

# Mass-deformed ABJ and ABJM theory, Meixner-Pollaczek polynomials, and $su(1,1)$ oscillators

Miguel Tierz\*

*Departamento de Matemática, Grupo de Física Matemática, Faculdade de Ciências,  
Universidade de Lisboa, Campo Grande, Edifício C6, 1749-016 Lisboa, Portugal*  
(Received 28 April 2016; published 15 June 2016)

We give explicit analytical expressions for the partition function of  $U(N)_k \times U(N+M)_{-k}$  ABJ theory at weak coupling ( $k \rightarrow \infty$ ) for finite and arbitrary values of  $N$  and  $M$  (including the ABJM case and its mass-deformed generalization). We obtain the expressions by identifying the one-matrix model formulation with a Meixner-Pollaczek ensemble and using the corresponding orthogonal polynomials, which are also eigenfunctions of a  $su(1,1)$  quantum oscillator. Wilson loops in mass-deformed ABJM are also studied in the same limit and interpreted in terms of  $su(1,1)$  coherent states.

DOI: 10.1103/PhysRevD.93.126003

## I. INTRODUCTION

In recent years there has been considerable progress in applications of the localization technique to the study of supersymmetric gauge theories in a number of dimensions,  $d \geq 2$ . In three dimensions in particular, results have been obtained for the partition functions and Bogomolny-Prasad-Sommerfield (BPS) Wilson loops of  $\mathcal{N} = 2$  supersymmetric Chern-Simons-matter (CSM) theories, including the  $\mathcal{N} = 6$  superconformal theories constructed by Aharony, Bergman, Jafferis and Maldacena (ABJM theory) [1]. We shall focus here in the 3d realm exclusively and, in particular and more specifically, on the partition function of the ABJ theory [2], which is the  $\mathcal{N} = 6$  supersymmetric  $U(N_1)_k \times U(N_2)_{-k}$  CSM theory that generalizes the equal rank  $N_1 = N_2$  case of the ABJM theory.

In addition to the extension of ABJM theory, it has also been conjectured that the ABJ theory at large  $N_2$  and  $k$  with  $N_2/k$  and  $N_1$  fixed finite is dual to the  $\mathcal{N} = 6$  parity-violating Vasiliev higher spin theory on  $AdS_4$  with  $U(N_1)$  gauge symmetry [3–5]. In this paper, we will be studying the ABJ theory at large  $k$  and hence we shall be commenting on this higher-spin scaling limit.

The application of the localization method results in the reduction of path integrals into eigenvalue integrals of the

matrix model type, which allows to obtain various exact results at strong coupling of supersymmetric gauge theories and helps in providing further tests of the AdS/CFT correspondence. The appearance of matrix models typically implies a connection with exactly solvable models and/or integrable systems. This will be the viewpoint of this paper, where we identify ABJ theory at large  $k$  with a Meixner-Pollaczek random matrix ensemble [6], solved completely in terms of Meixner-Pollaczek orthogonal polynomials [7], which are also eigenfunctions of the  $su(1,1)$  harmonic oscillator [8].

Using either the two-matrix model description of ABJ (M) theory or the Fermi gas formulation, both briefly described below, a large number of results have been obtained, mostly in studying nonperturbative corrections of the ABJ(M) matrix models. Both the 't Hooft expansion, which is the asymptotic expansion as  $N$  goes to infinity and the 't Hooft parameter of the model is kept fixed at large  $N$ , and the M-theory expansion, which is the asymptotic expansion as  $N$  goes to infinity in which  $k$  is fixed, have now been studied in detail, see [4,5,9–11] for example.

The ABJ(M)  $U(N)_k \times U(N+M)_{-k}$  two-matrix model is given by [9,12]

$$Z_{\text{ABJ(M)}}(N, k, M) = \mathcal{N}_{\text{ABJ}} \iint \frac{d^N \mu}{(2\pi)^N} \frac{d^{N+M} y}{(2\pi)^{N+M}} \frac{\prod_{i < j} 4 \sinh^2(\frac{1}{2}(\mu_i - \mu_j)) \prod_{i < j} 4 \sinh^2(\frac{1}{2}(y_i - y_j))}{\prod_{i=1}^N \prod_{j=1}^{N+M} (2 \cosh(\frac{1}{2}(\mu_i - y_j)))^2} \times e^{\frac{ik}{4\pi}(\sum_{i=1}^N \mu_i^2 - \sum_{i=1}^{N+M} y_i^2)}, \tag{1.1}$$

where

$$\mathcal{N}_{\text{ABJ}} = \frac{i^{-\kappa M(N-1/2)}}{N!(N+M)!} \quad \text{where } \kappa := \text{sgn}(k).$$

\*tierz@fc.ul.pt

The ABJM case corresponds to  $M = 0$  above. By identifying the Cauchy determinant inside the integrand of (1.1) and using Cauchy identity

$$\frac{\prod_{i<j} \sinh^2(\frac{1}{2}(\mu_i - \mu_j)) \prod_{i<j} \sinh^2(\frac{1}{2}(y_i - y_j))}{\prod_{i=1}^N \prod_{j=1}^{N+M} (\cosh(\frac{1}{2}(\mu_i - y_j)))^2} = \sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \prod_i \frac{1}{\cosh(\mu_i - y_{\sigma(i)})}, \quad (1.2)$$

one can write the matrix model as a sum of permutations [13,14]. In turn, in the final expression obtained, which is the basis of the Fermi gas method [14], one can apply again the Cauchy determinant identity, obtaining then the one-matrix model form of the ABJ(M) matrix model [11,12,15]

$$\tilde{Z}_{\text{ABJ}(M)}(N, k, M) = \frac{1}{N!} \int \prod_{j=1}^N \frac{dx_j}{4\pi k} \frac{\prod_{l=-(M-1)/2}^{(M-1)/2} \tanh \frac{x_j + 2\pi i l}{k}}{e^{x_j/2} + (-1)^M e^{-x_j/2}} \times \prod_{i<j} \tanh^2 \left( \frac{1}{2k} (x_i - x_j) \right). \quad (1.3)$$

Note that the matrix model integral has  $N$  eigenvalues and the dependence in  $M$  is through the potential. This is the matrix model we shall study, focusing on the large  $k$  limit. In the ABJM case,  $M = 0$  above, the matrix integral (1.3) gives the full partition function but in the more general ABJ case a somewhat lengthy numerical prefactor, which includes the partition function of  $U(M)$  Chern-Simons theory on  $S^3$  appears as well [15]. We shall focus on the matrix integral (1.3) first and will discuss these additional prefactors at the end of the next section.

This paper is organized as follows. In the next section, we show that the large  $k$  limit of the one matrix model formulation of ABJ theory is a Meixner-Pollaczek random matrix ensemble and compute the partition functions for finite  $N$  and  $M$  (from now on, we always write  $N_2 = N + M$ ). Some quotients between partition functions, relevant in the study of the higher-spin limit, are given and shown to coincide with the partition function of the Penner matrix model.

In Sec. III, we show that the identification with a solvable model also holds for the mass-deformed version of the theories, focusing on the mass-deformed ABJM theory. An analytical computation for this model is also given, using the orthogonal polynomials, which admit an interpretation as eigenfunctions of a  $su(1, 1)$  quantum oscillator. In this interpretation, the  $M$  parameter is equal to the Bargmann index of the positive discrete series representation of  $su(1, 1)$ . We also study 1/6-BPS Wilson loops in the fundamental representation with winding  $n$ , in the mass-deformed ABJM theory, giving an interpretation in terms of  $su(1, 1)$  coherent states. The discussion of Wilson loops in Sec. III is carried out in comparison with  $\frac{1}{2}$ -BPS Wilson

loop in  $\mathcal{N} = 4$  super Yang-Mills theory, whose coherent state interpretation is also put forward.

Finally, we succinctly conclude with some avenues for further research. In the Appendix we collect some technical details on the analytical continuations in the parameters (Secs. II and III) and on the analytical solution for Wilson loops (Sec. III).

## II. WEAK-COUPLING LIMIT AND MEIXNER-POLLACZEK POLYNOMIALS

Let us start focusing on (1.3) in the ABJM case,  $M = 0$ . Then, the weight function of the matrix model (1.3), which in general is

$$\omega(x; M) = e^{-V(x; M)} = \frac{\prod_{l=-(M-1)/2}^{(M-1)/2} \tanh \frac{x + 2\pi i l}{k}}{e^{x/2} + (-1)^M e^{-x/2}},$$

greatly simplifies to

$$\omega(x; M = 0) = \frac{1}{2 \cosh(x/2)}. \quad (2.1)$$

The most distinctive mathematical feature of the matrix model (1.3) is the interaction term between the eigenvalues. It admits, like the model with a standard Vandermonde term  $\prod_{i<j} (x_i - x_j)$  or with its hyperbolic version  $\prod_{i<j} \sinh(x_i - x_j)$ , a Coulomb gas interpretation [16], but in general there are no standardized analytical methods to solve the resulting matrix model. Consequently, the one matrix model formulation has been analyzed with less detail than the original two-matrix model formulation (1.1) or the Fermi gas formulation [14].

A simple yet fundamental regime which has only eventually been analyzed is the weak-coupling limit  $k \rightarrow \infty$ , when  $N$  is fixed. Expressions for the partition function in the literature include, for the ABJM case, given in [9] and a computation in Appendix C of [12] for ABJ theory, using (1.1) and the analytical continuation from a lens space Chern-Simons matrix model.

To study this regime, the consideration of the one-matrix model formulation (1.3) is especially useful. Indeed, the very simple dependence of  $k$  in the matrix model (1.3), immediately implies that, using the rescaled variable  $\pi y = x/2$

$$Z_{\text{ABJM}}(N, k \rightarrow \infty) = \frac{1}{N!} \frac{\pi^{N(N-1)}}{2^{2N} k^{N^2}} \int \prod_{i=1}^N \frac{dy_i}{\cosh(\pi y_i)} \times \prod_{i<j} (y_i - y_j)^2, \quad (2.2)$$

which is in the form of a standard random matrix ensemble. In addition, the weight function of the model (2.2) can also be easily identified with a particular case of

the Meixner-Pollaczek polynomials [6,7], which are polynomials orthogonal in the interval  $-\infty < x < \infty$ , with regards to the weight function [6]<sup>1</sup>

$$w^{(\lambda)}(x) = \frac{2^{2\lambda-1}}{\pi} |\Gamma(\lambda + ix)|^2 = \frac{2^{2\lambda-1} |\Gamma(\lambda)|^2}{\pi} \prod_{k=0}^{\infty} \left(1 + \frac{x^2}{(k + \lambda)^2}\right)^{-1}. \quad (2.3)$$

Notice that the Gamma function is evaluated in a complex argument and that

$$|\Gamma(\lambda + ix)|^2 = \Gamma(\lambda + ix)\Gamma(\lambda - ix) \quad \text{for } x, \lambda \in \mathbb{R}.$$

Therefore, for  $\lambda = 1/2$ , and also taking into account Euler's duplication formula  $\Gamma(z)\Gamma(1 - z) = \pi / \sin(\pi z)$ , then

$$w^{(1/2)}(x) = \frac{1}{\cosh(\pi x)} = \omega_{\text{ABJM}}(x),$$

and with the orthogonal polynomials one can compute immediately (2.2). We do that below, but before we discuss the more general ABJ case (1.3) because, interestingly enough, this identification with an exactly solvable model also holds with more generality, as it holds for the ABJ matrix model. In particular, we have that, again in terms of the  $y$  variable

$$\begin{aligned} \tilde{Z}_{\text{ABJ}}(N, k \rightarrow \infty, M) &= \frac{1}{N!} \frac{2^{N(M-1)} \pi^{N^2-N+MN}}{k^{N(N+M)}} \\ &\times \int \prod_{j=1}^N \frac{\prod_{l=-(M-1)/2}^{(M-1)/2} (y_j + il) dy_j}{e^{\pi y_j} + (-1)^M e^{-\pi y_j}} \prod_{i < j} (y_i - y_j)^2. \end{aligned} \quad (2.4)$$

To simplify the presentation, we focus first on the case when  $M = 2q$  and  $q \in \mathbb{N}$  and later on we show that the odd case  $M = 2q - 1$  works in the same way. Then, the weight function of the matrix model is

$$\omega_{\text{ABJ}}^{(\text{even})}(y) = \frac{\prod_{l=1/2}^{(M-1)/2} (y^2 + l^2)}{2 \cosh(\pi y)}.$$

This is precisely the Meixner-Pollaczek weight if we take its parameter  $\lambda$  to be  $\lambda = 1/2 + M/2 = 1/2 + q$  with  $q \in \mathbb{N}$ . This follows immediately from the form (2.3) and  $\Gamma(z + 1) = z\Gamma(z)$ . Specifically,

<sup>1</sup>This is actually not the most general weight function for this system of polynomials, since there is also a phase factor, but we will only need that case below, when we discuss the mass-deformed ABJM model.

$$\begin{aligned} \tilde{Z}_{\text{ABJ}}(N, k \rightarrow \infty, M) &= \frac{1}{N!} \frac{2^{N(M-2)} \pi^{N^2-N+MN}}{k^{N(N+M)}} \\ &\times \int \prod_{i=1}^N \frac{\prod_{l=1/2}^{(M-1)/2} (y_i^2 + l^2) dy_i}{\cosh(\pi y_i)} \prod_{i < j} (y_i - y_j)^2 \\ &= \frac{1}{N!} \frac{\pi^{N^2-N+MN}}{2^{2N} k^{N(N+M)}} \int \prod_{i=1}^N w^{(1/2+M/2)}(y_i) dy_i \prod_{i < j} (y_i - y_j)^2, \end{aligned} \quad (2.5)$$

where the weight function is (2.3) with  $\lambda = 1/2 + M/2$ .

Let us take into account then the orthogonality properties of the polynomials, which is all we need to obtain the partition functions/free energies. The polynomials  $P_n^{(\lambda)}(x)$  satisfy [6]

$$\int_{-\infty}^{+\infty} dx P_n^{(\lambda)}(x) P_m^{(\lambda)}(x) w^{(\lambda)}(x) = \delta_{nm} h_n^{(\lambda)}, \quad (2.6)$$

with

$$h_n^{(\lambda)} = \frac{\Gamma(n + 2\lambda)}{\Gamma(n + 1)}. \quad (2.7)$$

From the recurrence relation [7], the leading coefficient of the polynomial  $P_n^{(\lambda)}(x) = a_n x^n + \dots$  is also obtained:

$$a_n = \frac{2^n}{\Gamma(n + 1)}. \quad (2.8)$$

Then, we can use, exactly as with the Stieltjes-Wigert polynomials and the computation of the free energy of  $U(N)$  Chern-Simons theory on  $S^3$  [17], the explicit analytical expression for the partition function of the matrix model, using

$$Z = N! \prod_{j=0}^{N-1} \tilde{h}_j,$$

where  $Z$  denotes the partition function of a Hermitian matrix model with the Meixner-Pollaczek weight function and  $\tilde{h}_j$  denote the analogous of the  $h_j$  in (2.6), but for the *monic* orthogonal polynomials. Thus, we need the orthogonality properties of the polynomials  $Q_n^{(\lambda)}(x) = P_n^{(\lambda)}(x)/a_n$ , which are

$$\int_{-\infty}^{+\infty} dx Q_n^{(\lambda)}(x) Q_m^{(\lambda)}(x) w^{(\lambda)}(x) = \delta_{nm} \tilde{h}_n^{(\lambda)} = \delta_{nm} \frac{h_n^{(\lambda)}}{a_n^2}. \quad (2.9)$$

Therefore, we obtain for (2.2)

$$\begin{aligned}
Z_{\text{ABJM}}(N, k \rightarrow \infty) &= \frac{\pi^{N^2-N}}{2^{2N} k^{N^2}} \prod_{j=0}^{N-1} \frac{h_j^{(1/2)}}{a_j^2} = \frac{\pi^{N^2-N}}{2^{2N} k^{N^2}} \prod_{j=0}^{N-1} 4^{-j} \Gamma^2(j+1) \\
&= \frac{\pi^{N^2-N}}{2^{N(N+1)} k^{N^2}} G^2(N+1) = \left(\frac{\pi}{2k}\right)^{N^2} \frac{G^2(N+1)}{(2\pi)^N},
\end{aligned}$$

where  $G(z)$  is a Barnes G-function [18]. Hence, the free energy  $F = \ln Z$  is

$$\begin{aligned}
F_{\text{ABJM}}(N, k \rightarrow \infty) &= N^2 \log\left(\frac{\pi}{2k}\right) - N \log(2\pi) \\
&\quad + 2 \log G(N+1). \tag{2.10}
\end{aligned}$$

This expression is identical to the one given in Eq. (4.47) of [9], which is obtained there, since it is a perturbative computation, by evaluation of the determinants giving the one-loop contributions of two copies of pure Chern-Simons theory and the one-loop determinants of the matter fields. The free energy is the sum of both contributions and this is why the expression is organized in a slightly different way there. Thus, the Meixner-Pollaczek polynomials capture exactly both contributions together. For the ABJ partition function, we obtain

$$\begin{aligned}
\tilde{Z}_{\text{ABJ}}(N, M, k \rightarrow \infty) &= \frac{\pi^{N^2-N+MN}}{2^{2N} k^{N(N+M)}} \prod_{j=0}^{N-1} \frac{h_j^{(1/2+M/2)}}{a_j^2} \\
&= \frac{\pi^{N^2-N+MN}}{2^{N(N+1)} k^{N(N+M)}} \frac{G(N+M+1)G(N+1)}{G(M+1)}. \tag{2.11}
\end{aligned}$$

Notice that, in this weak-coupling limit, it holds that

$$\frac{\tilde{Z}_{\text{ABJ}}(N, M, k \rightarrow \infty)}{Z_{\text{ABJM}}(N, k \rightarrow \infty)} = \left(\frac{\pi}{k}\right)^{NM} \frac{G(N+M+1)}{G(N+1)G(M+1)}. \tag{2.12}$$

Below we will show how this quotient is modified when we consider the full ABJ partition function. Since  $\tilde{Z}_{\text{ABJ}}$  does not include the pure Chern-Simons partition function factor, the ratio (2.12) is precisely the one considered to be specially relevant in the higher-spin double scaling limit [Eq. (5.2) of [4]] (dubbed ‘‘vector model subsector’’ in [4]).

The rhs of (2.12) is also, exactly, the partition function of the Penner matrix model (a Laguerre random matrix ensemble), which is characterized by a weight function  $\omega(x) = x^M \exp(-x)$  for  $x > 0$ . It would be interesting to see if this can be related both to the results in Eq. (5.2) of [4] and, at the same time, with the well-known results on the  $c = 1$  string, topological strings, which are associated with the Penner matrix model (see [19] for a review).

We consider now the case of  $M$  odd since we restricted the discussion above to  $M$  being an even number just to simplify the presentation of the connection with Meixner-Pollaczek polynomials. It also holds for  $M$  odd in the same way. The difference being that, instead of half-integer values of  $\lambda$ , we will have integer values. This is immediate from, again, Euler’s duplication formula, because

$$|\Gamma(1+ix)|^2 = \Gamma(1+ix)\Gamma(1-ix) = \frac{\pi x}{\sinh \pi x},$$

therefore, increasing  $\lambda$  by 1 adds a  $(1+x^2)$  term, and so on. The correspondence with the semiclassical limit of the ABJ weight is then again exact,

$$\omega_{\text{ABJ}}^{(\text{odd})}(y) = \frac{\prod_{l=0}^{(M-1)/2} (y^2 + l^2)}{\sinh(\pi y)} = 2^{-(M-1)} w^{(1+(M-1)/2)}(y),$$

and, therefore, for the  $M$  odd case, we have

$$\begin{aligned}
\tilde{Z}_{\text{ABJ}}(N, k \rightarrow \infty, M) &= \frac{\pi^{N^2-N+MN}}{2^{2N} k^{N(N+M)}} \prod_{j=0}^{N-1} \frac{h_j^{(1+(M-1)/2)}}{a_j^2} \\
&= \frac{\pi^{N^2-N+MN}}{2^{2N} k^{N(N+M)}} \prod_{j=0}^{N-1} \frac{\Gamma(j+1+M)\Gamma^2(j+1)}{4^j},
\end{aligned}$$

which is the same result as for  $M$  even and, therefore (2.11) holds for  $M$  any natural number and the identification is always  $\lambda = 1/2 + M/2$  for  $M \in \mathbb{N}$ . Below we will see that this parameter can be interpreted as the Bargmann index of the positive discrete series representation of the  $su(1, 1)$  Lie algebra.

Let us now finally deal with the extra factors that multiply (1.3). The exact relationship between (1.1) and (1.3) is detailed in [15], with  $q = \exp(2i\pi/k)$ ,

$$\begin{aligned}
Z_{\text{ABJ}}^{(N, N+M)}(k) &= \frac{i^{-\frac{\text{sign}(k)}{2}(N^2+(N+M)^2)} (-1)^{\frac{N}{2}(N-1)+\frac{M}{2}(M-1)+NM} i^{N+\frac{M}{2}} q^{\frac{M}{12}(M^2-1)}}{k^{\frac{M}{2}}} \prod_{1 \leq l < m \leq M} 2i \sin \frac{\pi(l-m)}{k} \\
&\quad \times \frac{1}{N!} \int_{-\infty}^{\infty} \frac{d^N y}{(4\pi k)^N} \prod_{a < b} \tanh^2 \frac{y_a - y_b}{2k} \prod_{a=1}^N \frac{1}{2 \cosh \frac{y_a}{2}} \prod_{l=0}^{M-1} \tanh \frac{y_a + 2\pi i(l+1/2)}{2k}. \tag{2.13}
\end{aligned}$$

We can write, for simplicity, the  $k$ -independent prefactor in the first line in (2.13) as  $\tilde{\mathcal{N}}_{\text{ABJ}}(N, M)$  and the most interesting aspect of considering the full prefactor in (2.13) is that the  $k \rightarrow \infty$  of what is essentially the  $U(M)$

Chern-Simons partition function on  $S^3$  gives a  $G(M + 1)$  Barnes function which cancels the one in (2.11). More precisely,

$$\begin{aligned} \lim_{k \rightarrow \infty} Z_{\text{ABJ}}^{(N, N+M)}(k) &= k^{\frac{M(M-2)}{2}} \tilde{\mathcal{N}}_{\text{ABJ}}(N, M) G(M + 1) \tilde{Z}_{\text{ABJ}}(N, M, k \rightarrow \infty) \\ &= \frac{\tilde{\mathcal{N}}_{\text{ABJ}}(N, M) \pi^{N^2 - N + MN} k^{\frac{M(M-2)}{2}}}{2^{N(N+1)} k^{N(N+M)}} G(N + M + 1) G(N + 1), \end{aligned}$$

where

$$\tilde{\mathcal{N}}_{\text{ABJ}}(N, M) = i^{-\frac{\text{sign}(k)}{2}(N^2 + (N+M)^2)} (-1)^{\frac{N}{2}(N-1) + \frac{M}{2}(M-1) + NM} i^{N + \frac{M^2}{2}} (2\pi)^{\frac{M}{2}(M-1)}.$$

The quotient (2.12) between the full ABJ and ABJM partition function is now

$$\frac{Z_{\text{ABJ}}(N, M, k \rightarrow \infty)}{Z_{\text{ABJM}}(N, k \rightarrow \infty)} = \tilde{\mathcal{N}}_{\text{ABJ}} k^{\frac{M(M-2)}{2}} \left(\frac{\pi}{k}\right)^{NM} \frac{G(N + M + 1)}{G(N + 1)}. \tag{2.14}$$

In addition to the usual asymptotics of the  $G$  Barnes function, one can give an asymptotic expression for the quotient of  $G$  functions in (2.14) for  $N$  and  $M$  finite, but  $N$  much bigger than  $M$ . This estimates, in compact form, how the ABJ free energy differs from the ABJM one at small values of  $M$ . More precisely, from proposition A.1. (1) of [20], we have that, for  $M \leq N^{1/6}$  then

$$\frac{G(1 + M + N)}{G(1 + N)} = (2\pi)^{M/2} e^{-(N+1)M} (1 + N)^{M^2/2 + NM} \left(1 + \mathcal{O}\left(\frac{M^2 + M^3}{N}\right)\right). \tag{2.15}$$

Thus, we have seen that with Meixner-Pollaczek polynomials, which are actually eigenfunctions of the  $su(1, 1)$  harmonic oscillator as we shall see below, the finite  $N$  large  $k$  behavior of the ABJ free energy is easily obtained. This complements the varied number of results in the literature on ABJ(M) models and the expressions obtained are consistent with [9] and [12], giving also an alternative derivation for the ABJ case, to the one in [12], without having to rely on analytical continuation from the lens space matrix model. We can also apply the method to a mass-deformed version of the theory and that is the subject of the next section.

### III. MASS-DEFORMED THEORY, WILSON LOOPS AND $su(1,1)$ COHERENT STATES

More relevant than the application itself is the fact that, starting with the one matrix model formulation (1.3), the ABJ matrix model, for all  $M$  and  $N$ , is an exactly solvable model in the weak-coupling limit: the Meixner-Pollaczek model. Further results can be obtained from that identification. For example, it can be extended to the mass-deformed ABJM matrix model, which has been the subject of interest lately. It was first studied at large  $N$  in [21] (with Fayet-Iliopoulos parameter  $\zeta = 0$ ). The model is [22,23]

$$Z(N, k, m_1, m_2) = \frac{1}{N!^2} \iint \frac{d^N \mu}{(2\pi)^N} \frac{d^N y}{(2\pi)^N} \frac{\prod_{i < j} \sinh^2(\frac{\mu_i - \mu_j}{2}) \prod_{i < j} \sinh^2(\frac{y_i - y_j}{2}) e^{\frac{ik}{4\pi} \sum_{i=1}^N (\mu_i^2 - y_i^2)}}{\prod_{i,j=1}^N \cosh(\frac{1}{2}(\mu_i - y_j + m_1)) \cosh(\frac{1}{2}(\mu_i - y_j - m_2))}. \tag{3.1}$$

This extension of the ABJM matrix model actually also includes a Fayet-Iliopoulos parameter  $\zeta$ , which has been repackaged with the mass term deformation as  $m_1 = m + \zeta$  and  $m_2 = m - \zeta$  [22,23].

It also admits a sum over permutations form [22,23] and, if one again applies to the resulting expression the Cauchy determinant formula, we see that we also have a one matrix model form of the mass-deformed ABJM matrix model<sup>2</sup>

<sup>2</sup>A factor of 4 is added in the denominator of the matrix model with respect to the expressions in [22,23]. In this way, in the limit  $m_1 \rightarrow 0$  and  $m_2 \rightarrow 0$  it reduces to the ABJM matrix model above, which is as in [10], for example. The reason is that the two cosh in [10], unlike those in [22,23], have a 2 term in front.

$$\begin{aligned}
 Z_{\text{ABJM}}(N, k, m_1, m_2) &= \frac{1}{N!} \int \prod_{i=1}^N \frac{e^{-im_2 x_i} dx_i}{4k \cosh(\pi x_i)} \frac{\prod_{i<j} \sinh^2(\frac{\pi}{k}(x_i - x_j))}{\prod_{i,j} \cosh(\frac{\pi}{k}(x_i - x_j) - \frac{m_1}{2})} \\
 &= \frac{1}{N! \cosh(m_1/2)^N 2^{2N} k^N} \int \prod_{i=1}^N \frac{e^{-im_2 x_i} dx_i}{\cosh(\pi x_i)} \prod_{i<j} \frac{\sinh^2(\frac{\pi}{k}(x_i - x_j))}{\cosh^2(\frac{\pi}{k}(x_i - x_j) - \frac{m_1}{2})}. \tag{3.2}
 \end{aligned}$$

Thus, in first approximation in the semiclassical limit, we have

$$Z_{\text{ABJM}}(N, k \rightarrow \infty, m_1, m_2) = \frac{\pi^{N^2-N}}{N! \cosh(m_1/2)^{N^2} 4^N k^{N^2}} \int \prod_{i=1}^N \frac{e^{-im_2 x_i} dx_i}{\cosh(\pi x_i)} \prod_{i<j} (x_i - x_j)^2. \tag{3.3}$$

Notice that, to account for the  $m_1$  deformation is immediate in this way and, if we also want to account for the  $m_2$  parameter, we need to consider what is actually the most general form of the Meixner-Pollaczek (MP) weight function [7]

$$w_{\text{MP}}(x; \lambda, t) = \frac{2^{2\lambda-1} e^{tx}}{\pi} |\Gamma(\lambda + ix)|^2, \tag{3.4}$$

where  $t \in (-\pi, \pi)$ . This restriction on  $t$  is because, for fixed  $\lambda$ ,  $|\Gamma(\lambda + ix)|^2 \sim e^{-\pi|x|}$  as  $|x| \rightarrow \infty$ , therefore it ensures exponential convergence for  $x \rightarrow \pm\infty$ . Being  $t$  real we need to take its analytical prolongation to imaginary values at the end (or, alternatively, do the same to the physical parameter  $m_2$ ). Such a step has precedents in the application of orthogonal polynomial methods to the study of Chern-Simons theory, with or without matter [17,24]<sup>3</sup> and, as we shall see, it poses no problems here. In the case  $m_2 = 0$ , this continuation is not necessary and the parameter  $\phi$  below will be  $\phi = \pi/2$ , as is the case for the nondeformed ABJM theory. As explained in [23], this specific mass deformation is a fixed point of the symmetry  $Z(2\zeta, m; k) = Z(m, 2\zeta; k)$ , satisfied by the mass-deformed theory and, in the dual  $\mathcal{N} = 4$  supersymmetric super Yang-Mills theory,  $m_2 = 0$  corresponds to coupling the theory to a massless adjoint hypermultiplet.

In most references, the notation convention is  $t = 2\phi - \pi$ , where  $\phi \in (0, \pi)$  and the orthogonality properties now read

$$\begin{aligned}
 \frac{2^{2\lambda-1}}{\pi} \int_{-\infty}^{+\infty} e^{(2\phi-\pi)x} |\Gamma(\lambda + ix)|^2 P_n^{(\lambda)}(x; \phi) P_m^{(\lambda)}(x; \phi) dx \\
 = \frac{\Gamma(n+2\lambda)}{(\sin \phi)^{2\lambda} \Gamma(n+1)} \delta_{nm}. \tag{3.5}
 \end{aligned}$$

Notice that, if  $\phi = \pi/2$ , the weight function indeed reduces to (2.3). We need to see if the leading coefficients of the

<sup>3</sup>The works [25,26] also consider orthogonal polynomials in Chern-Simons-matter theory.

polynomial, which did not depend on  $\lambda$ , depend now on  $\phi$ . From the recurrence relationships [7], we quickly deduce that

$$a_n^{(\phi)} = \frac{(2 \sin \phi)^n}{\Gamma(n+1)}. \tag{3.6}$$

Thus, since the weight function of the mass-deformed ABJM model (3.3) is  $w_{\text{MP}}(x; \lambda=1/2, t=2\phi-\pi=-im_2)$ , then using (3.5) and (3.6), we have, in terms of the  $\phi$  parameter

$$\begin{aligned}
 Z_{\text{ABJM}}(N, m_1, \phi, k \rightarrow \infty) \\
 = \left( \frac{\pi}{k \cosh(m_1/2)} \right)^{N^2} \frac{1}{(4\pi)^N} \prod_{j=0}^{N-1} \frac{h_{j,\phi}^{(1/2)}}{a_{j,\phi}^2} \\
 = \frac{1}{(2\pi)^N} \left( \frac{\pi}{2k \cosh(m_1/2)} \right)^{N^2} \frac{G^2(N+1)}{(\sin \phi)^{N^2}}.
 \end{aligned}$$

The identification  $\phi = -im_2/2 + \pi/2$  poses no problems, as we shall explicitly check in the Appendix, and we have

$$\begin{aligned}
 Z_{\text{ABJM}}(N, m_1, m_2, k \rightarrow \infty) \\
 = \left( \frac{\pi}{2k \cosh(m_1/2) \cosh(m_2/2)} \right)^{N^2} \frac{G^2(N+1)}{(2\pi)^N}.
 \end{aligned}$$

Thus, in this limit, for the mass-deformed model we have that the free energy

$$\begin{aligned}
 F_{\text{ABJM}}(m_1, m_2) &= F_{\text{ABJM}} - N^2 (\ln \cosh(m_1/2) \\
 &\quad + \ln \cosh(m_2/2)).
 \end{aligned}$$

### A. Wilson loops in the mass-deformed case

We consider Wilson loops now. The idea is to see if the Wilson loop can be computed as an average using a density of states constructed from quantum oscillators, very much as the Drukker-Gross computation of the  $\frac{1}{2}$ -BPS Wilson

loop in  $\mathcal{N} = 4$  super Yang-Mills theory [27], which uses the ordinary quantum harmonic (Hermite) wave functions. To do so, we will use the Meixner-Pollaczek polynomials and their  $su(1, 1)$  quantum oscillator interpretation, giving also a coherent state interpretation, both for the mass-deformed ABJ Wilson loops at weak coupling and also for the  $\frac{1}{2}$ -BPS Wilson loop in  $\mathcal{N} = 4$  super Yang-Mills theory [27].

For this, one first needs to write down the one matrix model representation of the Wilson loop. We have not seen that result in the literature, at least not in a form convenient to our purpose and methods (there are essentially equivalent integral representations for Wilson loops in [15,28]). Thus, we work out the required one matrix model expression, by repeating the procedure in Eqs. (5)–(8) of [21] for the ABJM mass-deformed theory, but with a Wilson loop insertion. The Wilson loop averages we consider are the  $\frac{1}{6}$ -BPS Wilson loops in [29]

$$\langle W_R^{1/6} \rangle = \langle \text{Tr}_R(e^\mu) \rangle, \tag{3.7}$$

where the average is taken over the ensemble (1.1) with  $M = 0$  and one could equivalently consider the Wilson loop over the other gauge group. The  $\frac{1}{2}$ -BPS Wilson loops of Drukker and Trancanelli treat both gauge groups more symmetrically [30], but they can be expressed in terms of the  $\frac{1}{6}$ -BPS Wilson loops. We will consider (3.7) in the fundamental representation and with winding  $n$ ,

$$\langle W_n^{1/6} \rangle_m = \langle \text{Tr}(e^{n\mu}) \rangle_m, \tag{3.8}$$

and the ensemble average will be over the mass-deformed ABJM model (3.1), instead of the regular ABJM. This is denoted by the  $m$  subindex in the average.

The procedure in Eqs. (5)–(8) of [21] consists in using the Fourier transform identity

$$\int d\tau \frac{e^{i\tau\mu}}{\cosh(\pi\tau)} = \frac{1}{\cosh\frac{\mu}{2}}, \tag{3.9}$$

for (3.8) twice (as an integral representation; that is, from right to left), a Gaussian integration and a further immediate integration, using (3.9). We do exactly the same for (3.8), which is explicitly given by the matrix model (3.1) with a  $\sum_{j=1}^N e^{n\mu_j}$  insertion in the integrand. We obtain that the one-matrix model representation of the Wilson loop is

$$\begin{aligned} \langle W_n^{1/6} \rangle_m &= \frac{c_n}{Z} \int \prod_{i=1}^N \frac{e^{-im_2x_i} \sum_{j=1}^N e^{\frac{2\pi n}{k}x_j}}{4k \cosh(\pi x_i)} dx_i \\ &\times \frac{\prod_{i<j} \sinh^2(\frac{\pi}{k}(x_i - x_j))}{\prod_{i,j} \cosh(\frac{\pi}{k}(x_i - x_j) - \frac{m_1}{2} + \frac{i\pi n}{k} \delta_{ij})}, \end{aligned} \tag{3.10}$$

where the prefactor  $c_n$  is generated by the Wilson loop insertion and is given by  $c_n = e^{-in^2\pi/k-n\varsigma}$ . Notice the

appearance of  $k$  in the added exponential insertion in the integrand in (3.10) (this is due to the Gaussian integration in the procedure). In the  $k \rightarrow \infty$  we can then take the double-scaling limit with the winding  $n$  such that  $n/k = \beta$ , and therefore, we have that

$$\begin{aligned} \lim_{\substack{k \rightarrow \infty \\ n/k \rightarrow \beta}} \langle W_n^{1/6} \rangle_m &= \frac{\alpha_N(k, \beta, m_1) c_n}{Z} \int \prod_{i=1}^N \frac{e^{-im_2x_i} \sum_{j=1}^N e^{2\pi\beta x_j}}{\cosh(\pi x_i)} dx_i \\ &\times \prod_{i<j} (x_i - x_j)^2, \end{aligned} \tag{3.11}$$

where the prefactor  $\alpha_N(k, \beta, m_1)$  follows in the same way as for the partition function (3.3). Specifically,

$$\begin{aligned} \alpha_N(k, \beta, m_1) &= \frac{\pi^{N^2-N}}{\cosh(m_1/2 - i\pi\beta)^N \cosh(m_1/2)^{N(N-1)} 4^N k^{N^2}}. \end{aligned}$$

We first given an analytical expression for the matrix integral in (3.11) and, in the next subsection, a physical interpretation. Notice that, in general, for a Hermitian matrix model with weight function  $\omega(x)$ ,

$$\frac{\langle \text{Tr} e^{yM} \rangle}{Z} = \sum_{s=0}^{N-1} \frac{\int dx \omega(x) e^{yx} P_s^2(x)}{\int dx \omega(x) P_s^2(x)} = \sum_{s=0}^{N-1} \langle s | e^{yx} | s \rangle, \tag{3.12}$$

where

$$\langle x | s \rangle = \frac{\omega(x)^{1/2} P_s(x)}{[\int dx \omega(x) P_s^2(x)]^{1/2}} \tag{3.13}$$

and  $P_s(x)$  denotes the polynomial of order  $s$ , orthogonal with respect to  $\omega(x)$ . Thus, it is as in the  $\mathcal{N} = 4$  SYM/Hermite setting [27] but with an imaginary  $y$  parameter, instead of a real one. Hence, denoting by  $|n; \lambda, \phi\rangle$  the Meixner-Pollaczek eigenstates,<sup>4</sup> we have that the nontrivial piece of the Wilson loop (that is, up to normalization constants) is

$$\begin{aligned} \lim_{\substack{k \rightarrow \infty \\ n/k \rightarrow \mu}} \langle W_n^{1/6} \rangle_m &\propto \sum_{n=0}^{N-1} \langle n; \lambda = 1/2, \phi | \exp(-im_2 \hat{X}) | n; \\ &\lambda = 1/2, \phi \rangle, \end{aligned} \tag{3.14}$$

with  $\phi = i\pi(\frac{1}{2} - \beta)$  and  $\hat{X}$  is the position operator. Below, we will identify the operator insertion in (3.14) with a squeeze operator, but we focus first on obtaining an analytical expression. The explicit evaluation of (3.14) follows from a Fourier integral in [31], which generalizes the orthogonality identity of the MP polynomials (3.5) and

<sup>4</sup>These are the states such that the rhs of (3.13) is the normalized Meixner-Pollaczek polynomial of degree  $s$  and the weight function is (3.4).

is the analogue of the identity for Hermite polynomials which leads to (3.18) [27] (see the Appendix for further details). In particular, using [31], we have the following expression in terms of a (terminating) Gauss hypergeometric function:

$$\begin{aligned} & \langle n; \lambda = 1/2, \phi | \exp(-im_2 \hat{X}) | n; \lambda = 1/2, \phi \rangle \\ &= {}_2F_1 \left( \begin{matrix} -n, -n \\ 1 \end{matrix}; -\left(\frac{\cos(\pi\beta)}{\sinh(\frac{m_2}{2})}\right)^2 \right) \frac{\pi i (\sinh(\frac{m_2}{2}))^{2n}}{(\sinh(\frac{m_2}{2} + i\pi(\frac{1}{2} - \beta))^{2n+1}}. \end{aligned} \quad (3.15)$$

In this expression, we have identified the Fourier kernel in (A1) with the mass-deformed term in the matrix integral  $e^{-im_2 x}$ , and the Wilson loop insertion with the real exponential part  $e^{(2\phi - \pi)x}$  of the weight function. Hence, no analytical continuation is necessary and one can also give an expression for the Wilson loop in the standard ABJM theory with only the standard orthogonality relationship (3.5). Notice that, in contrast to the Hermite case, the evaluation of the summation in (3.14) of the hypergeometric terms (3.15), to obtain a more compact expression for the Wilson loop, is an open problem. On the other hand, specific cases at finite rank are immediate from (3.15), since the hypergeometric expression above is a terminating series, and a more detailed study, including extension to the ABJ setting and comparison with perturbative computations, will be given elsewhere.

### B. $su(1,1)$ oscillator, coherent states and photonic interpretation

We now exploit a physical interpretation of the Meixner-Pollaczek polynomials: they are eigenfunctions of a quantum oscillator model. Indeed, the  $su(1,1)$  model of a quantum oscillator is a model which obeys the dynamics of a harmonic oscillator, but with the position and momentum operators and the Hamiltonian being elements of the Lie algebra  $su(1,1)$  instead of the Heisenberg algebra [8]. For an oscillator model, one requires also that the spectrum of  $H$  in unitary representations of the Lie algebra is equidistant. The generators of the algebra are  $K_{\pm}$  and  $K_0$  and their commutation relations,

$$[K_-, K_+] = 2K_0 \quad \text{and} \quad [K_0, K_{\pm}] = \pm K_{\pm}. \quad (3.16)$$

The model is constructed using the positive discrete series representations of  $su(1,1)$ ,  $D^+(\lambda)$ , which are infinite dimensional and labeled by a positive number (Bargmann index)  $\lambda > 0$ . Then the spectrum of the position operator is  $\mathbb{R}$ , the spectra of the Hamiltonian is  $n + \lambda$  and the position wave function, when the oscillator is in the  $n$ th eigenstate of the Hamiltonian, is given by [8]

$$\phi_n(x; \lambda, \phi) = C_n e^{(2\phi - \pi)x} |\Gamma(\lambda + ix)|^2 P_n^{(\lambda)}(x; \phi), \quad (3.17)$$

where again  $P_n^{(\lambda)}(x; \phi)$  are Meixner-Pollaczek polynomials. Since the partition function of a Hermitian matrix model is

$$Z_N = \int \rho_N(x) dx = \int \lim_{x' \rightarrow x} K_N(x', x) dx,$$

where  $\rho_N(x)$  is the density of states and the two-point kernel is

$$K_N(x', x) = (\omega(x')\omega(x))^{1/2} \sum_{n=0}^{N-1} P_n^{(\lambda)}(x'; \phi) P_n^{(\lambda)}(x; \phi).$$

Therefore, we have an immediate interpretation in terms of a  $su(1,1)$  oscillator wave function overlap,

$$\tilde{Z}_{\text{ABJ}}(N, M, k \rightarrow \infty) = \langle \Phi | \Phi \rangle,$$

where  $|\Phi\rangle = \sum_{n=0}^{N-1} |n\rangle$  and

$$\langle x | n \rangle = \tilde{\phi}_n(x, \lambda = 1/2 + M/2, \phi = \pi/2),$$

where  $\tilde{\phi}_n(x, \lambda = 1/2 + M/2, \phi = \pi/2)$  is (3.17), properly normalized. That is, with  $C_n = \|\phi_n\|_2^{-1}$ . Likewise, the same expression holds for the mass-deformed ABJM case above, but with normalized eigenfunctions

$$\langle x | n \rangle = \tilde{\phi}_n(x, \lambda = 1/2, \phi = -im_2/2 + \pi/2).$$

From the results above, we thus have a half-integer Bargmann index for  $M$  even and integer for  $M$  odd. The case  $\lambda = 1/2$ , which is the one that corresponds to ABJM theory, including its mass-deformed version, is precisely the one that has the exact quantum oscillator spectra  $n + 1/2$ , without any shift in the ground state energy.

This relationship with a  $su(1,1)$  quantum oscillator can be further developed in the case of Wilson loops. For this, notice first that, for a Gaussian matrix model, the (3.13) are Hermite polynomials, and the resulting expression for the Wilson loop average (3.12) is that of Drukker and Gross [27]. In terms of the creation and annihilation operators  $a = \lambda/2 + d/d\lambda$  and  $a^\dagger = \lambda/2 - d/d\lambda$  the average (3.12) is that of the displacement operator [32],

$$D(y) \equiv \exp(y(a + a^\dagger)).$$

The displacement operator acting on the vacuum state generates a coherent state  $D(y)|0\rangle = |y\rangle$  and when it acts on an excited state  $D(y)|n\rangle$  these are displaced number states [33], which are also wave packets that keep their shape and follow classical motion. The evaluation of the matrix element in (3.12) is a central result in the theory of coherent states and quantum optics [32] and in laser cooling [34]. Its evaluation is classical (and can be completely algebraic, without relying on Hermite polynomials identities) [32,34]



$$\langle n' | e^{y(a+a^\dagger)} | n \rangle = e^{-y^2/2} y^{\Delta n} \left( \sqrt{\frac{n_{<}!}{n_{>}!}} \right) L_{n_{<}}^{\Delta n}(-y^2), \quad (3.18)$$

where  $n_{<} = \min(n, n')$ ,  $n_{>} = \max(n, n')$  and  $\Delta n = n_{>} - n_{<}$ . Comparing matrix models, the generic  $y$  parameter in (3.12) here is identified with the  $t'$  Hooft parameter in [27]  $y = \sqrt{\lambda/4N}$  and then (3.18) for  $n = n'$  coincides with the result in [27].<sup>5</sup> Because of the summation property of Laguerre polynomials  $\sum_{n=0}^{N-1} L_n^\alpha(x) = L_n^{\alpha+1}(x)$  (used in [27]), the Wilson loops can also be expressed as a single matrix element (3.18) between adjacent levels  $n' = n + 1 = N$ . That is

$$\langle W_{\mathcal{N}=4}^{1/2} \rangle = 2N\lambda^{-1/2} \langle N | e^{\sqrt{\lambda/4N}(a+a^\dagger)} | N-1 \rangle. \quad (3.19)$$

This displacement operator average in turn can be interpreted as the average of a unitary operator involving a Jaynes-Cummings Hamiltonian, but such a quantum optics interpretation of the  $\frac{1}{2}$ -BPS Wilson loop in  $\mathcal{N} = 4$  super Yang-Mills theory (3.19) shall be better discussed elsewhere.

Regarding our weak-coupling ABJ model, we have that the averages in (3.14) can be interpreted as averages of the  $su(1, 1)$  displacement operator,<sup>6</sup> which is, in terms of the generators (3.16) and with  $\xi \in \mathbb{C}$  [33],

$$S(\xi) = \exp(\xi^* K_- - \xi K_+), \quad (3.20)$$

acting on the states of the  $su(1, 1)$  quantum oscillator and with the identification  $\xi = -im_2/2$ , as explained above.

For a photonic interpretation and to further understand the meaning of the displacement operator average in the  $su(1, 1)$  setting, recall that there is a well-known two-mode realization<sup>7</sup> of  $su(1, 1)$  [the Holstein-Primakoff realization for  $su(1, 1)$ ] where

$$\begin{aligned} K_+ &= a^\dagger b^\dagger, \\ K_- &= ab, \\ K_0 &= \frac{1}{2}(a^\dagger a + b^\dagger b + 1). \end{aligned}$$

The states corresponding to the discrete positive series are then given by  $|n + n_0, n\rangle = |n + n_0\rangle \otimes |n\rangle$ , where the Bargmann index is  $\lambda = (|n_0| + 1)/2$ . Thus, the observables in the large  $k$  limit of the ABJ theory can be described in

<sup>5</sup>In the work [35] on half-BPS Wilson loops in  $\mathcal{N} = 4$  SYM theory, the displacement operator is introduced and coherent states mentioned, but mostly *en route* to obtain a normal matrix model description of the Wilson loops.

<sup>6</sup>The reason is simply the same that leads to (3.18) in the quantum harmonic oscillator case. Notice that the position operator is  $\hat{X} = (K_+ + K_-)/2$  and that these generators are raising and lowering operators acting on the  $su(1, 1)$  oscillator states [8].

<sup>7</sup>The one boson realization only allows for Bargmann index  $\lambda = 1/4$  or  $\lambda = 3/4$ .

terms of states of a two-mode photonic system with occupancies of  $n$  and  $n + n_0$  photons in each mode with  $n = 0, 1, 2, \dots, N-1$  and  $n_0 = 2M-1$  for  $M = 1, 2, \dots$  and  $n_0 = 0$  for  $M = 0$  (ABJM). Notice also that the  $su(1, 1)$  displacement operator (3.20) is  $e^{\xi ab - \xi^* a^\dagger b^\dagger}$  and hence it is a squeeze operator acting on these two-mode states [generalized coherent states for  $su(1, 1)$  are conventional squeezed states of quantum optics].

## IV. OUTLOOK

The connection with Meixner-Pollaczek polynomials can be extended to carry out more detailed computations of Wilson loops in mass-deformed ABJ(M) theory, in the weak-coupling limit (some technical aspects are discussed in more detail in the Appendix). As another open problem, recall that we have also seen that some of the quotients of partition functions studied above, such as (2.12), which we have related to the Penner matrix model and hence, in principle, with topological strings, are precisely the ones relevant in the higher-spin double scaling limit [4]. In this limit  $U(N)_k \times U(N+M)_{-k}$  ABJ theory with finite  $N$  and large  $M$  and  $k$  is conjectured to be dual to  $\mathcal{N} = 6$  parity-violating Vasiliev higher spin theory on  $AdS_4$  with  $U(N)$  gauge symmetry [3,4]. Recall that, in the oscillator interpretation put forward here, the  $M$  parameter is specifically the Bargmann index  $k$  of the positive discrete series representation of  $su(1, 1)$ . It would be also interesting, therefore, to use the exact solvability found in this paper to further study this double scaling limit.

Another possible open problem would be to give a physical interpretation of the appearance of  $su(1, 1)$  oscillators, for example in terms of motion in AdS space. In this sense, it is already known that some slightly different  $su(1, 1)$  quantum oscillators emerge as solutions of the Klein-Gordon equation in AdS space [36].

Finally, and more generally, it would be interesting if a deformation of the  $su(1, 1)$  oscillator eigenfunctions (or equivalently, of the Meixner-Pollaczek polynomials) can be found, such that the full theory, given by the matrix model (1.3). This deformation would be seemingly different from the usual  $q$ -deformation (which holds in the Gaussian unitary ensemble/Stieltjes-Wigert ensemble description of pure Chern-Simons theory [17]) because the Vandermonde determinant is deformed according to  $x \rightarrow \tanh x$ , instead of  $x \rightarrow \sinh x$ . Notice that different generalizations of the eigenfunctions (beyond the  $q$ -deformation) already exist [8].

## ACKNOWLEDGMENTS

Thanks to Jorge Russo and Francesco Aprile for reading the paper and making valuable comments. This work was supported by the Fundação para a Ciência e Tecnologia (FCT) through its program Investigador FCT IF2014, under Contract No. IF/01767/2014.

## APPENDIX: FOURIER INTEGRAL AND CONSISTENCY CHECK

The following Fourier integral [31],

$$\int_{-\infty}^{\infty} e^{-2ixt} P_n^\lambda(x, \phi) P_m^\lambda(x, \phi) e^{(2\phi - \pi)x} |\Gamma(\lambda + ix)|^2 dx = \frac{\Gamma(2\lambda + n)\Gamma(2\lambda + m)}{4^2 \Gamma(2\lambda) n! m!} \frac{2\pi e^{i\pi\lambda} (\sinh t)^{n+m}}{(\cos \phi \sinh t + i \sin \phi \cosh t)^{n+m+2\lambda}} {}_2F_1\left(\begin{matrix} -n, -m \\ 2\lambda \end{matrix}; -\left(\frac{\sin \phi}{\sinh t}\right)^2\right), \quad (\text{A1})$$

is an extension of the orthogonality relationship for Meixner-Pollaczek polynomials. It suggests that the analytical continuation, required by the presence of the term in the mass-deformed ABJM theory, namely  $e^{-im_2x}$ , can be accounted for with no problems, since (A1) contains both a real exponential factor and a Fourier kernel. However, we can explicitly check that this is the case by explicitly computing the mass-deformed free energy using an equivalent formulation of the matrix model in terms of a Hankel determinant and only using the Fourier transform identity (3.9) to compute the matrix elements. Indeed, since the integral in (3.3) is a determinant,

$$\frac{1}{N!} \int \prod_{i=1}^N \frac{e^{-im_2x_i} dx_i}{\cosh(\pi x_i)} \prod_{i<j} (x_i - x_j)^2 = \det(c_{i+j})_{i,j=0}^{N-1}$$

with matrix elements

$$c_n = \int dx \frac{e^{-im_2x} x^n}{\cosh(\pi x)}, \quad (\text{A2})$$

using (A1) and differentiation under the integral sign (with respect to  $m_2$ ), every matrix element is computed explicitly. For  $N = 2$  for example

$$\frac{1}{2} \int \prod_{i=1}^2 \frac{e^{-im_2x_i} dx_i}{\cosh(\pi x_i)} \prod_{i<j} (x_i - x_j)^2 = \det \begin{pmatrix} 1/\cosh(m_2/2) & \frac{\sinh(m_2/2)}{2i\cosh^2(m_2/2)} \\ \frac{\sinh(m_2/2)}{2i\cosh^2(m_2/2)} & \frac{1}{4} \left( \frac{1}{\cosh^3(m_2/2)} - \frac{\sinh^2(m_2/2)}{\cosh^3(m_2/2)} \right) \end{pmatrix} = \frac{1}{4\cosh^4(m_2/2)}. \quad (\text{A3})$$

The cross-diagonal terms nicely cancel with part of the diagonal term and the result coincides with the orthogonal polynomial computation [notice that in this case, Barnes function is  $G^2(3) = 1$ ]. It is also immediate to check that this determinant is equivalent to carrying out the integrations on the lhs of (A3) explicitly. Let us explicitly also check the  $N = 3$  case,

$$\det \begin{pmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{pmatrix} = \frac{1}{16} \operatorname{sech}^9(m_2/2),$$

where the entries are as in (A3) and the additional matrix entries are

$$\begin{aligned} c_3 &= \frac{i}{8} (-5\operatorname{sech}^3(m_2/2) \tanh(m_2/2) \\ &\quad + \operatorname{sech}(m_2/2) \tanh^3(m_2/2)), \\ c_4 &= \frac{5}{16} \operatorname{sech}^5(m_2/2) - \frac{9}{8} \operatorname{sech}^3(m_2/2) \tanh^2(m_2/2) \\ &\quad + \frac{1}{16} \operatorname{sech}(m_2/2) \tanh^4(m_2/2). \end{aligned}$$

Thus the same cancellation occurs and only the direct product of the leading term in the diagonals ends up contributing. This is remarkable but expected from the theory of orthogonal polynomials and we emphasize that the test was just to make sure that the Fourier kernel in (A2) (and in the matrix model) did not invalidate the orthogonal polynomial computation. Notice that the moments (A2) are all obtained from differentiation, under the integral sign, of  $a_0$ . This is related to the fact that this Hankel determinant satisfies a Toda lattice equation, which is a well-known result in the theory of the six vertex model [37], a problem where the Meixner-Pollaczek polynomials are well known to provide analytical solutions [38].

Finally, note that the argument put forward above, right below Eq. (A1), extends also to the analytical calculability of the Wilson loop with leads to an average with both types of factors present. Effectively, the identity (A1) is analogous to the existing expression for Hermite polynomials which leads to the celebrated expression for the  $\frac{1}{2}$ -BPS Wilson loop in  $\mathcal{N} = 4$  in terms of a Laguerre polynomial [27]. However, as we have pointed out above, the same result can be obtained (and was indeed obtained decades ago) by using the action of the creation and annihilation

operators on the Fock states. It would be interesting to do the same in our setting, since the action on the  $su(1, 1)$  Fock basis is equally well known [8].

Regarding the use of (A1) in the evaluation of the Wilson loop average which, in the mass-deformed case, contains both a factor of the type  $e^{-2ixt}$  and another one of the type  $e^{(2\phi-\pi)x}$ , notice that we chose to identify the former with the mass-deformed term  $e^{-im_2x}$ . This lead to the expression

(3.15). It would be interesting to study in further detail the convenience of this choice and also to see if there exists a summation analogous to the one that holds for the  $\frac{1}{2}$ -BPS Wilson loop in  $\mathcal{N} = 4$  SYM, which leads to the final result in [27] and also has lead us to the interpretation of the Wilson loop in terms of the overlap of two consecutive displaced number states of the harmonic oscillator (3.19).

- 
- [1] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena,  $N = 6$  superconformal Chern-Simons-matter theories, M2-branes and their gravity duals, *J. High Energy Phys.* **10** (2008) 091.
- [2] O. Aharony, O. Bergman, and D. L. Jafferis, Fractional M2-branes, *J. High Energy Phys.* **11** (2008) 043.
- [3] C. M. Chang, S. Minwalla, T. Sharma, and X. Yin, ABJ triality: From higher spin fields to strings, *J. Phys. A* **46**, 214009 (2013).
- [4] S. Hirano, M. Honda, K. Okuyama, and M. Shigemori, ABJ theory in the higher spin limit, [arXiv:1504.00365](https://arxiv.org/abs/1504.00365).
- [5] M. Honda, Identification of bulk coupling constant in higher spin/ABJ correspondence, *J. High Energy Phys.* **08** (2015) 110.
- [6] V. Freilikher, E. Kanzieper, and I. Yurkevich, Unitary random-matrix ensemble with governable level confinement, *Phys. Rev. E* **53**, 2200 (1996).
- [7] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their  $q$ -Analogues*, Springer Monographs in Mathematics (Springer-Verlag, Berlin, 2010).
- [8] E. I. Jafarov, N. I. Stoilova, and J. Van der Jeugt, Deformed  $su(1, 1)$  algebra as a model for quantum oscillators, *SIGMA* **8**, 025 (2012)
- [9] M. Marino, Lectures on localization and matrix models in supersymmetric Chern-Simons-matter theories, *J. Phys. A* **44**, 463001 (2011).
- [10] A. Grassi and M. Marino, M-theoretic matrix models, *J. High Energy Phys.* **02** (2015) 115.
- [11] S. Codesido, A. Grassi, and M. Mariño, Exact results in  $\mathcal{N} = 8$  Chern-Simons-matter theories and quantum geometry, *J. High Energy Phys.* **07** (2015) 011.
- [12] H. Awata, S. Hirano, and M. Shigemori, The partition function of ABJ theory, *Prog. Theor. Exp. Phys.* **2013**, 053B04 (2013).
- [13] A. Kapustin, B. Willett, and I. Yaakov, Nonperturbative tests of three-dimensional dualities, *J. High Energy Phys.* **10** (2010) 013.
- [14] M. Marino and P. Putrov, ABJM theory as a Fermi gas, *J. Stat. Mech.* **03** (2012) P03001.
- [15] M. Honda, Direct derivation of “mirror”, ABJ partition function, *J. High Energy Phys.* **12** (2013) 046.
- [16] P. J. Forrester, B. Jancovici, and G. Tellez, Universality in some classical Coulomb systems of restricted dimension, *J. Stat. Phys.* **84**, 359 (1996).
- [17] M. Tierz, Soft matrix models and Chern-Simons partition functions, *Mod. Phys. Lett. A* **19**, 1365 (2004).
- [18] E. W. Barnes, The theory of the G-function, *Quart. J. Math.* **31**, 264 (1899).
- [19] S. Mukhi, Topological matrix models, Liouville matrix model and  $c = 1$  string theory, [arXiv:hep-th/0310287](https://arxiv.org/abs/hep-th/0310287).
- [20] E. Kowalski and A. Nikeghbali, Mod-Gaussian convergence and the value distribution of  $\zeta(1/2 + it)$  and related quantities, *J. Lond. Math. Soc.* **86**, 291 (2012).
- [21] L. Anderson and K. Zarembo, Quantum phase transitions in mass-deformed ABJM matrix model, *J. High Energy Phys.* **09** (2014) 021.
- [22] L. Anderson and J. G. Russo, ABJM theory with mass and FI deformations and quantum phase transitions, *J. High Energy Phys.* **05** (2015) 064.
- [23] J. G. Russo and G. A. Silva, Exact partition function in  $U(2) \times U(2)$  ABJM theory deformed by mass and Fayet-Iliopoulos terms, *J. High Energy Phys.* **12** (2015) 092.
- [24] M. Tierz, Exact solution of Chern-Simons-matter matrix models with characteristic/orthogonal polynomials, *J. High Energy Phys.* **04** (2016) 168.
- [25] A. Amariti, On the exact R charge for  $N = 2$  CS theories, *J. High Energy Phys.* **06** (2011) 110.
- [26] A. Amariti and M. Siani, Z-extremization and F-theorem in Chern-Simons matter theories, *J. High Energy Phys.* **10** (2011) 016.
- [27] N. Drukker and D. J. Gross, An exact prediction of  $N = 4$  SUSYM theory for string theory, *J. Math. Phys. (N.Y.)* **42**, 2896 (2001).
- [28] S. Hirano, K. Nii, and M. Shigemori, ABJ Wilson loops and Seiberg duality, *Prog. Theor. Exp. Phys.* **2014**, 113B04 (2014).
- [29] A. Kapustin, B. Willett, and I. Yaakov, Exact results for Wilson loops in superconformal Chern-Simons theories with matter, *J. High Energy Phys.* **03** (2010) 089.
- [30] N. Drukker and D. Trancanelli, A supermatrix model for  $N = 6$  super Chern-Simons-matter theory, *J. High Energy Phys.* **02** (2010) 058.
- [31] M. Meiler, R. Cordero-Soto, and S. K. Suslov, Solution of the Cauchy problem for a time-dependent Schrödinger equation, *J. Math. Phys. (N.Y.)* **49**, 072102 (2008).
- [32] K. E. Cahill and R. J. Glauber, Ordered expansions in boson amplitude operators, *Phys. Rev.* **177**, 1857 (1969).

- [33] M. M. Nieto, Displaced and squeezed number states, *Phys. Lett. A* **229**, 135 (1997).
- [34] D. J. Wineland and W. M. Itano, Laser cooling of atoms, *Phys. Rev. A* **20**, 1521 (1979).
- [35] K. Okuyama and G. W. Semenoff, Wilson loops in  $N = 4$  SYM and fermion droplets, *J. High Energy Phys.* **06** (2006) 057.
- [36] D. J. Navarro and J. Navarro-Salas, Special-relativistic harmonic oscillator modeled by Klein-Gordon theory in anti-de Sitter space, *J. Math. Phys. (N.Y.)* **37**, 6060 (1996).
- [37] P. M. Bleher and K. Liechty, *Random Matrices and the Six-Vertex Model*, CRM Monograph Series Vol. 32 (American Mathematical Society, Providence, RI, 2014).
- [38] F. Colomo and A. G. Pronko, Square ice, alternating sign matrices and classical orthogonal polynomials, *J. Stat. Mech.* (2005) P01005.