Towards a complete $\Delta(27) \times SO(10)$ SUSY GUT

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We propose a renormalizable model based on $\Delta(27)$ family symmetry with an $SO(10)$ grand unified theory leading to a novel form of spontaneous geometrical CP violation. The symmetries, including $\Delta(27)$ and $\mathbb{Z}_9 \times \mathbb{Z}_{12} \times \mathbb{Z}_4^R$, are broken close to the grand unified theory breaking scale to yield the minimal supersymmetric standard model with the standard R parity. $SO(10)$ is broken via $SU(5)$ with doublet-triplet splitting achieved by a version of the Dimopoulos-Wilczek (missing vacuum expectation value) mechanism. Low-scale Yukawa structure is dictated by the coupling of matter to $\Delta(27)$ antitriplets $\bar{\phi}$ of which the vacuum expectation values are aligned in the constrained sequential dominance 3 directions by the superpotential. Light physical Majorana neutrinos masses emerge from a specific implementation of the seesaw mechanism within $SO(10)$. The model predicts a normal neutrino mass hierarchy with the best-fit lightest neutrino mass between 0.32 and 0.38 meV, CP-violating oscillation phase $\delta^l \approx (275\text{–}280)°$, and the remaining neutrino parameters all within 1σ of their best-fit experimental values.

I. INTRODUCTION

The Standard Model (SM) cannot possibly be a complete theory, since it does not provide an explanation for neutrino mass and mixing. In addition, it provides no glimmer of insight into the flavor and CP puzzles or the origin of three distinct gauge forces. A very ambitious approach, capable in principle of addressing all these questions, is the idea of a grand unified theory (GUT) combined with a family symmetry which can control the structure of the Yukawa couplings, leading to a predictive theory of flavor. In addition, supersymmetry (SUSY) is the most elegant way to ensure gauge coupling unification, also stabilizing the Higgs mass (for a review, see, e.g., Ref. [1]). The state of the art is to combine a realistic GUT (addressing issues like doublet-triplet splitting) with predictive flavor structures [2,3], and we proposed a fairly complete $A_4 \times SU(5)$ SUSY GUT of flavor along these lines [4]. However, the most ambitious, but also the most challenging, of such theories are those based on $SO(10)$ [5] where three right-handed neutrinos are predicted and neutrino mass is therefore inevitable. Typically, such theories are very difficult to reconcile with a family symmetry and generally involve rather large dimensional Higgs representations.

In this paper, we propose a realistic and fairly complete model, capable of addressing all the above questions unanswered by the SM, based on $\Delta(27) \times SO(10)$ with a CP symmetry at the high scale. The choice of $\Delta(27)$ is primarily due to its triplet and antitriplet representation,

such that there is no invariant between two triplets, which is convenient due to the SM fermions being placed all in a single $SO(10)$ **16**, $\Delta(27)$ triplet. In addition, the nontrivial singlets of $\Delta(27)$ are also useful, as they are used to give rise CP-violating phases that are related to the group rather than arbitrary parameters in the Lagrangian. We therefore describe this as spontaneous geometrical CP violation [6], in this model in a novel form, as it fixes relative of phases between distinct flavons. The model has many attractive features, including the use of only the lower-dimensional "named" representations of $SO(10)$, i.e., the singlet, fundamental, spinor, or adjoint representations. $SO(10)$ is broken via $SU(5)$ with doublet-triplet splitting achieved by a version of the Dimopoulos-Wilczek (DW) or missing vacuum expectation value (VEV) mechanism [7].

The renormalizable $\Delta(27) \times SO(10)$ model also involves a discrete $\mathbb{Z}_9 \times \mathbb{Z}_{12} \times \mathbb{Z}_4^R$. The family symmetries are broken close to the GUT breaking scale to yield the minimal supersymmetric standard model (MSSM) supplemented by a right-handed neutrino seesaw mechanism [8,9], where the $\mathbb{Z}_4^R$ is the origin of the MSSM R parity [10]. The model is realistic in the sense that it provides a successful (and natural) description of the quark and lepton (including neutrino) mass and mixing spectra, including spontaneous CP violation. The low-scale Yukawa structure is dictated by the coupling of matter to $\Delta(27)$ antitriplets $\bar{\phi}$ of which the VEVs are aligned in the Constrained Sequential Dominance 3 (CSD3) directions by a superpotential. Light physical Majorana neutrino masses emerge from a specific implementation of the seesaw mechanism within $SO(10)$. It is fairly complete in the sense that GUT and family symmetry breaking are addressed, including doublet-triplet splitting and the origin of the MSSM μ term.

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We emphasize the predictive nature of the model. Large lepton mixing is accounted for by the seesaw mechanism [8] with CSD [11]. The basic goal of the flavor sector in these models is to couple the SM fermions to flavons $\bar{\phi}_{\text{atm}}$, $\bar{\phi}_{\text{sol}}$, and $\bar{\phi}_{\text{dec}}$, VEVs of which are aligned in the CSD3 direction [12,13],[1] i.e., where

$$\bar{\phi}_{\text{atm}} \sim \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \bar{\phi}_{\text{sol}} \sim \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad \bar{\phi}_{\text{dec}} \sim \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (1.1)$$

We achieve this in a way that is compatible with an $SO(10)$ GUT, i.e., where all fermion states may be united in a **16** of $SO(10)$, and left- and right-handed fermions transform equally under the family symmetry. Since $SO(10)$ constrains the Dirac couplings of all leptons and quarks to be equal (within a family), it is actually rather nontrivial that the successful scheme in the lepton sector will translate to success in the quark sector. Remarkably, we find that we can attain good fits to data for quark and lepton masses, mixings, and phases. This is notably different from our previous work [4] based on $SU(5)$ with CSD3, wherein the three generations of fermions were not all unified into triplets of the family symmetry.

The full literature on flavored SUSY GUTs [9], i.e., which involve a family symmetry, is quite extensive (for an incomplete list, see, e.g., Refs. [15,16]), but there have been relatively few attempts in the literature to combine an $SO(10)$ GUT with a discrete non-Abelian family symmetry [16], and we would argue that none are as successful or complete as the present one. The goal of all these models is clear: to address the questions left unanswered by the SM. It will take some time and (experimental) effort to resolve these models. However, the most promising models are those that make testable predictions while being theoretically complete and consistent.

The layout of the remainder of the paper is as follows. In Sec. II, we present a renormalizable Yukawa superpotential and discuss how it leads to the fermion mass matrices. In Sec. III, we show how the CSD3 alignment is produced in $\Delta(27)$, how the flavon VEVs are driven, and how their relative phases are fixed. In Sec. IV, we show how $SO(10)$ is broken and how we achieve doublet-triplet splitting. In Sec. V, we give a numerical fit of model parameters to the masses and mixing parameters as given by data. Section VI concludes.

II. MODEL BASED ON $\Delta(\mathbf{27}) \times SO(\mathbf{10})$ WITH CSD3

A. Yukawa superpotential and field content

The most important field content is given in Table I. In Table II, we have the messengers with *R*-charge 1, which

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[1]CSD4 models have been discussed in Ref. [14].

TABLE I. Superfields important for quark and lepton Yukawa couplings.

Field	Representation				
	$\Delta(27)$	$SO(10)$	$\mathbb{Z}_9$	$\mathbb{Z}_{12}$	$\mathbb{Z}_4^R$
Ψ	3	16	0	0	1
H_{10}^u	1	10	6	0	0
H_{10}^d	1	10	5	0	0
H_{45}	1	45	0	0	0
H_{45}'	1	45	0	3	0
H_{DW}	1	45	6	0	2
Z	1	1	0	0	2
Z''	1	1	0	6	2
$H_{\overline{16}}$	1	$\overline{16}$	6	0	0
H_{16}	1	16	2	0	2
$\bar{\phi}_{\text{dec}}$	$\bar{3}$	1	6	0	0
$\bar{\phi}_{\text{atm}}$	$\bar{3}$	1	1	0	0
$\bar{\phi}_{\text{sol}}$	$\bar{3}$	1	5	6	0
ξ	1	1	1	0	0

result in the superpotential in Eq. (2.2). Higgs fields are typically denoted by their $SO(10)$ representation, with two **10**s that couple respectively to the up-type and down-type MSSM fields at the low scale. The fields $\bar{\phi}_i$ are flavons that are antitriplets under $\Delta(27)$ and are named in accordance with their respective roles in the CSD3 scheme. The messenger fields are typically indexed by their $\mathbb{Z}_9$ charge, while each prime tick corresponds to an additive $\mathbb{Z}_{12}$ charge of 3.

The MSSM matter content is collected in Ψ, a **16** of $SO(10)$ and a triplet under $\Delta(27)$. The two Higgs doublets arise from H_{10}^u and H_{10}^d, both **10** representations of $SO(10)$, where one only gets a VEV in the $[SU(2)]$ H_u direction and the other in the H_d direction. If we did not have the two H_{10}, we would get the erroneous relation

$$\tan\beta m_{ij}^d = m_{ij}^u, \quad (2.1)$$

which gives no Cabibbo-Kobayashi-Maskawa (CKM) mixing. The $H_{\overline{16}}$ breaks $SO(10) \rightarrow SU(5)$ and gives masses to right-handed neutrinos.

TABLE II. Messengers with unit R charge.

Field	Representation				
	$\Delta(27)$	$SO(10)$	$\mathbb{Z}_9$	$\mathbb{Z}_{12}$	$\mathbb{Z}_4^R$
χ_i	1	16	$i \in \{1, 5, 6, 7\}$	0	1
$\bar{\chi}_i$	1	$\overline{16}$	$i \in \{8, 4, 3, 2\}$	0	1
χ_i''	1	16	$i \in \{5, 6, 7\}$	6	1
$\bar{\chi}_i''$	1	$\overline{16}$	$i \in \{4, 3, 2\}$	6	1
χ_6'	1	16	6	3	1
$\bar{\chi}_3'''$	1	$\overline{16}$	3	9	1
Ω_i	1	1	$i \in \{0, \ldots, 8\}$	0	1
Ω_i''	1	1	$i \in \{3, 4, 5, 6\}$	6	1

The H_{45} obtains a VEV that breaks $SU(5)$ to the Standard Model group, i.e., $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$. It also gives the necessary Clebsch-Gordan coefficients to give the correct masses. Since it has no $\mathbb{Z}$ charge and the messengers should be in the **16**

representation, they can have a renormalizable mass or a mass depending on the VEV of the **45**. This is discussed further in Sec. II B 1.

The Yukawa superpotential that produces the quark and lepton mass matrices is

$$
\begin{aligned}
\mathcal{W}_Y = \Psi_i \Psi_j H_{10}^u \Bigg[& \bar{\phi}_{\text{dec}}^i \bar{\phi}_{\text{dec}}^j \sum_{n=0}^{2} \frac{\lambda_{\text{dec},n}^{(u)}}{\langle H_{45} \rangle^n M_\chi^{2-n}} + \bar{\phi}_{\text{atm}}^i \bar{\phi}_{\text{atm}}^j \xi \sum_{n=0}^{3} \frac{\lambda_{\text{atm},n}^{(u)}}{\langle H_{45} \rangle^n M_\chi^{3-n}} + \bar{\phi}_{\text{sol}}^i \bar{\phi}_{\text{sol}}^j \xi^2 \sum_{n=0}^{4} \frac{\lambda_{\text{sol},n}^{(u)}}{\langle H_{45} \rangle^n M_\chi^{4-n}} \\
& + \bar{\phi}_{\text{sol}}^i \bar{\phi}_{\text{dec}}^j \xi \left(\frac{\lambda_{\text{sd},1}^{(u)}}{\langle H_{45}' \rangle^2 M_\chi} + \frac{\lambda_{\text{sd},2}^{(u)}}{\langle H_{45}' \rangle^2 \langle H_{45} \rangle} \right) \Bigg] \\
+ \Psi_i \Psi_j H_{10}^d \Bigg[& \bar{\phi}_{\text{dec}}^i \bar{\phi}_{\text{dec}}^j \xi \sum_{n=0}^{3} \frac{\lambda_{\text{dec},n}^{(d)}}{\langle H_{45} \rangle^n M_\chi^{3-n}} + \bar{\phi}_{\text{atm}}^i \bar{\phi}_{\text{atm}}^j \xi^2 \sum_{n=0}^{4} \frac{\lambda_{\text{atm},n}^{(d)}}{\langle H_{45} \rangle^n M_\chi^{4-n}} + \bar{\phi}_{\text{sol}}^i \bar{\phi}_{\text{sol}}^j \xi^3 \sum_{n=0}^{5} \frac{\lambda_{\text{sol},n}^{(d)}}{\langle H_{45} \rangle^n M_\chi^{5-n}} \Bigg] \\
+ \Psi_i \Psi_j H_{\overline{16}} H_{\overline{16}} \Bigg[& \bar{\phi}_{\text{dec}}^i \bar{\phi}_{\text{dec}}^j \xi^3 \frac{\lambda_{\text{dec}}^{(M)}}{M_\chi^2 M_{\Omega_{\text{dec}}}^4} + \bar{\phi}_{\text{atm}}^i \bar{\phi}_{\text{atm}}^j \xi^4 \frac{\lambda_{\text{atm}}^{(M)}}{M_\chi^3 M_{\Omega_{\text{atm}}}^4} + \bar{\phi}_{\text{sol}}^i \bar{\phi}_{\text{sol}}^j \xi^5 \frac{\lambda_{\text{sol}}^{(M)}}{M_\chi^4 M_{\Omega_{\text{sol}}}^4} \Bigg],
\end{aligned} \tag{2.2}
$$

where $\lambda_{i,n}^{(f)}$ are constants; $\bar{\phi}_{\text{dec}}$, $\bar{\phi}_{\text{sol}}$, and $\bar{\phi}_{\text{atm}}$ are GUT singlets that are antitriplets under $\Delta(27)$ and acquire VEVs according to the CSD3 alignment shown in Eq. (1.1). The details of this alignment are discussed in Sec. III. The singlet field ξ acquires a VEV slightly below the GUT scale and is primarily responsible for the mass hierarchy between fermions through the Froggatt-Nielsen mechanism [17].

Each term in the above superpotential has an associated scale derived from the VEVs of the messengers that produce it. These are generally different, but for simplicity, we refer to them all as M_χ when they are produced by pairs of $SO(10)$ spinor messengers χ and $\bar{\chi}$. We make a special note of cases in which scale differences have important consequences for the model, in particular, writing $M_{\Omega_{\text{dec}}}$, $M_{\Omega_{\text{atm}}}$, and $M_{\Omega_{\text{sol}}}$ as the combinations of messenger masses that appear in these respective terms. This is discussed further in Sec. II C and Fig. 6.

B. Quarks and charged leptons

1. Diagrams and Clebsch-Gordan coefficients

The diagrams involving messengers that give the Yukawa terms in the up sector [first two lines of Eq. (2.2)] are shown in Fig. 1, while the diagrams for the down sector [which produce the third and fourth lines of Eq. (2.2)] are in Fig. 2. Note that in these and all future diagrams solid lines correspond to fields with odd R charge, while dashed lines signify even R charge.

There are several more diagrams that can be written wherein messenger pairs couple to the H_{45}. Specifically, since the H_{45} has no $\mathbb{Z}$ charge and is a real representation, it may replace a renormalizable mass diagram as in Fig. 3.

The H_{45} acquiring a VEV leads to Clebsch-Gordan relations, and it will be aligned in such a way that it only affects colored particles, as will be discussed later. This is a

GUT-scale VEV, that we will call v_{45}, and breaks $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$. As an example, consider the charged leptons and down quarks. At the low scale, the superpotential resembles

$$
\mathcal{W}_{\text{MSSM}} \sim \bar{d}_L d_R H_d \left(\frac{y_1}{M_\chi^2} + \frac{y_2}{v_{45} M_\chi} + \frac{y_3}{v_{45}^2} \right) + \bar{e}_L e_R H_d \frac{y_1}{M_\chi^2}. \tag{2.3}
$$

We may use the parameters y_i to fit all the masses.[2] As we take v_{45} to be complex (one of two possibilities; see Sec. IV A), the linear combinations of coefficients y_i yield a single effective complex coefficient which is typically different for each generation and different for each of the up, down, charged lepton, and neutrino sectors.

2. Mass matrices

As a consequence of $SO(10)$ unifying the quarks and leptons, all fermion Dirac matrices have the same generic structure. After the flavons acquire VEVs in the CSD3 alignment, the mass matrices are given by

$$
\begin{aligned}
m^f = {}& \mu_a^f \langle \bar{\phi}_{\text{atm}} \rangle^i \langle \bar{\phi}_{\text{atm}} \rangle^j + \mu_s^f \langle \bar{\phi}_{\text{sol}} \rangle^i \langle \bar{\phi}_{\text{sol}} \rangle^j + \mu_d^f \langle \bar{\phi}_{\text{dec}} \rangle^i \langle \bar{\phi}_{\text{dec}} \rangle^j \\
= {}& m_a^f e^{2i\rho_{\text{atm}}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_s^f e^{2i\rho_{\text{sol}}} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix} \\
& + m_d^f e^{2i\rho_{\text{dec}}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{2.4}
$$

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[2] For the third family, we have three y_i, with four for the second family and five for the first family.
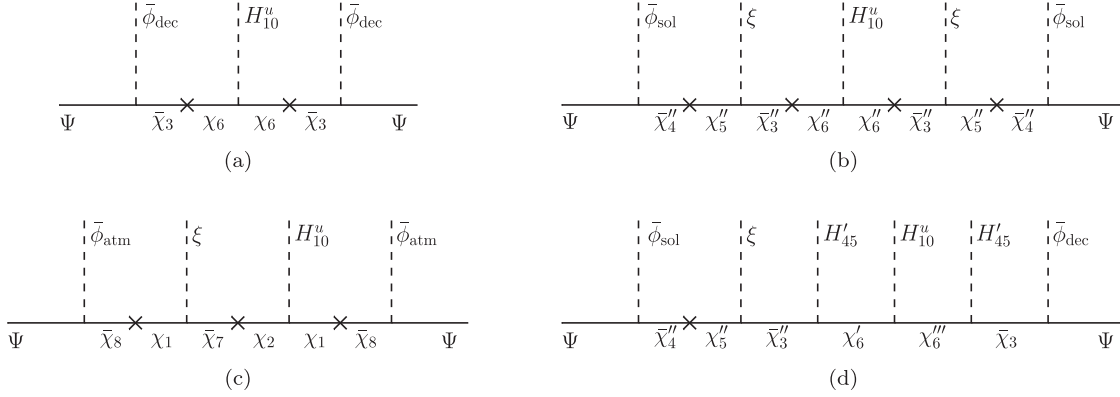
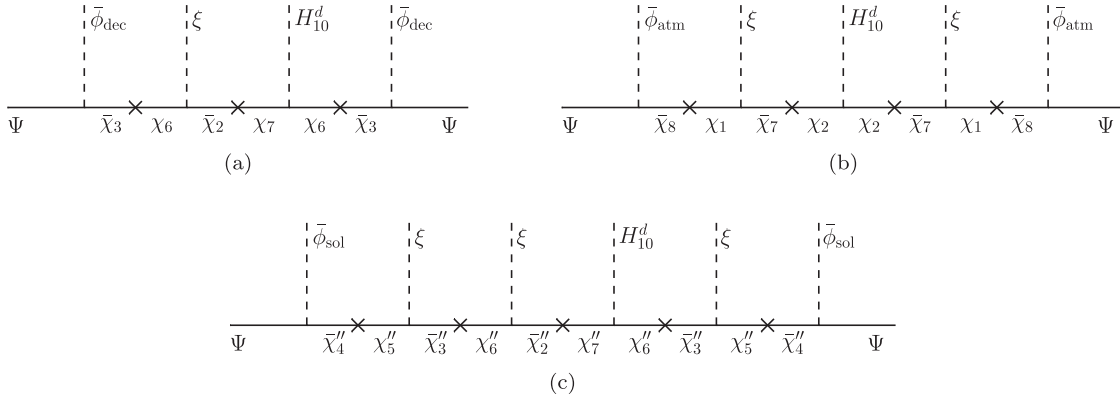FIG. 1.   Diagrams for up-type quark and Dirac neutrino Yukawa terms.

FIG. 2.   Diagrams for down-type and charged-lepton Yukawa terms.

where μ_i^f are coefficients derived from the $H_{10}^{u,d}$, H_{45}, and ξ VEVs and ρ_i are the phases of flavon VEVs. This structure does not include an additional contribution to the up-quark mass matrix, which arises from a term in $\mathcal{W}_Y$ [Eq. (2.2), line 2]. Allowed by the symmetries and messengers, it is proportional to $\bar{\phi}_{sol}\bar{\phi}_{dec}$ and couples to H_{10}^u but not H_{10}^d. This term leads to the additional contribution to the up-quark mass matrix

$$m_{sd}^u e^{i\rho_{sd}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 3 & 2 \end{pmatrix}. \qquad (2.5)$$
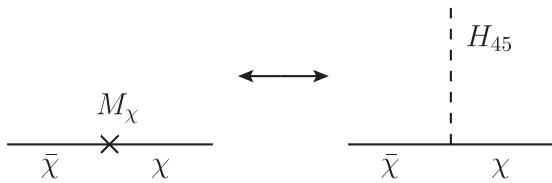

FIG. 3.   The model symmetries allow for any mass insertion M_χ to be replaced by an $H_{45}\bar{\chi}\chi$ vertex, leading to extra superpotential terms.

This mixed term is not allowed for the H_{10}^d due to a lack of messengers able to produce it. In Fig. 4, we see how this mixed term would have had to be built with an H_{10}^d. Since there is no field χ_7''' to build this diagram, it is not allowed. There are no messengers that allow us to build other mixed terms (involving different pairs of flavons); even if there were, they would be highly suppressed. Without the term in Eq. (2.5), the fit to CKM parameters is quite poor, whereas with this term included, a reasonable fit can be made (for more, see Sec. V).

The additional term in Eq. (2.2) does not contribute to down quarks or charged leptons, since it only involves H_{10}^u. Furthermore, because of its structure, it does not contribute to neutrino masses either. To see this, we may decompose the contribution to neutrinos from the fourth diagram in Fig. 1 in $SU(5)$ terms. We adopt the naming convention where the $SU(5)$ representation is labelled by its dimension, with its parent $SO(10)$ field given in parentheses. The left-handed neutrinos are in $\bar{\mathbf{5}}(\Psi)$, and the right-handed neutrinos are the $\mathbf{1}(\Psi)$. The diagram would be in Fig. 5. We see that the subdiagram that is emphasized involves one adjoint and two $SU(5)$ singlets, which is zero, and therefore the whole diagram is zero.
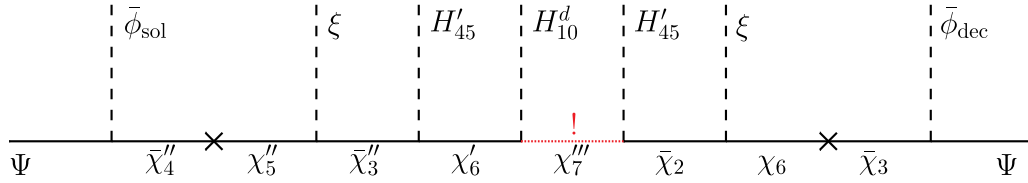
FIG. 4.   Hypothetical diagram that would produce a mixed term involving H_{10}^d, $\bar{\phi}_{\text{sol}}$, and $\bar{\phi}_{\text{dec}}$. Because of an absence of the field χ_7''', this term is forbidden.
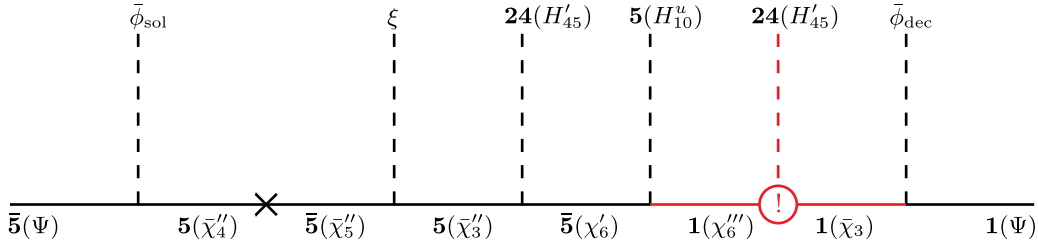

FIG. 5.   Null contribution from the $\bar{\phi}_{\text{sol}}\bar{\phi}_{\text{dec}}$ mixed term to neutrinos.

3. Relative phases of flavons

Since H_{45} acquires a VEV that only affect colored particles, the lepton and neutrino Dirac matrices will not depend on v_{45} (which is generally complex). As such, the only phases contributing to these matrices are ρ_{atm}, ρ_{sol}, and ρ_{dec}, the phases of $\langle\bar{\phi}_{\text{atm}}\rangle$, $\langle\bar{\phi}_{\text{sol}}\rangle$, and $\langle\bar{\phi}_{\text{dec}}\rangle$, respectively, as well as ρ_ξ, the phase of $\langle\xi\rangle$. We define the dominant phase as the phase of the subdominant (second) matrix in the seesaw basis where the dominant matrix is real, i.e.,

$$\eta \equiv -\arg\left[\frac{\langle\bar{\phi}_{\text{sol}}\rangle^2}{\langle\bar{\phi}_{\text{atm}}\rangle^2}\langle\xi\rangle\right] = -2(\rho_{\text{sol}} - \rho_{\text{atm}}) - \rho_\xi. \quad (2.6)$$

Similarly, the subdominant phase is

$$\eta' \equiv -\arg\left[\frac{\langle\bar{\phi}_{\text{dec}}\rangle^2}{\langle\bar{\phi}_{\text{atm}}\rangle^2}\frac{1}{\langle\xi\rangle}\right] = -2(\rho_{\text{dec}} - \rho_{\text{atm}}) + \rho_\xi. \quad (2.7)$$

Each mass matrix derived from the superpotential will have an overall phase dependent on the (generally different) phases of the Higgs VEVs, but these are not physical and may be factored out. We defined the phases in this way because, as we will see shortly, these definitions are the ones that apply for the effective neutrino mass matrix after the seesaw.

This phase structure does not exist in the quark mass matrices, as the factor in front of each submatrix is given as
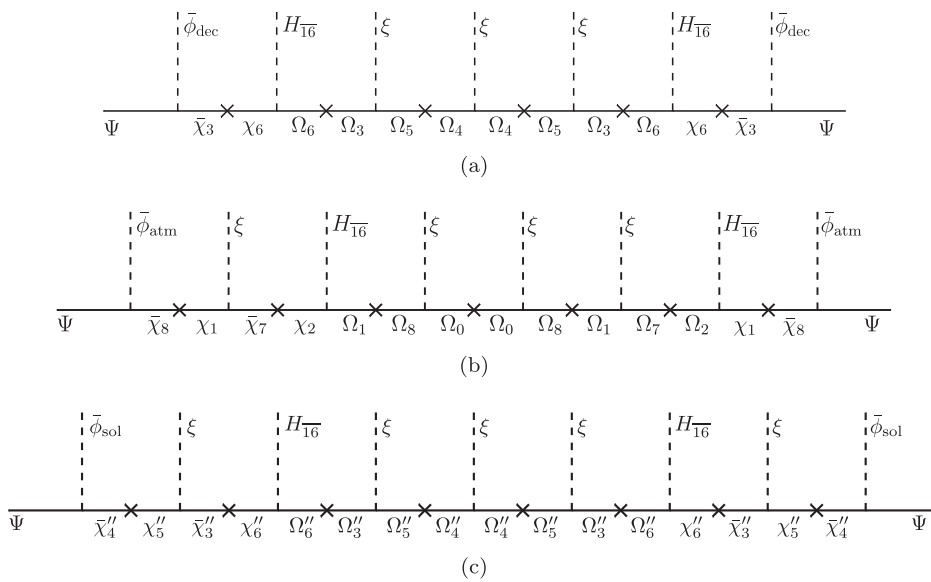

(a)

(b)

(c)

FIG. 6.   Diagrams for right-handed neutrino Majorana masses.

a linear combination of superpotential couplings [see Eq. (2.3)], which in turn depend on v_{45}. As such, the relative phases in the quark sector are arbitrary.

C. Neutrino masses

Finally, the right-handed neutrino Majorana terms [last line of Eq. (2.2)] are produced by the diagrams in Fig. 6. If we decompose these diagrams into $SU(5)$ components, the base line would be all singlets. Therefore, there can be no contribution coming from the H_{45} nor the H'_{45}, and there is no mixed term allowed.

Even though they seem quite suppressed, these terms get the correct order. It is usual for the right-handed neutrino masses to be in the range 10^{10}–10^{14} GeV. The VEV $H_{\overline{16}}$ breaks $SO(10) \rightarrow SU(5)$ and thus is higher than the GUT scale, while the scale M for the messengers is yet higher, such that they may be integrated out. Thus, we have $\xi < \langle H_{\overline{16}} \rangle \lesssim M$, and this way we may obtain the correct scale for right-handed neutrino masses.

It is true, though not immediately obvious, that the mass matrix structure given in Eq. (2.4) is true also for the effective neutrino masses after the seesaw. To show this, consider the neutrino sector after $SO(10) \rightarrow SU(5)$ breaking, where the left- and right-handed neutrinos ν and ν^c are contained, respectively, in a $\bar{\mathbf{5}}$ and $\mathbf{1}$ of $SU(5)$, in triplets of the family symmetry. We denote the $\bar{\mathbf{5}}$ by F and the singlet by N^c. The Dirac mass matrix is then sourced by the terms

$$H_{10}^u \left[\frac{\lambda_{\text{atm}}^{(\nu)}\xi}{M_\chi^3}(\bar{\phi}_{\text{atm}}F)(\bar{\phi}_{\text{atm}}N^c) + \frac{\lambda_{\text{sol}}^{(\nu)}\xi^2}{M_\chi^4}(\bar{\phi}_{\text{sol}}F)(\bar{\phi}_{\text{sol}}N^c) \right.$$
$$\left. + \frac{\lambda_{\text{dec}}^{(\nu)}}{M_\chi^2}(\bar{\phi}_{\text{dec}}F)(\bar{\phi}_{\text{dec}}N^c) \right], \qquad (2.8)$$

when the H_{10}^u, ξ, and $\bar{\phi}$ fields acquire VEVs. Pairs of terms in parentheses, like (ϕF) and (ϕN^c), signify a contraction of a $\Delta(27)$ triplet-antitriplet pair, yielding a flavor singlet. In a similar fashion, the right-handed Majorana matrix originates from the terms

$$\langle H_{\overline{16}}H_{\overline{16}} \rangle \left[\frac{\lambda_{\text{atm}}^{(M)}\xi^4}{M_\chi^3 M_{\Omega_{\text{atm}}}^4}(\bar{\phi}_{\text{atm}}N^c)(\bar{\phi}_{\text{atm}}N^c) \right.$$
$$+ \frac{\lambda_{\text{sol}}^{(M)}\xi^5}{M_\chi^4 M_{\Omega_{\text{sol}}}^4}(\bar{\phi}_{\text{sol}}N^c)(\bar{\phi}_{\text{sol}}N^c)$$
$$\left. + \frac{\lambda_{\text{dec}}^{(M)}\xi^3}{M_\chi^2 M_{\Omega_{\text{dec}}}^4}(\bar{\phi}_{\text{dec}}N^c)(\bar{\phi}_{\text{dec}}N^c) \right], \qquad (2.9)$$

where we have made a distinction between the average scales of the messengers that produce each of the above three terms, giving us three distinct mass scales for the Ω-type messengers, denoted $M_{\Omega_{\text{atm}}}$, $M_{\Omega_{\text{sol}}}$, and $M_{\Omega_{\text{dec}}}$. We will see that the best fit to data suggests that the third effective neutrino mass is small. Implementing the seesaw mechanism, requiring the third right-handed neutrino to be decoupled, means that the last term in Eq. (2.9), leading to a very large third right-handed neutrino mass M_{dec}, can be achieved if $M_{\Omega_{\text{dec}}} < M_{\Omega_{\text{atm}}}, M_{\Omega_{\text{sol}}}$.

We now demonstrate how the seesaw mechanism is implemented in our model.[3]

Collecting the Higgs and ξ fields along with λ coefficients into generic parameters κ (with dimensions of inverse mass), we can write Eqs. (2.8) and (2.9) in the simplified form

$$\kappa_{\text{atm}}^\nu(\bar{\phi}_{\text{atm}}F)(\bar{\phi}_{\text{atm}}N^c) + \kappa_{\text{sol}}^\nu(\bar{\phi}_{\text{sol}}F)(\bar{\phi}_{\text{sol}}N^c) + \kappa_{\text{dec}}^\nu(\bar{\phi}_{\text{dec}}F)(\bar{\phi}_{\text{dec}}N^c)$$
$$+ \kappa_{\text{atm}}^M(\bar{\phi}_{\text{atm}}N^c)(\bar{\phi}_{\text{atm}}N^c) + \kappa_{\text{sol}}^M(\bar{\phi}_{\text{sol}}N^c)(\bar{\phi}_{\text{sol}}N^c) + \kappa_{\text{dec}}^M(\bar{\phi}_{\text{dec}}N^c)(\bar{\phi}_{\text{dec}}N^c), \qquad (2.10)$$

noting also that generically $\kappa^\nu \ll \kappa^M$. This can be written in matrix form as

$$\begin{array}{c} \\ (\bar{\phi}_{\text{atm}}F) \\ (\bar{\phi}_{\text{sol}}F) \\ (\bar{\phi}_{\text{dec}}F) \\ (\bar{\phi}_{\text{atm}}N^c) \\ (\bar{\phi}_{\text{sol}}N^c) \\ (\bar{\phi}_{\text{dec}}N^c) \end{array} \begin{array}{cccccc} (\bar{\phi}_{\text{atm}}F) & (\bar{\phi}_{\text{sol}}F) & (\bar{\phi}_{\text{dec}}F) & (\bar{\phi}_{\text{atm}}N^c) & (\bar{\phi}_{\text{sol}}N^c) & (\bar{\phi}_{\text{dec}}N^c) \\ \left(\begin{array}{cccccc} 0 & 0 & 0 & \kappa_{\text{atm}}^\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa_{\text{sol}}^\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa_{\text{dec}}^\nu \\ \kappa_{\text{atm}}^\nu & 0 & 0 & \kappa_{\text{atm}}^M & 0 & 0 \\ 0 & \kappa_{\text{sol}}^\nu & 0 & 0 & \kappa_{\text{sol}}^M & 0 \\ 0 & 0 & \kappa_{\text{dec}}^\nu & 0 & 0 & \kappa_{\text{dec}}^M \end{array} \right). \end{array} \qquad (2.11)$$

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[3]This is a variation of the mechanism described in Ref. [18].

Diagonalization gives, to $\mathcal{O}((\kappa^\nu/\kappa^M)^2)$, the effective Majorana mass terms

$$-\frac{(\kappa^\nu_{\text{atm}})^2}{\kappa^M_{\text{atm}}}(\bar{\phi}_{\text{atm}}F)(\bar{\phi}_{\text{atm}}F) - \frac{(\kappa^\nu_{\text{sol}})^2}{\kappa^M_{\text{sol}}}(\bar{\phi}_{\text{sol}}F)(\bar{\phi}_{\text{sol}}F)$$

$$-\frac{(\kappa^\nu_{\text{dec}})^2}{\kappa^M_{\text{dec}}}(\bar{\phi}_{\text{dec}}F)(\bar{\phi}_{\text{dec}}F). \qquad (2.12)$$

These in turn reproduce a light neutrino Majorana mass matrix of the form given in Eq. (2.4), when the flavons acquire the CSD3 VEVs.

One final step that is particularly relevant to determining the physical phases is to change to the seesaw basis, as in Ref. [4]. From the neutrino superpotential terms, Eqs. (2.8) and (2.9), we define the neutrino Yukawa and right-handed Majorana matrices λ^ν and M^c,

$$W_\nu = \lambda^\nu H^u_{10} F N^c + M^c N^c N^c, \qquad (2.13)$$

with the structure of λ^ν and M^c arising directly from the flavon VEVs, as in Eq. (2.4). This is the SUSY basis. In the so-called seesaw basis, in a left-right (LR) convention,

the Yukawa and Majorana matrices Y^ν and M_R are instead defined by the Lagrangian

$$\mathcal{L}^{LR} = -H^u_{10} Y^\nu_{ij} \bar{L}^i_L \nu^j_R - \frac{1}{2} M_R \bar{\nu}^c_R \nu_R + \text{H.c.}, \qquad (2.14)$$

where the three families are labeled by $i, j = 1, 2, 3$; L_i are the lepton doublets; and ν^j_R are the right-handed neutrinos below the GUT scale. The light effective Majorana neutrino mass matrix m^ν, defined by

$$\mathcal{L}^{LL}_\nu = -\frac{1}{2} m^\nu \bar{\nu}_L \nu^c_L + \text{H.c.}, \qquad (2.15)$$

is then determined by the seesaw mechanism

$$m^\nu = v^2_u Y^\nu M^{-1}_R Y^{\nu\text{T}}. \qquad (2.16)$$

The matrices in the seesaw basis are obtained by complex conjugation of the matrices in the SUSY basis, i.e.,

$$Y^\nu = (\lambda^\nu)^*, \qquad M_R = (M^c)^*. \qquad (2.17)$$

We proceed in the seesaw basis, wherein

$$Y^\nu = \kappa^{\nu*}_{\text{atm}} v^{*2}_{\text{atm}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \kappa^{\nu*}_{\text{sol}} v^{*2}_{\text{sol}} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix} + \kappa^{\nu*}_{\text{dec}} v^{*2}_{\text{dec}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$M_R = \kappa^{M*}_{\text{atm}} v^{*2}_{\text{atm}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \kappa^{M*}_{\text{sol}} v^{*2}_{\text{sol}} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix} + \kappa^{M*}_{\text{dec}} v^{*2}_{\text{dec}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (2.18)$$

using the effective parameters introduced in Eq. (2.10).

To verify that the relative phases are again η and η' [as defined in Eqs. (2.6) and (2.7)], we may insert VEVs of all fields (denoted v_f for given field f) to give

$$m^\nu = \frac{(v^*_{H^u_{10}})^2}{(v^*_{H_{\bar{16}}})^2} \left[\frac{(\lambda^{(\nu)}_{\text{atm}})^2 M^4_{\Omega_{\text{atm}}}}{\lambda^{(M)}_{\text{atm}} M^3_\chi} \frac{(v^{*2}_{\text{atm}} v^*_\xi)^2}{v^{*2}_{\text{atm}} v^{*4}_\xi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \frac{(\lambda^{(\nu)}_{\text{sol}})^2 M^4_{\Omega_{\text{sol}}}}{\lambda^{(M)}_{\text{sol}} M^4_\chi} \frac{(v^{*2}_{\text{sol}} v^{*2}_\xi)^2}{v^{*2}_{\text{sol}} v^{*5}_\xi} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix} \right.$$

$$\left. + \frac{(\lambda^{(\nu)}_{\text{dec}})^2 M^4_{\Omega_{\text{dec}}}}{\lambda^{(M)}_{\text{dec}} M^2_\chi} \frac{v^{*4}_{\text{dec}}}{v^{*2}_{\text{dec}} v^{*3}_\xi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$$\therefore m^\nu \equiv \mu_a e^{i\alpha} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \mu_b e^{i\beta} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix} + \mu_c e^{i\gamma} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (2.19)$$

where the conjugation of the VEVs is due to changing to the seesaw basis (described above) and we define

$$\mu_a \equiv \left| \frac{(v_{H_{10}^u})^2}{(v_{H_{\overline{16}}})^2} \frac{(\lambda_{\text{atm}}^{(\nu)})^2 M_{\Omega_{\text{atm}}}^4}{\lambda_{\text{atm}}^{(M)} M_\chi^3} \frac{(v_{\text{atm}}^2 v_\xi)^2}{v_{\text{atm}}^2 v_\xi^4} \right|,$$

$$\alpha \equiv -\arg\left[\frac{(v_{H_{10}^u})^2}{(v_{H_{\overline{16}}})^2} \frac{v_{\text{atm}}^2}{v_\xi^2} \right],$$

$$\mu_b \equiv \left| \frac{(v_{H_{10}^u})^2}{(v_{H_{\overline{16}}})^2} \frac{(\lambda_{\text{sol}}^{(\nu)})^2 M_{\Omega_{\text{sol}}}^4}{\lambda_{\text{sol}}^{(M)} M_\chi^4} \frac{(v_{\text{sol}}^2 v_\xi^2)^2}{v_{\text{sol}}^2 v_\xi^5} \right|,$$

$$\beta \equiv -\arg\left[\frac{(v_{H_{10}^u})^2}{(v_{H_{\overline{16}}})^2} \frac{v_{\text{sol}}^2}{v_\xi} \right],$$

$$\mu_c \equiv \left| \frac{(v_{H_{10}^u})^2}{(v_{H_{\overline{16}}})^2} \frac{(\lambda_{\text{dec}}^{(\nu)})^2 M_{\Omega_{\text{dec}}}^4}{\lambda_{\text{dec}}^{(M)} M_\chi^2} \frac{v_{\text{dec}}^4}{v_{\text{dec}}^2 v_\xi^3} \right|,$$

$$\gamma \equiv -\arg\left[\frac{(v_{H_{10}^u})^2}{(v_{H_{\overline{16}}})^2} \frac{v_{\text{dec}}^2}{v_\xi^3} \right], \tag{2.20}$$

where messenger masses and λ couplings are all real due to *CP* conservation. The remarkable fact that the effective left-handed Majorana neutrino mass matrix has the same structure as the neutrino Yukawa matrix and heavy right-handed Majorana neutrino mass matrix can be understood from the argument presented in Eqs. (2.10)–(2.12).

As before, the physical phases η and η' are defined as the relative phases between the dominant (atm) matrix and, respectively, the subdominant (sol) matrix and sub-subdominant (dec) matrix, i.e.,

$$\eta \equiv \beta - \alpha = -\arg\left[\frac{v_{\text{sol}}^2}{v_\xi} \right] + \arg\left[\frac{v_{\text{atm}}^2}{v_\xi^2} \right]$$

$$= -2(\rho_{\text{sol}} - \rho_{\text{atm}}) - \rho_\xi,$$

$$\eta' \equiv \gamma - \alpha = -\arg\left[\frac{v_{\text{dec}}^2}{v_\xi^3} \right] + \arg\left[\frac{v_{\text{atm}}^2}{v_\xi^2} \right]$$

$$= -2(\rho_{\text{dec}} - \rho_{\text{atm}}) + \rho_\xi, \tag{2.21}$$

which is identical to Eqs. (2.6) and (2.7).

D. Renormalizability of the top

The terms in the superpotential in Eq. (2.2) that are primarily responsible for the masses of the third family of fermions are, naively,

$$\Psi_i \Psi_j H_{10}^u \bar{\phi}_{\text{dec}}^i \bar{\phi}_{\text{dec}}^j \sum_{n=0}^{2} \frac{\lambda_{\text{dec},n}^{(u)}}{\langle H_{45} \rangle^n M_\chi^{2-n}}. \tag{2.22}$$

When $\bar{\phi}_{\text{dec}}$ gets a VEV like $(0, 0, v_{\text{dec}})$, with v_{dec} assumed to be near the GUT scale, these terms reduce to

$$v_{\text{dec}}^2 \Psi_3 \Psi_3 H_{10}^u \sum_{n=0}^{2} \frac{\lambda_{\text{dec},n}^{(u)}}{\langle H_{45} \rangle^n M_\chi^{2-n}}. \tag{2.23}$$

In fact, we can only consistently write these nonrenormalizable terms when $\langle \bar{\phi}_{\text{dec}} \rangle \ll M_\chi$, but as we will justify in Sec. III C, we actually have $\langle \bar{\phi}_{\text{dec}} \rangle \approx M_\chi$, so the simple integrating out of the messengers is not possible. We actually need to work out the mixing between the messengers and the field Ψ. This is only necessary for this term since all of the others involve other flavons that have a VEV $\langle \bar{\phi}_{\text{atm,sol}} \rangle \ll \langle \bar{\phi}_{\text{dec}} \rangle \approx M_\chi$ and powers of $\langle \xi \rangle \ll M_\chi$ that allow a consistent integrating out of the messengers.

To prove that this in fact gives us a renormalizable top mass, it is sufficient to examine the first term in the above sum (with $n = 0$). It is sourced by the renormalizable terms

$$\mathcal{W} \sim \Psi \bar{\phi}_{\text{dec}} \bar{\chi}_3 + M_\chi \chi_6 \bar{\chi}_3 + H_{10}^u \chi_6 \chi_6. \tag{2.24}$$

In matrix form, this gives

$$\mathcal{W} \sim (\,\Psi_3 \quad \chi_6 \quad \bar{\chi}_3\,) \begin{pmatrix} 0 & 0 & v_{\text{dec}}/2 \\ 0 & \langle H_{10}^u \rangle & M_\chi/2 \\ v_{\text{dec}}/2 & M_\chi/2 & 0 \end{pmatrix} \begin{pmatrix} \Psi_3 \\ \chi_6 \\ \bar{\chi}_3 \end{pmatrix}. \tag{2.25}$$

Since $\langle H_{10}^u \rangle \ll v_{\text{dec}} \sim M_\chi$, diagonalizing this mass matrix reveals two heavy and one light eigenstates, the latter being at the electroweak scale and which we can associate with the third family, and crucially with the top quark. Supposing $v_{\text{dec}} \approx M_\chi$ (as justified in Sec. III C), the electroweak scale eigenstate is

$$t \approx \frac{1}{\sqrt{2}} (\Psi_3 + \chi_6); \tag{2.26}$$

i.e., the third family up-type fermion, specifically the top quark, is a linear combination of Ψ_3 and χ_6, where the latter has a renormalizable coupling to the Higgs. The other eigenstates have a mass at the GUT scale and are therefore identified as messenger eigenstates.

III. VACUUM ALIGNMENT IN $\Delta(27)$

In this section, we describe the basic properties of the $\Delta(27)$ group and how the CSD3 alignment is produced by *F*-term alignment and orthogonality arguments. We further write down a superpotential which drives the VEVs of the flavons, such that they acquire expectation values at a fixed scale (slightly below the GUT scale), with phases fixed to discrete roots of unity. In particular, the relative phases between $\bar{\phi}_{\text{atm}}$, $\bar{\phi}_{\text{sol}}$, and $\bar{\phi}_{\text{dec}}$ are constrained to discrete choices, which subsequently fixes the physical phases η, η' in the lepton mass matrices to exact values.

A. Group products

The $\Delta(27)$ rules for taking the product of a triplet A and an antitriplet $\bar{B}$ are

$$[A\bar{B}]_{00} \equiv (a_1\bar{b}^1 + a_2\bar{b}^2 + a_3\bar{b}^3)_{00}$$
$$[A\bar{B}]_{01} \equiv (a_1\bar{b}^3 + a_2\bar{b}^1 + a_3\bar{b}^2)_{01}$$
$$[A\bar{B}]_{02} \equiv (a_1\bar{b}^2 + a_2\bar{b}^3 + a_3\bar{b}^1)_{02}$$
$$[A\bar{B}]_{10} \equiv (a_1\bar{b}^1 + \omega^2 a_2\bar{b}^2 + \omega a_3\bar{b}^3)_{10}$$
$$[A\bar{B}]_{11} \equiv (\omega a_1\bar{b}^3 + a_2\bar{b}^1 + \omega^2 a_3\bar{b}^2)_{11}$$
$$[A\bar{B}]_{12} \equiv (\omega^2 a_1\bar{b}^2 + \omega a_2\bar{b}^3 + a_3\bar{b}^1)_{12}$$
$$[A\bar{B}]_{20} \equiv (a_1\bar{b}^1 + \omega a_2\bar{b}^2 + \omega^2 a_3\bar{b}^3)_{20}$$
$$[A\bar{B}]_{21} \equiv (\omega^2 a_1\bar{b}^3 + a_2\bar{b}^1 + \omega a_3\bar{b}^2)_{21}$$
$$[A\bar{B}]_{22} \equiv (\omega a_1\bar{b}^2 + \omega^2 a_2\bar{b}^3 + a_3\bar{b}^1)_{22}, \qquad (3.1)$$

where $\omega \equiv e^{i2\pi/3}$. The product of two triplets or two antitriplets yields, respectively, an antitriplet and a triplet. There are three possible products that can be made in each case, labelled I (identity), S (symmetric), and A (antisymmetric). Defining triplets $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$ and antitriplets $\bar{A} = (\bar{a}^1, \bar{a}^2, \bar{a}^3)$, $\bar{B} = (\bar{b}^1, \bar{b}^2, \bar{b}^3)$, their products are given by

$$[AB]_I \equiv (a_1 b_1, a_2 b_2, a_3 b_3)_{02}$$
$$[\bar{A}\bar{B}]_I \equiv (\bar{a}^1\bar{b}^1, \bar{a}^2\bar{b}^2, \bar{a}^3\bar{b}^3)_{01}$$
$$[AB]_S \equiv (a_2 b_3 + a_3 b_2, a_3 b_1 + a_1 b_3, a_1 b_2 + a_2 b_1)_{02}$$
$$[\bar{A}\bar{B}]_S \equiv (\bar{a}^2\bar{b}^3 + \bar{a}^3\bar{b}^2, \bar{a}^3\bar{b}^1 + \bar{a}^1\bar{b}^3, \bar{a}^1\bar{b}^2 + \bar{a}^2\bar{b}^1)_{01}$$
$$[AB]_A \equiv (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)_{02}$$
$$[\bar{A}\bar{B}]_A \equiv (\bar{a}^2\bar{b}^3 - \bar{a}^3\bar{b}^2, \bar{a}^3\bar{b}^1 - \bar{a}^1\bar{b}^3, \bar{a}^1\bar{b}^2 - \bar{a}^2\bar{b}^1)_{01}.$$
$$(3.2)$$

Note that the bar on antitriplet fields serves merely a reminder of their assignment under $\Delta(27)$.

B. CSD3 directions in $\Delta(27)$

The special directions for $\Delta(27)$ are VEVs with two zeros and VEVs with three equal magnitudes, with phases that are powers of $\omega = e^{i2\pi/3}$. There are three distinct ways to obtain either the $(0,0,1)$ class of VEVs or the $(1,1,1)$ class of VEVs [19]. One of the possibilities that we make use of here uses invariants built out of an antitriplet and triplet, and out of three triplets, of the type

$$c[A\bar{\phi}]_{00} + c_I[A[\phi\phi]_I]_{00} + c_S[A[\phi\phi]_S]_{00}, \qquad (3.3)$$

where $\bar{\phi}$ is an antitriplet unrelated with triplet ϕ and A is itself a triplet, giving rise to three F terms,

$$c\bar{\phi}^1 + c_I\phi_1\phi_1 + 2c_S\phi_2\phi_3 = 0$$
$$c\bar{\phi}^2 + c_I\phi_2\phi_2 + 2c_S\phi_3\phi_1 = 0$$
$$c\bar{\phi}^3 + c_I\phi_3\phi_3 + 2c_S\phi_1\phi_2 = 0. \qquad (3.4)$$

To obtain the VEVs we require in the $(0,0,1)$ and $(1,1,1)$ direction class of VEVs, an economical solution is the superpotential,

$$\mathcal{W}_{V0} = c_a[\phi_0\bar{A}_0]_{00}\sigma_{00}^0 + c_b[\phi_0\bar{A}_0]_{02}\sigma_{01}^0$$
$$+ c_c[A_1\bar{\phi}_1]_{00}M + c_d[A_1\bar{\phi}_1]_{02}\sigma_{01}^1$$
$$+ c_e[A_3\bar{\phi}_3]_{00}M + c_f[A_3[\phi_4\phi_4]_I]_{00} + c_g[A_3[\phi_4\phi_4]_S]_{00}$$
$$+ c_h[\phi_4\bar{A}_4]_{00}M + c_i[[\bar{\phi}_3\bar{\phi}_3]_I\bar{A}_4]_{00} + c_j[[\bar{\phi}_3\bar{\phi}_3]_S\bar{A}_4]_{00}$$
$$+ O_{02}[\phi_2\bar{\phi}_3]_{01} + O_{00}[\phi_2\bar{\phi}_1]_{00}, \qquad (3.5)$$

where the c_x $(x = a, \ldots, j)$ are coefficients that we show explicitly and the coefficients for the other terms are not shown as they are not relevant when taking the respective F term. The triplet flavon ϕ_0 is aligned to $(1, \omega, \omega^2)$ similarly to how the antitriplet flavon $\bar{\phi}_1$ is aligned to $(1,1,1)$, through the alignment antitriplet $\bar{A}_0$ or triplet A_1 and flavon singlets σ_{00}^0, σ_{01}^0 VEVs with a relative phase of ω and σ_{01}^1 taking a real VEV.

The antitriplet flavon $\bar{\phi}_3$ is aligned in a $(0,0,1)$ direction together with triplet ϕ_4. This proceeds from the F terms of the components of A_3 and $\bar{A}_4$, which are of the type shown in Eq. (3.4). Taken together, the six equations only allow a discrete set of solutions where both flavons are aligned in the same direction. One of the solutions has them aligned like $(0, 0, v_3)$ and $(0, 0, v_4)$,[4] with their magnitudes v_3 and v_4 fixed. The relevant VEV magnitudes are

$$v_3^3 = -\frac{c_e c_h^2}{c_f c_i^2}M^3, \qquad v_4^3 = -\frac{c_e^2 c_h}{c_f^2 c_i}M^3,$$
$$\langle\sigma_{01}^1\rangle = -\frac{c_c}{c_d}M. \qquad (3.6)$$

We impose trivial CP symmetry on the fields, including the triplets and antitriplets. This is consistent with the contractions that make invariants with the 1_{0i} set of singlets that we are using. Since the coupling constants c_x are forced to be real by CP conservation, up to minus signs (which can be reabsorbed into the real coefficients), the VEVs $v_{3,4}$ can have a phase only as a third root of unity while $\langle\sigma_{01}^1\rangle$ has to be real. We expect this mass scale M to be around the GUT scale, and with $\mathcal{O}(1)$ c parameters, these VEVs should be at this scale also.

The triplet ϕ_2 is then forced into the $(0, y_2, z_2)$ direction due to the alignment singlet O_{02}, and the alignment singlet O_{00} ensures $y_2 = -z_2$ by orthogonality with $(1,1,1)$.

To have CSD3, we want the directions $(0,1,1)$ and $(1,3,1)$. We can use a chain of orthogonality relations, where in $\Delta(27)$ they must be between the triplet and antitriplet. Using the three directions above, we can arrive relatively easily at $(0,1,1)$, through orthogonality with ϕ_2 and ϕ_4,

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[4]The phenomenologically viable solution is where both flavons are aligned in the $(0,0,1)$ direction; another possibility is that they would both be aligned in the $(1,1,1)$ direction.

TABLE III.   Flavon fields.

	Representation				
Field	$\Delta(27)$	$SO(10)$	$\mathbb{Z}_9$	$\mathbb{Z}_{12}$	$\mathbb{Z}_4^R$
$\bar{\phi}_{\rm dec}$	$\bar{3}$	1	6	0	0
$\bar{\phi}_{\rm atm}$	$\bar{3}$	1	1	0	0
$\bar{\phi}_{\rm sol}$	$\bar{3}$	1	5	6	0
$\bar{\phi}_1$	$\bar{3}$	1	0	4	0
$\bar{\phi}_7$	$\bar{3}$	1	0	5	0
ϕ_0	3	1	2	6	0
ϕ_2	3	1	3	7	0
ϕ_8	3	1	1	8	0
ϕ_4	3	1	3	0	0
ϕ_6	3	1	0	11	0
σ_{00}^0	1_{00}	1	0	1	0
σ_{01}^0	1_{01}	1	0	1	0
σ_{01}^1	1_{01}	1	0	0	0

$$\mathcal{W}_{V1} = O_{00}^1[\phi_2\bar{\phi}_5]_{00} + O_{01}^1[\phi_4\bar{\phi}_5]_{02}. \qquad (3.7)$$

With this, we obtain a $\bar{\phi}_5$ antitriplet in the (0,1,1) direction (note the $[]_{02}$ contraction matches the first component of the antitriplet with the third component of the triplet, putting the zero in the right place in $\bar{\phi}_5$).

To get to (1,3,1), we require a $(2,-1,1)$ direction, which itself requires $(1,1,-1)$. To obtain the latter, we also duplicate the $\bar{\phi}_5$ direction into a triplet ϕ_6, which is a different field and unrelated to $\bar{\phi}_5$ other than the fact that they both have VEVs in the same direction. A way to do both things in one step is

$$\mathcal{W}_{V2} = O_{02}^2[\phi_6\bar{\phi}_3]_{01} + O_{02}'^2[\phi_2\bar{\phi}_7]_{01}$$
$$+ O_{00}^2[\phi_6\bar{\phi}_7]_{00} + O_{01}^2[\phi_6\bar{\phi}_7]_{02}. \qquad (3.8)$$

Starting with the first two orthogonalities, we ensure the zero is in a specific component for ϕ_6 as $(0, y_6, z_6)$ and that $\bar{\phi}_7$ is in the (x_7, x_7, z_7) direction. The other two mutual orthogonalities give $0x_7 + y_6x_7 + z_6z_7 = 0$ and $0x_7 + y_6z_7 + z_6x_7 = 0$, which completes the (0,1,1) and $(1,1,-1)$ alignments. Strictly speaking, this alignment allows both an undesired solution where we get $(0,1,-1)$ with (1,1,1) and the desired solution of (0,1,1) with $(1,1,-1)$.

The next step is obtaining the $(2,-1,1)$ as a triplet. For this, we want to use the (0,1,1) antitriplet direction and the antitriplet with the recently obtained $(1,1,-1)$ direction,

$$\mathcal{W}_{V3} = O_{00}^3[\phi_8\bar{\phi}_7]_{00} + O_{00}'^3[\phi_8\bar{\phi}_5]_{00}. \qquad (3.9)$$

Finally, by orthogonality,

$$\mathcal{W}_{V4} = O_{01}^4[\phi_2\bar{\phi}_9]_{02} + O_{00}^4[\phi_8\bar{\phi}_9]_{00}, \qquad (3.10)$$

TABLE IV.   Alignment field content.

	Representation				
Field	$\Delta(27)$	$SO(10)$	$\mathbb{Z}_9$	$\mathbb{Z}_{12}$	$\mathbb{Z}_4^R$
A_1	3	1	0	8	2
A_3	3	1	3	0	2
$\bar{A}_0$	$\bar{3}$	1	7	5	2
$\bar{A}_4$	$\bar{3}$	1	6	0	2
O_{02}	1_{02}	1	0	5	2
O_{00}	1_{00}	1	6	1	2
O_{00}^1	1_{00}	1	5	5	2
O_{01}^1	1_{01}	1	5	0	2
O_{02}^2	1_{02}	1	3	1	2
$O_{02}'^2$	1_{02}	1	6	0	2
O_{00}^2	1_{00}	1	0	8	2
O_{01}^2	1_{01}	1	0	8	2
O_{00}^3	1_{00}	1	8	11	2
$O_{00}'^3$	1_{00}	1	7	4	2
O_{01}^4	1_{01}	1	1	11	2
O_{00}^4	1_{00}	1	3	10	2

TABLE V.   Field content for driving the flavon VEVs.

	Representation				
Field	$\Delta(27)$	$SO(10)$	$\mathbb{Z}_9$	$\mathbb{Z}_{12}$	$\mathbb{Z}_4^R$
P_1	1_{00}	1	8	1	2
P_2	1_{00}	1	1	6	2
P_3	1_{01}	1	2	0	2
ζ_i	1_{00}	1	$i \in \{0,1,2,3\}$	1	2
$\bar{\zeta}_i$	1_{00}	1	$i \in \{0,6,7,8\}$	11	0
ζ_i'	1_{01}	1	$i \in \{3,4,5\}$	0	2
$\bar{\zeta}_i'$	1_{02}	1	$i \in \{4,5,6\}$	0	0

one obtains the (1,3,1) direction as an antitriplet. We did not need to align a $(1,0,-1)$ direction as the $[\ldots]_{02}$ contraction with the triplet $(0,1,-1)$ (ϕ_2) puts its zero together with the second component of the antitriplet $\bar{\phi}_9$.

Noting now that the VEVs of antitriplets $\bar{\phi}_3$, $\bar{\phi}_5$, and $\bar{\phi}_9$ are the desired directions for $\bar{\phi}_{\rm dec}$, $\bar{\phi}_{\rm atm}$, and $\bar{\phi}_{\rm sol}$, respectively, we now rename these fields to match the notation used in other sections, so $v_3 = v_{\rm dec}$ and

$$\bar{\phi}_3 \equiv \bar{\phi}_{\rm dec}, \qquad \bar{\phi}_5 \equiv \bar{\phi}_{\rm atm}, \qquad \bar{\phi}_9 \equiv \bar{\phi}_{\rm sol}. \qquad (3.11)$$

This notation is also used in Table III, which together with Table IV summarizes the field content and the representation of each field under the symmetries. For the sake of completeness, we collect all alignment terms into one superpotential,

$$\mathcal{W}_V = \mathcal{W}_{V0} + \mathcal{W}_{V1} + \mathcal{W}_{V2} + \mathcal{W}_{V3} + \mathcal{W}_{V4}, \qquad (3.12)$$

such that, omitting the coefficients, we have

$$\mathcal{W}_V \sim [\phi_0 \bar{A}_0]_{00} \sigma_{00}^0 + [\phi_0 \bar{A}_0]_{02} \sigma_{01}^0 + [A_1 \bar{\phi}_1]_{00} M + [A_1 \bar{\phi}_1]_{02} \sigma_{01}^1 + [A_3 \bar{\phi}_3]_{00} M + [A_3 [\phi_4 \phi_4]_I]_{00} + [A_3 [\phi_4 \phi_4]_S]_{00}$$
$$+ [\phi_4 \bar{A}_4]_{00} M + [[\bar{\phi}_3 \bar{\phi}_3]_I \bar{A}_4]_{00} + [[\bar{\phi}_3 \bar{\phi}_3]_S \bar{A}_4]_{00} + O_{02}[\phi_2 \bar{\phi}_3]_{01} + O_{00}[\phi_2 \bar{\phi}_1]_{00} + O_{00}^1 [\phi_2 \bar{\phi}_{\text{atm}}]_{00} + O_{01}^1 [\phi_4 \bar{\phi}_{\text{atm}}]_{02}$$
$$+ O_{02}^2 [\phi_6 \bar{\phi}_{\text{dec}}]_{01} + O_{02}'^2 [\phi_2 \bar{\phi}_7]_{01} + O_{00}^2 [\phi_6 \bar{\phi}_7]_{00} + O_{01}^2 [\phi_6 \bar{\phi}_7]_{02} + O_{00}^3 [\phi_8 \bar{\phi}_7]_{00} + O_{00}'^3 [\phi_8 \bar{\phi}_{\text{atm}}]_{00}$$
$$+ O_{01}^4 [\phi_2 \bar{\phi}_{\text{sol}}]_{02} + O_{00}^4 [\phi_8 \bar{\phi}_{\text{sol}}]_{00}. \tag{3.13}$$

We summarize the alignments produced by the above superpotential as follows:

$$\langle \phi \rangle_0 \propto (1, \omega, \omega^2), \qquad \langle \bar{\phi}_1 \rangle \propto (1, 1, 1),$$
$$\langle \phi_2 \rangle \propto (0, 1, -1), \qquad \langle \bar{\phi}_{\text{dec}} \rangle \propto (0, 0, 1),$$
$$\langle \phi_4 \rangle \propto (0, 0, 1), \qquad \langle \bar{\phi}_{\text{atm}} \rangle \propto (0, 1, 1),$$
$$\langle \phi_6 \rangle \propto (0, 1, 1), \qquad \langle \bar{\phi}_7 \rangle \propto (1, 1, -1),$$
$$\langle \phi_8 \rangle \propto (2, -1, 1), \qquad \langle \bar{\phi}_{\text{sol}} \rangle \propto (1, 3, 1). \tag{3.14}$$

C. Driving flavon VEVs and phases

To drive the flavon VEVs, we introduce a set of fields given in Table V. They are GUT singlets with nontrivial representations under $\Delta(27)$ and the $\mathbb{Z}$ symmetries and couple to the flavon fields.

To obtain the necessary superpotential, we need to add more messengers $\zeta, \bar{\zeta}$, with a characteristic mass M_ζ, also listed in Table V. The superpotential which drives the flavons is

$$\mathcal{W}_\phi = P_1 \left[\kappa_1 \left(\frac{\xi}{M_\zeta} \right)^4 \bar{\phi}_{\text{dec}} \phi_6 - \kappa_2 \bar{\phi}_{\text{atm}} \phi_6 \right]$$
$$+ P_2 [\kappa_3 \bar{\phi}_{\text{sol}} \phi_4 - \kappa_4 \bar{\phi}_{\text{dec}} \phi_0]$$
$$+ P_3 \left[\kappa_5 \bar{\phi}_{\text{sol}} \phi_0 - \kappa_6 \left(\frac{\xi}{M_\zeta} \right)^3 \bar{\phi}_{\text{atm}} \phi_4 \right], \tag{3.15}$$

where κ_i are real dimensionless constants. As discussed in Sec. V, to acquire a good fit to the data without tuning, we need to assume that $\langle \xi \rangle \lesssim M_\zeta$. The F-term equations for the P fields give relationships between the VEVs of the flavons that couple to the SM fields. The (nontrivial) representations of the P fields under $\Delta(27)$ are chosen specifically so that the pairs of flavon VEVs they are multiplied by do not give zero when they acquire VEVs.

The constants κ_i are forced to be real by CP conservation, but the VEV $\langle \phi_0 \rangle$ has complex components that introduce phases to the other VEVs. Specifically, the terms multiplied by the constants $\kappa_{4,5}$ obtain the factors

$$[\langle \bar{\phi}_{\text{sol}} \rangle \langle \phi_0 \rangle]_{02} = 2 v_{\text{sol}} v_0, \qquad [\langle \bar{\phi}_{\text{dec}} \rangle \langle \phi_0 \rangle]_{00} = \omega^2 v_{\text{dec}} v_0 \tag{3.16}$$

when contracting the $\Delta(27)$ triplets, so we may effectively treat as κ_4 in Eq. (3.15) carrying a factor of ω^2.

We proceed to solve the F-term equations coming from the P fields, yielding VEVs for the important flavons $\bar{\phi}_{\text{sol}}$ and $\bar{\phi}_{\text{atm}}$, while $\langle \bar{\phi}_{\text{dec}} \rangle$ is given in Eq. (3.6) (recall that $v_3 \equiv v_{\text{dec}}$). It is useful to note the relation $v_4 = c_j v_{\text{dec}}^2/(c_h M)$, which can be seen from comparing the VEVs in Eq. (3.6). We obtain

$$v_{\text{sol}}^2 = \omega^2 \frac{\kappa_4 \kappa_6 c_h}{2 \kappa_3 \kappa_5 c_j} \left(\frac{\xi}{M_\zeta} \right)^7 v_{\text{dec}}^2, \qquad v_{\text{atm}}^2 = \frac{\kappa_1^2}{4 \kappa_2^2} \left(\frac{\xi}{M_\zeta} \right)^8 v_{\text{dec}}^2, \tag{3.17}$$

where, since $\langle \xi \rangle / M_\zeta < 1$, we conclude that $v_{\text{dec}} \gg v_{\text{atm}} \sim v_{\text{sol}}$. Given these VEVs, the physical phases defined in Eqs. (2.6) and (2.7) are given by

$$\eta = -\arg \left[\frac{v_{\text{sol}}^2}{v_{\text{atm}}^2} \langle \xi \rangle \right] = -\arg[\omega^2],$$
$$\eta' = -\arg \left[\frac{v_{\text{dec}}^2}{v_{\text{atm}}^2} \frac{1}{\langle \xi \rangle} \right] = 9 \arg[\langle \xi \rangle], \tag{3.18}$$

where the real coupling constants c_x, κ_i do not contribute to phases. These phases are in fact completely fixed. As will be shown in Eq. (4.2), the phase of $\langle \xi \rangle$ is a ninth root of unity; by the cancellation of this phase, we finally have

$$\eta = \frac{2\pi}{3}, \qquad \eta' = 0. \tag{3.19}$$

Strictly speaking, these phases are fixed only up to a relative phase π, depending on the signs of the real constants. However, this additional phase is unphysical, as it may always be subsumed into other real parameters at the low scale.

VEVs of other flavons that are not set by the superpotential terms described above are driven through radiative breaking [20].

IV. GUT BREAKING

In this section, we detail how the $SO(10)$ GUT is broken down to the MSSM via $SU(5)$, how doublet-triplet splitting is achieved, and how only two light Higgs doublets are present below the GUT scale, as in the MSSM. We note that, due to $\Delta(27) \times SO(10)$, at the GUT scale, there is a

small number of relevant soft parameters: m_0, the universal soft mass for the $\Psi \sim (3, 16)$; nonuniversal soft Higgs masses m_{H^u} and m_{H^d}; and the universal gaugino mass $M_{1/2}$.

A. Breaking potential and diagrams

The superpotential that breaks $SO(10)$ is given by

$$\mathcal{W}_{\mathrm{GUT}} = M^2 Z + \lambda_1 Z^3 + \lambda_2 Z Z''^2 + \lambda_3 Z'' H_{45}'^2 + \lambda_4 Z \frac{H_{45}'^4}{M_{\Upsilon}^2}$$

$$+ \frac{Z}{M_{\Sigma}} \left(\lambda_5 H_{16} H_{16} H_{10}^d + \lambda_6 \frac{\xi^8}{M_{\Sigma}^8} H_{16} H_{16} H_{10}^u + \lambda_7 H_{\overline{16}} H_{\overline{16}} H_{10}^u + \lambda_8 \frac{\xi}{M_{\Sigma}} H_{\overline{16}} H_{\overline{16}} H_{10}^d \right) + \lambda_9 Z H_{\mathrm{DW}}^2 \frac{\xi^6}{M_Z^6}$$

$$+ H_{\mathrm{DW}} \frac{\xi^3}{M_Z^2} \left(\lambda_{10} H_{45} + \lambda_{11} \frac{H_{45}^3}{M_{\Upsilon}^2} \right) + H_{16} H_{\overline{16}} \left(\lambda_{12} \xi + \frac{\lambda_{13}}{M_Z} \bar{\phi}_1 \phi_8 \right)$$

$$+ Z \left(\lambda_{14} \frac{\xi^6}{M_Z^6} \bar{\phi}_7 \phi_2 + \lambda_{15} \frac{\xi^8}{M_Z^8} \bar{\phi}_1 \phi_8 + \lambda_{16} \frac{\xi^5}{M_Z^5} \bar{\phi}_{\mathrm{sol}} \phi_4 + \lambda_{17} \frac{\xi^2}{M_Z^2} \bar{\phi}_{\mathrm{sol}} \phi_0 + \lambda_{18} \bar{\phi}_{\mathrm{dec}} \phi_4 \right). \quad (4.1)$$

The renormalizable diagrams that give rise to this superpotential are given[5] in Fig. 7 (giving lines 1 and 3) and Fig. 8 (giving line 2), and the corresponding messenger fields (Σ, Υ, and Z_i) are detailed in Table VI. Most fields are familiar from the Yukawa sector discussed previously, while the field H_{DW} is an $SO(10)$ adjoint that governs doublet-triplet splitting, as we will see shortly. Requiring that every field's F term vanishes yields a set of equations that fixes the VEVs of the above fields.

The first line contains terms involving different powers of Z, Z'', and H_{45}', which ensures that their corresponding F-term conditions fix all VEVs to be nonzero. The exact expressions for the VEVs are complicated and thus are not shown, since they are not enlightening.

The second line has terms involving the fields $H_{10}^{u,d}$ that will be discussed carefully in the next section on doublet-triplet splitting. At this level, the fields $H_{10}^{u,d}$ have a zero VEV, so any term involving two of them does not contribute to the F-term equations. The F-term conditions coming from $H_{10}^{u,d}$ themselves relate the $H_{16,\overline{16}}$ VEVs and also fix the VEV of ξ to be

$$\langle \xi \rangle = \left(\frac{\lambda_5 \lambda_7}{\lambda_8 \lambda_6} \right)^{1/9} M_{\Sigma}, \quad (4.2)$$

which subsequently fixes the phase of $\langle \xi \rangle$ to be one of the ninth roots of unity.

At this stage, it is relevant to consider superfields H_{DW} and Υ_6, which have the same quantum numbers. In terms of superfields Υ_6^a, Υ_6^b, the mass term for the messenger pair reads $M_{\Upsilon}(c_a \Upsilon_6^a + c_b \Upsilon_6^b) \bar{\Upsilon}_3$. We define $\Upsilon_6 \equiv (c_a \Upsilon_6^a + c_b \Upsilon_6^b)$ and H_{DW} as the orthogonal combination. The F term

with respect to $\bar{\Upsilon}_3$ forces Υ_6 to have a zero VEV, meaning it will not contribute elsewhere and justifies identifying it as half of the messenger pair. Therefore, the third line contains different powers of H_{DW} and H_{45} and gives them VEVs.

The model actually allows an infinity of terms involving H_{45}, each with a higher power of this field. We keep only the first two terms since they are enough to give the H_{45} a general VEV, whereas adding the other terms will make its VEV look more complicated but will not affect the physics. Its own F-term equation fixes its VEV to be

$$v_{45} = \sqrt{-\frac{\lambda_{10}}{\lambda_{11}}} M_{\Upsilon}, \quad (4.3)$$

which must define the GUT scale, while we may choose the signs of $\lambda_{10,11}$ so that it is imaginary (this is the phenomenologically favored solution). The F term for ξ will fix the VEV of $H_{16,\overline{16}}$. The F terms coming from $H_{16,\overline{16}}$ will drive the VEVs of the flavons $\bar{\phi}_7$ and ϕ_2 (seen on line 2).

The last line, allowed by the symmetries and messengers, only adds terms to the F terms for Z and ξ, relating their VEVs to the flavon ones. The flavon F terms will fix some of the O field VEVs.

The VEVs $\langle H_{16,\overline{16}} \rangle$ specifically break $SO(10) \to SU(5)$. The VEVs $\langle H_{45}, H_{45}', H_{\mathrm{DW}} \rangle$ specifically break $SU(5) \to SM$. The VEV $\langle \xi \rangle$ completely breaks $\mathbb{Z}_9$. Finally, the VEVs $\langle Z, Z'' \rangle$, carrying two units of charge under $\mathbb{Z}_4^R$, break it into the usual $\mathbb{Z}_2^R$ R parity at the GUT scale.

B. Obtaining two light Higgs doublets

In this section, we show how the $SU(2)$ doublets and $SU(3)$ triplet Higgses contained within the $H_{16,\overline{16}}$ and $H_{10}^{u,d}$ acquire masses, as dictated by the model. They do so in such a way that all triplets are heavy, while only two light Higgs doublets remain at low scales, which we may associate with the MSSM Higgs doublets. This is important because any light colored Higgs states would lead to very rapid proton decay, so we need a justification for why

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[5]In Eq. (4.1), we include all the terms allowed by the symmetries and messenger fields. We omit those diagrams with seven or eight powers of ξ, as they are constructed in a similar way using the same messengers but are not particularly illuminating. Similarly, we omit the diagrams that produce the final line of Eq. (4.1), as these terms, while allowed, have no significant impact on physics.
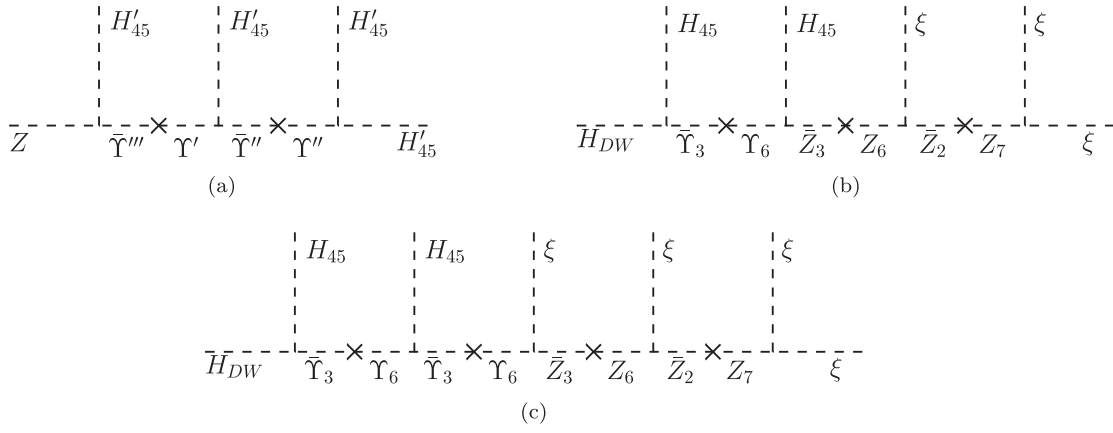
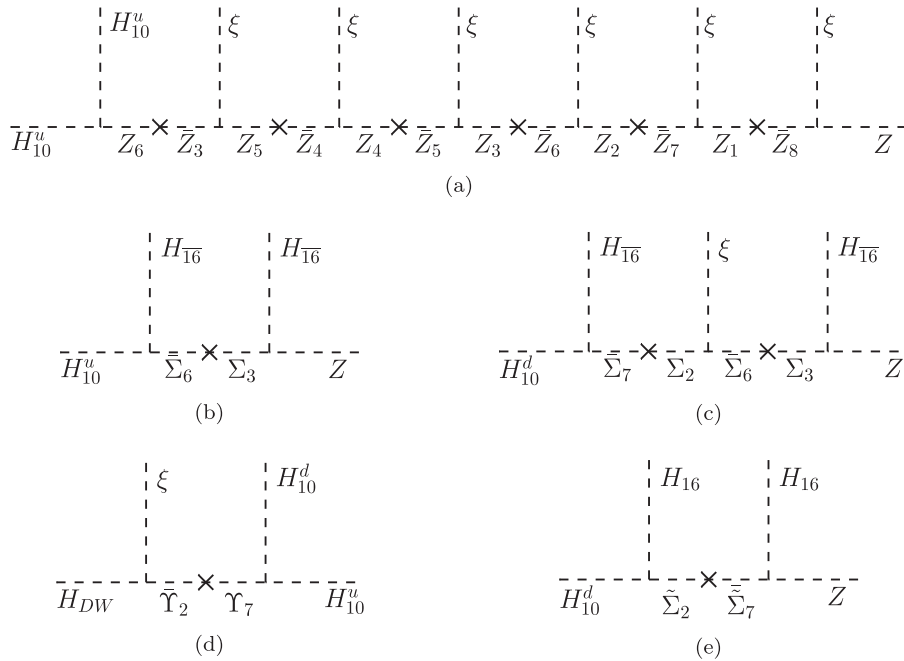FIG. 7. Diagrams that give rise to GUT breaking terms.

FIG. 8. Diagrams that give rise to doublet-triplet splitting.

certain doublet states remain light, while triplets remain heavy, a problem known as doublet-triplet splitting. We solve this á la Dimopolous-Wilczek [7]. In $SO(10)$, there is a further complication, where each H_{10} has two $SU(2)$ doublet states within it, and we have two additional doublets from the $H_{16,\overline{16}}$. Only two of these six doublets should be light, so as to reduce to the MSSM. We also show how we achieve this.

1. Doublet-doublet splitting

The superpotential that gives masses to doublets is

$$\mathcal{W}_\mu \sim Z H_{10}^u H_{10}^u \frac{\xi^6}{M_Z^6} + Z H_{10}^u H_{10}^d \frac{\xi^7}{M_Z^7} + Z H_{10}^d H_{10}^d \frac{\xi^8}{M_Z^8} + \xi H_{16} H_{\overline{16}}$$

$$+ \frac{Z}{M_\Sigma} \left(H_{16} H_{16} H_{10}^d + \frac{\xi^8}{M_\Sigma^8} H_{16} H_{16} H_{10}^u + H_{\overline{16}} H_{\overline{16}} H_{10}^u + \frac{\xi}{M_\Sigma} H_{\overline{16}} H_{\overline{16}} H_{10}^d \right), \tag{4.4}$$

where the corresponding diagrams that produce the nonrenormalizable terms are given in Fig. 8. The second line is a reproduction of the second line in Eq. (4.1), while the first line includes those terms involving more than one insertion of $H_{10}^{u,d}$, which had previously been omitted.

The fields $H_{16,\overline{16}}$ contain doublets that mix with the ones in $H_{10}^{u,d}$ and will also contribute to masses, since they both get VEVs above the GUT scale. Defining $\tilde{H}_{16,\overline{16}} = \langle H_{16,\overline{16}}\rangle / M_{\Sigma}$, we assume it to be reasonably close to 1. We similarly define $\tilde{\xi} \sim \langle \xi \rangle / M_Z \sim \langle \xi \rangle / M_{\Sigma}$, where M_Z and M_{Σ} are the typical scales of the messengers that produce Eq. (4.4). We will find that they are necessarily different from the scale M_{ζ} that governs the flavon driving potential.

To make the connection to the two MSSM Higgs doublets, typically denoted H_u and H_d in reference to their behavior under $SU(2)$, we call the H_u-like doublets inside the H_{10}^u, H_{10}^d, and $H_{\overline{16}}$ fields, respectively, H_u^u, H_u^d, and $H_u^{\overline{16}}$. The H_d-like doublets are named similarly, replacing the subindex $u \to d$. In other words,

$$\mathbf{2}_u(H_{10}^u) \equiv H_u^u, \qquad \mathbf{2}_d(H_{10}^u) \equiv H_d^u,$$
$$\mathbf{2}_u(H_{10}^d) \equiv H_u^d, \qquad \mathbf{2}_d(H_{10}^d) \equiv H_d^d,$$
$$\mathbf{2}_u(H_{\overline{16}}) \equiv H_u^{\overline{16}}, \qquad \mathbf{2}_d(H_{\overline{16}}) \equiv H_d^{\overline{16}}. \tag{4.5}$$

The doublet mass matrix can then be written in matrix form as

$$M_D \sim \begin{array}{c} \\ H_d^u \\ H_d^d \\ H_d^{\overline{16}} \end{array} \begin{array}{ccc} H_u^u & H_u^d & H_u^{\overline{16}} \end{array} \atop \left(\begin{array}{ccc} \tilde{\xi}^6 & \tilde{\xi}^7 & \tilde{H}_{\overline{16}} \\ \tilde{\xi}^7 & \tilde{\xi}^8 & \tilde{\xi}\tilde{H}_{\overline{16}} \\ \tilde{H}_{16}\tilde{\xi}^8 & \tilde{H}_{16} & \xi/\langle Z \rangle \end{array} \right) \langle Z \rangle. \tag{4.6}$$

Its eigenvalues m_D are

$$m_D \sim \tilde{\xi}\langle Z \rangle, \qquad \tilde{\xi}\langle Z \rangle, \qquad \tilde{\xi}^8\langle Z \rangle. \tag{4.7}$$

Two doublets receive large masses, which we assume are slightly larger than M_{GUT} such that they do not upset gauge coupling unification. The remaining eigenvalue is suppressed by a factor $\tilde{\xi}^8$. We can choose the mass of the Z and Σ messengers so that $\tilde{\xi} \sim 0.03$, i.e., $\tilde{\xi}^8\langle Z \rangle \sim 1$ TeV. This generates the MSSM μ term $\tilde{\xi}^8\langle Z \rangle H_u^u H_d^d$ at the correct scale, where we make the connection to MSSM Higgs doublets by

$$H_u \sim H_u^u, \qquad H_d \sim H_d^d. \tag{4.8}$$

2. Doublet-triplet splitting

The Dimopoulos-Wilczek mechanism [7] is based on having an $SO(10)$ **45**, which we call H_{DW}, that obtains a VEV with the structure

$$\langle H_{\text{DW}} \rangle = \begin{pmatrix} 0 & \langle H_{U(5)} \rangle \\ -\langle H_{U(5)} \rangle & 0 \end{pmatrix}, \tag{4.9}$$

TABLE VI. Messenger superfields required for the doublet and triplet mass terms. Note that the model also includes Z_0 (but not $\bar{Z}_0$), which we considered earlier as Z, and that Υ_6 has the same quantum number as H_{DW} (see Table I).

			Representation		
Field	$\Delta(27)$	$SO(10)$	$\mathbb{Z}_9$	$\mathbb{Z}_{12}$	$\mathbb{Z}_4^R$
Z_i	1	1	$i \in \{1, ..., 8\}$	0	2
$\bar{Z}_i$	1	1	$i \in \{1, ..., 8\}$	0	0
$\bar{\Sigma}_{6,7}$	1	$\overline{16}$	6,7	0	2
$\Sigma_{3,2}$	1	16	3,2	0	0
$\tilde{\Sigma}_i$	1	16	$i \in \{0, ..., 8\}$	0	2
$\tilde{\bar{\Sigma}}_i$	1	$\overline{16}$	$i \in \{0, ..., 8\}$	0	0
$\bar{\Upsilon}_{3,2}$	1	45	3,2	0	0
$\Upsilon_{6,7}$	1	45	6,7	0	2
$\bar{\Upsilon}'''$	1	45	0	9	0
Υ'	1	45	0	3	2
$\bar{\Upsilon}''$	1	45	0	6	0
Υ''	1	45	0	6	2

which is traceless regardless of the structure of $\langle H_{U(5)} \rangle$. We can actually choose $\langle H_{U(5)} \rangle = v_{45}\text{diag}(1, 1, 1, 0, 0)$ such that it contributes only to the mass of the triplets [7]. This VEV alignment is not possible in an $SU(5)$ adjoint representation but is possible in the $SO(10)$ one. The field H_{DW} has an R charge of 2 and a $\mathbb{Z}_9$ charge of 6, allowing us to write the term

$$\mathcal{W}_{DT} \sim H_{\text{DW}} H_{10}^u H_{10}^d \frac{\xi}{M_{\Upsilon}}, \tag{4.10}$$

where, due to the antisymmetry of $\langle H_{\text{DW}} \rangle$, only the mixed term is possible. The renormalizable diagram that produces this term is given in Fig. 8.

In analogy to Eq. (4.5), we define Higgs triplets T arising from H_{10}^u, H_{10}^d, and $H_{\overline{16}}$ by

$$\mathbf{3}_u(H_{10}^u) \equiv T_u^u, \qquad \mathbf{3}_d(H_{10}^u) \equiv T_d^u,$$
$$\mathbf{3}_u(H_{10}^d) \equiv T_u^d, \qquad \mathbf{3}_d(H_{10}^d) \equiv T_d^d,$$
$$\mathbf{3}_u(H_{\overline{16}}) \equiv T_u^{\overline{16}}, \qquad \mathbf{3}_d(H_{\overline{16}}) \equiv T_d^{\overline{16}}. \tag{4.11}$$

The terms involving these triplets arising from the superpotential in Eq. (4.4) produces the mass matrix

$$M_T \sim \begin{array}{c} \\ T_d^u \\ T_d^d \\ T_d^{\overline{16}} \end{array} \begin{array}{ccc} T_u^u & T_u^d & T_u^{\overline{16}} \end{array} \atop \left(\begin{array}{ccc} \tilde{\xi}^6 & \tilde{\xi}\langle H_{\text{DW}} \rangle / \langle Z \rangle & \tilde{H}_{\overline{16}} \\ \tilde{\xi}\langle H_{\text{DW}} \rangle / \langle Z \rangle & \tilde{\xi}^8 & \tilde{\xi}\tilde{H}_{\overline{16}} \\ \tilde{H}_{16}\tilde{\xi}^8 & \tilde{H}_{16} & \xi/\langle Z \rangle \end{array} \right) \langle Z \rangle, \tag{4.12}$$

where the only structural difference between this and Eq. (4.6) is in the (12) and (21) entries, which arise from

Eq. (4.10). All the eigenvalues of this matrix are at the scale $\tilde{\xi}\langle Z\rangle \lesssim M_{\mathrm{GUT}}$, i.e., there are no light triplet eigenstates, which gives doublet-triplet splitting.

Finally, we remark that the mechanisms shown above will fix the mass scale of most fields. The messenger fields have renormalizable masses that we expect to be all above the GUT scale, and we have shown how most of the remaining fields receive GUT scale masses when the GUT breaks. The MSSM fields are the only $SO(10)$-charged fields that remain light, while a few flavon fields receive masses below the GUT scale radiatively. Collectively, this ensures that gauge coupling unification is as good as in the MSSM.

C. Proton decay

A classic problem in GUT theories, including $SO(10)$, is the prediction of excessively fast proton decay. The most dangerous processes come from the "dimension 5" operators $\Psi\Psi\Psi\Psi$ (for a discussion of dimension 6 operators, we refer the reader to Ref. [3]). These dimension 5 operators are forbidden by the symmetries of the model, but related higher-order operators of the form

$$\Psi\Psi\Psi\Psi\frac{Z\bar{\phi}}{M^3}\left(\frac{\xi}{M}\right)^n \tag{4.13}$$

are allowed, where n is some positive integer. Since we are working with the renormalizable theory, in order for this type of effective term to be present with $M \sim M_{\mathrm{GUT}}$, there would have to be GUT scale messengers allowing them. Specifically, to produce the above term, we would need messengers that are $\Delta(27)$ triplets, which are completely absent from our model. Hence, such terms cannot be produced at (or below) the GUT scale.

Such operators may, however, arise with Planck scale suppression, i.e., $M \sim M_P$. Specifically, the lowest-order term arising from fields that acquires nonvanishing VEVs and therefore contributes to proton decay in our model is

$$\Psi\Psi\Psi\Psi\frac{Z\bar{\phi}_{\mathrm{dec}}\xi^3}{M_P^6}, \tag{4.14}$$

which would generate dangerous proton decay terms of the type

$$gQQQL\frac{\langle X\rangle}{M_P^2}, \tag{4.15}$$

where g is a dimensionless coupling and $\langle X\rangle$ is a generic VEV of a field, as discussed in Ref. [21]. These terms must be suppressed enough to generate a proton lifetime $\tau_p > 10^{32}$ yrs, which is achieved when

$$g\langle X\rangle < 3 \times 10^9 \text{ GeV}. \tag{4.16}$$

In our model,

$$\langle X\rangle = \frac{\langle Z\rangle v_{\mathrm{dec}}\langle \xi\rangle^3}{M_P^4} \sim 150 \text{ GeV}, \tag{4.17}$$

such that with $\mathcal{O}(1)$ dimensionless couplings it predicts very suppressed proton decay. This suppression means proton decay rates are several orders of magnitude below the current experimental bounds. This is to be compared to a model such as Ref. [22], where observation would be possible with an improvement of experimental sensitivity of an order of magnitude, making threshold corrections extremely relevant in order to make an accurate prediction of the amplitudes. In our case, a robust order of magnitude estimate is possible even though a treatment of threshold corrections is beyond the scope of the current paper—the conclusion remains that proton decay is well beyond experimental sensitivity in our model.

V. NUMERICAL FIT

A. Low-scale mass matrices

At the low scale, the VEVs of flavons and messenger fields combine to give the mass matrices

$$m^u = v^u \left[y_1^u \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + y_2^u e^{i\eta_2^u} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix} \right.$$
$$\left. + y_3^u e^{i\eta_3^u} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + y_4^u e^{i\eta_4^u} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right] \tag{5.1a}$$

$$m^d = v^d \left[y_1^d \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + y_2^d e^{i\eta_2^d} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix} \right.$$
$$\left. + y_3^d e^{i\eta_3^d} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \tag{5.1b}$$

$$m^e = v^d \left[y_1^e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + y_2^e e^{i\eta} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix} \right.$$
$$\left. + y_3^e e^{i\eta'} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \tag{5.1c}$$

$$m^\nu = \mu_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + \mu_b e^{i\eta} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix}$$

$$+ \mu_c e^{i\eta'} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.1d}$$

where $v^u = v \sin\beta$ and $v^d = v \cos\beta$ are the VEVs of the MSSM Higgs fields and $v = 174$ GeV. We recall from Eq. (3.19) that $\eta = 2\pi/3$, while $\eta' = 0$, while the remaining phases are free. Hence, there are 12 free parameters in the quark matrices and six in the lepton matrices.

Assuming all superpotential terms have $\mathcal{O}(1)$ couplings, we may derive a "natural" scale for each of the coefficients y_i^f. First, we recall that there are several messenger scales present in our model. The ones that appear in y_i^f are M_χ, M_ζ, $M_{\Omega_{\text{dec}}}$, $M_{\Omega_{\text{sol}}}$, and $M_{\Omega_{\text{atm}}}$. As previously established, we have $\langle\xi\rangle \lesssim M_\zeta < M_\chi$. More specifically, we will assume the following ratios:

$$\frac{\langle\xi\rangle}{M_\zeta} \gtrsim 0.5, \qquad \frac{\langle\xi\rangle}{M_\chi} \lesssim 0.1. \tag{5.2}$$

We further define the GUT scale by $M_{\text{GUT}} \equiv v_{45} \lesssim M_\chi$. Finally, as discussed previously, we assume that $M_{\Omega_{\text{atm}}} \approx M_{\Omega_{\text{sol}}} > M_{\Omega_{\text{dec}}}$, by roughly 1 order of magnitude.

The coefficients y_i^f derive from terms in Eq. (2.2), which take a generic form,

$$y_1^f = \bar\phi_{\text{atm}}\bar\phi_{\text{atm}}\xi^{N-2} \sum_{n=0}^{N} \frac{\lambda_{X,n}^{(f)}}{\langle H_{45}\rangle^n M_\chi^{N-n}},$$

$$y_2^f = \bar\phi_{\text{sol}}\bar\phi_{\text{sol}}\xi^{N-2} \sum_{n=0}^{N} \frac{\lambda_{X,n}^{(f)}}{\langle H_{45}\rangle^n M_\chi^{N-n}},$$

$$y_3^f = \bar\phi_{\text{dec}}\bar\phi_{\text{dec}}\xi^{N-2} \sum_{n=0}^{N} \frac{\lambda_{X,n}^{(f)}}{\langle H_{45}\rangle^n M_\chi^{N-n}}, \tag{5.3}$$

where λ are $\mathcal{O}(1)$ couplings and N is a number between 2 and 5. We will assume there are no large cancellations between terms in the sums. The flavon VEVs are discussed in Sec. III C, from which we may approximate their VEVs by

$$\langle\bar\phi_{\text{dec}}\rangle \sim M_{\text{GUT}}, \qquad \langle\bar\phi_{\text{atm}}\rangle \sim \frac{\langle\xi\rangle^4}{M_\zeta^4} M_{\text{GUT}},$$

$$\langle\bar\phi_{\text{sol}}\rangle \sim \frac{\langle\xi\rangle^{7/2}}{M_\zeta^{7/2}} M_{\text{GUT}}. \tag{5.4}$$

We note immediately that these VEVs have large powers of $\langle\xi\rangle/M_\zeta$, which is primarily bounded below [see Eq. (5.2)].

This translates to only a loose upper bound on the fitting parameters. From Eqs. (5.2) and (5.3), we expect the following scales for the fitted coefficients:

$$y_1^u \gtrsim 4\times10^{-4}, \quad y_1^e \gtrsim 4\times10^{-5}, \quad y_1^d \gtrsim 4\times10^{-5}, \quad \mu_a \sim 10^{-2}\,\text{eV},$$

$$y_2^u \gtrsim 8\times10^{-5}, \quad y_2^e \gtrsim 8\times10^{-6}, \quad y_2^d \gtrsim 8\times10^{-6}, \quad \mu_b \sim 10^{-3}\,\text{eV},$$

$$y_3^u \sim 1, \quad y_3^e \sim 10^{-1}, \quad y_3^d \sim 10^{-1}, \quad \mu_c \sim 10^{-3}\,\text{eV}, \quad y_4^u \gtrsim 5\times10^{-4}. \tag{5.5}$$

B. Fitting procedure

To fit the real coefficients $y_{1,2,3,4}^u$, $y_{1,2,3}^d$, $y_{1,2,3}^e$, and $\mu_{a,b,c}$ as well as phases $\eta_{2,3,4}^u$ and $\eta_{2,3}^d$, we wish to minimize a χ^2 function that relates the N physical predictions $P_i(\{x\})$ for a given set of input parameters $\{x\}$ to their current best-fit values μ_i and their associated 1σ errors, denoted σ_i. It is defined by

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{P_i(\{x\}) - \mu_i}{\sigma_i}\right)^2. \tag{5.6}$$

The errors σ_i are equivalent to the standard deviation of the experimental fits to a Gaussian distribution. For most parameters, their distribution is essentially Gaussian, with the exception of the (lepton) atmospheric angle θ_{23}^l.

For a normal hierarchy (as predicted by the model), the distribution is roughly centered on the maximal atmospheric angle, i.e., $(\theta_{23}^l)^{\text{best-fit}} \sim 45°$, while the best-fit value is given by $\theta_{23}^l = 42.3°$; i.e., there is a small preference for θ_{23}^l to be in the first octant. As such, there are two possible scenarios to consider when performing our fit.

  (i) Scenario 1: We assume that the (weak) preference for $\theta_{23}^l < 45°$ is true and approximate its distribution by a Gaussian about 42.3°, setting $\sigma_{\theta_{23}^l} = 1.6°$ as the error.

TABLE VII. Model predictions in the quark sector, for $\tan\beta = 5$. The quark contribution to the total χ^2 is 2.0. The observables are at the GUT scale.

Observables	Model	Data fit 1σ range (from Ref. [25])
$\theta_{12}^q/°$	13.024	$12.985 \to 13.067$
$\theta_{13}^q/°$	0.1984	$0.1866 \to 0.2005$
$\theta_{23}^q/°$	2.238	$2.202 \to 2.273$
$\delta^q/°$	69.32	$66.12 \to 72.31$
m_u/MeV	0.575	$0.351 \to 0.666$
m_c/MeV	248.4	$240.1 \to 257.5$
m_t/GeV	92.79	$89.84 \to 95.77$
m_d/MeV	0.824	$0.744 \to 0.929$
m_s	15.55	$15.66 \to 17.47$
m_b/GeV	0.939	$0.925 \to 0.948$
$\delta\chi^2$	2.0	

TABLE VIII.   Quark sector input parameter values.

Parameter	Fitted value
y_1^u	3.314×10^{-5}
y_2^u	2.060×10^{-4}
y_3^u	5.503×10^{-1}
y_4^u	7.423×10^{-3}
η_2^u	0.617π
η_3^u	1.047π
η_4^u	1.718π
y_1^d	3.288×10^{-4}
y_2^d	3.308×10^{-5}
y_3^d	2.785×10^{-2}
η_2^d	0.521π
η_3^d	1.065π

   (ii) Scenario 2: We remain octant agnostic by assuming a Gaussian distribution centred at the midpoint between the two 1σ bounds, i.e., $(\theta_{23}^l)^{\text{best-fit}} = 45.9°$ with $\sigma_{\theta_{23}^l} = 3.5°$.

Below, we present the results of our fit in each of these two scenarios.

In this analysis, $N = 18$, corresponding to six mixing angles θ_{ij}^l (neutrinos) and θ_{ij}^q (quarks), the CKM phase δ^q, nine Yukawa eigenvalues for the quarks and charged leptons, and two neutrino mass-squared differences Δm_{21}^2 and Δm_{31}^2. In the lepton sector, we use the Particle Data Group (PDG) parametrization of the Pontecorvo-Maki-Nakagawa-Sakata

(PMNS) matrix [23] $U_{\text{PMNS}} = R_{23}^l U_{13}^l R_{12}^l P_{\text{PDG}}$ in terms of $s_{ij} = \sin\theta_{ij}^l$, $c_{ij} = \cos\theta_{ij}^l$, the Dirac CP-violating phase δ^l, and further Majorana phases contained in $P_{\text{PDG}} = \text{diag}(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}})$. Experimentally, the leptonic phase δ^l is poorly constrained at 1σ (and completely unconstrained at 3σ), so is not fitted, and left as a pure prediction of the model, as are the (completely unconstrained) Majorana phases α_{21} and α_{31}.

We wish to emphasize that, of the above 18 observed parameters, ten belong to the quark sector, fitted by 12 free parameters, while eight belong to the lepton sector, fitted by six free parameters. These two sectors are distinct; the model is predictive in the lepton sector but not in the quark sector.

The running of best-fit and error values to the GUT scale are generally dependent on SUSY parameters, notably $\tan\beta$, as well as contributions from SUSY threshold corrections, which are relevant even for small $\tan\beta$ [24]. We extract the GUT-scale CKM parameters and all Yukawa couplings (with associated errors) from Ref. [25] for $\tan\beta = 5$. The value of $\tan\beta$ does not have a significant impact on the quality of our model, so we only present results for $\tan\beta = 5$ here. Given that our model is not predictive in the quark sector (and also does not predict the charged lepton masses), the effect of threshold corrections can be subsumed into a redefinition of the model input parameters. In further reference to Ref. [25], this is equivalent to setting the parameters $\bar\eta_i$ to zero. Experimental neutrino parameters are extracted from Ref. [26].

TABLE IX.   Model predictions in the lepton sector, for $\tan\beta = 5$. The observables are at the GUT scale. The lepton contributions to the total χ^2 are 1.3 and 0.7 in scenario 1 and 2, respectively. Note the two different data fit 1σ ranges for θ_{23}^l, depending on the choice of scenario, as discussed in Sec. V B.

Observables	Model		Data fit 1σ range
	Scenario 1 ($\theta_{23}^{\text{exp,bf}} = 42.3°$)	Scenario 2 ($\theta_{23}^{\text{exp,bf}} = 45.9°$)	(from Refs. [25–27])
$\theta_{12}^l/°$	33.13	32.94	$32.83 \to 34.27$
$\theta_{13}^l/°$	8.59	8.55	$8.29 \to 8.68$
$\theta_{23}^l/°$	40.81		$40.63 \to 43.85$
$\cdots$	$\cdots$	46.65	$42.40 \to 49.40$
$\delta^l/°$	280	275	$192 \to 318$
m_e/MeV	0.342	0.342	$0.340 \to 0.344$
m_μ/MeV	72.25	72.25	$71.81 \to 72.68$
m_τ/GeV	1.229	1.229	$1.223 \to 1.236$
$\Delta m_{21}^2/\text{eV}^2$	7.58×10^{-5}	7.46×10^{-5}	$(7.33 \to 7.69) \times 10^{-5}$
$\Delta m_{31}^2/\text{eV}^2$	2.44×10^{-3}	2.47×10^{-3}	$(2.41 \to 2.50) \times 10^{-3}$
m_1/meV	0.32	0.38	$\cdots$
m_2/meV	8.64	8.65	$\cdots$
m_3/meV	49.7	49.7	$\cdots$
$\sum m_i/\text{meV}$	58.7	59.4	< 230
$\alpha_{21}/°$	264	264	$\cdots$
$\alpha_{31}/°$	323	333	$\cdots$
$|m_{ee}|/\text{meV}$	2.46	2.42	$\cdots$
$\delta\chi^2$	1.3	0.7	

TABLE X.　Lepton input parameter values (with η, η' fixed by the theory).

Parameter	Fitted value	
	Scenario 1 $(\theta_{23}^{\text{exp,bf}} = 42.3°)$	Scenario 2 $(\theta_{23}^{\text{exp,bf}} = 45.9°)$
y_1^e	2.217×10^{-3}	-1.966×10^{-3}
y_2^e	-1.025×10^{-5}	1.027×10^{-5}
y_3^e	3.366×10^{-2}	3.790×10^{-2}
μ_a/meV	26.60	25.90
μ_b/meV	2.571	2.546
μ_c/meV	2.052	2.461
η	$2\pi/3$	
η'	0	

Tables VII and VIII show the numerical fit of all relevant parameters to quark mass and mixing data, while Tables IX and X show the fit to lepton mass and mixing. The fit gives $\chi^2 \approx 3.3$ and $\chi^2 \approx 2.7$ in scenarios 1 and 2, respectively.

Although the model is not technically predictive in the quark sector due to an excess of free parameters, the structure of the mass matrices forces some tension between parameters, notably the $\sim 1\sigma$ deviation from the experimental fit value in the strange quark mass, as seen in Table VII. The best-fit GUT scale mass $m_s = 15.55$ GeV corresponds to a pull of -1.12, while the (absolute magnitude of) pulls for all other parameters are less than 1.

In the lepton sector, there are two fixed discrete phases plus six continuous input parameters that we fit to three charged lepton masses, two neutrino mass-squared differences, and three mixing angles (a total of eight observables), while predicting the CP phase δ^l, two Majorana phases, and the effective neutrino mass $|m_{ee}|$. Although our fit does not constitute a full analysis of the parameter space, it agrees with the results of a more dedicated numerical analysis of CSD(n) models [13].

VI. CONCLUSION

We have proposed a renormalizable $\Delta(27) \times SO(10)$ SUSY GUT of flavor. All symmetries, including an additional $\mathbb{Z}_9 \times \mathbb{Z}_{12} \times \mathbb{Z}_4^R$ discrete symmetry, are broken close to the GUT breaking scale due to the action of explicit superpotential terms to yield the MSSM with the standard R parity as the surviving theory at low energies.

The model is very ambitious since it is not only a full $SO(10)$ SUSY GUT theory, with GUT symmetry breaking sectors including doublet-triplet splitting and guaranteeing the absence of extra light doublets, but also addresses the flavor problem due to additional commuting discrete family symmetries. The mystery of why there are three families of quarks and leptons including their observed pattern of masses and mixing angles is addressed, and a novel form of spontaneous geometrical CP violation arises from the nature of the $\Delta(27)$ group.

In many respects $SO(10)$ is the "holy grail" of GUT groups since it involves probably the most elegant unification of quarks and leptons, including a right-handed neutrino, making neutrino mass and mixing inevitable [unlike $SU(5)$ where neutrino masses could quite happily be set to zero]. When combined with the family symmetry $\Delta(27)$, all quarks and leptons are unified into a single multiplet (3,16) providing a very elegant and simple unification of all matter. Such a complete flavor unification has been attempted before, but until now, the technicalities involved have led to only partial success. The contribution of the present paper lies in showing how many of these technical difficulties may be successfully overcome within a fully fledged $\Delta(27) \times SO(10)$ SUSY GUT of flavor.

We emphasize that in our model all quark and lepton (including neutrino) mass matrices take a particularly simple universal form, with a small correction to the up-type quark mass matrix being responsible for quark mixing. The heavy right-handed neutrino Majorana matrix also has the same universal form, and even including the seesaw mechanism, the low-energy effective light left-handed Majorana neutrino mass matrix also has this form, indeed corresponding to CSD3, leading to a highly predictive scheme for leptonic mixing. The model predicts a normal neutrino mass hierarchy with the best-fit lightest neutrino mass $m_1 \approx 0.32$ or $m_1 \approx 0.38$ meV and all neutrino parameters fitted to within 1σ of the values predicted by global fits to experiment. In particular, we predict a CP-violating oscillation phase $\delta^l \approx 280°$ or $\delta^l \approx 275°$, in agreement with current experimental hints.

The model has the following virtues:

(i) It is fully renormalizable at the GUT scale, with an explicit $SO(10)$ breaking sector and a spontaneously broken CP symmetry.

(ii) It involves only the smaller named representations of $SO(10)$.

(iii) The MSSM is reproduced below the GUT scale, with R parity emerging from a discrete $\mathbb{Z}_4^R$.

(iv) Doublet-triplet splitting is achieved through the Dimopoulos-Wilczek mechanism.

(v) A μ term is generated at the correct scale.

(vi) Proton decay is sufficiently suppressed.

(vii) $\Delta(27)$ justifies the CSD3 alignment.

(viii) Spontaneous geometrical CP violation, where the input phase is the cube root of unity $\omega = e^{2i\pi/3}$, originates from the $\Delta(27)$.

(ix) We successfully fit all quark and lepton masses, with the PMNS mixing matrix predicted (with no free parameters) once the physical neutrino masses are specified.

These features are desirable for any flavor or GUT model. Achieving them all in the same model represents a significant step toward a complete flavored $SO(10)$ GUT. At the cost of its large field content, the model is rather successful and fairly complete. Nevertheless, some

relevant topics remain beyond the scope of the paper. Notably, while we discuss how the model leads to an effective MSSM after the symmetries are broken, we do not discuss the details of SUSY breaking nor the mass spectrum of SUSY partners at low energy. Because the field content is reduced to the MSSM, a Landau pole would not be produced below the GUT scale. Beyond the MSSM fields, all other gauge multiplets have masses close to or at the Planck scale. To avoid the use of global symmetries, we may envisage that gauged $SU(3)$ and $U(1)$ symmetries are broken at the Planck scale to give rise to the $\Delta(27)$ and $\mathbb{Z}_N$ groups, respectively. In such a case, anomaly cancellation imposes constraints on the charges which we did not consider, but which can always be satisfied by adding superfields at the Planck scale. Furthermore, we have not considered GUT threshold corrections and their effect at high energy on gauge coupling unification nor a possible string theory completion for this model.

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