 \equiv \frac{\bar{p}_Q'}{\bar{\rho}_Q'} < 0. \tag{81}$$

For the Navier-Stokes or Euler equation (21) to be stable,

$$c_{Qa}^2 \neq \frac{\delta p_Q}{\delta \rho_Q} \equiv c_{Qs}^2 > 0, \tag{82}$$

where s denotes synchronous gauge. Thus, an internal nonadiabatic stress Γ_Q is required for dynamical dark energy [33].

In terms of the scalar field, internal nonadiabatic stress arises from the separate kinetic and potential contributions to the energy density and pressure. In that case, the local energy density does not uniquely specify the local pressure as it is possible to specify a sound speed that is independent of $w_Q(a)$. For definiteness, we take the Lagrangian

$$P(X, Q) = \Lambda_X \left(\frac{X}{\Lambda_X}\right)^n - V(Q) \tag{83}$$

as an example.

The sound speed for the fluid is best defined in the rest frame "r" or equivalently the constant field gauge [35]

$$c_Q^2 \equiv \frac{\delta p_{Qr}}{\delta \rho_{Qr}} = \frac{P_{,X}}{2P_{,XX}X + P_{,X}} = \frac{1}{2n-1}, \tag{84}$$

where

$$\delta p_{Qr} = \delta p_Q - \bar{p}_Q' \frac{aH}{k}\frac{u_Q}{1+w_Q},$$
$$\delta \rho_{Qr} = \delta \rho_Q - \bar{\rho}_Q' \frac{aH}{k}\frac{u_Q}{1+w_Q}. \tag{85}$$

The sound horizon of this system is then defined by the wave number k_Q where

$$\frac{c_Q k_Q}{aH} = 1. \tag{86}$$

The pressure fluctuation therefore carries internal nonadiabatic stress

$$p_Q \Gamma_Q = \delta p_Q - c_{Qa}^2 \delta \rho_Q, \tag{87}$$

which we shall see also generates an entropy fluctuation S_{Qm} dynamically.

For the two-component system, the equations of motion for the synchronous gauge perturbations (20) and (21) become

$$\delta_Q' + 3(c_{Qs}^2 - w_Q)\delta_Q = -\frac{k}{aH}u_Q + (1+w_Q)\delta', \tag{88}$$

$$u_Q' + (1 - 3w_Q)u_Q = \frac{k}{aH}c_{Qs}^2 \delta_Q \tag{89}$$

for the scalar field and

$$\delta'' + \left(2 + \frac{H'}{H}\right)\delta' = \frac{3}{2}\frac{H_0^2}{H^2}\sum_J \Omega_J F_J (1 + 3c_{Js}^2)\delta_J \tag{90}$$

for the matter, where $\delta_m = \delta$, $c_{ms}^2 = 0$, $F_m = a^{-3}$, and

$$c_{Qs}^2\delta_Q = c_Q^2\delta_Q + 3(c_Q^2 - c_{Qa}^2)\frac{aH}{k}u_Q. \tag{91}$$

To specify the background evolution, we can either fix w_Q by hand and leave the corresponding potential implicit or solve the Q background equations of motion

$$\bar{Q}'' + \frac{3}{2}(2c_Q^2 - 1 - w_T)\bar{Q}' + \frac{c_Q^2}{H^2}\frac{V_{,Q}}{P_{,X}} = 0, \tag{92}$$

for a given potential V. Here, the total equation of state $w_T = p_Q/(\rho_m + \rho_Q)$ for the two-component system, and we construct

$$w_Q = \frac{\bar{p}_Q}{\bar{\rho}_Q} = \frac{P}{2P_{,X}X - P}\bigg|_{\bar{Q}} \tag{93}$$

from the background solution $\bar{Q}$.

Below the sound horizon, Q density fluctuations are pressure supported and become negligible compared with the dark matter. Above the sound horizon, dark energy fluctuations influence the growth of matter density perturbations and thus the separate universe construction. Above the total Jeans scale, the separate universe construction provides a real separate universe. The correspondence between the total Jeans scale and the sound horizon depends on the initial conditions as we shall see next.

B. Initial curvature perturbations

For the usual case of adiabatic perturbations that originate from initial comoving curvature fluctuations $\mathcal{R}$ in the matter dominated epoch, the matter density fluctuation growth takes on a scale-free form when the mode is either well above or below the Q sound horizon. In this case, the sound horizon defined in the constant field gauge corresponds to the total Jeans scale during the acceleration epoch [36].

For simplicity, let us assume that $P(X,Q)$ has been constructed so that in the global background $0 > w_Q = c_{Qa}^2 = \text{const}$. Then, at the initial epoch a_i, the universe is matter dominated, and the growing mode of adiabatic or initial curvature fluctuations $\mathcal{R}$ is

$$\delta(a_i) = \frac{2}{5}\left(\frac{k}{a_iH_i}\right)^2\mathcal{R} \propto a_i. \tag{94}$$

Inspecting the equations of motion (88), (89), and (90), we obtain

$$\delta_Q(a_i) = \frac{(5-6c_Q^2)(1+w_Q)}{5+9c_Q^2-15w_Q}\delta(a_i),$$
$$u_Q(a_i) = \frac{k}{a_iH_i}\frac{2c_Q^2(1+w_Q)}{5+9c_Q^2-15w_Q}\delta(a_i). \tag{95}$$

Note that there is an entropy fluctuation induced by the curvature fluctuations

$$S_{Qm} = \frac{15w_Q - 15c_Q^2}{5+9c_Q^2-15w_Q}\delta(a_i) \tag{96}$$

and the internal nonadiabatic stress. Here, $S_{Qm} \propto \delta$, and so $|S_{Qm}| \ll |\mathcal{R}| \sim |\eta_T|$ outside the horizon when $k/aH \ll 1$ unlike the isocurvature conditions discussed in the next section. Equivalently, the entropy fluctuation vanishes as $a_i \to 0$, and the background dark energy density in the separate and global universe are initially the same.

For the sub-Jeans case of $k \gg k_Q$, the dark energy perturbations are negligible compared with the matter, and the system reduces to the familiar case $\delta \approx \delta_-$,

$$\delta_-'' + \left(2 + \frac{H'}{H}\right)\delta_-' = \frac{3}{2}\frac{H_0^2}{H^2}\frac{\Omega_m}{a^3}\delta_-. \tag{97}$$

We show this growth function in Fig. 1 for various w_Q. As w_Q increases, dark energy domination occurs earlier for the same Ω_m, and pressure support in Q has a larger impact on the matter growth.

For the super-Jeans case of $k \ll k_Q$, the growth of δ in the dark energy dominated epoch depends on the sound speed if relativistic $c_Q \sim 1$. In this case, the Jeans scale is near the horizon. The super- and sub-Jeans scale differences are difficult to measure and also involve relativistic effects in relating synchronous gauge quantities to direct observables.

It is therefore interesting to consider the $c_Q \ll 1$ limit where we solve

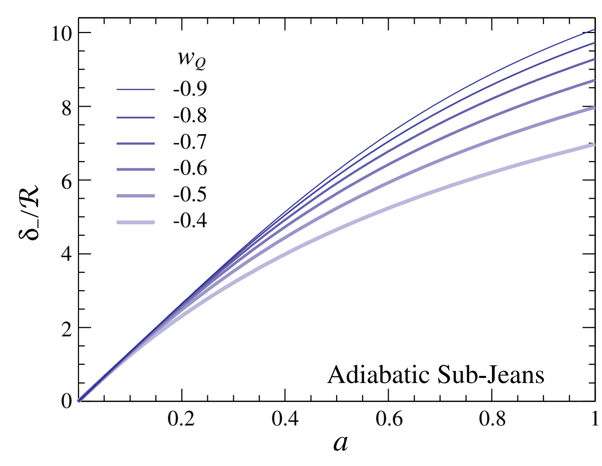

FIG. 1. Adiabatic matter density perturbation evolution δ relative to the initial curvature fluctuation $\mathcal{R}$ for an arbitrary wavenumber on sub-Jeans scales. Pressure support in the scalar field Q slows the growth of fluctuations as w_Q increases and dark energy domination occurs earlier. Here and throughout the figures, we take $\Omega_m = 1 - \Omega_Q = 0.3$.

$$\delta'_Q - 3w_Q\delta_Q = (1+w_Q)\delta'_+,$$
$$\delta''_+ + \left(2+\frac{H'}{H}\right)\delta'_+ = \frac{3}{2}\frac{H_0^2}{H^2}\left[\frac{\Omega_m}{a^3}\delta_+ + \Omega_Q F_Q \delta_Q\right] \quad (98)$$

and take $\delta \approx \delta_+$. Notice that in this limit the Euler equation (89) for u_Q has a negligible source from gradients in δ_Q, and with the initial conditions (95), its value remains negligible. Thus, we have a real separate universe above the sound horizon which plays the role of the Jeans scale for initial curvature fluctuations [36].

This sets the change in the separate universe scale factor $\delta \ln a = \ln a_W - \ln a$. In Fig. 2 (top), we show the ratio of this change above to below the Jeans scale for the same value of $\delta(a=1) = \delta_0$,

$$R_a \equiv \frac{\delta \ln a_+}{\delta \ln a_-} = \frac{\delta_+(a)}{\delta_-(a)}\frac{\delta_-(1)}{\delta_+(1)}. \quad (99)$$

For definiteness, we take $\Omega_m = 1 - \Omega_Q = 0.3$ here and throughout the examples. Since above the Jeans scale δ grows more relative to its initial value, it is actually smaller in value at $a < 1$ once normalized to today. This difference goes to zero as $w_Q \to -1$ since the gravitational source to δ_Q in Eq. (88) $(1+w_Q)\delta' \to 0$.

The difference in $a_W(a)$ between the super- and sub-Jeans scale separate universes implies that the Hubble rate also differs. Using Eq. (8), for the same value of δ_0, the ratio is

$$R_H \equiv \frac{\delta \ln H_+}{\delta \ln H_-} = \frac{\delta'_+(a)}{\delta'_-(a)}\frac{\delta_-(1)}{\delta_+(1)}. \quad (100)$$

In the Hubble rate, the super-Jeans scale separate universe is closer to the global universe than the sub-Jeans scale one at early times and farther at late times as the universe begins to accelerate. The latter reflects the enhanced growth rate above the Jeans scale required to produce the same δ_0 today.

An interesting consequence of this behavior is that objects formed at high redshift will differ in their response to a long-wavelength mode than those formed at low redshift. We will address the implications for halo bias in future work.

This dependence on scale and redshift also changes the response of the linear growth $d\ln D_W/d\ln\delta$ of Eq. (76) above and below the Jeans scale. This is shown in Fig. 3. Since, relative to the same δ today, its amplitude at high redshift is smaller above the Jeans scale as shown in Fig. 2, the linear growth response is also smaller. In principle, this would lead to an observable change in the matter bispectrum, trispectrum, and supersample power spectrum

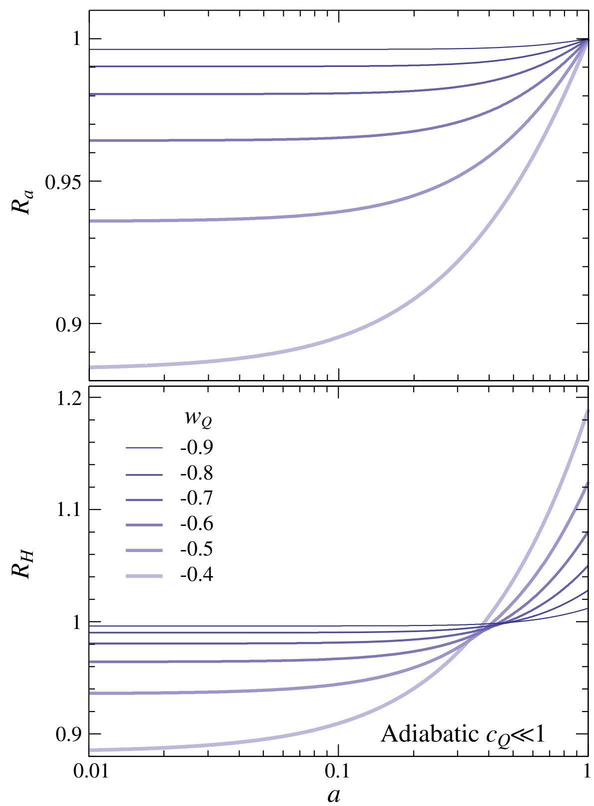

FIG. 2. Ratios of super- to sub-Jeans scale separate universe changes in the scale factor $\delta \ln a$ [top; Eq. (99)] and Hubble rate $\delta \ln H$ [bottom; Eq. (100)] for the same long-wavelength density δ_0 today. Above the Jeans scale, the scale factor is always closer to global, whereas the Hubble rate is closer at early times and further at late times. Observable responses to δ_0 therefore depend on its scale, but as $w_Q \to -1$, this difference vanishes. Here, initial conditions are adiabatic, and $c_Q \ll 1$.

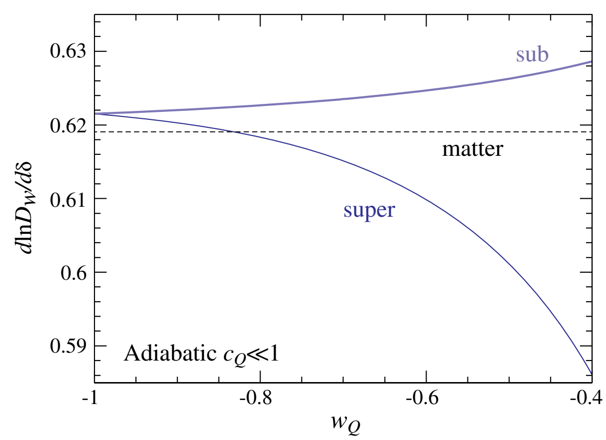

FIG. 3. Response of the short-wavelength linear growth function D_W to the long-wavelength density field δ at $a = 1$ for adiabatic initial conditions and scalar field dark energy with $c_Q \ll 1$. This response depends on whether the long wavelength is super- or sub-Jeans scale. Long-wavelength scale dependence of the squeezed bispectrum and trispectrum would result but vanishes if $w_Q \to -1$. For reference, the second order response in the matter dominated regime 13/21 is also shown (dashed).

covariance, but the size of this change is small for observationally viable values of w_Q.

C. Initial dark energy perturbations

For pure curvature initial conditions, we have seen that the difference between super- and sub-Jeans scale growth and hence the scale dependencies of separate universe responses of observables vanish as $w_Q \to -1$. This is because the gravitational source to these dark energy density fluctuations vanishes in this limit. The difference can be much larger if instead these fluctuations were provided by the initial conditions.

If the field fluctuations associated with these initial conditions were nearly frozen outside the horizon, then large scale fluctuations would survive to the current epoch [31]. In the field equation (92), this occurs when the slope of the potential $V_{,Q}$ is too small to overcome the Hubble drag. The field then only rolls by a small amount during its cosmic evolution. In that case, $V_{,Q} \approx \text{const}$ for any smooth potential, and the field equation becomes a Bernoulli equation for Q' which does not depend on the field value itself. By making $V_{,Q}$ arbitrarily small compared with V, we can bring the expansion history as close to ΛCDM as desired. We therefore consider for simplicity the limiting case where

$$w_T(a) = -\frac{\Omega_Q}{\Omega_m a^{-3} + \Omega_Q}, \tag{101}$$

in a flat $\Omega_m + \Omega_Q = 1$ universe.

In this case, Q' reaches a terminal velocity independent of its initial value, and the adiabatic sound speed becomes

$$c_{Qa}^2 = \frac{x(c_Q^2 - x^2)\sqrt{1+x^2} - c_Q^2(1+x^2)\sinh^{-1}x}{(1+x^2)(x\sqrt{1+x^2} - \sinh^{-1}x)}, \tag{102}$$

where

$$x = a^{3/2}\sqrt{\frac{\Omega_Q}{\Omega_m}} \tag{103}$$

parametrizes the transition between matter and dark energy domination.

Note that in the matter dominated epoch $x \ll 1$ and

$$c_{Qa}^2 = -\frac{c_Q^2 + 3}{2}. \tag{104}$$

Since the adiabatic sound speed $c_{Qa}^2 = p_Q'/\rho_Q'$, the small evolution of the energy density and pressure makes $c_{Qa}^2 \neq w_Q \approx -1$.

Inspecting the equations of motion, we find that initial conditions in the matter dominated epoch are

$$\begin{aligned}\delta_Q(a_i) &= \mathcal{I},\\ u_Q(a_i) &= -\frac{2}{9}\frac{k}{a_i H_i}\mathcal{I},\\ \delta(a_i) &= -\frac{1}{3}\frac{\Omega_Q}{\Omega_m}a_i^3\mathcal{I}.\end{aligned} \tag{105}$$

The constant $\mathcal{I}$ is equivalent to an initial entropy fluctuation $\mathcal{I} = (1+w_Q)S_{Qm}$, but given that $w_Q \approx -1$, this notation is more convenient. Note that

$$\eta_T(a_i) = -\frac{1}{9}\frac{\Omega_Q}{\Omega_m}a_i^3\mathcal{I}, \tag{106}$$

reflecting the isocurvature initial conditions $|\eta_T| \ll |\mathcal{I}|$. These quantities should be added to those generated by the initial curvature fluctuations $\mathcal{R}$, and so their relative strengths are determined by $\mathcal{I}/\mathcal{R}$ and their correlation.

In Fig. 4 (top), we show the evolution of δ_Q from these isocurvature conditions as a function of scale for an example with $c_Q = 0.1$. On scales larger than the current horizon k/H_0, the density perturbations are frozen even through the acceleration epoch as expected. We can analytically verify this behavior by dropping the velocity divergence source ku_Q/aH in the Q continuity equation (88). The Q system is then solved by

$$\begin{aligned}\delta_Q(a) &= \mathcal{I},\\ u_Q(a) &= -\frac{1+c_Q^2}{3(c_Q^2 - c_{Qa}^2)}\frac{k}{aH}\mathcal{I}, \qquad \frac{k}{aH} \ll 1.\end{aligned} \tag{107}$$

These dark energy perturbations induce a growing mode in the matter fluctuations of opposite sign (see Fig. 4, bottom). In terms of the separate universe, a positive change in the dark energy density acts like a universe with a larger cosmological constant. In the construction of Eq. (65), we introduce an entropy component with

$$\Omega_S F_S(a) = \Omega_Q(1+w_Q)S_{Qm} = \Omega_Q\mathcal{I}, \tag{108}$$

and this represents a real separate universe with a constant change to the cosmological constant as expected. Technically, the curvature fluctuation is changing at the Hubble rate during matter domination and growing logarithmically during dark energy domination, but its value is suppressed by $(k/aH)^2$ and has a negligible impact since initial curvature fluctuations must also exist.

Between the horizon and sound horizon, the dark energy density fluctuation grows due to the nongravitational velocity divergence in its continuity equation. Unlike in the case of curvature fluctuations, the total Jeans scale in the acceleration epoch is the horizon and not the sound horizon. In the Euler equation (89), even though density gradients from δ_Q do not source nongravitational flows for

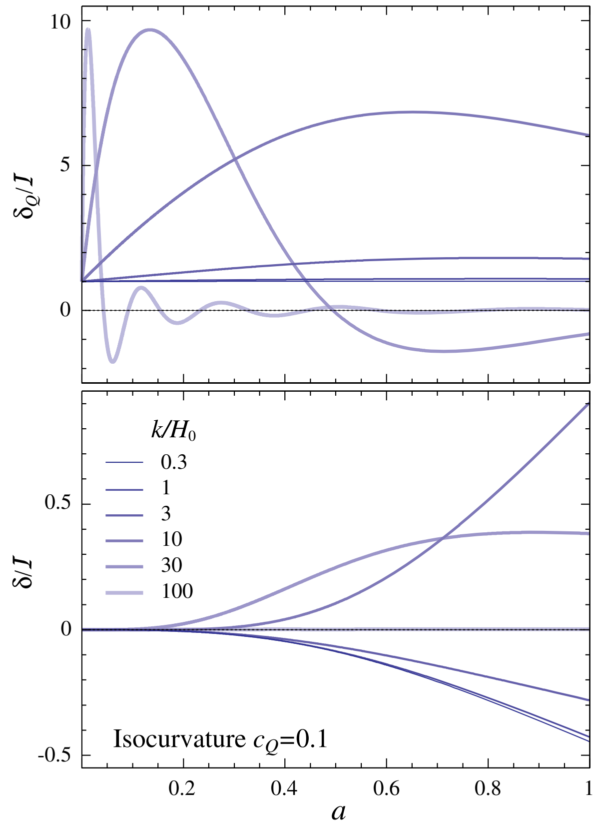

FIG. 4. Growth of the dark energy and matter density perturbations δ_Q and δ from dark energy isocurvature initial conditions $\mathcal{I}$ for $c_Q = 0.1$. Dark energy perturbations (top) are frozen outside the horizon $k/H_0 < 1$, grow between the horizon and sound horizon, and oscillate below the sound horizon. Matter perturbations (bottom) are anticorrelated with $\mathcal{I}$ outside the horizon, grow in correlation with it between the horizon and sound horizon, and are independent of it below the sound horizon. By adding isocurvature modes to the adiabatic modes, the separate universe construction and observable response can depend on scale even for $w_Q \approx -1$.

$c_Q \ll 1$, their initial values are set by the isocurvature conditions (105), and they grow until horizon crossing. At this point, they become large enough to cause a violation of the real separate universe. The divergence of the flow then impacts δ_Q through the continuity equation (88) which sources δ through Eq. (90) causing it to change signs. In this regime, a positive initial dark energy density fluctuation leads to a positive and growing matter fluctuation.

Below the sound horizon, the dark energy density fluctuation oscillates and decays. The matter fluctuations then are also suppressed, and the separate and global universe coincide in parameters. In principle, a small $c_Q \ll 1$ allows these effects to produce novel separate universe responses between the horizon and sound horizon. However, the matter density perturbations associated with the curvature fluctuations $\mathcal{R}$ also grow during this regime for $\Omega_m \sim 0.3$ where the universe is not completely dark energy dominated. One must therefore arrange the initial spectrum of $\mathcal{I}$ to produce a sizable change in δ of the desired wavelength at the current epoch. Finding an early universe mechanism to generate such dark energy isocurvature fluctuations is beyond the scope of this work.

VI. DISCUSSION

In this work, we have shown how to construct a separate universe to absorb the entire growth history of a long-wavelength density perturbation of a multicomponent system into the cosmological background from the perspective of the nonrelativistic matter. By exactly matching the acceleration equation to the synchronous gauge matter density fluctuations, we extend the validity of the approach to scales smaller than the Jeans length where nongravitational effects play a role. Above the Jeans scale, the construction also satisfies the Friedmann equation with real energy densities and a curvature that is constant in comoving coordinates. Below the Jeans scales, the curvature evolves and in the separate universe Friedmann equation acts like a fake density component. In both cases, the matter evolution on small scales is correctly modeled.

Once the long-wavelength density fluctuation is absorbed into the background, we can assess its impact on small scale cosmological observables as a change in the expansion rate or the cosmological parameters that drive it. Our construction highlights the fact that its influence is nonlocal in time. For the same long-wavelength density fluctuation, its impact on small scale observables at the same epoch depends on its entire growth history. If this growth depends on scale as in the case of the super- and sub-Jeans scale fluctuations, then the response also becomes dependent on the scale of the long-wavelength mode.

As a concrete illustration, scalar field dark energy with a finite sound speed introduces its sound horizon to the Jeans scale of the system. For the same long-wavelength density perturbation today, the different growth histories imply different separate universes and hence different responses in short-wavelength observables. In particular, we have highlighted the scale dependent response to the linear growth rate for adiabatic fluctuations and the novel changes that can occur if initial dark energy isocurvature perturbations are also present.

By employing cosmological simulations of the separate universe, this technique should prove useful for studying the analogous scale dependent responses in the nonlinear matter and halo power spectrum; supersample covariance; bispectrum, trispectrum, and halo abundance and bias. Likewise, other systems such as massive neutrino and modified gravity models possess scale dependent long-wavelength growth that can also be studied with these methods. We leave these topics for future work.

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