# Electroweak theory based on $SU(4)_L \otimes U(1)_X$ gauge group

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(Received 25 May 2016; published 6 July 2016)

This paper includes two main parts. In the first part, we present generalized gauge models based on the $SU(3)_C \otimes SU(4)_L \otimes U(1)_X$ (3-4-1) gauge group with arbitrary electric charges of exotic leptons. The mixing matrix of neutral gauge bosons is analyzed, and the eigenmasses and eigenstates are obtained. The anomaly-free as well as matching conditions are discussed precisely. In the second part, we present a new development of the original 3-4-1 model [R. Foot, H. N. Long, and T. A. Tran, Phys. Rev. D **50**, R34 (1994), F. Pisano and V. Pleitez, Phys. Rev. D **51**, 3865 (1995).]. Different from previous works, in this paper the neutrinos, with the help of the scalar decuplet $H$, get the Dirac masses at the tree level. The vacuum expectation value (VEV) of the Higgs boson field in the decuplet $H$ acquiring the VEV responsible for neutrino Dirac mass leads to mixing in separated pairs of singly charged gauge bosons, namely the Standard Model (SM) $W$ boson and $K$, the new gauge boson acting in the right-handed lepton sector, as well as the singly charged bileptons $X$ and $Y$. Due to the mixing, there occurs a right-handed current carried by the $W$ boson. From the expression of the electromagnetic coupling constant, ones get the limit of the sine-squared of the Weinberg angle, $\sin^2\theta_W < 0.25$, and a constraint on electric charges of extra leptons. In the limit of lepton number conservation, the Higgs sector contains all massless Goldstone bosons for massive gauge bosons and the SM-like Higgs boson. Some phenomenology is discussed.

DOI: 10.1103/PhysRevD.94.015007

## I. INTRODUCTION

The current status of particle physics leads to widespread evidence for extending the SM. The recently observed 750 GeV diphoton excess [1] can be explained as an existence of a new neutral scalar that couples to extra heavy quarks or, in some cases, to new leptons and bosons. In this sense, the 3-3-1 models [2–4] seem to be good candidates since they contain all ingredients such as extra quarks and new scalar fields. However, the problem is that satisfying the LHC diphoton excess and the experimental value of the muon anomalous magnetic moment $(g-2)_\mu$ requires that the 3-3-1 scale be low $\omega \approx 400$ GeV, while the flavor-changing neutral current (FCNC) requires a high scale $\omega \approx 2$ TeV. To solve this puzzle for the 3-3-1 model with right-handed neutrinos, one must introduce an inert scalar triplet [5], extend the gauge group to a larger one such as $SU(3)_C \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_X$ [6], or introduce new charged scalars [7].

The above situation is also correct for the other SM extensions; hence, the search for the models satisfying current experimental data is needed. It is known that the $SU(4)$ is the highest symmetry group in the electroweak sector [8]. There have been some gauge models based on the $SU(3)_C \otimes SU(4)_L \otimes U(1)_X$ [2,9,10]; however, the Higgs physics—currently the most important sector—has not received enough attention. In some versions, this puzzle was not studied in much detail at all. In light of the current status of particle physics, the Higgs sector should be considered with as much detail as possible, especially in the neutral scalar sector where the SM-like Higgs boson is contained. Thus, in this work, we will focus on the Higgs sector. As in the 3-3-1 models, the electric charge quantization was also explained in the framework of the 3-4-1 model [11].

The aim of this paper is to present the 3-4-1 models which are able to deal with current Higgs physics. It is well known that if scalar sector contains many neutral scalar fields the situation is very complicated. In addition, we will pay attention to the gauge boson sector where new physics is quite rich and explore some interesting features of phenomenology. The 3-4-1 model we consider here may

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be considered as combination of the minimal 3-3-1 model [3] and an alternative version with right-handed neutrinos [2,4]. Hence, the derived model is quite interesting and deserves further study.

Our work is arranged as follows. Section II will present conditions of anomaly cancellation and particle content, where extra novel induced leptons have arbitrary electric charges $q$ and $q'$. We will show that the 3-4-1 models are anomaly free only if there are an equal number of quadruplets (4) and antiquadruplets ($4^*$). Simply speaking, this condition requires the sum of all fermion charges to vanish (see subsection II A). In subsection II B, the Higgs bosons needed for fermion mass production are discussed. We want to avoid nonrenormalizable effective couplings, so every Higgs multiplet has only one component with nonzero VEV. Subsection IIC focuses on the gauge boson fields, especially neutral ones. Section III is devoted to the original 3-4-1 model [2,9]. As in [9], to produce the mass for leptons, the Higgs decuplet is introduced. However, as will be seen below, it has to be redefined. In this work, we will build a lepton number operator from which the lepton flavor number violating (LFV) processes will be pointed out. In this section, mixing of the singly charged gauge bosons and the currents will get more attention. For completeness, the neutral gauge boson sector will also be presented, though it is quite similar to the previous analysis. From an expression of the ratio of two gauge couplings $t$, it follows the bound on the sine-squared of the Weinberg angle $\sin^2\theta_W < 0.25$. In section V the bounds on masses of new gauge bosons are roughly derived, based on the data of $W$ bosons, rare muon decays and $\mu - e$ conversion. Finally, we present our conclusion in Sec. VI.

## II. THE MODEL

As above mentioned, we first check the conditions for anomaly free of the models based on $\mathrm{SU}(3)_C \otimes \mathrm{SU}(4)_L \otimes \mathrm{U}(1)_X$ gauge group.

### A. Anomaly cancellation and fermion content

For the class of the models constructed from the gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(3)_L \times \mathrm{U}(1)_N$, the conditions for anomaly cancellation were discussed in detail in [12]. Similarly for the class of the $\mathrm{SU}(3)_C \times \mathrm{SU}(4)_L \times \mathrm{U}(1)_X$ (3-4-1) models the following gauge anomalies must vanish: (i) $[SU(3)_C]^2 \times U(1)_X$, (ii) $[SU(4)_L]^3$, (iii) $[SU(4)_L]^2 \times U(1)_X$; (iv) $[\mathrm{Grav}]^2 \times U(1)_X$; and (v) $[U(1)_X]^3$. Being different from the studies of anomaly cancellation for the 3-3-1 models [12,13], we will exploit the relation between charge operator and diagonal generators of the gauge symmetry $SU(4)_L$ to prove that the five conditions will reduce to two conditions only: $[SU(4)_L]^3$ and $[SU(4)_L]^2 \times U(1)_X$.

For a general 3-4-1 model, the electric charge operator is in the form

$$Q = T_3 + bT_8 + cT_{15} + X, \tag{1}$$

where the coefficient in front of $T_3$ equaling 1, is chosen to ensure that the SM group is a subgroup of the model under consideration: $SU(2)_L \otimes U(1)_Y \subset SU(4)_L \otimes U(1)_X$.

The leptons are in quadruplet

$$f_{aL} = (\nu_a, l_a, E_a^q, E_a^{\prime q'})_L^T, \quad a = e, \mu, \tau, \tag{2}$$

where $q$ and $q'$ are electric charges of associated extra leptons. Applying Eq. (1) to Eq. (2) we obtain

$$b = \frac{-2q-1}{\sqrt{3}}, \quad c = \frac{q-3q'-1}{\sqrt{6}}, \quad X_{f_{aL}} = \frac{q+q'-1}{4}, \tag{3}$$

or

$$\begin{aligned} q &= -\frac{1}{2} - \frac{\sqrt{3}b}{2}, \\ q' &= -\frac{1}{2} - \frac{b}{2\sqrt{3}} - \frac{\sqrt{2}c}{\sqrt{3}} \quad \text{and} \\ X_{f_{aL}} &= -\frac{1}{2} - \frac{b}{2\sqrt{3}} - \frac{c}{2\sqrt{6}}. \end{aligned} \tag{4}$$

Before discussing anomaly cancellation, we remember that the fermion representations of $SU(3)_C$ and $SU(4)_L$ in the 3-4-1 models are all $SU(3)_C$ triplets, $SU(4)_L$ (anti) quadruplets, and singlets. All singlets do not contribute to anomalies so in consideration we omit them. The representative matrices of generators corresponding to the $SU(3)_C$ triplets are denoted as $T_C^a$ $(a = 1, 2, \ldots, 8)$ , and the $SU(4)_L$ (anti)quadruplets are $T_L^a(\bar{T}_L^a)$, $a = 1, 2, \ldots, 15$.

Now we consider the 3-4-1 model with $M$ and $N$ families of leptons and quarks, respectively. In addition, the number of $SU(4)_L$ quadruplets of quark families is $K$. For simplicity, we assume that all left-handed leptons are in the quadruplets. The general case is derived easily. All of them respect the gauge symmetry $SU(3)_C \times SU(4)_L \times U(1)_X$ as follows. For leptons, we have

$$\begin{aligned} &f_{iL} = (\nu_{iL}, l_{iL}, E_{iL}^q, E_{iL}^{\prime q'})^T \sim (1, 4, X_{f_L}), \\ &\nu_{iR} \sim (1, 1, 0), \quad l_{iR} \sim (1, 1, -1), \\ &E_{iR}^q \sim (1, 1, q), \quad E_{iR}^{\prime q'} \sim (1, 1, q'), \\ &\quad i = 1, 2, \ldots, M. \end{aligned} \tag{5}$$

These left-handed leptons are generalized from (2). As in the 3-3-1 models [2–4], the parameters $b$ and $c$ are closely connected with $q$ and $q'$; in [14,15], the parameters $b$, $c$

have been used. In our point of view, the clearer way is using $q$ and $q'$. We note that $\nu_{iR}$ is an option, while $E^q_{iR}$ and $E^{q'}_{iR}$ may disappear in the specific case of the minimal 3-4-1 model. If the left-handed leptons are in antiquadruplet, then $b$ and $c$ will be replaced by $-b$ and $-c$, respectively.

The quark sector is

$$\begin{aligned}&Q_{mL}=(u_{mL},d_{mL},T_{mL},T'_{mL})^T\sim(3,4,X_{q_L}),\\&\quad m=1,2,\ldots,K,\\&Q_{nL}=(d_{nL},-u_{nL},D_{nL},D'_{nL})^T\sim(3,4^*,X_{\bar{q}_L}),\\&u_{pR}\sim(3,1,2/3),\quad d_{pR}\sim(3,1,-1/3),\\&T_{mR}\sim(3,1,X_{T_R}),\quad T'_{mR}\sim(3,1,X_{T'_R}),\\&D_{nR}\sim(3,1,X_{D_R}),\quad D'_{nR}\sim(3,1,X_{D'_R}),\\&\quad n=K+1,..,N,\quad p=1,2,..,N.\end{aligned}\tag{6}$$

Let us first consider the anomaly of $[SU(4)_L]^3$. Each of the $SU(4)_L$ quadruplets $4_L$ or antiquadruplets $4^*_L$ contributes a well-known quantity $\mathcal{A}^{abc}(4_L)=\mathrm{Tr}(T^a_L\{T^b_L,T^c_L\})$ or $\mathcal{A}^{abc}(4^*_L)=\mathrm{Tr}(\bar{T}^a_L\{\bar{T}^b_L,\bar{T}^c_L\})$, where $a$, $b$, and $c$ mean three $SU(4)_L$ gauge bosons related to the triangle diagrams. Because $\mathcal{A}^{abc}(4_L)=-\mathcal{A}^{abc}(4^*_L)$, the total contribution to the $[SU(4)_L]^3$ anomaly can be written as

$$\mathcal{A}^{abc}(4_L)\left(\sum_{Q_{mL},f_{iL}}4_L-\sum_{Q_{nL}}4^*_L\right)=\mathcal{A}^{abc}(4_L)(n_{4_L}-n_{4^*_L}),\tag{7}$$

where $n_{4_L}$ and $n_{4^*_L}$ are the number of fermion quadruplets and antiquadruplets, respectively. This means that the above anomaly cancels only if the number of quadruplets is equal to the number of antiquadruplets, namely,

$$M+6K=3N,\tag{8}$$

where the factor 3 appears because the quarks are in $SU(3)_C$ triplets while leptons are in $SU(3)_C$ singlets.

Next, we consider the anomaly of $[SU(4)_L]^2\times U(1)_X$. The $[SU(4)_L]^2$ gives the same factor for both quadruplets and antiquadruplets, i.e., $\mathrm{Tr}[T^aT^b]=\mathrm{Tr}[\bar{T}^a\bar{T}^b]=\delta_{ab}/2$, where $a$ and $b$ relate to two $SU(4)_L$ gauge bosons. Hence, this anomaly is free if the sum of all $U(1)_X$ charges of the $SU(4)_L$ chiral multiplets is zero, namely,

$$\sum_{f_{iL},Q_{pL}}X_L=MX_{f_L}+3KX_{q_L}+3(N-K)X_{\bar{q}_L}=0,\tag{9}$$

where $X_L$ denotes the $U(1)_X$ charge of an arbitrary left-handed (anti)quadruplet in the model.

Now we turn to the anomaly of $[SU(3)_C]^2\times U(1)_X$. This case is similar to the case of $[SU(4)_L]^2\times U(1)_X$, but now only the $SU(3)_C$ quark triplets contribute to the mentioned anomaly. The anomaly-free condition is

$$\sum_{Q_{mL}}4X_{q_L}+\sum_{Q_{nL}}4X_{\bar{q}_L}-\sum_{q_R}X_{q_R}=0,\tag{10}$$

where $X_{q_R}$ and $X_{q_L,\bar{q}_L}$ are $U(1)_X$ charges of the right-handed $SU(4)_L$ quark singlets $q_R$ and left-handed $SU(4)_L$ quark (anti)quadruplets $Q_{m_L}(Q_{n_L})$, respectively. The factors 4 appear in Eq. (10) because we take four components of every $SU(4)_L$ (anti)quadruplet into account. The minus sign implies the opposite contributions of left- and right-handed fermions to gauge anomalies. Because all $q_R$ are singlets of the $SU(4)_L\times U(1)_X$, their $U(1)_X$ charges are always equal to the electric charges $q_{q_R}$, leading to $\sum_{q_R}X_{q_R}=\sum_{q_R}q_{q_R}$. On the other hand, from the definition of the charge operators $Q$ given in (1), it can be seen that

$$\sum_{Q_{mL}}4X_{q_L}=\sum_{Q_{mL}}\mathrm{Tr}(X_L\times I_4)=\sum_{Q_{mL}}\mathrm{Tr}[Q],$$

where $\mathrm{Tr}[Q]$ implies the sum over the electric charges of all components of the quark quadruplet $Q_{mL}$. Note that we have used the traceless property of the $SU(4)_L$ generators: $\mathrm{Tr}(T^a)=0$. Doing this the same way for the case of quark antiquadruplets, the condition (10) can be rewritten in terms of the electric charges of left-and right-handed quarks,

$$\sum_{Q_{pL}}\sum_{i=1}^{4}q_{q_L}-\sum_{q_R}q_{q_R}=0,\tag{11}$$

where the first sum implies that all $SU(4)_L$ quark (anti) quadruplets and their components are counted. The equality (11) is always correct because every left-handed quark always has its right-handed partner with the same electric charge.

The above discussion on the anomaly cancellation of $[SU(3)_C]^2\times U(1)_X$ can be applied for the case of the anomaly cancellation of $[\mathrm{Grav}]^2\times U(1)_X$, but the electric charges must be counted for all components of the $SU(3)_C$ and $SU(4)_L$ multiplets of all quarks and leptons. The proof of the zero contribution of the quark sector is very easy, while that of the lepton sector needs more explanation. Although the presence of right-handed neutrinos is optional, they are neutral leptons and, therefore, do not contribute to this anomaly. If right-handed charged leptons are arranged into components of left-handed (anti)quadruplets, they must be changed into their charge conjugations. These new forms of right-handed leptons have the opposite signs of electric charges compared to the respective left-handed partners. Hence, the total contribution to the considered anomaly is still zero.

Cancellation of the $[U(1)_X]^3$ anomaly, which relates to the triangle diagram having three $B''$ gauge bosons, is expressed by the following condition,

$$\sum_{F_L} X_{F_L}^3 - \sum_{F_R} X_{F_R}^3 = 0, \tag{12}$$

where $F_L$ and $F_R$ are any components of the fermion (quark and lepton) representations of the $SU(3)_C$ and $SU(4)_L$ gauge symmetries. Hence, the sum is taken over all components of these representations. Because the $U(1)_X$ and the electric charges relate to each other through the definition of the charge operator (1), we can write the left-hand side of (12) as a function of the electric charges. Because all right-handed fermions are $SU(4)_L$ singlets, $X_{F_R} = q_{F_R}$; therefore,

$$\sum_{F_R} X_{F_R}^3 = \sum_{F_R} q_{F_R}^3. \tag{13}$$

In contrast, all left-handed fermions are (anti)quadruplets, and we can write the left-handed term in (12) as a sum over all fermion (anti)quadruplets, namely,

$$\sum_{F_L} X_{F_L}^3 = \sum_{4_L, 4_L^*} 4X_{F_L}^3. \tag{14}$$

Now we come back to the formula of the charge operator, where we denote $Q_{F_L}$ as the charge operator of the left-handed fermion (anti)quadruplets. If we denote $T^{(3,8,15)} \equiv T^3 + bT^8 + cT^{15}$, the $U(1)_X$ charge part of each $4_L$ representation can be written as

$$\begin{aligned} X_{F_L} I_4 &= Q_{F_L} - T^{(3,8,15)} \to (X_{F_L} I_4)^3 = (Q_{F_L} - T^{(3,8,15)})^3 \\ \to \mathrm{Tr}[X_{F_L}^3 I_4] &= \mathrm{Tr}[Q_{F_L}^3] - 3\mathrm{Tr}[Q_{F_L} T^{(3,8,15)} (Q_{F_L} - T^{(3,8,15)})] - \mathrm{Tr}[(T^{(3,8,15)})^3] \\ \to 4X_{F_L}^3 &= \mathrm{Tr}[Q_{F_L}^3] - 3\mathrm{Tr}[(T^{(3,8,15)} + X_{F_L} I_4) T^{(3,8,15)} X_{F_L} I_4] - \mathrm{Tr}[(T^{(3,8,15)})^3] \\ &= \mathrm{Tr}[Q_{F_L}^3] - 3X_{F_L} \mathrm{Tr}[(T^{(3,8,15)})^2] - \mathrm{Tr}[(T^{(3,8,15)})^3], \end{aligned} \tag{15}$$

where we have used the fact that both $Q$ and $T^{(3,8,15)}$ are diagonal so they commute with each other, and $T^{(3,8,15)}$ is traceless. Remember that $4X_{F_L}^3$ is the contribution of four components in one quadruplet. Then the contribution to $[U(1)_X]^3$ of all quadruplets is

$$\begin{aligned} \sum_{4_L} X_{F_L}^3 &= \sum_{F_L} q_{F_L}^3 - 3\sum_{4_L} X_{F_L} [(T^{(3,8,15)})^2] \\ &\quad - n_{4_L} \mathrm{Tr}[(T^{(3,8,15)})^3]. \end{aligned} \tag{16}$$

The same proof can be applied for the case of antiquadruplets with generators $\bar{T}^a = -T^a$, $(a = 3,\ 8,\ 15)$ and $\bar{T}^{(3,8,15)} = -T^{(3,8,15)}$. From this, it can be proved that

$$\begin{aligned} \mathrm{Tr}[(\bar{T}^{(3,8,15)})^2] &= \mathrm{Tr}[(T^{(3,8,15)})^2], \\ \mathrm{Tr}[(\bar{T}^{(3,8,15)})^3] &= -\mathrm{Tr}[(T^{(3,8,15)})^3]. \end{aligned}$$

The above discussion is enough to write (12) in the following new form:

$$\begin{aligned} &\left(\sum_{F_L} q_{F_L}^3 - \sum_{F_R} q_{F_R}^3\right) - 3\mathrm{Tr}[(T^{(3,8,15)})^2] \sum_{4_L, 4_L^*} X_{F_L} \\ &- \mathrm{Tr}[(T^{(3,8,15)})^3](n_{4_L} - n_{4_L^*}) = 0. \end{aligned} \tag{17}$$

The equality (17) is satisfied as a consequence of the two anomaly-free conditions (7) and (9). The equality (9) also implies that the sum over the electric charges of left-handed fermions is zero.

Finally, we conclude that the anomaly-free conditions of the 3-4-1 models are as follows: (i) the number of fermion quadruplets is equal to that of fermion antiquadruplets and (ii) the sum over electric charges of all left-handed fermions is zero.

The 3-4-1 models of concern here are all satisfied with these two conditions.

### B. Yukawa couplings and masses for fermions

Since the leptons are arranged as

$$\begin{aligned} f_{aL} &= (\nu_a, l_a, E_a^q, E_a'^{q'})_L^T \sim \left(1, 4, \frac{1}{4}(q + q' - 1)\right), \\ l_{aR} &\sim (1, 1, -1), \qquad E_{aR}^q \sim (1, 1, q), \qquad E_{aR}'^{q'} \sim (1, 1, q'), \end{aligned} \tag{18}$$

the mass of $E_a'^{q'}$ is obtained from the Yukawa coupling,

$$-L_{\mathrm{Yukawa}}^{E'} = h_{ab}^{E'} \overline{f_{aL}} \Phi_1 E_{bR}'^{q'} + \mathrm{H.c.}, \tag{19}$$

where

$$\Phi_1 \sim \left(1, 4, \frac{(q - 3q' - 1)}{4}\right) = (\Phi_1^{(-q')}, \Phi_1^{(-q'-1)}, \Phi_1^{(q-q')}, \Phi_1^0)^T. \tag{20}$$

Hence, if $\Phi_1^0$ has a VEV $\frac{V}{\sqrt{2}}$ then $E_a'^{q'}$ gets mass from a mass matrix

$$(m_{E'})_{ab} = h_{ab}^{E'}\frac{V}{\sqrt{2}}. \tag{21}$$

The mass of $E_a^q$ is obtained from the following Yukawa term,

$$-L_{\text{Yukawa}}^{E} = h_{ab}^{E}\overline{f_{aL}}\Phi_2 E_{bR}^{q} + \text{H.c.}, \tag{22}$$

where

$$\Phi_2 \sim \left(1, 4, -\frac{(1+3q-q')}{4}\right) = (\Phi_2^{(-q)}, \Phi_2^{(-q-1)}, \Phi_2^0, \Phi_2^{(q'-q)})^T. \tag{23}$$

Thus, if $\Phi_2^0$ has a VEV $\frac{\omega}{\sqrt{2}}$ then $E_a^q$ gets mass from a matrix:

$$(m_E)_{ab} = h_{ab}^{E}\frac{\omega}{\sqrt{2}}. \tag{24}$$

Finally, the ordinary lepton masses come from the following Yukawa term,

$$-L_{\text{Yukawa}}^{l} = h_{ab}^{l}\overline{f_{aL}}\Phi_3 l_{bR} + \text{H.c.}, \tag{25}$$

where

$$\Phi_3 \sim \left(1, 4, \frac{(3+q+q')}{4}\right) = (\Phi_3^{(+)}, \Phi_3^0, \Phi_3^{(q+1)}, \Phi_3^{(q'+1)})^T. \tag{26}$$

If $\Phi_3^0$ has a VEV $\frac{v}{\sqrt{2}}$ then the mass matrix related to masses of $l_a$ is

$$(m_l)_{ab} = h_{ab}^{l}\frac{v}{\sqrt{2}}. \tag{27}$$

We turn now to the quark sector where

$$\begin{aligned} &Q_{3L} = (u_3, d_3, T, T')_L^T \sim \left(3, 4, \frac{5+3(q+q')}{12}\right),\\ &u_{3R} \sim (3,1,2/3), \qquad d_{3R} \sim (3,1,-1/3),\\ &T_R \sim \left(3, 1, \frac{2+3q}{3}\right), \qquad T'_R \sim \left(3, 1, \frac{2+3q'}{3}\right). \end{aligned} \tag{28}$$

The $u_3$ gets mass through the Yukawa part,

$$-L_{\text{Yukawa}}^{t} = h^{t}\overline{Q_{3L}}\Phi_4 u_{3R} + \text{H.c.}, \tag{29}$$

where

$$\Phi_4 \sim \left(1, 4, \frac{(q+q'-1)}{4}\right) = (\Phi_4^0, \Phi_4^-, \Phi_4^{(q)}, \Phi_4^{(q')})^T. \tag{30}$$

If $\Phi_4^0$ has a VEV $\frac{u}{\sqrt{2}}$ then the mass term of $u_3$ is

$$m_{u_3} = h^{t}\frac{u}{\sqrt{2}}. \tag{31}$$

The other Yukawa terms related to $Q_{3L}$ are

$$\begin{aligned} -L_{\text{Yukawa}}^{g3} = {}& h^{b}\overline{Q_{3L}}\Phi_3 d_{3R} + h^{T}\overline{Q_{3L}}\Phi_2 T_R\\ &+ h^{T'}\overline{Q_{3L}}\Phi_1 T'_R + \text{H.c.}, \end{aligned} \tag{32}$$

which give three mass terms:

$$m_{d_3} = h^{b}\frac{v}{\sqrt{2}}, \qquad m_T = h^{T}\frac{\omega}{\sqrt{2}}, \qquad m_{T'} = h^{T'}\frac{V}{\sqrt{2}}. \tag{33}$$

Two other quark generations are

$$\begin{aligned} &Q_{\alpha L} = (d_\alpha, -u_\alpha, D_\alpha, D'_\alpha)_L^T \sim \left(3, 4^*, -\frac{1+3(q+q')}{12}\right),\\ &\quad \alpha = 1, 2,\\ &u_{\alpha R} \sim (3,1,2/3), \qquad d_{\alpha R} \sim (3,1,-1/3),\\ &D_{\alpha R} \sim \left(3, 1, -\frac{1+3q}{3}\right), \qquad D'_{\alpha R} \sim \left(3, 1, -\frac{1+3q'}{3}\right). \end{aligned} \tag{34}$$

The relevant Yukawa terms are

$$\begin{aligned} -L_{\text{Yukawa}}^{12} = {}& h_{\alpha\beta}^{d2}\overline{Q_{\alpha L}}\Phi_4^\dagger d_{\beta R} + h_{\alpha\beta}^{u2}\overline{Q_{\alpha L}}\Phi_3^\dagger u_{\beta R}\\ &+ h_{\alpha\beta}^{D2}\overline{Q_{\alpha L}}\Phi_2^\dagger D_{\beta R} + h_{\alpha\beta}^{D'2}\overline{Q_{\alpha L}}\Phi_1^\dagger D'_{\beta R} + \text{H.c.}, \end{aligned} \tag{35}$$

from which it follows that

$$\begin{aligned} &(m_{d_2})_{\alpha\beta} = h_{\alpha\beta}^{d2}\frac{u}{\sqrt{2}}, \qquad (m_{u_2})_{\alpha\beta} = -h_{\alpha\beta}^{u2}\frac{v}{\sqrt{2}},\\ &(m_{D_2})_{\alpha\beta} = h_{\alpha\beta}^{D2}\frac{\omega}{\sqrt{2}}, \qquad (m_{D'_2})_{\alpha\beta} = h_{\alpha\beta}^{D'2}\frac{V}{\sqrt{2}}. \end{aligned} \tag{36}$$

We emphasize that if all fermions except neutrinos have the right-handed counterparts, then only four Higgs quadruplets are needed. Because the sum of the $X$ charges over four Higgs quadruplets vanishes, in the Higgs potential, there always exists an antisymmetric term $\epsilon_{ijkl}\Phi_1^i\Phi_2^j\Phi_3^k\Phi_4^l$.

### C. Gauge boson masses

Gauge boson masses arise from the covariant kinetic term of the Higgses,

$$L_{\mathrm{Higgs}} = \sum_{i=1}^{4} (D^\mu \langle \Phi_i \rangle)^\dagger D_\mu \langle \Phi_i \rangle. \quad (37)$$

The covariant derivative is defined as

$$\begin{aligned} D_\mu &= \partial_\mu - ig \sum_{a=1}^{15} A_{a\mu} T_a - ig' X B''_\mu T_{16} \\ &\equiv \partial_\mu - ig P^{NC}_\mu - ig P^{CC}_\mu, \end{aligned} \quad (38)$$

where $g$, $g'$ and $A_{a\mu}$, $B''_\mu$ are gauge couplings and fields of the gauge groups $SU(4)_L$ and $U(1)_X$, respectively. For the quadruplet, $T_{16} = \frac{1}{2\sqrt{2}} \mathrm{diag}(1,1,1,1)$, and the part relating to neutral currents is

$$\begin{aligned} P^{NC}_\mu = \frac{1}{2} \mathrm{diag}\Bigg( & A_3 + \frac{A_8}{\sqrt{3}} + \frac{A_{15}}{\sqrt{6}} + Xt \frac{B''}{\sqrt{2}}, \\ & -A_3 + \frac{A_8}{\sqrt{3}} + \frac{A_{15}}{\sqrt{6}} + Xt \frac{B''}{\sqrt{2}}, \\ & -\frac{2A_8}{\sqrt{3}} + \frac{A_{15}}{\sqrt{6}} + Xt \frac{B''}{\sqrt{2}}, -\frac{3A_{15}}{\sqrt{6}} + Xt \frac{B''}{\sqrt{2}} \Bigg)_\mu, \end{aligned} \quad (39)$$

where the spacetime indices of gauge fields are omitted for compactness, and $t \equiv g'/g$. The part associated with charged currents is

$$\begin{aligned} P^{CC}_\mu &= \frac{1}{2} \sum_a \lambda_a A_{a\mu}; \quad a = 1,2,4,5,6,7,9,10,11,12,13,14 \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & W^+ & W^{-q}_{13} & W^{-q'}_{14} \\ W^- & 0 & W^{-(1+q)}_{23} & W^{-(1+q')}_{24} \\ W^q_{13} & W^{(1+q)}_{23} & 0 & W^{(q-q')}_{34} \\ W^{q'}_{14} & W^{(1+q')}_{24} & W^{-(q-q')}_{34} & 0 \end{pmatrix}_\mu, \end{aligned} \quad (40)$$

where we have denoted $\sqrt{2} W^\mu_{13} \equiv A^\mu_1 - i A^\mu_3$, and so forth. The upper subscripts label the electric charges of gauge bosons. We note that this part does not depend on the $X$-charges of quadruplets.

To summarize, with the following Higgs vacuum structure,

$$\begin{aligned} \langle \Phi_1 \rangle &= \left(0,0,0,\frac{V}{\sqrt{2}}\right)^T, & \langle \Phi_2 \rangle &= \left(0,0,\frac{\omega}{\sqrt{2}},0\right)^T, \\ \langle \Phi_3 \rangle &= \left(0,\frac{v}{\sqrt{2}},0,0\right)^T, & \langle \Phi_4 \rangle &= \left(\frac{u}{\sqrt{2}},0,0,0\right)^T, \end{aligned} \quad (41)$$

masses of non-Hermitian (charged) gauge bosons are given by

$$\begin{aligned} m^2_W &= \frac{g^2(v^2+u^2)}{4}, & m^2_{W_{13}} &= \frac{g^2(u^2+\omega^2)}{4}, \\ m^2_{W_{23}} &= \frac{g^2(v^2+\omega^2)}{4}, \\ m^2_{W_{14}} &= \frac{g^2(u^2+V^2)}{4}, & m^2_{W_{24}} &= \frac{g^2(v^2+V^2)}{4}, \\ m^2_{W_{34}} &= \frac{g^2(\omega^2+V^2)}{4}. \end{aligned} \quad (42)$$

By spontaneous symmetry breaking (SSB), the following relation should be in order: $V \gg \omega \gg u, v$, and from (42) one gets

$$u^2 + v^2 = v^2_{SM} = 246^2\ \mathrm{GeV}^2. \quad (43)$$

### D. Neutral gauge bosons

Inserting Eq. (39) into the Higgs multiplets, we get the mass terms

$$\begin{aligned} M^{2NG}_{\mathrm{mass}} = \frac{g^2}{4} \Bigg[ & u^2 \left( A_3 + \frac{A_8}{\sqrt{3}} + \frac{A_{15}}{\sqrt{6}} + X_{\Phi_4} t \frac{B''}{\sqrt{2}} \right)^2 \\ & + v^2 \left( -A_3 + \frac{A_8}{\sqrt{3}} + \frac{A_{15}}{\sqrt{6}} + X_{\Phi_3} t \frac{B''}{\sqrt{2}} \right)^2 \\ & + \omega^2 \left( -\frac{2A_8}{\sqrt{3}} + \frac{A_{15}}{\sqrt{6}} + X_{\Phi_2} t \frac{B''}{\sqrt{2}} \right)^2 \\ & + V^2 \left( -\frac{3A_{15}}{\sqrt{6}} + X_{\Phi_1} t \frac{B''}{\sqrt{2}} \right)^2 \Bigg]. \end{aligned} \quad (44)$$

In the basis $(A_{3\mu}, A_{8\mu}, A_{15\mu}, B''_\mu)$, the respective squared mass matrix is given by

$$M^{2NG}_{\mathrm{mass}} = \frac{g^2}{4} \begin{pmatrix} u^2+v^2 & \frac{1}{\sqrt{3}}(u^2-v^2) & \frac{1}{\sqrt{6}}(u^2-v^2) & \frac{t}{\sqrt{2}}(X_{\Phi_4}u^2 - X_{\Phi_3}v^2) \\ & \frac{1}{3}(u^2+v^2+4\omega^2) & \frac{1}{3\sqrt{2}}(u^2+v^2-2\omega^2) & \frac{t}{\sqrt{6}}(X_{\Phi_4}u^2 + X_{\Phi_3}v^2 - 2X_{\Phi_2}\omega^2) \\ & & \frac{1}{6}(u^2+v^2+\omega^2+9V^2) & \frac{t}{2\sqrt{3}}(X_{\Phi_4}u^2 + X_{\Phi_3}v^2 + X_{\Phi_2}\omega^2 - 3X_{\Phi_1}V^2) \\ & & & \frac{t^2}{2}(X^2_{\Phi_4}u^2 + X^2_{\Phi_3}v^2 + X^2_{\Phi_2}\omega^2 + X^2_{\Phi_1}V^2) \end{pmatrix}. \quad (45)$$

Following the above assumption, the SSB following the pattern

$$SU(4)_L \otimes U(1)_X \xrightarrow{V} SU(3)_L \otimes U(1)_N \xrightarrow{\omega} SU(2)_L \otimes U(1)_Y \xrightarrow{u,v} U(1)_Q$$

will be used for constructing the matching relation of the gauge couplings and $U(1)$ charges of the group $SU(4)_L \times U(1)_X$ and those of the SM gauge group $SU(2)_L \times U(1)_Y$. Corresponding to each step of the breaking, the neutral gauge boson states will be changed as follows:

$$\begin{aligned} & SU(4)_L \otimes U(1)_X \overset{A_3,A_8,A_{15},B''}{\longrightarrow} SU(3)_L \otimes U(1)_N \\ & \overset{A_3,A_8,B',Z_4''}{\longrightarrow} SU(2)_L \otimes U(1)_Y \overset{A_3,B,Z_3',Z_4'}{\longrightarrow} U(1)_Q : A, Z, Z_3', Z_4'. \end{aligned} \tag{46}$$

At the first step of breaking, the nonzero VEV $V \neq 0$ just results in $M^{2NG}_{\text{mass}} \to M^2_{43} = M^{2NG}_{\text{mass}}|_{w=v=u=0}$, where the $M^2_{43}$ is

$$M^2_{43} = \frac{g^2}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3V^2}{2} & \frac{-3ctV^2}{4\sqrt{2}} \\ 0 & 0 & \frac{-3ctV^2}{4\sqrt{2}} & \frac{3c^2t^2V^2}{16} \end{pmatrix}. \tag{47}$$

The transformation $C_{43}$ relating the two before- and after-breaking bases, namely, $(A_3, A_8, A_{15}, B'')^T = C^T_{43} \times (A_3, A_8, B', Z''_4)^T$, is given by

$$C_{43} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{43} & s_{43} \\ 0 & 0 & -s_{43} & c_{43} \end{pmatrix}, \tag{48}$$

with

$$c_{43} \equiv \frac{ct}{\sqrt{8+c^2t^2}} \quad \text{and} \quad s_{43} = \frac{2\sqrt{2}}{\sqrt{8+c^2t^2}}. \tag{49}$$

After this step, only $A_{15}$ and $B''$ mix with each other to create the $U(1)_N$ gauge boson, denoted as $B'$, of the $SU(3)_L \times U(1)_N$ group. Also, the case $c = 0$ leads to $c_{43} = 0$ and $s_{43} = 1$, implying that the $SU(4)_L$ decouples from the $U(1)_X$. The diagonal squared mass matrix can be found as $M^2_{43d} = C_{43} M^2_{43} C^T_{43} = \text{diagonal}(0, 0, 0, \frac{3V^2}{2s^2_{43}})$, including three massless and one massive values.

Similarly, the second step of the breaking from $SU(3)_L \times U(1)_N \to SU(2)_L \times U(1)_Y$ can be done by the second transformation $C_{32}$ satisfying $(A_3, A_8, B', Z''_4)^T = C^T_{32} \times (A_3, B, Z''_3, Z''_4)^T$, where $Z''_3$ and $Z''_4$, at this moment, are not mass eigenstates. The squared mass matrix now is $M^{2NG}_{\text{mass}} \to M^2_{42} = M^{2NG}_{\text{mass}}|_{v=u=0}$ so that $(A_3, A_8, A_{15}, B'')^T = C^T_{42} \times (A_3, B, Z''_3, Z''_4)^T$. The concrete transformation of the two steps of the breaking are $C_{42} = C_{32}.C_{43}$. These transformations are given as follows,

$$C_{32} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{32} & s_{32} & 0 \\ 0 & -s_{32} & c_{32} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C_{42} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{32} & c_{43}s_{32} & s_{43}s_{32} \\ 0 & -s_{32} & c_{43}c_{32} & s_{43}c_{32} \\ 0 & 0 & -s_{43} & c_{43} \end{pmatrix}, \tag{50}$$

where

$$\begin{aligned} s_{32} &= \frac{2\sqrt{2}}{\sqrt{b^2t^2s^2_{43}+8}} = \frac{\sqrt{8+c^2t^2}}{\sqrt{8+(b^2+c^2)t^2}}, \\ c_{32} &= \frac{bts_{43}}{\sqrt{b^2t^2s^2_{43}+8}} = \frac{bt}{\sqrt{8+(b^2+c^2)t^2}}. \end{aligned} \tag{51}$$

After two steps of breaking, the squared mass matrix,

$$\begin{aligned} M'^2_{42} &= C_{42} M^2_{42} C^T_{42} \\ &= \frac{g^2}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4w^2}{3s^2_{32}} & \frac{\sqrt{2}(-1+c_{43}s_{43}bt)w^2}{3s_{43}s_{32}} \\ 0 & 0 & \frac{\sqrt{2}(-1+c_{43}s_{43}bt)w^2}{3s_{43}s_{32}} & \frac{(-1+c_{43}s_{43}bt)^2w^2+9V^2}{6s^2_{43}} \end{pmatrix}, \end{aligned} \tag{52}$$

contains two massless eigenvalues corresponding to the $A_3$ and $B$ gauge bosons of the SM gauge group $SU(2)_L \times U(1)_Y$. The two states $Z''_3$ and $Z''_4$ are still not eigenstates, but they decouple from the SM gauge bosons. They can be easily diagonalized, but that will be done later. To find the matching condition, we pay attention to the $B_\mu$ state involved with the neutral part of the covariant derivative that changes following the steps of the breaking, namely,

$$\begin{aligned} D^{(41)}_{\text{neutral}\mu} &= \partial_\mu - ig(A_{3\mu}T^3 + A_{8\mu}T^8 + A_{15\mu}T^{15} \\ &\quad + t\times X \times B''_\mu T^{16}) \\ \to D^{(21)}_{\text{neutral}\mu} &= \partial_\mu - ig[A_{3\mu}T^3 + c_{32}B_\mu T^8 + c_{43}s_{32}B_\mu T^{15} \\ &\quad + t\times X\times s_{43}s_{32}B_\mu T^{16}] \\ &= \partial_\mu - igA_{3\mu}T^3 - i\frac{gt}{\sqrt{8+(b^2+c^2)t^2}} \\ &\quad \times B_\mu(bT^8 + cT^{15} + XI_4), \end{aligned} \tag{53}$$

where all $A_{8,15}$ and $B''$ are replaced by $B$ based on (50). Identifying the $D^{(21)}_\mu$ in (53) with the covariant derivative defined in the SM, we derive that the gauge coupling of the $SU(4)_L$ is the $SU(2)_L$ coupling $g$. The other two important equalities are

$$\frac{gt}{\sqrt{8+(b^2+c^2)t^2}} = g_1, \quad \text{and} \quad \frac{\hat{Y}}{2} = bT^8 + cT^{15} + XI_4, \tag{54}$$

where $g_1$ and $\hat{Y}$ are the coupling and charge operator of the SM $U(1)_Y$ gauge group. The second formula in (54) is consistent with the identification of $\hat{Y}$ from the definition of the electric charge operator (1). Furthermore, it can be seen that $\hat{N} \equiv cT^{15} + X$ and $b \equiv \beta/\sqrt{3}$ are relations between parameters defined in the gauge groups $SU(4)_L \times U(1)_X$ and $SU(3)_L \times U(1)_N$.

From $g_1/g = s_W/c_W$, where $s_W^2 = 0.231$, we find

$$t = \frac{g'}{g} = \frac{2\sqrt{2}s_W}{\sqrt{1-(1+b^2+c^2)s_W^2}}. \tag{55}$$

We emphasize the ratio of two couplings—the parameter $t$ is symmetric in changing from $q$ to $q'$. To see this, with the help of Eq. (3), the above formula can be written in terms of $q$ and $q'$ as follows:

$$t = \frac{2\sqrt{2}s_W}{\sqrt{1-[q+q'-qq'+\frac{3}{2}(1+q^2+q'^2)]s_W^2}}. \tag{56}$$

Keeping in mind that $s_W^2 \simeq 0.23$, from (56) we get the constraint on electric charges of the new exotic leptons $E^q_a$ and $E'^{q'}_a$, namely,

$$q+q'-qq'+\frac{3}{2}(1+q^2+q'^2) \leq 4. \tag{57}$$

The squared mass matrix (52) can be diagonalized by a matrix $C'_{32}$, which gives two mass eigenstates $Z'_3$ and $Z'_4$ in the second step of the breaking. While the breaking from $SU(2)_L \times U(1)_Y$ to $U(1)_Q$ is taken by the well-known transformation $C_{21}$, the two transformative matrices are

$$C_{21} = \begin{pmatrix} s_W & c_W & 0 & 0 \\ c_W & -s_W & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$C'_{32} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_\alpha & s_\alpha \\ 0 & 0 & -s_\alpha & c_\alpha \end{pmatrix}, \tag{58}$$

where $c_\alpha = \cos\alpha$ and $s_\alpha = \sin\alpha$ satisfy

$$t_{2\alpha} \equiv -\frac{2\sqrt{2}(-1+c_{43}s_{43}bt)w^2}{s_{43}s_{32}\left[\frac{(-1+c_{43}s_{43}bt)^2w^2+9V^2}{2s_{43}^2} - \frac{4w^2}{s_{32}^2}\right]} = \frac{4\sqrt{2}s_{43}s_{32}(-1+c_{43}s_{43}bt)w^2}{8s_{43}^2w^2 - s_{32}^2[(-1+c_{43}s_{43}bt)^2w^2+9V^2]}. \tag{59}$$

Then we have $M^2_{42d} = C'_{32}M^2_{42}C'^T_{32} = \text{diag}(0,0,m^2_{Z'_3}, m^2_{Z'_4})$, where

$$\begin{aligned} m^2_{Z'_3} &= (M^2_{42d})_{33} = \frac{g^2}{4}\left[\frac{3s_\alpha^2V^2}{2s_{43}^2} + \frac{[2\sqrt{2}c_\alpha s_{43} + s_\alpha s_{32}(-1+bs_{43}c_{43}t)]^2w^2}{6s_{43}^2s_{32}^2}\right], \\ m^2_{Z'_4} &= (M^2_{42d})_{44} = \frac{g^2}{4}\left[\frac{3c_\alpha^2V^2}{2s_{43}^2} + \frac{[2\sqrt{2}s_\alpha s_{43} - c_\alpha s_{32}(-1+bs_{43}c_{43}t)]^2w^2}{6s_{43}^2s_{32}^2}\right]. \end{aligned} \tag{60}$$

The total transformation after all steps of breaking is $C = C_{21}.C'_{32}.C_{32}.C_{43}$. The squared mass matrix of the neutral gauge boson transformed under this rotation is derived as

$$M^2_{41} = C.M^{2NG}_{\text{mass}}.C^T = \text{diag}(0,0,m^2_{Z'_3}, m^2_{Z'_4}) + \delta M^2_{41}, \tag{61}$$

where $\delta M^2_{41}$ is a $4\times 4$ matrix having the property that $(\delta M^2_{41})_{ij} = \mathcal{O}(m_W^2)$ with all $i, j = 1, 2, 3, 4$. In addition, $(\delta M^2_{41})_{i0} = (\delta M^2_{41})_{0i} = 0$ with any $i = 1, 2, 3, 4$, and $(\delta M^2_{41})_{22} = m_Z^2$. We can approximately consider $M^2_{41}$ as the diagonal matrix, where $C$ is the transformation relating the original and physical bases of neutral gauge bosons $(A_3, A_8, A_{15}, B'')^T$ and $(A, Z, Z_3, Z_4)^T$; precisely,

$$\begin{aligned} A_\mu &= s_W A_{3\mu} + c_W(c_{32}A_{8\mu} + c_{43}s_{32}A_{15\mu} + s_{43}s_{32}B''_\mu), \\ Z_\mu &\simeq c_W A_{3\mu} - s_W(c_{32}A_{8\mu} + c_{43}s_{32}A_{15\mu} + s_{43}s_{32}B''_\mu), \\ Z_{3\mu} &\simeq Z'_{3\mu} = -s_{32}c_\alpha A_{8\mu} + (c_{43}c_{32}c_\alpha - s_{43}s_\alpha)A_{15\mu} + (s_{43}c_{32}c_\alpha + c_{43}s_\alpha)B''_\mu, \\ Z_{4\mu} &\simeq Z'_{4\mu} = s_{32}s_\alpha A_{8\mu} - (c_{43}c_{32}s_\alpha + s_{43}c_\alpha)A_{15\mu} + (c_{43}c_\alpha - s_{43}c_{32}s_\alpha)B''_\mu. \end{aligned} \tag{62}$$

Now we return to the Higgs content of the model. From the above presentation, we explicitly see that

(1) If $q, q' \neq 0$, $q, q' \neq -1$, and $q \neq q'$, then the Higgs sector is smallest containing only four neutral Higgs fields.
(2) The case $q = q' = 0$ has been considered in [16], where the Higgs sector contains ten neutral Higgs fields, and there are three non-Hermitian neutral gauges. This case is extremely complicated.
(3) The case $q = q' = -1$ has been considered in [14–20].
(4) The case $q = 0$, $q' = 1$ has been considered in [2,9,21].
(5) The case $q = -1$, $q' = 0$ has been considered in [21].
(6) $\mathrm{SU}(4)(\mathrm{L}) \times \mathrm{U}(1)(\mathrm{X})$ models with a little Higgs have been presented in [22].

Let us summarize the SSB pattern. At the first step of symmetry breaking through $V$, only the following fields get masses: the prime fermions including the exotic leptons $E'_i$, quarks $T'$ and quarks $D'_\alpha$; and the gauge bosons $W_{34}$ and $Z''$. At the second step of SSB through $\omega$, all remain exotic fermions—exotic leptons and quarks get masses. The charged gauge bosons in the top right corner of the non-Hermitian gauge boson matrix [see, (40)] and the extra $Z'$ obtain masses. Finally, the last step is possible through $u$ and $v$ and all the SM fermions and gauge bosons get masses.

Let us take time to briefly review the development of the $\mathrm{SU}(4) \otimes \mathrm{U}(1)$ models. To our knowledge, the first attempt by Fayyazuddin and Riazuddin [23] introduced the decuplet. With the electric charges of leptons $q = 0$, $q' = 1$, the limit on the sine-squared of the Weinberg angle was obtained: $\sin^2\theta_W = 0.25$ and the bound for unification mass is as follows: $3.3 \times 10^4 \geq m_X \geq 6.4 \times 10^3$ GeV. At that time, the particle arrangement in Ref. [23] was not correct. The next step belongs to M. B. Voloshin [8] who attempted to solve the problem related to the realizability of the small mass and large magnetic moment of neutrinos. For this purpose, the author focused on the lepton sector only where the particle arrangement is the same as ours (see below).

The 3-4-1 model in the form discussed here was proposed in [2,9]. The questions concerning anomaly cancellation and quantization of electric charge and the neutrino and generation nonuniversality were addressed in [13] and [24], respectively. In [25], the neutrinos and electromagnetic gauge invariance were discussed, while the Majoron emitting neutrinoless double beta decay in the minimal 3-4-1 model with right-handed neutrino containing a decuplet were addressed in [25,26]. In association with discrete $Z_2$ symmetry, the model without exotic electric charges providing a consistent mass spectrum was proposed in [27]. The $\mathrm{SU}(4)_{(\mathrm{EW})} \times \mathrm{U}(1)_{(B-l)}$ model with left-right symmetry was proposed in [28]. It is interesting that the electroweak unification of quarks and leptons in a gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(4) \times \mathrm{U}(1)$ was built in [29]. The muon anomalous magnetic moment in the $\mathrm{SU}(4) \times \mathrm{U}(1)_N$ model was considered in [30]. The neutrino mass and mixing in the special formalism were presented in [10].

It is worth mentioning that, except for the suspersymmetric 3-4-1 model [31], the Higgs potential containing a decuplet is presented for the first time in this paper.

Now we turn to the model similar to the one originally built in [2,9], with $q = 1$, $q' = 0$.

## III. MINIMAL 3-4-1 WITH RIGHT-HANDED NEUTRINOS

Let us consider a model in which leptons are arranged as

$$f_{aL} = (\nu_a, l_a, l_a^c, \nu_a^c)_L^T \sim (1, 4, 0), \quad a = e, \mu, \tau, \tag{63}$$

where we have in mind that $\nu_L^c \equiv (\nu_R)^c$ and the charge conjugation of $f_{aL}$: $f_{aR}^c \equiv (f_{aL})^c = (\nu_{aR}^c, l_{aR}^c, l_{aR}, \nu_{aR})^T$.

One quark generation is in quadruplet:

$$\begin{aligned} &Q_{3L} = (u_3, d_3, T, T')_L^T \sim \left(3, 4, \frac{2}{3}\right), \\ &u_{3R} \sim (3, 1, 2/3), \qquad d_{3R} \sim (3, 1, -1/3), \\ &T_R \sim \left(3, 1, \frac{5}{3}\right), \qquad T'_R \sim \left(3, 1, \frac{2}{3}\right). \end{aligned} \tag{64}$$

The exotic quarks have electric charges: $q_T = \frac{5}{3}$, $q_{T'} = \frac{2}{3}$. Two other quark generations are in antiquadruplets:

$$Q_{\alpha L} = (d_\alpha, -u_\alpha, D_\alpha, D'_\alpha)^T_L \sim \left(3, 4^*, -\frac{1}{3}\right), \quad \alpha = 1, 2,$$
$$u_{\alpha R} \sim (3, 1, 2/3), \qquad d_{\alpha R} \sim (3, 1, -1/3),$$
$$D_{\alpha R} \sim \left(3, 1, -\frac{4}{3}\right), \qquad D'_{\alpha R} \sim \left(3, 1, -\frac{1}{3}\right). \tag{65}$$

The exotic quarks have electric charges: $q_{D_\alpha} = -\frac{4}{3}$, $q_{D'_\alpha} = -\frac{1}{3}$.

Applying Eq. (1) to Eq. (63), we obtain

$$b = -\sqrt{3}, \qquad c = 0, \qquad X_{f_{aL}} = 0. \tag{66}$$

Then the electric charge operator, for the quadruplet, has the form

$$Q = \mathrm{diag}(X, -1 + X, 1 + X, X). \tag{67}$$

For SSB, we need four Higgs quadruplets, namely,

$$\chi = (\chi_1^0, \chi_2^-, \chi_3^+, \chi_4^0)^T \sim (1, 4, 0),$$
$$\phi = (\phi_1^-, \phi^{--}, \phi^0, \phi_2^-)^T \sim (1, 4, -1),$$
$$\rho = (\rho_1^+, \rho^0, \rho^{++}, \rho_2^+)^T \sim (1, 4, 1),$$
$$\eta = (\eta_1^0, \eta_2^-, \eta_3^+, \eta_4^0)^T \sim (1, 4, 0). \tag{68}$$

In [19], the Higgs sector contains only three Higgs quadruplets, but to produce masses of charged leptons and neutrinos, the nonrenormalizable effective dimension-five and -nine operators were used. Here we prefer the original way in [2,9].

The Yukawa couplings for the quark sector are

$$-L^q_{\mathrm{Yukawa}} = h^t \overline{Q_{3L}} \eta u_{3R} + h^b \overline{Q_{3L}} \rho d_{3R} + h^T \overline{Q_{3L}} \phi T_R + h^{T'} \overline{Q_{3L}} \chi T'_R + h^{d2}_{\alpha\beta} \overline{Q_{\alpha L}} \eta^\dagger d_{\beta R} + h^{u2}_{\alpha\beta} \overline{Q_{\alpha L}} \rho^\dagger u_{\beta R} + h^{D2}_{\alpha\beta} \overline{Q_{\alpha L}} \phi^\dagger D_{\beta R} + h^{D'2}_{\alpha\beta} \overline{Q_{\alpha L}} \chi^\dagger D'_{\beta R} + \mathrm{H.c.} \tag{69}$$

If the Higgs sector has VEV structure as

$$\langle \chi \rangle = \left(0, 0, 0, \frac{V}{\sqrt{2}}\right)^T, \qquad \langle \phi \rangle = \left(0, 0, \frac{\omega}{\sqrt{2}}, 0\right)^T,$$
$$\langle \rho \rangle = \left(0, \frac{v}{\sqrt{2}}, 0, 0\right)^T, \qquad \langle \eta \rangle = \left(\frac{u}{\sqrt{2}}, 0, 0, 0\right)^T, \tag{70}$$

then the quarks get masses as follows:

$$m_{u_3} = h^t \frac{u}{\sqrt{2}}, \qquad m_{d_3} = h^b \frac{v}{\sqrt{2}},$$
$$m_T = h^T \frac{\omega}{\sqrt{2}}, \qquad m_{T'} = h^{T'} \frac{V}{\sqrt{2}},$$
$$(m_{d_2})_{\alpha\beta} = h^{d2}_{\alpha\beta} \frac{u}{\sqrt{2}}, \qquad (m_{u_2})_{\alpha\beta} = -h^{u2}_{\alpha\beta} \frac{v}{\sqrt{2}},$$
$$(m_{D_2})_{\alpha\beta} = h^{D2}_{\alpha\beta} \frac{\omega}{\sqrt{2}}, \qquad (m_{D'_2})_{\alpha\beta} = h^{D'2}_{\alpha\beta} \frac{V}{\sqrt{2}}. \tag{71}$$

Until now, the leptons were massless. To produce masses for leptons from the renormalizable Yukawa interations, we base it on the product $\overline{f_{aL}} f^c_{bR} \sim 6_A \oplus 10^*_S$. If an antisymmetric **6** is used, then the lepton mass matrix will be antisymmetric; consequently, one lepton is still massless. So, the better way is to introduce a symmetric decuplet ($\mathbf{10}_S$) given by

$$H' \sim (1, \mathbf{10}, 0) = \begin{pmatrix} H_1^0 & H_1^- & H_2^+ & H_2^0 \\ H_1^- & H_1^{--} & H_3^0 & H_3^- \\ H_2^+ & H_3^0 & H_2^{++} & H_4^+ \\ H_2^0 & H_3^- & H_4^+ & H_4^0 \end{pmatrix}. \tag{72}$$

The gauge-invariant Lagrangian of the 10-plet is given by

$$L_0^{H'} = \mathrm{Tr}[(D_\mu H')^\dagger D^\mu H'] - V. \tag{73}$$

We will show that the Higgs content written in (72) should be *redefined*. To clarify this, let us consider the kinetic part of $H'$ in (73)

$$\begin{aligned} L^{H'}_{\mathrm{kinetic}} &= \mathrm{Tr}[(\partial_\mu H')^\dagger \partial^\mu H'] \\ &= [\partial_\mu H_1^{0*} \partial^\mu H_1^0 + \partial_\mu H_4^{0*} \partial^\mu H_4^0 + \partial_\mu H_1^{++} \partial^\mu H_1^{--} \\ &\quad + \partial_\mu H_2^{++} \partial^\mu H_2^{--} + 2(\partial_\mu H_2^{0*} \partial^\mu H_2^0 + \partial_\mu H_3^{0*} \partial^\mu H_3^0 \\ &\quad + \partial_\mu H_1^+ \partial^\mu H_1^- + \partial_\mu H_2^+ \partial^\mu H_2^- \\ &\quad + \partial_\mu H_3^+ \partial^\mu H_3^- + \partial_\mu H_4^+ \partial^\mu H_4^-)]. \end{aligned} \tag{74}$$

The factor 2 in the second line of (74) shows that nondiagonal fields in (73) must be redefined as follows:

$$H' \to H = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} H_1^0 & H_1^- & H_2^+ & H_2^0 \\ H_1^- & \sqrt{2} H_1^{--} & H_3^0 & H_3^- \\ H_2^+ & H_3^0 & \sqrt{2} H_2^{++} & H_4^+ \\ H_2^0 & H_3^- & H_4^+ & \sqrt{2} H_4^0 \end{pmatrix}. \tag{75}$$

If so, we have

$$\mathrm{Tr}[(H)^\dagger H] = H_1^{0*}H_1^0 + H_4^{0*}H_4^0 + H_2^{0*}H_2^0 + H_3^{0*}H_3^0 + H_1^{++}H_1^{--} + H_2^{++}H_2^{--} + H_1^+H_1^- + H_2^+H_2^- + H_3^+H_3^- + H_4^+H_4^-. \quad (76)$$

In what follows, we will use $H$ only.

The Yukawa interaction for the lepton is given by

$$\begin{aligned} -L^l_{\mathrm{Yukawa}} &= h^l_{ab}\overline{f_{aL}}Hf^c_{bR} + \mathrm{H.c.} \\ &= \frac{h^l_{ab}}{\sqrt{2}}[\overline{\nu_{aL}}(\sqrt{2}\nu^c_{bR}H_1^0 + l^c_{bR}H_1^- + l_{bR}H_2^+ + \nu_{bR}H_2^0) \\ &\quad + \overline{l_{aL}}(\nu^c_{bR}H_1^- + \sqrt{2}l^c_{bR}H_1^{--} + l_{bR}H_3^0 + \nu_{bR}H_3^-) \\ &\quad + \overline{l^c_{aL}}(\nu^c_{bR}H_2^+ + l^c_{bR}H_3^0 + \sqrt{2}l_{bR}H_2^{++} + \nu_{bR}H_4^+) \\ &\quad + \overline{\nu^c_{aL}}(\nu^c_{bR}H_2^0 + l^c_{bR}H_3^- + l_{bR}H_4^+ + \sqrt{2}\nu_{bR}H_4^0)] \\ &\quad + \mathrm{H.c.} \end{aligned} \quad (77)$$

As usual, assuming an expansion of the neutral Higgs fields as follows,

$$H_3^0 = \frac{v' + R_{H_3^0} - iI_{H_3^0}}{\sqrt{2}}, \qquad H_2^0 = \frac{\epsilon + R_{H_2^0} - iI_{H_2^0}}{\sqrt{2}}, \quad (78)$$

then

$$\langle H\rangle = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & \epsilon \\ 0 & 0 & v' & 0 \\ 0 & v' & 0 & 0 \\ \epsilon & 0 & 0 & 0 \end{pmatrix}. \quad (79)$$

The charged leptons get mass matrix given by

$$(m_l)_{ab} = \frac{h^l_{ab}}{\sqrt{2}}\langle H_3^0\rangle = \frac{h^l_{ab}v'}{2}. \quad (80)$$

The neutrinos obtain the Dirac mass by $\langle H_2^0\rangle$ in the same Yukawa coupling matrix:

$$(m_\nu)_{ab} = \frac{h^l_{ab}}{\sqrt{2}}\langle H_2^0\rangle = \frac{h^l_{ab}\epsilon}{2}. \quad (81)$$

The neutrino Majorana mass will follow from $\langle H_1^0\rangle$ and $\langle H_4^0\rangle$.

Noting that the mixing parameters among the charged leptons are tiny, while those among the neutrinos are large, the matrix in (81) must be modified. It is hoped that the radiative corrections will provide the mixing that matches the current experimental data [32]. We will return to this problem in our future work.

It is emphasized that there are flavor-lepton-violating interactions in Eq. (77).

The lepton number operator is constructed from the diagonal generators as follows:

$$L = \alpha T_3 + \beta T_8 + \gamma T_{15} + \mathcal{L}. \quad (82)$$

For the general case, let us assume that the new extra leptons $E$ and $E'$ acquire lepton number $l$ and $l'$, respectively,

$$f_{aL} = (\nu_a, l_a, E_a, E_a{}')^T_L, \qquad a = e, \mu, \tau. \quad (83)$$

Applying (82) for (83), we obtain

$$\alpha = 0, \qquad \mathcal{L}_{f_{aL}} = \frac{1}{2} + \frac{1}{4}(l + l'), \qquad \beta = \frac{2(1-l)}{\sqrt{3}}, \qquad \gamma = \frac{(2 + l - 3l')}{\sqrt{6}}. \quad (84)$$

The vanishing of the coefficient $\alpha$ is a consequence of the lepton number conservation in the SM. Thus,

$$L = \frac{2(1-l)}{\sqrt{3}}T_8 + \frac{(2 + l - 3l')}{\sqrt{6}}T_{15} + \mathcal{L}. \quad (85)$$

The above formula is useful for the extensions where the flavor discrete symmetries, such as $A_4$, $S_3$, etc., are implemented.

Now we return to our model, where the lepton quadruplet in (63) contains $l^c_a$ and $\nu^c_a$ with lepton number $(-1)$. We then get

$$\beta = \frac{4}{\sqrt{3}}, \qquad \gamma = \frac{2\sqrt{6}}{3}. \quad (86)$$

Hence, the lepton number operator in the minimal 3-4-1 model with right-handed neutrinos gets the form

$$L = \frac{4}{\sqrt{3}}\left(T_8 + \frac{1}{\sqrt{2}}T_{15}\right) + \mathcal{L}. \quad (87)$$

This formula is an extension of that in the 3-3-1 model [33]. For the quadruplet, this operator has the form

$$L = \mathrm{diag}(1 + \mathcal{L}, 1 + \mathcal{L}, -1 + \mathcal{L}, -1 + \mathcal{L}). \quad (88)$$

The fields with nonzero lepton numbers are listed in Tables I, II, and III.

Now we turn to the gauge boson sector. The contribution to the gauge boson masses from $H$ arises from a piece,

$$\begin{aligned} \mathcal{L}^H_{\mathrm{mass}} &= \mathrm{Tr}[(D_\mu\langle H\rangle)^+(D^\mu\langle H\rangle)] \\ &= g^2\mathrm{Tr}[(P^{CC}_\mu\langle H\rangle)^\dagger(P^{\mu CC}\langle H\rangle) \\ &\quad + (P^{NC}_\mu\langle H\rangle)^\dagger(P^{\mu NC}\langle H\rangle)], \end{aligned} \quad (89)$$

where

TABLE I. $\mathcal{B}$ and $\mathcal{L}$ charges for the multiplets in the 3-4-1 model with right-handed neutrinos.

| Multiplet | $\chi$ | $\phi$ | $\eta$ | $\rho$ | $H$ | $Q_{3L}$ | $Q_{\alpha L}$ | $u_{aR}$ | $d_{aR}$ | $T_R$ | $T'_R$ | $D_{\alpha R}$ | $D'_{\alpha R}$ | $f_{aL}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathcal{B}$ charge | 0 | 0 | 0 | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 |
| $\mathcal{L}$ charge | 1 | 1 | −1 | −1 | 0 | −1 | 1 | 0 | 0 | −2 | −2 | 2 | 2 | 0 |

TABLE II. Nonzero lepton number $L$ of the Higgs fields in the 3-4-1 model with right-handed neutrinos.

| Fields | $\chi_1^0$ | $\chi_2^-$ | $\phi_1^-$ | $\phi^{--}$ | $\rho^{++}$ | $\rho_2^+$ | $\eta_3^+$ | $\eta_4^0$ | $H_1^0$ | $H_4^0$ | $H_1^+$ | $H_4^+$ | $H_1^{++}$ | $H_2^{++}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $L$ | 2 | 2 | 2 | 2 | −2 | −2 | −2 | −2 | 2 | −2 | −2 | −2 | −2 | −2 |

$$D_\mu = \partial_\mu - ig\sum_{a=1}^{15} A_{a\mu}T_a - ig'XB''_\mu T_{16} \equiv \partial_\mu - igP_\mu \equiv \partial_\mu - igP_\mu^{NC} - igP_\mu^{CC}. \tag{90}$$

As a result of the symmetric form of the two quadruplets, we have (for details, see [34])

$$(P_\mu H)_{ij} = (P_\mu)_i^k H_{kj} + (P_\mu)_j^k H_{ki}. \tag{91}$$

For the gauge boson masses, one needs to calculate

$$(P_\mu \langle H\rangle)_{ij} = (P_\mu)_i^k \langle H\rangle_{kj} + (P_\mu)_j^k \langle H\rangle_{ki}. \tag{92}$$

We first deal with the charged gauge boson masses being defined through

$$P_\mu^{CC} = \frac{1}{2}\sum_a \lambda_a A_a, \quad a = 1,2,4,5,6,7,9,10,11,12,13,14$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & W'^+ & W_{13}^- & W_{14}^0 \\ W'^- & 0 & W_{23}^{--} & W_{24}^- \\ W_{13}^+ & W_{23}^{++} & 0 & W_{34}^+ \\ (W_{14}^0)^* & W_{24}^+ & W_{34}^- & 0 \end{pmatrix}_\mu = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & W'^+ & Y'^- & N^0 \\ W'^- & 0 & U^{--} & X'^- \\ Y'^+ & U^{++} & 0 & K'^+ \\ (N^0)^* & X'^+ & K'^- & 0 \end{pmatrix}_\mu, \tag{93}$$

where we have denoted $\sqrt{2}W'^+_\mu \equiv A_{1\mu} - iA_{2\mu}$, $Y'^- \equiv W_{13}^-$, $X'^- \equiv W_{24}^-$, $K'^+ \equiv W_{34}^+$, $U^{--} \equiv W_{23}^{--}$ and $N^0 \equiv W_{14}^0$.

The masses of the non-Hermitian neutral $N^0$ and doubly charged $U^{\pm\pm}$ gauge bosons are

$$m^2_{U^{\pm\pm}} = \frac{g^2(\omega^2 + v^2 + 4v'^2)}{4}, \qquad m^2_{N^0} = \frac{g^2(V^2 + u^2 + 4\epsilon^2)}{4}, \tag{94}$$

where the $N^0$ and $N^{0*}$ gauge bosons do not mix with other Hermitian neutral gauge bosons. The squared mass matrix of the singly charged gauge bosons is rewritten in the basis $(W'^\pm, K'^\pm, X'^\pm, Y'^\pm)^T$ as follows:

$$M^2_{G^\pm} = \frac{g^2}{4}\begin{pmatrix} v^2+u^2+v''^2 & 2v'\epsilon & 0 & 0 \\ & w^2+V^2+v''^2 & 0 & 0 \\ & & v^2+V^2+v''^2 & 2v'\epsilon \\ & & & u^2+w^2+v''^2 \end{pmatrix}, \tag{95}$$

where $v''^2 \equiv v'^2 + \epsilon^2$. In the limit $\epsilon = 0$, all of the above masses are the same as those given in [9] (with unique differences associated with $v''$, since in [9] the authors did not use the redefined decuplet). Note that all nondiagonal elements of the matrix (95) are proportional to $v'\epsilon$, which are much smaller than the diagonal ones; therefore, the mass eigenstates of the singly charged gauge bosons can be reasonably identified with those given in [9].

TABLE III. Nonzero lepton number $L$ of fermion in the 3-4-1 model with right-handed neutrinos.

| Fields | $l_a$ | $\nu_a$ | $T$ | $T'$ | $D_\alpha$ | $D'_\alpha$ |
|---|---|---|---|---|---|---|
| $L$ | 1 | 1 | −2 | −2 | 2 | 2 |

The physical states are determined as

$$W_\mu = \cos\theta W'_\mu - \sin\theta K'_\mu, \qquad K_\mu = \sin\theta W'_\mu + \cos\theta K'_\mu, \tag{96}$$

where the $W - K$ mixing angle $\theta$ characterizing lepton number violation is given by

$$\tan 2\theta = \frac{4v'\epsilon}{V^2 + \omega^2 - u^2 - v^2}. \tag{97}$$

For the $X - Y$ mixing, we obtain the physical states,

$$Y_\mu = \cos\theta' Y'_\mu - \sin\theta' X'_\mu, \qquad X_\mu = \sin\theta' Y'_\mu + \cos\theta' X'_\mu, \tag{98}$$

with the mixing angle $\theta'$ defined as

$$\tan 2\theta' = \frac{4v'\epsilon}{V^2 - \omega^2 - u^2 + v^2}. \tag{99}$$

The masses of the physical states are determined as

$$m^2_{W^\pm} \simeq \frac{g^2}{4}(v^2 + u^2 + v''^2), \qquad m^2_{K^\pm} \simeq \frac{g^2}{4}(V^2 + w^2 + v''^2),$$
$$m^2_{X^\pm} \simeq \frac{g^2}{4}(V^2 + v^2 + v''^2), \qquad m^2_{Y^\pm} \simeq \frac{g^2}{4}(w^2 + u^2 + v''^2). \tag{100}$$

From (100), it follows that

$$v^2 + u^2 + v''^2 \simeq v^2_{\rm SM} = (246\ {\rm GeV})^2, \tag{101}$$

and bounds on the singly charged gauge boson mass splitting are

$$|m^2_K - m^2_X| \leq m^2_Y, \qquad |m^2_K - m^2_X - m^2_Y| \leq m^2_W. \tag{102}$$

Comparing (97) with (99), we see that the mixing angle between the lightest $W$ and heaviest $K$ is smaller than the $X - Y$ mixing angle. The mixing angle is quite small and can be constrained from the $W$ decay width (as in the economical 3-3-1 model [35,36]).

From the experimental point of view, the approximation in previous works [9,17], $V = \omega$, creates the difficulty in distinguishing between the bileptons $X$ and $Y$. So the natural way is to assume $V \gg \omega$. Note that $U^{\pm\pm}$ and $Y^\pm$ are similar to the singly charged gauge bosons in the minimal 3-3-1 model [3], while $N^0$ and $X^\pm$ play the similar role in the 3-3-1 model with right-handed neutrinos [2,4]. The heaviest singly charged gauge bosons $K^\pm$ are the completely new ones that couple with the exotic quarks and right-handed leptons only (see Sec. III A). In our assignment (and also in Voloshin's paper [8]), particles belonging to the minimal version are lighter than those in the 3-3-1 model with right-handed neutrinos [i.e., Eqs. (94) and (100)]. For the original 3-4-1 model [2,9], the above consequence is the opposite.

Now we turn to the neutral gauge boson sector. In the basis $(A_{3\mu}, A_{8\mu}, A_{15\mu}, B''_\mu)$, the squared mass matrix for the neutral gauge bosons is given by

$$M^{2NG}_{\rm mass} = \frac{g^2}{4}\begin{pmatrix} u^2+v^2+v''^2 & \frac{1}{\sqrt{3}}(u^2-v^2+v''^2) & \frac{1}{\sqrt{6}}(u^2-v^2-2v''^2) & -\frac{t}{\sqrt{2}}v^2 \\ & \frac{1}{3}(u^2+v^2+4\omega^2+v''^2) & \frac{1}{3\sqrt{2}}(u^2+v^2-2\omega^2-2v''^2) & \frac{t}{\sqrt{6}}(v^2+2\omega^2) \\ & & \frac{1}{6}(u^2+v^2+\omega^2+9V^2+4v''^2) & \frac{t}{2\sqrt{3}}(v^2-\omega^2) \\ & & & \frac{t^2}{2}(v^2+\omega^2) \end{pmatrix}. \tag{103}$$

It has a property that $\mathrm{Det}(M^{2NG}_{\rm mass}) = 0$, implying a massless state of the photon.

The mass eigenvalues of this matrix can be done the same way as in the general case. For this particular case, we have

$$s_{43} = 1, \quad c_{43} = 0, \quad s_{32} = \frac{2\sqrt{2}}{\sqrt{8+3t^2}}, \quad c_{32} = \frac{-\sqrt{3}t}{\sqrt{8+3t^2}},$$
$$t_{2\alpha} = \frac{2\sqrt{8+3t^2}w^2}{9V^2 - (7+3t^2)w^2}. \tag{104}$$

The transformative matrix $C_{41}$ satisfying $(A, Z, Z'_3, Z'_4)^T = C_{41}.(A_3, A_8, A_{15}, B'')^T$ now has the form

$$C_{41} = \begin{pmatrix} s_W & c_W c_{32} & 0 & c_W s_{32} \\ c_W & -s_W c_{32} & 0 & -s_W s_{32} \\ 0 & -c_\alpha s_{32} & -s_\alpha & c_\alpha c_{32} \\ 0 & s_\alpha s_{32} & -c_\alpha & -s_\alpha c_{32} \end{pmatrix}. \tag{105}$$

The squared mass matrix in the new basis is obtained as

$$M^{2NG}_{41} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & m^2_Z & m^2_{23} & m^2_{24} \\ & & m^2_{Z'_3} & m^2_{34} \\ & & & m^2_{Z'_4} \end{pmatrix}, \tag{106}$$

where

$$
\begin{aligned}
m_Z^2 &= \frac{g^2(v^2+u^2+v''^2)}{4c_W^2} = \frac{m_W^2}{c_W^2},\\
m_{Z'_3}^2 &= \frac{g^2}{24}\left[9s_\alpha^2V^2 + \left(s_\alpha - c_\alpha\sqrt{8+3t^2}\right)^2 w^2\right] + \frac{g^2}{24}\left[(\sqrt{2}c_\alpha s_{32}+s_\alpha)^2u^2\right.\\
&\quad \left. + \left(s_\alpha + \frac{(3t^2+4)c_\alpha s_{32}}{2\sqrt{2}}\right)^2 v^2 + 2(\sqrt{2}s_\alpha - c_\alpha s_{32})^2 v''^2\right],\\
m_{Z'_4}^2 &= \frac{g^2}{24}\left[9c_\alpha^2V^2 + \left(c_\alpha + s_\alpha\sqrt{8+3t^2}\right)^2 w^2\right] + \frac{g^2}{24}\left[(c_\alpha - \sqrt{2}s_\alpha s_{32})^2u^2\right.\\
&\quad \left. + \left(c_\alpha - \frac{(3t^2+4)s_\alpha s_{32}}{2\sqrt{2}}\right)^2 v^2 + 2(\sqrt{2}c_\alpha + s_\alpha s_{32})^2 v''^2\right],\\
m_{23}^2 &= \frac{g^2}{4}\left[-\left(\frac{c_\alpha\sqrt{1-4s_W^2}}{c_W^2\sqrt{3}} + \frac{s_\alpha}{\sqrt{6}c_W}\right)u^2 + \left(\frac{c_\alpha(3-2c_W^2)}{c_W^2\sqrt{3(1-4s_W^2)}} + \frac{s_\alpha}{\sqrt{6}c_W}\right)v^2\right.\\
&\quad \left. + \left(-\frac{c_\alpha\sqrt{1-4s_W^2}}{c_W^2\sqrt{3}} + \frac{\sqrt{2}s_\alpha}{\sqrt{3}c_W}\right)v''^2\right] \sim \mathcal{O}(m_W^2),\\
m_{24}^2 &= \frac{g^2}{4}\left[\frac{c_\alpha(v^2-u^2+2v''^2)}{\sqrt{6}c_W} + \frac{s_\alpha[(4c_W^2-3)(u^2+v''^2)+(2c_W^2-3)v^2]}{c_W^2\sqrt{3(4c_W^2-3)}}\right] \sim \mathcal{O}(m_W^2),\\
m_{34}^2 &= \frac{g^2}{24}\left[\frac{c_{2\alpha}}{\sqrt{3t^2+8}}[4(u^2-2v''^2)+(3t^2+4)v^2]\right.\\
&\quad \left. + \frac{s_{2\alpha}}{2(3t^2+8)}[(3t^2-8)u^2 - (8+21t^2+9t^4)v^2 + 4(3t^2+4)v''^2]\right] \sim \mathcal{O}(m_W^2).
\end{aligned}
\tag{107}
$$

It can be seen that all of the nondiagonal elements are in the order of $\mathcal{O}(m_W^2)$. Therefore, they are much smaller than $m_{Z'_3}^2$ and $m_{Z'_4}^2$, implying that these two values can be approximately the eigeinvalues of the matrix (106). Furthermore, the largest contribution to the $m_Z^2 = (M_{41}^{2NG})_{22}$ in the final diagonal matrix, which is proportional to $m_W^2 \times \mathcal{O}(\frac{u^2+v^2}{V^2+w^2})$, is also small. In conclusion, the matrix (106) can be considered as the diagonal matrix where the eigenvalues correspond to the diagonal elements, and the matrix $C_{41}$ is the one relating the two original and mass bases of the neutral gauge bosons.

### A. Currents

From the Lagrangian of the fermion,

$$
L_{\text{fermion}} = i\sum_f \bar{f}\gamma^\mu D_\mu f, \tag{108}
$$

one gets the interactions of the charged gauge bosons with the leptons in the following Lagrangian part:

$$
\begin{aligned}
L_{\text{leptons}} &= \frac{g}{\sqrt{2}}[\overline{\nu_{aL}}\gamma^\mu(W'^+_\mu l_{aL} + Y'^-_\mu l^c_{aL} + N^0_\mu \nu^c_{aL}) + \overline{l_{aL}}\gamma^\mu(U^{--}_\mu l^c_{aL} + X'^-_\mu \nu^c_{aL}) + \overline{l^c_{aL}}\gamma^\mu K'^+_\mu \nu^c_{aL}] + \text{H.c.}\\
&= \frac{g}{\sqrt{2}}[\overline{\nu_{aL}}\gamma^\mu(W'^+_\mu l_{aL} + N^0_\mu \nu^c_{aL}) - \overline{l_{aR}}\gamma^\mu Y'^-_\mu \nu^c_{aR} + \overline{l_{aL}}\gamma^\mu(U^{--}_\mu l^c_{aL} + X'^-_\mu \nu^c_{aL}) - \overline{\nu_{aR}}\gamma^\mu K'^+_\mu l_{aR}] + \text{H.c.}\\
&= \frac{g}{\sqrt{2}}[\overline{\nu_{aL}}\gamma^\mu N^0_\mu \nu^c_{aL} + \overline{l_{aL}}\gamma^\mu U^{--}_\mu l^c_{aL} + \overline{\nu_a}\gamma^\mu(c_\theta P_L + s_\theta P_R)l_a W^+_\mu + \overline{\nu_a}\gamma^\mu(s_\theta P_L - c_\theta P_R)l_a K^+_\mu\\
&\quad + \overline{l_a}\gamma^\mu(c_{\theta'}P_L - s_{\theta'}P_R)\nu^c_a X^-_\mu - \overline{l_a}\gamma^\mu(s_{\theta'}P_L + c_{\theta'}P_R)\nu^c_a Y^-_\mu] + \text{H.c.},
\end{aligned}
\tag{109}
$$

where we have used

$$
\overline{l^c_{aL}}\gamma^\mu \nu^c_{aL} = -\overline{\nu_{aR}}\gamma^\mu l_{aR}.
$$

From (109), we see that new gauge bosons $K^{\pm}$ play a similar role of the SM $W^{\pm}$ for right-handed leptons with just opposite sign of coupling constant $g$. This right-handed current also appeared in [37]. The gauge bosons carrying lepton number 2 (called bilepton gauge bosons) include $Y^{\pm}$, $N^0$, $U^{\pm\pm}$, and $X^{\pm}$.

For quarks, we have

$$\begin{aligned}L_{\text{quarks}} = \frac{g}{\sqrt{2}}[\bar{u_{3L}}\gamma^\mu(W'^+_\mu d_{3L} + Y'^-_\mu T_L + N^0_\mu T'_L) \\ + \bar{d_{3L}}\gamma^\mu(U^{--}_\mu T_L + X'^-_\mu T'_L) + \bar{T}_L\gamma^\mu K'^+_\mu T'_L \\ + \bar{d_{\alpha L}}\gamma^\mu(-W'^-_\mu u_{\alpha L} + Y'^+_\mu D_{\alpha L} + N^{0*}_\mu D'_{\alpha L}) \\ - \bar{u_{\alpha L}}\gamma^\mu(U^{++}_\mu D_{\alpha L} + X'^+_\mu D'_{\alpha L}) \\ + \bar{D_{\alpha L}}\gamma^\mu K'^-_\mu D'_{\alpha L}] + \text{H.c.}\end{aligned} \tag{110}$$

Taking into account the mixing among singly charged gauge bosons, we can express the above expression as follows,

$$\begin{aligned}-\mathcal{L}^{\text{CC}} = \frac{g}{\sqrt{2}}(J^{\mu-}_W W^+_\mu + J^{\mu-}_K K^+_\mu + J^{\mu-}_X X^+_\mu + J^{\mu-}_Y Y^+_\mu \\ + J^{\mu0*}_N N^0_\mu + J^{\mu--}_U U^{++}_\mu + \text{H.c.}),\end{aligned} \tag{111}$$

where

$$\begin{aligned}J^{\mu-}_W &= c_\theta(\bar{\nu}_{\alpha L}\gamma^\mu l_{aL} + \bar{u_{3L}}\gamma^\mu d_{3L} - \bar{u}_{\alpha L}\gamma^\mu d_{\alpha L}) \\ &\quad - s_\theta(-\bar{\nu_{aR}}\gamma^\mu l_{aR} + \bar{T}_L\gamma^\mu T'_L + \bar{D'_{\alpha L}}\gamma^\mu D_{\alpha L}),\end{aligned} \tag{112}$$

$$\begin{aligned}J^{\mu-}_K &= c_\theta(-\bar{\nu_{aR}}\gamma^\mu l_{aR} + \bar{T}_L\gamma^\mu T'_L + \bar{D'_{\alpha L}}\gamma^\mu D_{\alpha L}) \\ &\quad + s_\theta(\bar{\nu}_{aL}\gamma^\mu l_{aL} + \bar{u_{3L}}\gamma^\mu d_{3L} - \bar{u}_{\alpha L}\gamma^\mu d_{\alpha L}), \\ J^{\mu-}_X &= c_{\theta'}(\bar{\nu^c_{aL}}\gamma^\mu l_{aL} + \bar{T'_L}\gamma^\mu d_{3L} - \bar{u_{\alpha L}}\gamma^\mu D'_{\alpha L}) \\ &\quad + s_{\theta'}(\bar{l^c_{aL}}\gamma^\mu \nu_{aL} + \bar{T}_L\gamma^\mu u_{3L} + \bar{d_{\alpha L}}\gamma^\mu D_{\alpha L}), \\ J^{\mu-}_Y &= c_{\theta'}(\bar{l^c_{aL}}\gamma^\mu \nu_{aL} + \bar{T}_L\gamma^\mu u_{3L} + \bar{d_{\alpha L}}\gamma^\mu D_{\alpha L}) \\ &\quad - s_{\theta'}(\bar{\nu^c_{aL}}\gamma^\mu l_{aL} + \bar{T'_L}\gamma^\mu d_{3L} - \bar{u_{\alpha L}}\gamma^\mu D'_{\alpha L}), \\ J^{\mu--}_U &= \bar{l^c_{aL}}\gamma^\mu l_{aL} + \bar{T}_L\gamma^\mu d_{3L} - \bar{u_{\alpha L}}\gamma^\mu D_{\alpha L}, \\ J^{\mu0*}_N &= \bar{\nu_{aL}}\gamma^\mu\nu^c_{aL} + \bar{u_{3L}}\gamma^\mu T'_L + \bar{D'_{\alpha L}}\gamma^\mu d_{\alpha L}.\end{aligned} \tag{113}$$

It is emphasized that in the above expression, all fermions are in the weak states. For precision, they should be in the mass states. For the latter case, in the quark sector, the CKM matrix will appear. In the model under consideration, due to the neutrino Dirac mass matrix, the lepton mixing matrix $V_{\text{PMNS}}$ will appear in the $J^{\mu-}_W$. So in terms of mass eigenstates, the current in (112) has a new form:

$$J^{\mu-}_W = c_\theta(\bar{\nu}_{iL}\gamma^\mu V^{ij}_{\text{PMNS}} l_{jL} + s_\theta\bar{\nu_{iR}}\gamma^\mu V^{ij}_{\text{PMNS}} l_{jR}) + \cdots. \tag{114}$$

The neutral currents, including the electromagnetic current, are

$$\begin{aligned}-\mathcal{L}^{\text{NC}} = eJ^\mu_{em}A_\mu + \frac{g_4}{2c_W}\sum_{i=1}^{3} Z^i_\mu \sum_f \{\bar{f}\gamma^\mu[g^{(V)}(f)_{iV} \\ - g^{(A)}(f)_{iA}\gamma_5]f\},\end{aligned} \tag{115}$$

where

$$e = g\sin\theta_W, \qquad t = \frac{g'}{g} = \frac{2\sqrt{2}\sin\theta_W}{\sqrt{1-4\sin^2\theta_W}}, \tag{116}$$

and $Z^{1,2,3}$, which can be identified as $Z^1 \simeq Z$ and $Z^{2,3} \simeq Z'_{3,4}$, are exact eigenstates of the matrix (106).

The neutral currents are similar to that shown in Ref. [9], so the reader is referred to the mentioned work. Similar to the 3-3-1 models [38], in the model under consideration, there are FCNCs at the tree level due to $Z^2$ and $Z^3$.

The formula (116) leads to a consequence,

$$\sin^2\theta_W < 0.25, \tag{117}$$

which is the same as in the minimal 3-3-1 model. As mentioned in (56), this constraint is the same for the original version [2,9] and for the version we are considering now.

With the particle content in both the fermion and Higgs sectors similar to that in [23], we suggest that the unification mass is in the range of $\mathcal{O}(10)$ TeV. The possible Landau pole similar to those in the minimal 3-3-1 model [39] will be considered in our future work. We note that some interesting aspects relating to the Landau poles of both the minimal 3-3-1 models and 3-4-1 models were indicated in [19].

## IV. HIGGS POTENTIAL

The most general potential can then be written in the following form:

$$V(\eta,\rho,\phi,\chi,H) = V(\eta,\rho,\phi,\chi) + V(H),$$

where

$$\begin{aligned}V(\eta,\rho,\phi,\chi) = \mu_1^2\eta^\dagger\eta + \mu_2^2\rho^\dagger\rho + \mu_3^2\phi^\dagger\phi + \mu_4^2\chi^\dagger\chi \\ + \lambda_1(\eta^\dagger\eta)^2 + \lambda_2(\rho^\dagger\rho)^2 + \lambda_3(\phi^\dagger\phi)^2 + \lambda_4(\chi^\dagger\chi)^2 \\ + (\eta^\dagger\eta)[\lambda_5(\rho^\dagger\rho) + \lambda_6(\phi^\dagger\phi) + \lambda_7(\chi^\dagger\chi)] \\ + (\rho^\dagger\rho)[\lambda_8(\phi^\dagger\phi) + \lambda_9(\chi^\dagger\chi)] + \lambda'_9(\phi^\dagger\phi)(\chi^\dagger\chi) \\ + \lambda_{10}(\rho^\dagger\eta)(\eta^\dagger\rho) + \lambda_{11}(\rho^\dagger\phi)(\phi^\dagger\rho) \\ + \lambda_{12}(\rho^\dagger\chi)(\chi^\dagger\rho) + \lambda_{13}(\phi^\dagger\eta)(\eta^\dagger\phi) \\ + \lambda_{14}(\chi^\dagger\eta)(\eta^\dagger\chi) + \lambda_{15}(\chi^\dagger\phi)(\phi^\dagger\chi) \\ + (f\epsilon^{ijkl}\eta_i\rho_j\phi_k\chi_l + \text{H.c.}),\end{aligned} \tag{118}$$

and $V(H)$ consists of the lepton-number-conserving (LNC) and -violating (LNV) parts, namely,

$$V(H) \equiv V_{\mathrm{LNC}} + V_{\mathrm{LNV}}, \tag{119}$$

$$\begin{aligned} V_{\mathrm{LNC}} &= \mu_5^2 \mathrm{Tr}(H^\dagger H) + [f_4 \chi^\dagger H \eta^* + \mathrm{H.c.}] + \lambda_{16} \mathrm{Tr}[(H^\dagger H)^2] + \lambda_{17}[\mathrm{Tr}(H^\dagger H)]^2 \\ &\quad + \mathrm{Tr}(H^\dagger H)[\lambda_{18}(\eta^\dagger \eta) + \lambda_{19}(\rho^\dagger \rho) + \lambda_{20}(\phi^\dagger \phi) + \lambda_{21}(\chi^\dagger \chi)] \\ &\quad + \lambda_{22}(\chi^\dagger H)(H^\dagger \chi) + \lambda_{23}(\eta^\dagger H)(H^\dagger \eta) + \lambda_{24}(\rho^\dagger H)(H^\dagger \rho) + \lambda_{25}(\phi^\dagger H)(H^\dagger \phi), \\ V_{\mathrm{LNV}} &= f_2 \chi^\dagger H \chi^* + f_3 \eta^\dagger H \eta^* + \lambda_{26} \mathrm{Tr}(H^\dagger H)(\chi^\dagger \eta) + \lambda_{27}(\chi^\dagger H)(H^\dagger \eta) + \mathrm{H.c.} \end{aligned} \tag{120}$$

In the below illustration for the Higgs spectrum, we consider only the LNC part of $V(H)$. The minimum conditions correspond to the six linear coefficients of the neutral Higgs bosons with nonzero VEVs that vanish, leading to the six equalities shown in the appendix.

The squared mass matrix $\mathcal{M}^2_{\mathrm{DCH}}$ of the doubly charged Higgs (DCH) bosons is shown in the appendix. It can be checked that $\det \mathcal{M}^2_{\mathrm{DCH}} = 0$, so that there exist two Goldstone bosons of the $U^{\pm\pm}$ bosons. They can be found exactly as follows:

$$G_U^{\pm\pm} = \frac{\sqrt{2}v' H_1^{\pm\pm} - \sqrt{2}v' H_2^{\pm\pm} - v\rho^{\pm\pm} + w\phi^{\pm\pm}}{\sqrt{w^2 + v^2 + 4v'^2}}. \tag{121}$$

There are three physical masses. In the limit $v'^2 \simeq 0$, these masses are

$$\begin{aligned} m^2_{h_1^{\pm\pm}} &= \frac{1}{4}(-\lambda_{25} w^2 + \lambda_{24} v^2) = -m^2_{h_2^{\pm\pm}}, \\ m^2_{h_3^{\pm\pm}} &= \frac{w^2 + v^2}{2}\left(\lambda_{11} - \frac{fVu}{wv}\right), \end{aligned} \tag{122}$$

where $h_i^{\pm\pm}$, $i = 1, 2, 3$ are mass eigenstates of the DCHs. Hence, in the limit $v' = 0$, there always exists a negative value of $-|\frac{1}{4}(-\lambda_{25} w^2 + \lambda_{24} v^2)|$, implying a negative squared mass at the tree level. On the other hand, when $v' \neq 0$, the matrix $\mathcal{M}^2_{\mathrm{DCH}}$ in (A2) gives a tree-level mass relation, $\mathrm{Tr}(\mathcal{M}^2_{\mathrm{DCH}}) = \sum_{i=1}^3 m^2_{h_i^{\pm\pm}}$, or equivalently,

$$m^2_{h_1^{\pm\pm}} + m^2_{h_2^{\pm\pm}} + m^2_{h_3^{\pm\pm}} = \lambda_{16} v'^2 + \frac{w^2 + v^2}{2}\left(\lambda_{11} - \frac{fVu}{wv}\right). \tag{123}$$

As a consequence of the vacuum stabilities that the Higgs potential must be bounded from below, we have $\lambda_{16} > 0$. Then the sum of the two squared DCH masses (123) is in the order of $\mathcal{O}(\lambda_{16} v'^2)$. Because the DCH are solutions of the equation $\det(\mathcal{M}^2_{\mathrm{DCH}} - I_4 \times m^2_{h^{\pm\pm}}) = 0$, we have another relation:

$$m^2_{h_1^{\pm\pm}} m^2_{h_2^{\pm\pm}} m^2_{h_3^{\pm\pm}} = -\frac{1}{16}(-\lambda_{25} w^2 + \lambda_{24} v^2)^2 \times \frac{w^2 + v^2 + 4v'^2}{2}\left(\lambda_{11} - \frac{fVu}{wv}\right). \tag{124}$$

The right-hand side of (124) is nonpositive because the factor $(\lambda_{11} - \frac{fVu}{wv})$ has the same positive sign with squared masses of the heavy DCH $h_3^{\pm\pm}$. Hence, there is always a negative squared mass of DCH at the tree level. To avoid DCH tachyons, the $|\lambda_{25} w^2 - \lambda_{24} v^2|$ should be small so that the loop contributions can raise the DCH mass to positive values. As a result, the parameter $|\lambda_{25}|$ should be very small, and the model predicts the existence of rather light DCHs.

There are 12 pairs of singly charged Higgs (SCH) components in the original basis. In the mass basis, there are four massless pairs which are Goldstone bosons of $W$, $X$, $Y$, and $K$ gauge bosons. In the limit $\epsilon = 0$, the squared mass matrix in the original basis decomposes into four independent $3 \times 3$ matrices; see the details in the Appendix. Each of them has only one zero eigenvalue, implying the massless state, and two other massive values. All of the massless states are

$$\begin{aligned} G_1^\pm &= \frac{(-v' H_1^\pm - w\phi_1^\pm + u\eta_3^\pm)}{\sqrt{w^2 + u^2 + v'^2}}, \qquad G_2^\pm = \frac{(-V\chi_2^\pm + v' H_4^\pm + v\rho_2^\pm)}{\sqrt{V^2 + v^2 + v'^2}}, \\ G_3^\pm &= \frac{(w\phi_2^\pm - V\chi_3^\pm + v' H_3^\pm)}{\sqrt{V^2 + w^2 + v'^2}}, \qquad G_4^\pm = \frac{(-u\eta_2^\pm + v\rho_1^\pm + v' H_2^\pm)}{\sqrt{u^2 + v^2 + v'^2}}. \end{aligned} \tag{125}$$

In this limit, it is easily to identify that $G_Y^\pm \equiv G_1^\pm$, $G_X^\pm \equiv G_2^\pm$, $G_K^\pm \equiv G_3^\pm$, and $G_W^\pm \equiv G_4^\pm$ which are respective Goldstone bosons absorbed by $Y^\pm$, $X^\pm$, $K^\pm$, and $W^\pm$ bosons.

With $v' \neq 0$, the masses as well as mass eigenstates are a bit complicated. For illustration, it is enough to consider here the limit $v'$, $\epsilon \to 0$. The mass eigenvalues of the other eight pairs of SCHs are

$$\begin{aligned} m^2_{h_1^\pm} &= \frac{1}{4}(\lambda_{23}u^2 - \lambda_{25}w^2), \qquad m^2_{h_2^\pm} = \frac{1}{4}(\lambda_{23}u^2 - \lambda_{24}v^2),\\ m^2_{h_3^\pm} &= \frac{u^2+v^2}{2}\left(\lambda_{10} - f\frac{wV}{uv}\right), \qquad m^2_{h_4^\pm} = \frac{u^2+\omega^2}{2}\left(\lambda_{13} - \frac{fvV}{wu}\right),\\ m^2_{h_5^\pm} &= \frac{V^2+\omega^2}{2}\left(\lambda_{15} - \frac{fvu}{Vw}\right), \qquad m^2_{h_6^\pm} = \frac{V^2+v^2}{2}\left(\lambda_{12} - \frac{fwu}{Vv}\right),\\ m^2_{h_7^\pm} &= \frac{1}{4}(\lambda_{22}V^2 - \lambda_{25}w^2), \qquad m^2_{h_8^\pm} = \frac{1}{4}(\lambda_{22}V^2 - \lambda_{24}v^2), \end{aligned} \tag{126}$$

with the respective mass states as follows:

$$h_1^\pm \equiv H_1^\pm, \qquad h_2^\pm = H_2^\pm, \qquad h_3^\pm = \frac{v\eta_2^\pm + u\rho_1^\pm}{\sqrt{u^2+v^2}}, \qquad h_4^\pm = \frac{u\phi_1^\pm + w\eta_3^\pm}{\sqrt{u^2+w^2}},$$
$$h_5^\pm = \frac{V\phi_2^\pm + w\chi_3^\pm}{\sqrt{V^2+w^2}}, \qquad h_6^\pm = \frac{v\chi_2^\pm + V\rho_2^\pm}{\sqrt{V^2+v^2}}, \qquad h_7^\pm \equiv H_3^\pm, \qquad h_8^\pm \equiv H_4^\pm.$$

The model predicts two rather light SCHs, $h_1^\pm$ and $h_2^\pm$, because $\lambda_{ij}$ should be in the order of $\mathcal{O}(1)$, $|\lambda_{25}w|$ is not too large, and $u$, $v'$, and $v$ are in the electroweak scale.

There are ten neutral Higgs components in the original basis. Four of the ten CP-odd neutral Higgses are massless, of which the four independent combinations of them are four Goldstone bosons of the $Z$, $Z^2, Z^3$, and $N^0$ gauge bosons. But there is still one more massless state, which is exactly $H_3^0$, at the tree level. In the limit $\epsilon \to 0$, the mass eigenstates of the CP-odd neutral Higgs are shown explicitly in the appendix. There are five massive states, with eigenstates denoted as $H_{A_i}$ $(i = 1, 5)$, where three imagined parts of $H^0_{1,2,4}$ are approximate egeinstates. The condition of the positive mass of the CP-odd neutral Higgs $H_{A_5}$ shows that $f < 0$. There also exists one light CP-odd neutral Higgs boson.

In the neutral sector, one of the ten CP-even neutral Higgs bosons is the Goldstone boson of the $N^{0*}$ boson. The squared mass matrix separates into two submatrices, namely the $4\times 4$ and $6\times 6$ matrices. They are denoted as $\mathcal{M}^2_{1H^0}$ and $\mathcal{M}^2_{2H^0}$, corresponding to the two respective sub-bases $(\mathrm{Re}[H_1^0], \mathrm{Re}[H_4^0], \mathrm{Re}[\chi_1^0], \mathrm{Re}[\eta_4^0])^T$ and $(\mathrm{Re}[H_3^0], \mathrm{Re}[\chi_4^0], \mathrm{Re}[\phi_3^0], \mathrm{Re}[\rho_2^0], \mathrm{Re}[\eta_1^0], \mathrm{Re}[H_2^0])^T$. The massless values are contained in $\mathcal{M}^2_{1H^0}$. In the limit $\epsilon \to 0$, three other mass values are

$$\begin{aligned} m^2_{h_1^0} &= \frac{1}{4}(2\lambda_{23}u^2 - \lambda_{24}v^2 - 2\lambda_{16}v'^2 - \lambda_{25}w^2), \qquad m^2_{h_2^0} = \frac{V^2+u^2}{2}\left(\lambda_{14} - \frac{fwv}{Vu}\right),\\ m^2_{h_3^0} &= \frac{1}{4}(2\lambda_{22}V^2 - \lambda_{24}v^2 - 2\lambda_{16}v'^2 - \lambda_{25}w^2). \end{aligned} \tag{127}$$

The mass eigenstates $(h_1^0, h_3^0, h_3^0)$ and the Goldstone bosons $G_{N^{0*}}$ in this case are

$$h_1^0 \equiv \mathrm{Re}[H_1^0], \qquad h_3^0 \equiv \mathrm{Re}[H_4^0], \qquad \begin{pmatrix} G_{N^{0*}} \\ h_2^0 \end{pmatrix} = \begin{pmatrix} -\frac{V}{\sqrt{V^2+u^2}} & \frac{u}{\sqrt{V^2+u^2}} \\ \frac{u}{\sqrt{V^2+u^2}} & \frac{V}{\sqrt{V^2+u^2}} \end{pmatrix} \begin{pmatrix} \mathrm{Re}[\chi_1^0] \\ \mathrm{Re}[\eta_4^0] \end{pmatrix}.$$

It can be seen that while the two last are very heavy with the order of the $SU(3)_L$ and $SU(4)_L$ breaking scales, the first Higgs boson may be lighter because $|\lambda_{25}w^2|$ should not be large as discussed above. So it may be the SM Higgs boson or that in the 750 GeV diphoton excess.

Now consider the second mass matrix $\mathcal{M}^2_{2H^0}$. From our investigation, in general it is easy to check that $\det \mathcal{M}^2_{2H^0} \neq 0$. But if $v' = 0$, $\mathcal{M}^2_{2H^0}$ has a massless value. In addition, if $v' = v = u = 0$, the matrix has two massless values, implying that there may be two light CP-even neutral Higgs bosons. Hence one of them can be identified with the SM Higgs boson, meaning that the Higgs sector of the model under consideration is reliable. The main contributions to the four heavy Higgs bosons are

$$m_{h_4^0}^2 = -fwV, \qquad m_{h_5^0}^2 = \frac{1}{4}(\lambda_{22}V^2 - \lambda_{25}w^2),$$
$$m_{h_{6,7}^0}^2 = \lambda_3 w^2 + \lambda_4 V^2 \pm \sqrt{(-\lambda_3 w^2 + \lambda_4 V^2)^2 + \lambda_9'^2 w^2 V^2}. \tag{128}$$

To conclude, in the Higgs sector, we would like to emphasize two important results. First, the above investigation can be applied for the models where the $10_S$ $H$ is not included. Second, the model predicts many Higgs bosons with masses near the TeV range that today colliders can detect. Hence, this aspect of the Higgs sector should be explored in more detail.

## V. PHENOMENOLOGY

Our aim in this section is to find some constraints on the parameters of the model. From mixing of the singly charged gauge bosons, we have some special features related to the SM $W$ boson.

(1) In the model under consideration, the $W$ boson has the following normal main decay modes:

$$W^- \to l\tilde{\nu}_l (l = e, \mu, \tau),$$
$$\searrow u^c d, u^c s, u^c b, (u \to c), \tag{129}$$

which are the same as in the SM. Due to the $W - K$ mixing, we have other modes related to right-handed lepton counterparts, namely,

$$W^- \to l_R \tilde{\nu}_{lR} \quad (l = e, \mu, \tau). \tag{130}$$

It is easy to compute the tree-level decay widths as follows [40]. The predicted total width for the $W$ decay into fermions is

$$\Gamma_W^{\text{tot}} = 1.04 \frac{\alpha M_W}{2 s_W^2}(1 - s_\theta^2) + \frac{\alpha M_W}{4 s_W^2}. \tag{131}$$

This is quite similar to the case of the economical 3-3-1 model [36]. From the recent data of the $W^\pm$ boson [32]: $\alpha(m_Z) \simeq 1/128$, $m_W = 80.385 \pm 0.015$ GeV, $\Gamma_W^{\text{tot}} = 2.085 \pm 0.042$ GeV. The $s_\theta$ is less constrained than that of [36], with an upper bound of $s_\theta \leq 0.19$.

(2) In the model under consideration, the muon decay,

$$\mu^- \to e^- + \tilde{\nu}_e + \nu_\mu,$$

consists of two diagrams mediated by $W$ and $K$. The Feynman diagrams are on the left in Fig. 1.

The decay width is given by

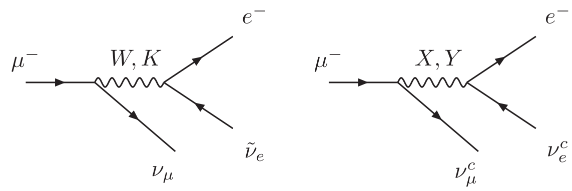


FIG. 1. Feynman diagram giving contribution to muon decay. The left and right diagrams present the main and wrong decay channels, respectively.

$$\Gamma(\mu^- \to e^- + \tilde{\nu}_e + \nu_\mu) = \frac{g^4 m_\mu^5}{6144\pi^2} \times \left(\frac{1}{m_W^4} + \frac{1}{m_K^4}\right). \tag{132}$$

Because the model predicts the wrong decay $\mu^- \to e^- + \nu_e + \tilde{\nu}_\mu$, we assume that the total decay of the muon is $\Gamma_\mu^{\text{total}} = \Gamma(\mu^- \to e^- + \tilde{\nu}_e + \nu_\mu) + \Gamma(\mu^- \to e^- + \tilde{\nu}_\mu + \nu_e)$. This result will be used for in a future study.

(3) The wrong decay of the muon is

$$\mu^- \to e^- + \nu_e + \tilde{\nu}_\mu,$$

where the Feynman diagram is on the right side of Fig. 1. The branching ratio of the wrong decay $\text{Br}(\mu \to e \nu_e \bar{\nu}_\mu) < 0.012$ leads to the following constraint:

$$\frac{\frac{1}{m_X^4} + \frac{1}{m_Y^4}}{\frac{1}{m_W^4} + \frac{1}{m_K^4} + \frac{1}{m_X^4} + \frac{1}{m_Y^4}} < 0.012 \quad \text{or}$$
$$\frac{1}{m_X^4} + \frac{1}{m_Y^4} - 0.012\left(\frac{1}{m_K^4} + \frac{1}{m_X^4} + \frac{1}{m_Y^4}\right) < \frac{0.012}{m_W^4}. \tag{133}$$

In the limit $V \gg w$, i.e., $1/m_K^4$, $1/m_X^4 \ll 1/m_Y^4$, we obtain the below constraint of $m_Y$: $m_Y > m_W \times \sqrt[4]{82.333} \simeq 242$ GeV. This is consistent with those in Ref. [41]. In the limit of $V \simeq w$, implying that $m_X^2 \simeq m_Y^2 \simeq m_K^2/2$, the constraint is more strict: $m_Y > m_W \times \sqrt[4]{164.417} \simeq 287$ GeV.

(4) The $\mu - e$ conversion: The charged current of the model under consideration has a similar structure as the ones discussed in [41–44], so these have similar results as the $\mu - e$ conversion. In particular, the mass of the doubly charged bileptons $U^{\pm\pm}$ satisfies $m_{U^{\pm\pm}} \geq 135$ GeV [43]. Note that this result was concluded for the $SU(3)_L$, but it is the same for the $SU(4)_L$ because they have the same gauge couplings. This constraint is less strict than the constraint from the wrong muon decay because $m_{Y^\pm}$ and $m_{U^{\pm\pm}}$ are related with only the $SU(3)_L$ breaking scale $w$.

## VI. CONCLUSION

In this paper, we have analyzed the 3-4-1 model with arbitrary electric charges of the extra leptons. The scalar and gauge boson sectors are presented in detail. The mixing matrix, the eigenmasses, and the eigenstates of the neutral gauge bosons are analyzed. For future studies, we will focus on the scalar sector, especially the neutral one, in which the SM-like Higgs boson is contained.

Next, we have presented a new development of the original 3-4-1 model. Different from previous works [2,9], in this paper, with the assumption of a new nonzero VEV of a neutral Higgs component in the decuplet $H$, some interesting new features occur: (i) the neutrinos get Dirac masses at tree level and (ii) the mixing among the singly charged gauge bosons leads to the appearance of a new small contribution to the decay width of the $W$ boson, the same as that shown in the economical 3-3-1 model. But under the present $W$ data, the constraint of the mixing angle is less strict. The model also predicts the existence of many bileptons, including new quarks and gauge and Higgs bosons, as well as the LFV interactions. Like the 3-3-1 models, many of these bileptons contribute to the LFV processes, such as the wrong muon decay and the $\mu - e$ conversion. If the $SU(4)_L$ breaking scale is much larger than the $SU(3)_L$ breaking scale, so that it gives suppressed contributions to the LFV processes, the constraint of the $SU(3)_L$ breaking is the same as what was indicated in the 3-3-1 models. In contrast, if the two breaking scales are close together, the lower bounds of the $SU(3)_L$ breaking scale significantly increase. We see this point in the case of the wrong muon decay, where the lower constraints of $m_Y$ are 242 and 287 GeV, corresponding to the two mentioned cases. We have also derived the lepton number operator, and the lepton number of the fields in the model is presented.

As in the minimal 3-3-1 model, in the 3-4-1 model with right-handed neutrino considered here, there exists a bound on the sine-squared of the Weinberg angle, namely, $\sin^2\theta_W < 0.25$. The constraint on the electric charges of extra leptons has been obtained as well.

The Higgs sector is roughly studied. In the limit of lepton number conservation, the Higgs sector contains all massless Goldstone bosons for massive gauge bosons and the SM-like Higgs boson. The model we have considered is quite interesting and deserves further study.

## ACKNOWLEDGMENTS

L. T. H. thanks Le Duc Ninh for useful discussions on gauge anomalies and SM matching conditions. This research is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No. 103.01-2014.51.

## APPENDIX: HIGGS SPECTRUM

This section pays attention to the Higgs potential satisfying the lepton number conservation. In addition, the squared mass matrices are written in the limit of $\epsilon = 0$, except the squared mass matrix of the doubly charged Higgs which is independent of $\epsilon$.

### 1. Minimal conditions of the Higgs potential

We list here six equalities for minimal conditions of the Higgs potential:

$$
\begin{aligned}
\mu_1^2 &= -\frac{1}{4}[2\lambda_{16}\epsilon^2(v'^2-\epsilon^2) - \lambda_{22}\epsilon^2 V^2 + \lambda_{25}\epsilon^2 w^2 + \lambda_{24}\epsilon^2 v^2 + 2fVwvu] \\
&\quad -\frac{1}{2}[\lambda_{18}v''^2 + 2\lambda_1 u^2 + \lambda_5 v^2 + \lambda_6 w^2 + \lambda_7 V^2], \\
\mu_2^2 &= -\frac{1}{2}\left[2\lambda_2 v^2 + \lambda_5 u^2 + \lambda_8 w^2 + \lambda_9 V^2 + \frac{fwVu}{v} + \lambda_{19}v''^2 + \frac{1}{2}\lambda_{24}v'^2\right], \\
\mu_3^2 &= -\frac{1}{2}\left[2\lambda_3 w^2 + \lambda_6 u^2 + \lambda_8 v^2 + \lambda'_9 V^2 + \frac{fuVv}{w} + \lambda_{20}v''^2 + \frac{1}{2}\lambda_{25}v'^2\right], \\
\mu_4^2 &= -\frac{1}{2}[2\lambda_4 V^2 + \lambda_7 u^2 + \lambda_9 v^2 + \lambda'_9 w^2 + \lambda_{21}v''^2] \\
&\quad -\frac{1}{4V^2}[2fVwuv + 2\lambda_{16}\epsilon^2(v'^2-\epsilon^2) - \lambda_{23}\epsilon^2 u^2 + \lambda_{24}\epsilon^2 v^2 + \lambda_{25}\epsilon^2 w^2], \\
\mu_5^2 &= -\frac{1}{2}\left[\lambda_{16}v'^2 + 2\lambda_{17}v''^2 + \lambda_{18}u^2 + \left(\lambda_{19} + \frac{1}{2}\lambda_{24}\right)v^2 + \lambda_{20}w^2 + \lambda_{21}V^2 + \frac{1}{2}\lambda_{25}w^2\right], \\
f_4 &= \frac{\epsilon}{2Vu}[2\lambda_{16}(v'^2-\epsilon^2) - \lambda_{22}V^2 - \lambda_{23}u^2 + \lambda_{24}v^2 + \lambda_{25}w^2].
\end{aligned} \tag{A1}
$$

### 2. Squared mass matrix of doubly charged Higgses

Squared mass matrix of the doubly charged Higgses in the basis $(H_1^{\pm\pm}, H_2^{\pm\pm}, \rho^{\pm\pm}, \phi^{\pm\pm})^T$ is given by

$$\mathcal{M}^2_{\mathrm{DCH}} = \frac{1}{4}\begin{pmatrix} 2\lambda_{16}v'^2 + \lambda_{24}v^2 - \lambda_{25}w^2 & 2\lambda_{16}v'^2 & \sqrt{2}\lambda_{24}vv' & \sqrt{2}\lambda_{25}wv' \\ & 2\lambda_{16}v'^2 - \lambda_{24}v^2 + \lambda_{25}w^2 & \sqrt{2}\lambda_{24}vv' & \sqrt{2}\lambda_{25}wv' \\ & & 2\lambda_{11}w^2 - \frac{2fwVu}{v} & 2\lambda_{11}wv - 2fVu \\ & & & 2\lambda_{11}v^2 - \frac{2fVvu}{w} \end{pmatrix}. \quad (A2)$$

### 3. Squared mass matrices of singly charged Higgses

The squared mass matrix of the singly charged Higgs consists of two independent $6\times 6$ matrices. They are denoted as $\mathcal{M}^2_{1\mathrm{sch}}$ and $\mathcal{M}^2_{2\mathrm{sch}}$ with respective sub-bases $(H_1^{\pm}, \phi_1^{\pm}, \eta_3^{\pm}, \chi_2^{\pm}, H_4^{\pm}, \rho_2^{\pm})^T$ and $(\phi_2^{\pm}, \chi_3^{\pm}, H_3^{\pm}, \eta_2^{\pm}, \rho_1^{\pm}, H_2^{\pm})^T$. In the limit $\epsilon = 0$, they divide into four independent $3\times 3$ submatrices, denoted as

$$\mathcal{M}^2_{1\mathrm{sch}} = \begin{pmatrix} \mathcal{M}'^2_{1\mathrm{sch}} & 0 \\ 0 & \mathcal{M}''^2_{1\mathrm{sch}} \end{pmatrix}, \quad \text{and} \quad \mathcal{M}^2_{2\mathrm{sch}} = \begin{pmatrix} \mathcal{M}'^2_{2\mathrm{sch}} & 0 \\ 0 & \mathcal{M}''^2_{2\mathrm{sch}} \end{pmatrix},$$

where

$$(\mathcal{M}'^2_{1\mathrm{sch}}) = \frac{1}{4}\begin{pmatrix} \lambda_{23}u^2 - \lambda_{25}w^2 & \lambda_{25}wv' & \lambda_{23}uv' \\ & 2\lambda_{13}u^2 - \lambda_{25}v'^2 - \frac{2fVvu}{w} & 2(\lambda_{13}wu - fVv) \\ & & 2\lambda_{13}w^2 + \lambda_{23}v'^2 - \frac{2fVwv}{u} \end{pmatrix}, \quad (A3)$$

$$(\mathcal{M}''^2_{1\mathrm{sch}}) = \frac{1}{4}\begin{pmatrix} \lambda_{22}v'^2 + 2\lambda_{12}v^2 - \frac{2fwuv}{V} & \lambda_{22}Vv' & 2(\lambda_{12}Vv - fwu) \\ & \lambda_{22}V^2 - \lambda_{24}v^2 & \lambda_{24}vv' \\ & & 2\lambda_{12}V^2 - \lambda_{24}v'^2 - \frac{2fVwu}{v} \end{pmatrix}, \quad (A4)$$

$$(\mathcal{M}'^2_{2\mathrm{sch}}) = \frac{1}{4}\begin{pmatrix} 2\lambda_{15}V^2 - \lambda_{25}v'^2 - \frac{2fVvu}{w} & 2(\lambda_{15}Vw - fvu) & \lambda_{25}v'w \\ & 2\lambda_{15}w^2 + \lambda_{22}v'^2 - \frac{2fwvu}{V} & \lambda_{22}Vv' \\ & & \lambda_{22}V^2 - \lambda_{25}w^2 \end{pmatrix}, \quad (A5)$$

and

$$(\mathcal{M}''^2_{2\mathrm{sch}}) = \frac{1}{4}\begin{pmatrix} \lambda_{23}v'^2 + 2\lambda_{10}v^2 - \frac{2fVwv}{u} & 2(\lambda_{10}uv - fVw) & \lambda_{23}v'u \\ & 2\lambda_{10}u^2 - \lambda_{24}v'^2 - \frac{2fVuw}{v} & \lambda_{24}vv' \\ & & \lambda_{23}u^2 - \lambda_{24}v^2 \end{pmatrix}. \quad (A6)$$

### 4. Squared mass matrices of CP-odd neutral Higgses

This $10\times 10$ matrix has a massless state of $\mathrm{Im}[H_3^0]$, even $\epsilon \neq 0$. Furthermore, the remaining part separates into two $4\times 4$ and $5\times 5$ matrices, corresponding to the bases of $(\mathrm{Im}[\chi_1^0], \mathrm{Im}[\eta_4^0], \mathrm{Im}[H_4^0], \mathrm{Im}[H_1^0])^T$ and $(\mathrm{Im}[\rho_2^0], \mathrm{Im}[\phi_3^0], \mathrm{Im}[\chi_4^0], \mathrm{Im}[\eta_1^0], \mathrm{Im}[H_2^0])^T$. In the limit $\epsilon = 0$, the following Higgses are identified with mass eigenstates:

$$\mathrm{Im}[H_4^0] \equiv H_{A_1}, m^2_{A_1} = \frac{1}{4}(2\lambda_{22}V^2 - 2\lambda_{16}v'^2 - \lambda_{24}v^2 - \lambda_{25}w^2),$$
$$\mathrm{Im}[H_1^0] \equiv H_{A_2}, m^2_{A_2} = \frac{1}{4}(2\lambda_{23}u^2 - 2\lambda_{16}v'^2 - \lambda_{24}v^2 - \lambda_{25}w^2),$$
$$\mathrm{Im}[H_2^0] \equiv H_{A_3}, m^2_{A_3} = \frac{1}{4}(\lambda_{22}V^2 + \lambda_{23}u^2 - 2\lambda_{16}v'^2 - \lambda_{24}v^2 - \lambda_{25}w^2). \quad (A7)$$

The nontrivial parts of the two matrices are now $2\times 2$ and $4\times 4$ which relate to the four Goldstone bosons, including $G_{N^0}$ and $G_{Z_i}$ ($i=1,\,2,\,3$), and the two massive CP-odd neutral Higgses, in particular,

$$\begin{pmatrix} G_{N_0} \\ H_{A_4} \end{pmatrix} = \begin{pmatrix} \frac{V}{\sqrt{V^2+u^2}} & \frac{u}{\sqrt{V^2+u^2}} \\ -\frac{u}{\sqrt{V^2+u^2}} & \frac{V}{\sqrt{V^2+u^2}} \end{pmatrix} \begin{pmatrix} \mathrm{Im}[\chi_1^0] \\ \mathrm{Im}[\eta_4^0] \end{pmatrix} \tag{A8}$$

and

$$\begin{pmatrix} G_{Z_1} \\ G_{Z_2} \\ G_{Z_3} \\ H_{A_5} \end{pmatrix} = \begin{pmatrix} -\frac{v}{\sqrt{v^2+u^2}} & 0 & 0 & \frac{v}{\sqrt{v^2+u^2}} \\ -\frac{vu^2}{\sqrt{A(v^2+u^2)}} & 0 & \frac{\sqrt{(v^2+u^2}V}{\sqrt{A}} & -\frac{uv^2}{\sqrt{A(v^2+u^2)}} \\ -\frac{V^2u^2v}{\sqrt{AB}} & \frac{\sqrt{A}w}{\sqrt{B}} & -\frac{Vv^2u^2}{\sqrt{AB}} & -\frac{V^2v^2u}{\sqrt{AB}} \\ \frac{Vwu}{\sqrt{B}} & \frac{Vvu}{\sqrt{B}} & \frac{wvu}{\sqrt{B}} & \frac{Vwv}{\sqrt{B}} \end{pmatrix} \begin{pmatrix} \mathrm{Im}[\rho_2^0] \\ \mathrm{Im}[\phi_3^0] \\ \mathrm{Im}[\chi_4^0] \\ \mathrm{Im}[\eta_1^0] \end{pmatrix}, \tag{A9}$$

where $A = V^2v^2 + u^2(V^2+v^2)$, $B = V^2v^2(w^2+u^2) + w^2u^2(V^2+v^2)$. We note that three Goldstone bosons absorbed by the three Hermitian gauge bosons, $Z_i\,(i=1,2,3)$ are linear combinations of the above massless states, $G_{Z_i}$. But the $G_{Z_1}$ mainly contributes to the Goldstone boson of the SM $Z$ boson. The masses of the $H_{A_{4,5}}$ are

$$m_{A_4}^2 = \frac{(V^2+u^2)}{2}\left(\lambda_{14} - \frac{fwv}{Vu}\right), \qquad m_{A_5}^2 = -\frac{f}{2}\left[\frac{Vvu}{w} + w\left(\frac{Vv}{u} + \frac{u(V^2+v^2)}{Vv}\right)\right]. \tag{A10}$$


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