Adler function and Bjorken polarized sum rule: Perturbation expansions in powers of the $SU(N_c)$ conformal anomaly and studies of the conformal symmetry limit

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We consider a new form of analytical perturbation theory expansion in the massless $SU(N_c)$ theory, for the nonsinglet part of the e^+e^--annihilation to hadrons Adler function D^{ns} and of the Bjorken sum rule of the polarized lepton-hadron deep-inelastic scattering C_{ns}^{Bjp}, and demonstrate its validity at the $O(\alpha_s^4)$-level at least. It is a two-fold series in powers of the conformal anomaly and of $SU(N_c)$ coupling α_s. Explicit expressions are obtained for the $\{\beta\}$-expanded perturbation coefficients at $O(\alpha_s^4)$ level in $\overline{\mathrm{MS}}$ scheme, for both considered physical quantities. Comparisons of the terms in the $\{\beta\}$-expanded coefficients are made with the corresponding terms obtained by using extra gluino degrees of freedom, or skeleton-motivated expansion, or R_δ-scheme motivated expansion in the Principle of Maximal Conformality. Relations between terms of the $\{\beta\}$-expansion for the D^{ns}- and C_{ns}^{Bjp}-functions, which follow from the conformal symmetry limit and its violation, are presented. The relevance to the possible new analyses of the experimental data for the Adler function and Bjorken sum rule is discussed.

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It was demonstrated in [1] that, in the $SU(N_c)$ model of strong interactions, the generalized $\overline{\mathrm{MS}}$ scheme Crewther relation between the analytically evaluated perturbative expression for the nonsinglet (ns) contributions to the Adler function and the Bjorken sum rule of the polarized lepton-hadron deep-inelastic scattering (DIS) can be written as

$$D^{ns}(a_s)C_{ns}^{Bjp}(a_s) = 1 + \Delta_{csb}(a_s), \quad (1)$$

where $\Delta_{csb} \sim a_s^2$, $a_s \equiv \alpha_s(Q^2)/\pi$, and Q^2 is the physical scale of both D^{ns} and C_{ns}^{Bjp}. The unity on the rhs corresponds to the original Crewther relation, derived in [2] in the massless quark-parton model by applying the operator product expansion method to the $\pi^0 \to \gamma\gamma$ decay AVV-triangle amplitude in the conformal symmetry (CS) limit. It was shown in [1,3] that in $\overline{\mathrm{MS}}$ scheme the CS-breaking (CSB) term Δ_{csb} can be presented as a product of the conformal anomaly $\beta(a_s)/a_s$ and a polynomial $P(a_s)$ ($\sim a_s$). In $\overline{\mathrm{MS}}$ scheme the renormalization group (RG) β-function is defined as

$$\beta(a_s) = \mu^2 \frac{\partial a_s(\mu^2)}{\partial \mu^2} = -\sum_{j\geq 0} \beta_j a_s(\mu^2)^{j+2}.$$

When $\sim a_s^4$ contributions to D^{ns} and C_{ns}^{Bjp} [4] are included, the validity of the generalized Crewther relation (1) [1] at this level gets confirmed [4]. The $O(a_s^3)$ expression for the Δ_{csb}-term, fixed in [1], is proportional to the two-loop expressions of the conformal anomaly, multiplied by a polynomial $P(a_s)$ fixed in $\overline{\mathrm{MS}}$ scheme. The term at a_s^2 in $P(a_s)$ contains three $SU(N_c)$ group monomials C_F^2, $C_F C_A$, $C_F T_F n_f$ of total power 2, composed of the Casimir operators C_F, C_A and the flavor-dependent factor $T_F n_f$ (with $T_F = 1/2$).

The expression for Δ_{csb} obtained in [4] is proportional to the three-loop expression of the conformal anomaly, multiplied by the same polynomial $P(a_s)$, which has the third coefficient (at a_s^3) composed of six $SU(N_c)$ group monomials C_F^3, $C_F^2 C_A$, $C_F C_A^2$, $C_F^2(T_F n_f)$, $C_F(T_F n_f)^2$, $C_F C_A(T_F n_f)$ of total power 3. In [5,6] concrete theoretical arguments were presented showing that in $\overline{\mathrm{MS}}$ scheme the conformal anomaly is factorized in all orders of perturbation theory for the Δ_{csb}-term in Eq. (1), and therefore one should have

$$\Delta_{csb} = \left(\frac{\beta(a_s)}{a_s}\right) P(a_s) = \left(\frac{\beta(a_s)}{a_s}\right) \sum_{m\geq 1} K_m a_s^m. \quad (2)$$

In [7] a new form of the $\overline{\mathrm{MS}}$-scheme expression for the CSB term (2) of the generalized Crewther relation was proposed. It is written as the two-fold series

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$$\Delta_{csb}(a_s) = \sum_{n\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s) = \sum_{n\geq 1}\sum_{r\geq 1}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n^{(r)}[k,m]C_F^k C_A^m a_s^r. \quad (3)$$

Here, $r = k+m$ with $k\geq 1$ and $m\geq 0$, while the coefficients $P_n^{(r)}[k,m]$ contain rational numbers and transcendental Riemann ζ_{2l+1} functions with $l\geq 1$. The $SU(N_c)$ monomials in Eq. (3) *do not* contain terms proportional to $T_F n_f$, in contrast to the less detailed expression in Eq. (2) where the coefficients K_m $(m\geq 2)$ do depend on $T_F n_f$ (see [1,4] for explicit $O(a_s^2)$ and $O(a_s^3)$ results). In the postulated representation (3) the dependence on $T_F n_f$ appears in the powers of β-function. The validity and unambiguity of Eq. (3) was checked in [7] at the $O(a_s^4)$ level.

One can ask whether it is possible to formulate the analogous two-fold $\overline{\text{MS}}$-scheme perturbation expansion for D^{ns} and C_{ns}^{Bjp} separately, at least at the analytically available [4] $O(a_s^4)$ level. Here, we present the positive answer to this question, and then discuss the main consequences of this new QCD resummation procedure. In this procedure the expansions for $D^{ns}(a_s)$ and $C_{ns}^{Bjp}(a_s)$ take the following form:

$$D^{ns}(a_s) = 1 + \sum_{n=0}^{3}\left(\frac{\beta(a_s)}{a_s}\right)^n D_n(a_s), \quad (4)$$

$$C_{ns}^{Bjp}(a_s) = 1 + \sum_{n=0}^{3}\left(\frac{\beta(a_s)}{a_s}\right)^n C_n(a_s), \quad (5)$$

where for $0\leq n\leq 3$, at the $O(a_s^4)$ level, the polynomials $D_n(a_s)$ and $C_n(a_s)$ are defined as

$$D_n(a_s) = \sum_{r=1}^{4-n} a_s^r \sum_{k=1}^{r} D_n^{(r)}[k,r-k]C_F^k C_A^{r-k} + a_s^4\delta_{n0}\left(D_0^{(4)}[F,A]\frac{d_F^{abcd}d_A^{abcd}}{d_R} + D_0^{(4)}[F,F]\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f\right), \quad (6)$$

$$C_n(a_s) = \sum_{r=1}^{4-n} a_s^r \sum_{k=1}^{r} C_n^{(r)}[k,r-k]C_F^k C_A^{r-k} + a_s^4\delta_{n0}\left(C_0^{(4)}[F,A]\frac{d_F^{abcd}d_A^{abcd}}{d_R} + C_0^{(4)}[F,F]\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f\right). \quad (7)$$

The double sum expressions for Eqs. (6) and (7) are motivated by the form for the polynomials $P_n(a_s)$ in Eq. (3) introduced in [7]. In $SU(N_c)$ theory and $\overline{\text{MS}}$ scheme they have the unambiguous form determined by the system of linear equations, analogous to the system presented in [7]. The coefficients with the structures $d_F^{abcd}d_A^{abcd}/d_R$ and $d_F^{abcd}d_F^{abcd}/d_R$ appear at the $O(a_s^4)$ level [4]. These structures were defined first in [8], where the four-loop coefficient of the QCD β-function was evaluated. Since $(d_F^{abcd}d_F^{abcd}/d_R)a_s^4$ terms in (6) and (7) are proportional to $T_F n_f$, which also enters the β_0-coefficient of the QCD β-function, one may propose to move them into $D_1(a_s)$- and $C_1(a_s)$-polynomials. We explain below that such a redefinition of Eqs. (4) and (5) is not supported by the QED limit. Thus, the following $\overline{\text{MS}}$-scheme $O(a_s^{4-n})$ expressions for $D_n(a_s)$ $(0\leq n\leq 3)$ are valid:

$$\begin{aligned}
D_0(a_s) = {}& \frac{3}{4}C_F a_s + \left[-\frac{3}{32}C_F^2 + \frac{1}{16}C_F C_A\right]a_s^2 + \left[-\frac{69}{128}C_F^3 - \left(\frac{101}{256} - \frac{33}{16}\zeta_3\right)C_F^2 C_A - \left(\frac{53}{192} + \frac{33}{16}\zeta_3\right)C_F C_A^2\right]a_s^3 \\
& + \left[\left(\frac{4157}{2048} + \frac{3}{8}\zeta_3\right)C_F^4 - \left(\frac{3509}{1536} + \frac{73}{128}\zeta_3 + \frac{165}{32}\zeta_5\right)C_F^3 C_A + \left(\frac{9181}{4608} + \frac{299}{128}\zeta_3 + \frac{165}{64}\zeta_5\right)C_F^2 C_A^2\right. \\
& - \left(\frac{30863}{36864} + \frac{147}{128}\zeta_3 - \frac{165}{64}\zeta_5\right)C_F C_A^3 + \left(\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right)\frac{d_F^{abcd}d_A^{abcd}}{d_R} \\
& \left. + \left(-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5\right)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f\right)\Bigg]a_s^4, \quad (8)
\end{aligned}$$

$$D_1(a_s) = \left(-\frac{33}{8} + 3\zeta_3\right)C_F a_s + \left[\left(\frac{111}{64} + 12\zeta_3 - 15\zeta_5\right)C_F^2 - \left(\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5\right)C_F C_A\right]a_s^2$$
$$+ \left[\left(\frac{758}{128} + \frac{9}{16}\zeta_3 - \frac{165}{2}\zeta_5 + \frac{315}{4}\zeta_7\right)C_F^3 + \left(\frac{3737}{144} - \frac{3433}{64}\zeta_3 + \frac{99}{4}\zeta_3^2 + \frac{615}{16}\zeta_5 - \frac{315}{8}\zeta_7\right)C_F^2 C_A\right.$$
$$\left. + \left(\frac{2695}{384} + \frac{1987}{64}\zeta_3 - \frac{99}{4}\zeta_3^2 - \frac{175}{32}\zeta_5 + \frac{105}{16}\zeta_7\right)C_F C_A^2\right]a_s^3, \tag{9}$$

$$D_2(a_s) = \left(\frac{151}{6} - 19\zeta_3\right)C_F a_s + \left[\left(-\frac{4159}{384} - \frac{2997}{16}\zeta_3 + 27\zeta_3^2 + \frac{375}{2}\zeta_5\right)C_F^2\right.$$
$$\left. + \left(\frac{14615}{256} + \frac{39}{16}\zeta_3 - \frac{9}{2}\zeta_3^2 - \frac{185}{4}\zeta_5\right)C_F C_A\right]a_s^2, \tag{10}$$

$$D_3(a_s) = \left(-\frac{6131}{36} + \frac{203}{2}\zeta_3 + 45\zeta_5\right)C_F a_s. \tag{11}$$

Analogous expressions for the polynomials in (5) read:

$$C_0(a_s) = -\frac{3}{4}C_F a_s + \left[\frac{21}{32}C_F^2 - \frac{1}{16}C_F C_A\right]a_s^2 + \left[-\frac{3}{128}C_F^3 + \left(\frac{125}{256} - \frac{33}{16}\zeta_3\right)C_F^2 C_A + \left(\frac{53}{192} + \frac{33}{16}\zeta_3\right)C_F C_A^2\right]a_s^3$$
$$+ \left[\left(-\frac{4823}{2048} - \frac{3}{8}\zeta_3\right)C_F^4 + \left(\frac{605}{384} + \frac{469}{128}\zeta_3 + \frac{165}{32}\zeta_5\right)C_F^3 C_A + \left(-\frac{11071}{4608} - \frac{695}{128}\zeta_3 - \frac{165}{64}\zeta_5\right)C_F^2 C_A^2\right.$$
$$+ \left(\frac{30863}{36864} + \frac{147}{128}\zeta_3 - \frac{165}{64}\zeta_5\right)C_F C_A^3 + \left(-\frac{3}{16} + \frac{1}{4}\zeta_3 + \frac{5}{4}\zeta_5\right)\frac{d_F^{abcd}d_A^{abcd}}{d_R}$$
$$\left. + \left(\frac{13}{16} + \zeta_3 - \frac{5}{2}\zeta_5\right)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f\right)\right]a_s^4, \tag{12}$$

$$C_1(a_s) = \frac{3}{2}C_F a_s + \left[-\left(\frac{349}{192} + \frac{5}{4}\zeta_3\right)C_F^2 + \left(\frac{155}{96} + \frac{9}{4}\zeta_3 - \frac{5}{2}\zeta_5\right)C_F C_A\right]a_s^2$$
$$+ \left[\left(\frac{997}{384} + \frac{481}{32}\zeta_3 - \frac{145}{8}\zeta_5\right)C_F^3 + \left(-\frac{85801}{4608} - \frac{169}{24}\zeta_3 + \frac{365}{48}\zeta_5 + \frac{105}{4}\zeta_7\right)C_F^2 C_A\right.$$
$$\left. + \left(\frac{931}{768} - \frac{955}{192}\zeta_3 - \frac{895}{96}\zeta_5 - \frac{105}{16}\zeta_7\right)C_F C_A^2\right]a_s^3, \tag{13}$$

$$C_2(a_s) = \left(-\frac{151}{24}\right)C_F a_s + \left[\left(\frac{261}{64} + \frac{87}{8}\zeta_3\right)C_F^2 - \left(\frac{3151}{256} + \frac{43}{16}\zeta_3 + \frac{3}{2}\zeta_3^2 - \frac{15}{4}\zeta_5\right)C_F C_A\right]a_s^2, \tag{14}$$

$$C_3(a_s) = \frac{605}{36}C_F a_s. \tag{15}$$

The singlet (si) corrections to the Adler function and to the Bjorken sum rule should be considered separately (see [9,10]). In the Adler function they appear first in $O(a_s^3)$ [11–13] and are known up to a_s^4 [14]. For the Bjorken sum rule they start to contribute at $O(a_s^4)$ [15,16]. For $n_f = 3, 6$ the si contributions to both quantities are equal to zero. For the cases of $n_f = 4, 5$ they are significantly smaller then the ns-effects.

We explain how the results (8)–(11) and (12)–(15) were obtained. The coefficients β_0, β_1, β_2 of the RG β-function on the rhs of (4) and (5) are known in terms of powers of C_F, C_A and $T_F n_f$. The β_0-term was evaluated in [17,18], β_1 in [19–21], β_2 in $\overline{\text{MS}}$ in [22,23]. To determine the coefficients $D_n^{(r)}[k,m]$, $C_n^{(r)}[k,m]$ in (6) and (7) the lhs of Eqs. (4) and (5) is expressed as

$$D^{ns}(a_s) = 1 + d_1 a_s + d_2 a_s^2 + d_3 a_s^3 + d_4 a_s^4 + O(a_s^5), \tag{16}$$

$$C_{ns}^{Bjp}(a_s) = 1 + c_1 a_s + c_2 a_s^2 + c_3 a_s^3 + c_4 a_s^4 + O(a_s^5), \tag{17}$$

and the $\overline{\rm MS}$-coefficients expanded in color structures of the $SU(N_c)$ group. The coefficients d_1–d_4 are known from the works [24,25], [11] and [4], correspondingly, while c_1–c_4 were evaluated in [26,27], [28] and [4], respectively. Following the logic of [7], we used in Eq. (4) on the lhs the expansion (16), and on the rhs the expansions (6) for $D_n(a_s)$ and the expansions in terms of C_F, C_A and $T_F n_f$ of the RG β-function coefficients. Equating the expressions at all monomials in C_F, C_A and $T_F n_f$ at each power of a_s on both sides of Eq. (4) leads to a complete system of 22 linear equations, analogous to the (smaller) system in [7]. Its unique solution determines the polynomials $D_n(a_s)$ ($0 \le n \le 3$) in Eqs. (8)–(11). To get the results (12)–(15), the analogous procedure is applied to $C_{ns}^{Bjp}(a_s)$. As a cross-check we reproduced the results of [7] for Eq. (3).[1]

In the CS limit, i.e., when $\beta \mapsto 0$ in Eqs. (4) and (5), we get (cf. an analogous identity in [10]):

$$(1 + D_0(a_s(Q^2)) \times (1 + C_0(a_s(Q^2)) = 1, \qquad (18)$$

where $D_0(a_s)$ and $C_0(a_s)$ are given in (8) and (12). The terms proportional to $d_F^{abcd} d_A^{abcd}/d_R$ and $n_f d_F^{abcd} d_F^{abcd}/d_R$, in (8) and (12), cancel out in Eq. (18). This identity is an extension of the Crewther relation, derived in [2] in the Born approximation.

We can now fix the $\{\beta\}$-expansion structure (proposed in [29]) of the coefficients d_j of D^{ns},

$$d_1 = d_1[0], \qquad d_2 = \beta_0 d_2[1] + d_2[0], \qquad (19)$$

$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0], \qquad (20)$$

$$d_4 = \beta_0^3 d_4[3] + \beta_1 \beta_0 d_4[1,1] + \beta_2 d_4[0,0,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 d_4[1] + d_4[0]. \qquad (21)$$

In [29], this was performed up to $O(a_s^3)$ level only, with the $SU(N_c)$ model supplemented by a multiplet of gluino degrees of freedom of SUSY QCD.[2]

Applying the two-fold expansion (4) and the $SU(N_c)$ results (8)–(11), we obtain all $\{\beta\}$-expanded terms in d_2, d_3 and even d_4 $\overline{\rm MS}$-scheme coefficients:

$$\begin{aligned}
d_1[0] &= \frac{3}{4} C_F, \qquad d_2[0] = \left(-\frac{3}{32} C_F^2 + \frac{1}{16} C_F C_A\right),\\
d_2[1] &= \left(\frac{33}{8} - 3\zeta_3\right) C_F, \qquad d_3[0] = \left[-\frac{69}{128} C_F^3 - \left(\frac{101}{256} - \frac{33}{16}\zeta_3\right) C_F^2 C_A - \left(\frac{53}{192} + \frac{33}{16}\zeta_3\right) C_F C_A^2\right],\\
d_3[1] &= \left[\left(-\frac{111}{64} - 12\zeta_3 + 15\zeta_5\right) C_F^2 + \left(\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5\right) C_F C_A\right],\\
d_3[0,1] &= \left(\frac{33}{8} - 3\zeta_3\right) C_F, \qquad d_3[2] = \left(\frac{151}{6} - 19\zeta_3\right) C_F,\\
d_4[0] &= \left[\left(\frac{4157}{2048} + \frac{3}{8}\zeta_3\right) C_F^4 - \left(\frac{3509}{1536} + \frac{73}{128}\zeta_3 + \frac{165}{32}\zeta_5\right) C_F^3 C_A + \left(\frac{9181}{4608} + \frac{299}{128}\zeta_3 + \frac{165}{64}\zeta_5\right) C_F^2 C_A^2\right.\\
&\quad \left. - \left(\frac{30863}{36864} + \frac{147}{128}\zeta_3 - \frac{165}{64}\zeta_5\right) C_F C_A^3 + \left(\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right) \frac{d_F^{abcd} d_A^{abcd}}{d_R} + \left(-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5\right) \frac{d_F^{abcd} d_F^{abcd}}{d_R} n_f,\right.\\
d_4[1] &= \left(-\frac{785}{128} - \frac{9}{16}\zeta_3 + \frac{165}{2}\zeta_5 - \frac{315}{4}\zeta_7\right) C_F^3 - \left(\frac{3737}{144} - \frac{3433}{64}\zeta_3 + \frac{99}{4}\zeta_3^2 + \frac{615}{16}\zeta_5 - \frac{315}{8}\zeta_7\right) C_F^2 C_A\\
&\quad - \left(\frac{2695}{384} + \frac{1987}{64}\zeta_3 - \frac{99}{4}\zeta_3^2 - \frac{175}{32}\zeta_5 + \frac{105}{16}\zeta_7\right) C_F C_A^2,\\
d_4[0,1] &= \left[\left(-\frac{111}{64} - 12\zeta_3 + 15\zeta_5\right) C_F^2 + \left(\frac{83}{32} + \frac{5}{4}\zeta_3 - \frac{5}{2}\zeta_5\right) C_F C_A\right],\\
d_4[2] &= \left(-\frac{4159}{384} - \frac{2997}{16}\zeta_3 + 27\zeta_3^2 + \frac{375}{2}\zeta_5\right) C_F^2 + \left(\frac{14615}{256} + \frac{39}{16}\zeta_3 - \frac{9}{2}\zeta_3^2 - \frac{185}{4}\zeta_5\right) C_F C_A,\\
d_4[0,0,1] &= \left(\frac{33}{8} - 3\zeta_3\right) C_F, \qquad d_4[1,1] = \left(\frac{151}{3} - 38\zeta_3\right) C_F, \qquad d_4[3] = \left(\frac{6131}{36} - \frac{203}{2}\zeta_3 - 45\zeta_5\right) C_F. \qquad (22)
\end{aligned}$$

[1] Note that Eq. (15) in [7] contains a misprint. The $C_F C_A^2 a_s^3$ contribution to $P_1(a_s)$, defined in Eq. (3), should contain an extra $3/4$ factor.

[2] The validity of the $O(a_s^3)$ of the $\{\beta\}$-expansion results of [29] was confirmed recently in [30].

The $\{\beta\}$-expanded coefficients of C_{ns}^{Bjp} have the same structure as Eqs. (19)–(21):

$$c_1 = c_1[0], \qquad c_2 = \beta_0 c_2[1] + c_2[0], \tag{23}$$

$$c_3 = \beta_0^2 c_3[2] + \beta_1 c_3[0,1] + \beta_0 c_3[1] + c_3[0], \tag{24}$$

$$c_4 = \beta_0^3 c_4[3] + \beta_1\beta_0 c_4[1,1] + \beta_2 c_4[0,0,1] + \beta_0^2 c_4[2] + \beta_1 c_4[0,1] + \beta_0 c_4[1] + c_4[0]. \tag{25}$$

Using the two-fold series (5) and Eqs. (12)–(15), we get

$$\begin{aligned}
c_1[0] &= -\frac{3}{4}C_F, \qquad c_2[0] = \left(\frac{21}{32}C_F^2 - \frac{1}{16}C_F C_A\right),\\
c_2[1] &= -\frac{3}{2}C_F, \qquad c_3[0] = \left[-\frac{3}{128}C_F^3 + \left(\frac{125}{256} - \frac{33}{16}\zeta_3\right)C_F^2 C_A + \left(\frac{53}{192} + \frac{33}{16}\zeta_3\right)C_F C_A^2\right],\\
c_3[1] &= \left(\frac{349}{192} + \frac{5}{4}\zeta_3\right)C_F^2 - \left(\frac{155}{96} + \frac{9}{4}\zeta_3 - \frac{5}{2}\zeta_5\right)C_F C_A,\\
c_3[0,1] &= -\frac{3}{2}C_F, \qquad c_3[2] = -\frac{115}{24}C_F,\\
c_4[0] &= \left[\left(-\frac{4823}{2048} - \frac{3}{8}\zeta_3\right)C_F^4 + \left(\frac{605}{384} + \frac{469}{128}\zeta_3 + \frac{165}{32}\zeta_5\right)C_F^3 C_A + \left(-\frac{11071}{4608} - \frac{695}{128}\zeta_3 - \frac{165}{64}\zeta_5\right)C_F^2 C_A^2\right.\\
&\quad \left. + \left(\frac{30863}{36864} + \frac{147}{128}\zeta_3 - \frac{165}{64}\zeta_5\right)C_F C_A^3 + \left(-\frac{3}{16} + \frac{1}{4}\zeta_3 + \frac{5}{4}\zeta_5\right)\frac{d_F^{abcd}d_A^{abcd}}{d_R} + \left(\frac{13}{16} + \zeta_3 - \frac{5}{2}\zeta_5\right)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f\right],\\
c_4[1] &= \left[\left(-\frac{997}{384} - \frac{481}{32}\zeta_3 + \frac{145}{8}\zeta_5\right)C_F^3 + \left(\frac{85801}{4608} + \frac{169}{24}\zeta_3 - \frac{365}{48}\zeta_5 - \frac{105}{4}\zeta_7\right)C_F^2 C_A\right.\\
&\quad \left. - \left(\frac{931}{768} - \frac{955}{192}\zeta_3 - \frac{895}{96}\zeta_5 - \frac{105}{16}\zeta_7\right)C_F C_A^2\right],\\
c_4[0,1] &= \left(\frac{349}{192} + \frac{5}{4}\zeta_3\right)C_F^2 - \left(\frac{155}{96} + \frac{9}{4}\zeta_3 - \frac{5}{2}\zeta_5\right)C_F C_A,\\
c_4[2] &= \left[\left(\frac{261}{64} + \frac{87}{8}\zeta_3\right)C_F^2 - \left(\frac{3151}{256} + \frac{43}{16}\zeta_3 + \frac{3}{2}\zeta_3^2 - \frac{15}{4}\zeta_5\right)C_F C_A\right]\\
c_4[0,0,1] &= -\frac{3}{2}C_F, \qquad c_4[1,1] = -\frac{115}{12}C_F,\\
c_4[3] &= -\frac{605}{36}C_F.
\end{aligned} \tag{26}$$

Note that specific contributions to d_3 and c_3 differ from those given in [7,10,29]. The results for the $\{\beta\}$-expansion of d_4 and c_4 are new.

As mentioned, formally it is possible to rewrite the $a_s^4\delta_{n0}(d_F^{abcd}d_F^{abcd}/d_R)n_f$ contribution to Eqs. (6) and (7),

$$a_s^4\delta_{n0}D_0^{(4)}[F,F]\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f \mapsto a_s^4\left(\delta_{n0}\frac{11C_A}{4T_F}D_0^{(4)}[F,F] + \delta_{n1}\frac{3}{T_F}D_1^{(4)}[F,F]\right) \times \frac{d_F^{abcd}d_F^{abcd}}{d_R}, \tag{27}$$

where $D_0^{(4)}[F,F] = D_1^{(4)}[F,F]$. This leads to rearrangements of the $a_s^4(d_F^{abcd}d_F^{abcd}/d_R)$ terms in (8) between the a_s^4 terms of Eqs. (8) and (9), and to the redefinitions of the terms $d_4[0]$ and $d_4[1]$ in the $\{\beta\}$-expansion of the coefficient d_4,

$$d_4^{\mathrm{mod}}[0] = d_4[0] - D_0^{(4)}[F,F]\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f + \frac{11}{4}D_0^{(4)}[F,F]\frac{C_A d_F^{abcd}d_F^{abcd}}{T_F d_R}, \tag{28}$$

$$d_4^{mod}[1] = d_4[1] - 3D_1^{(4)}[F,F]\frac{d_F^{abcd}d_F^{abcd}}{T_F d_R}, \tag{29}$$

where $D_0^{(4)}[F,F] = D_1^{(4)}[F,F] = (-13/16 - \zeta_3 + 5\zeta_5/2)$. This gives the n_f-independent term $d_4^{mod}[0]$. However, this rearrangement is not supported by the QED limit, which should be valid in the case of theoretically self-consistent definition of the new resummed representations of Eq. (4) and of the related $\{\beta\}$-expanded expressions for the coefficients d_i. This QED limit is realized by fixing $C_A = 0$, $T_F = 1$, $d_F^{abcd} d_F^{abcd}/d_R = 1$ and $n_f = N$, where N is the number of leptons. In QED the remaining $D_0^{(4)}[F, F]$-contribution arises from the five-loop Feynman diagram with light-by-light scattering internal subgraph, contributing to the photon vacuum polarization function. However, this subgraph is convergent and does not give extra β_0-dependent (or N-dependent) contribution to the coefficient d_4. Therefore, we prefer the definitions of Eqs. (6) and (7) without applying to them the rearrangements of Eq. (27). Note also that $d_F^{abcd} d_F^{abcd}$ structure is contributing the n_f^2 part of the four-loop coefficient of the RG β-function in $SU(N_c)$ theory [8,31], which is manifesting itself in Eqs. (6) and (7) only starting from the unknown a_s^5 corrections. This is an extra argument which disfavors the a_s^4 rearrangements (28) and (29).

We now discuss common features and differences between the results for the $\{\beta\}$-expanded coefficients d_i and c_i, obtained with various perturbative approaches for $D^{ns}(a_s)$ and $C_{ns}^{Bjp}(a_s)$. Consider the $\{\beta\}$-expansion results obtained with (I) the $\{\beta\}$-expansion formalism [29] (cf. also [7,10]), (II) the $\{\beta\}$-expansion formalism based on the resummed Eqs. (4) and (5) proposed here, (III) skeleton-motivated expansion [32] (Sec. IV there), and (IV) R_δ-scheme motivated expansion of the Principle of Maximal Conformality [9,33,34].

In all four approaches the leading β_0-terms $d_n[n-1]\beta_0^{n-1}$ (and $c_n[n-1]\beta_0^{n-1}$) coincide. They coincide also with the leading β_0-terms in the β_0-expansion of [35], and with the corresponding terms of the large β_0-extension [36] of Brodsky-Lepage-Mackenzie (BLM) approach [37]. This feature is a consequence of a direct relation of these terms with the renormalon contributions [1] to the expressions for $D^{ns}(a_s)$ and $C_{ns}^{Bjp}(a_s)$.

Further, the approaches I–IV generate the same structure of $\{\beta\}$-expansion of the coefficients d_i and c_i, cf. Eqs. (19)–(21) and Eqs. (23)–(25).

However, specific coefficients in the $\{\beta\}$-expanded expressions of d_3 and c_3 obtained here do not coincide with those obtained in [7,10,29]. Only the C_F^3-terms coincide. The latter is a consequence of realization of the CS and therefore of the Crewther relation of Eq. (18) in the perturbative quenched QED approximation (cf. discussions in [38,39]). The analytical expressions for the $C_F^2 C_A$, $C_F C_A^2$ contributions to β_i-independent $d_3[0]$ and $c_3[0]$ components of d_3 and c_3 in (20) and (24), and for the terms $d_3[0,1]$, $d_3[1]$, and $c_3[0,1]$, $c_3[1]$, differ from the expressions obtained in [29] [cf. Eqs. (22) and (26) with the corresponding results in [7,10,29]]. This difference arises because the $\{\beta\}$-expansion formalism in [10,29] was performed in a gauge model which, in addition to $SU(N_c)$, contains a gluino multiplet, while the QCD results obtained here, including the identities

$$d_2[1] = d_3[0,1] = d_4[0,0,1] = \left(\frac{33}{8} - 3\zeta_3\right), \quad (30)$$

$$c_2[1] = c_3[0,1] = c_4[0,0,1] = \left(-\frac{3}{2} C_F\right), \quad (31)$$

use special resummation approach of Eqs. (4) and (5). This approach is unambiguously defined up to $O(a_s^4)$ within the $SU(N_c)$ gauge model, while the approach of [7,10,29] is at the moment defined only up to $O(a_s^3)$. We note that the identities (30) and (31) hold in the resummation approaches (III) and (IV) as well. Moreover, it turns out that at $O(a_s^4)$ level the $\{\beta\}$-expansions of perturbation coefficients in the approaches (III) and (IV), i.e., in the skeleton method [32], and R_δ-scheme method [9,33] are similar to each other.[3] The relations between these methods III and IV and the method developed here reside in a careful application of the RG method (for the stages of its development see [40]).

Note that in this work the concept of CS and the effects of CSB were essential to obtain new analytical results of Eqs. (22) and (26). These concepts allowed us to derive in [7,10] the number of relations from formulated in [7] Eq. (3). Therefore, the results obtained above satisfy them:

$$\begin{aligned} c_3[0] + d_3[0] &= 2d_1 d_2[0] - d_1^3 = -\frac{9}{16} C_F^3 + \frac{3}{32} C_F^2 C_A, \\ c_4[0] + d_4[0] &= 2d_1 d_3[0] - 3d_1^2 d_2[0] + d_2[0]^2 + d_1^4 \\ &= -\frac{333}{1024} C_F^4 + \left(-\frac{363}{512} + \frac{99}{32}\zeta_3\right) C_F^3 C_A \\ &\quad - \left(\frac{105}{256} + \frac{99}{32}\zeta_3\right) C_F^2 C_A^2, \end{aligned} \quad (32)$$

$$\begin{aligned} c_2[1] + d_2[1] &= c_3[0,1] + d_3[0,1] = c_4[0,0,1] + d_4[0,0,1] \\ &= \left(\frac{21}{8} - 3\zeta_3\right) C_F, \\ c_3[1] + d_3[1] &+ d_1(c_2[1] - d_2[1]) \\ &= c_4[0,1] + d_4[0,1] + d_1(c_3[0,1] - d_3[0,1]) \\ &= -\left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right) C_F^2 \\ &\quad + \left(\frac{47}{48} - \zeta_3\right) C_F C_A. \end{aligned} \quad (33)$$

We note, that these relations and expressions are model-independent and scheme-independent. They are also valid in the approaches III and IV. These expressions may be

[3] The details of these formulations and comparisons will be considered elsewhere.

used as a check if the $\{\beta\}$-expansion formalism in QCD with additional degrees of freedom [29], also considered in [7,10], is extended to d_4 and c_4.

The results obtained in this work may be used in future phenomenologically oriented studies of various resummation procedures and of their relations to generalizations of the BLM approach, related to Principle of Maximal Conformality [9,34], i.e. the ones considered recently in [10,30], and to the skeleton-motivated approach [32]. Here, we comment on a link of our studies with a specific result of the generalized BLM method, written in the form of commensurate scale relations [41], namely with the expression [42]

$$(1 + a_s^{Dns}(Q^*_{Dns}))(1 + a_s^{Bns}(Q^*_{Bns})) = 1. \qquad (34)$$

This expression follows from the generalized Crewther relation of [1] after defining the effective charges of the nonsinglet contributions to the Adler function and to the Bjorken polarized sum rule using the effective-charge approach [43] and absorbing the β-function dependent terms into the effective scales of the running effective charges a_s^{Dns} and a_s^{Bns}. The expression (34) is similar in its form to the QCD relation (18) derived here in the conformal invariant limit. The CSB effects are manifested in Eq. (34) in the (different) values of the effective scales Q^*_{Dns} and Q^*_{Bns}. The empirical, experimentally motivated, consideration for the importance of these CSB effects at sufficiently high energies was presented in [42].

We hope that the representation for the Adler function obtained here can be used in a more detailed comparison with the expression for the Adler function obtained in [44] from the available data for the e^+e^--annihilation to hadrons total cross section. Analogous comparison can be performed for the obtained Bjorken sum rule representation with the Bjorken sum rule most recent data, determined in [45] for the $Q^2 \leq 4.8$ GeV^2 region.

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