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Scaling laws in chiral hydrodynamic turbulence

Naoki Yamamoto
Department of Physics, Keio University, Yokohama 223-8522, Japan
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We study the turbulent regime of chiral (magneto)hydrodynamics for charged and neutral matter with chirality imbalance. We find that the chiral magnetohydrodynamics for charged plasmas possesses a unique scaling symmetry, only without fluid helicity under the local charge neutrality. We also find a different type of unique scaling symmetry in the chiral hydrodynamics for neutral matter with fluid helicity in the inertial range. We show that these symmetries dictate the self-similar inverse cascade of the magnetic and kinetic energies. Our results imply the possible inverse energy cascade in core-collapse supernovae due to the chiral transport of neutrinos.

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I. INTRODUCTION

Recently, chiral transport phenomena related to quantum anomalies [1,2] have attracted much attention both theoretically and experimentally, especially in heavy ion physics [3] and a new type of materials named the Weyl (semi) metals [4]. Of particular interest is the possible observation of the current along a magnetic field in the presence of the chirality imbalance, called the chiral magnetic effect (CME) [5–8]. Such unusual transport phenomena could potentially lead to some physical consequences in other relativistic systems, such as the electroweak plasmas in the early Universe [9,10], electromagnetic plasmas in neutron stars [11] and core-collapse supernovae [12,13], and so on.

To describe these chiral transport phenomena in nonequilibrium situations, hydrodynamics and kinetic theory have been reformulated, which are now referred to as the chiral (or anomalous) hydrodynamics [14] and chiral kinetic theory [15–17], respectively. However, the evolutions of chiral matter when *nonlinear* effects of the fluid velocity and/or *dynamical* electromagnetic fields become important have not been fully understood so far; see Refs. [18,19] for recent related works. For analytical and numerical analyses of chiral hydrodynamics in *external* electromagnetic fields, see, e.g., Refs. [20,21] and Ref. [22], respectively.

In this paper, we study the generic properties of the chiral (magneto)hydrodynamics describing the evolutions of charged and neutral matter at finite chiral chemical potential μ_5 and finite temperature T. We find that the chiral magnetohydrodynamics (ChMHD), together with the chiral anomaly relation, possesses a *unique* scaling symmetry for $\mu_5 \ll T$ under the local charge neutrality without fluid helicity. We also find a different type of unique scaling symmetry in the chiral hydrodynamics for neutral matter at finite chemical potential μ in the presence of fluid helicity in the so-called "inertial range" where dissipation is negligible. We stress that the presence of quantum anomalies and chiral transport phenomena is important for these scaling symmetries.

From these scaling symmetries, we derive the self-similar scaling laws of the magnetic and kinetic energies [see Eqs. (38) and (39)] and the scaling laws of the magnetic and kinetic correlation lengths in chiral matter [see Eqs. (47), (48), and (59)]. These results show that the inverse energy cascade—the process that transfers the energy from small to larger scales—occurs in the turbulent regime of both ChMHD and neutral chiral hydrodynamics under the conditions above. In particular, it implies that the chiral transport of neutrinos [13] neglected so far may lead to the inverse energy cascade in core-collapse supernovae, instead of the direct energy cascade observed in the conventional neutrino transport theory [23]. This qualitative modification may be potentially important to understanding the origin of supernova explosions.

The paper is organized as follows: In Sec. II, after reviewing the ChMHD equations, we discuss its applicability and the conservation laws.[1] We then study the scaling symmetry of the ChMHD and its physical consequences. In Sec. III we discuss the scaling symmetry of the neutral chiral hydrodynamics and its physical applications. Section IV is devoted to summary and discussion.

In the following, we use the natural units $\hbar = c = 1$.

II. CHIRAL MAGNETOHYDRODYNAMICS FOR CHARGED PLASMAS

Let us first consider the ChMHD for plasmas of a Dirac fermion at finite chiral chemical potential $\mu_5 \equiv (\mu_R - \mu_L)/2$. We will be interested in the time scale larger than $1/\sigma$ (with σ being the electrical conductivity), during which the electric charge diffuses immediately. Then, we can assume the local charge neutrality, $n = 0$ or $\mu \equiv (\mu_R + \mu_L)/2 = 0$. On the other hand, the chiral charge n_5 can be generally finite

[1] To our knowledge, the applicability of the ChMHD has not been appreciated earlier, except for Refs. [24,25]. The regime of applicability will be essential for determining the scaling symmetry of the ChMHD below.

in this regime. We will see in Sec. II B that there is actually some constraint for μ_5 to treat it as a hydrodynamic (slow) variable.

A. Hydrodynamic equations

The ChMHD equations are obtained by promoting external electromagnetic fields in the chiral hydrodynamic equations of Ref. [14] to dynamical ones. Note here that the chiral charge n_5 also evolves in time and space, according to the chiral anomaly relation in the presence of electromagnetic fields [see Eq. (3)], and it should be regarded as a *dynamical* variable $n_5(t, \boldsymbol{x})$ as well. As the electromagnetic fields and the chiral charge vary much faster (and at a shorter length scale) than T, we assume that T is static and homogeneous in the regime of our interest. Keeping the main applications of the ChMHD to chiral plasmas in the early Universe and astrophysical systems in mind, we also assume the bulk fluid velocity to be nonrelativistic, $v \equiv |\boldsymbol{v}| \ll 1$, and we only retain the terms to the leading order in v below. We will discuss the case with (ultra) relativistic bulk fluid velocity $v \sim 1$ in Sec. IV, which may be relevant to quark-gluon plasmas in heavy ion collisions.

The hydrodynamic equations in the Landau-Lifshitz frame are given by [13,14][2]

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda, \tag{1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{j} = 0, \tag{2}$$

$$\partial_t (n_5 + \kappa \boldsymbol{v} \cdot \boldsymbol{\omega}) + \boldsymbol{\nabla} \cdot \boldsymbol{j}_5 = C \boldsymbol{E} \cdot \boldsymbol{B}, \tag{3}$$

together with Maxwell's equations,

$$\partial_\nu F^{\nu\mu} = j^\mu. \tag{4}$$

Here the energy-momentum tensor $T^{\mu\nu}$, and the vector and axial currents, $\boldsymbol{j}$ and $\boldsymbol{j}_5$, are given by [14][3]

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu - P g^{\mu\nu} + \tau^{\mu\nu}, \tag{5}$$

$$\boldsymbol{j} = \sigma (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + \kappa_B \boldsymbol{B}, \tag{6}$$

$$\boldsymbol{j}_5 = n_5 \boldsymbol{v} + \kappa \boldsymbol{\omega}, \tag{7}$$

where ϵ is the energy density, P is the pressure, σ is the electrical conductivity, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$ is the vorticity, and $\tau^{\mu\nu}$ expresses the dissipative effects like viscosity. Equation (3) expresses the violation of the axial current conservation by the chiral anomaly [1,2] and the mixed gauge gravitational anomaly [13], where $C = e^2/(2\pi^2)$ is the coefficient of the chiral anomaly. The anomalous transport coefficients κ_B and κ can be expressed from symmetry consideration (parity and charge conjugation symmetries) as

$$\kappa_B = \tilde{\kappa}_B e^2 \mu_5, \qquad \kappa = \tilde{\kappa} T^2, \tag{8}$$

where $\tilde{\kappa}_B$ and $\tilde{\kappa}$ are some constants related to the coefficients of the chiral anomaly and mixed gauge gravitational anomaly [14,27]. (The complete expressions of κ and κ_B can be found, e.g., in Ref. [27], but they are unimportant for our purpose in this paper.) The currents proportional to $\boldsymbol{B}$ and $\boldsymbol{\omega}$ in Eqs. (6) and (7) are the CME [5–8] and the chiral vortical effect (CVE) [14,27–29], respectively.

Note that we used the local charge neutrality, $\mu = 0$, in Eqs. (6), (7), and (8) to ignore the part of the chiral separation effect (CSE) [30,31] and the CVE whose transport coefficients include a factor of μ: $\boldsymbol{j}^5_{\rm CSE} \propto \mu \boldsymbol{B}$, $\boldsymbol{j}_{\rm CVE} \propto \mu\mu_5 \boldsymbol{\omega}$, and $\boldsymbol{j}^5_{\rm CVE} \propto \mu^2 \boldsymbol{\omega}$. We also dropped the contribution of the cross helicity $\propto \mu \boldsymbol{v} \cdot \boldsymbol{B}$ [13] in Eq. (3).

Under the local charge neutrality, the displacement current $\partial_t \boldsymbol{E}$ is negligible, and Ampère's law becomes $\boldsymbol{j} = \boldsymbol{\nabla} \times \boldsymbol{B}$ [32]. Then, Eq. (2) is automatically satisfied. By eliminating $\boldsymbol{j}$ and $\boldsymbol{E}$, the ChMHD equations for an incompressible fluid ($\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$) above reduce to [13]

$$(\epsilon + P)(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}) = -\frac{1}{2} \boldsymbol{\nabla} \boldsymbol{B}^2 + (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B} + \nu \boldsymbol{\nabla}^2 \boldsymbol{v}, \tag{9}$$

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) + \kappa_B \eta \boldsymbol{\nabla} \times \boldsymbol{B} + \eta \boldsymbol{\nabla}^2 \boldsymbol{B}, \tag{10}$$

$$\partial_t (n_5 + \kappa \boldsymbol{v} \cdot \boldsymbol{\omega}) + \boldsymbol{v} \cdot \boldsymbol{\nabla} n_5 = -C\eta [\kappa_B \boldsymbol{B}^2 - (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \boldsymbol{B}], \tag{11}$$

where $\eta \equiv 1/\sigma$ is the resistivity. We here ignored the contribution $-\boldsymbol{\nabla} P$ on the right-hand side of Eq. (9), because, as we will show that $T \gg \mu_5$ in Sec. II B, the dominant contribution to P is the homogeneous T. This set of coupled equations is closed for dynamical variables, $\boldsymbol{v}(t, \boldsymbol{x})$, $\boldsymbol{B}(t, \boldsymbol{x})$, and $\mu_5(t, \boldsymbol{x})$. [As we will discuss in Eq. (14) below, n_5 is related to μ_5.] The electric field is given by using these variables as

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} - \kappa_B \eta \boldsymbol{B} + \eta \boldsymbol{\nabla} \times \boldsymbol{B}. \tag{12}$$

Equations (9)–(11) can be regarded as an extension of the usual MHD equations for relativistic fluids [26] to the ones with anomalous parity-violating effects [13]. Indeed, in the absence of anomalous effects (setting $n_5 = \kappa_B = 0$ and disregarding the terms with the coefficient C), they reduce to the usual MHD equations. These ChMHD equations describe the charged plasmas in the early Universe [9,10] and (proto)neutron stars [11,12] where matter with chirality imbalance may be realized.

[2]Precisely speaking, we have an additional charge density due to the CME as $\Delta n = \kappa_B \mu_5 \boldsymbol{v} \cdot \boldsymbol{B}$ [13]. However, it can be shown that its contribution to the right-hand side of Eq. (1), expressed by $\Delta n \boldsymbol{E}$, is negligibly small compared with the term $\boldsymbol{j} \times \boldsymbol{B}$ under the condition (13) derived below.

[3]We denote the anomalous transport coefficients by κ and κ_B instead of ξ and ξ_B in Refs. [13,14], as ξ will be used for the correlation length later.

B. Regime of applicability

Before proceeding further, we first clarify the regime of the applicability of the ChMHD above. It is known that the ChMHD has a plasma instability at finite μ_5 [9,10], called the chiral plasma instability (CPI), whose length scale is microscopically estimated as $l_{\rm CPI} \sim (e^2\mu_5)^{-1}$ [24]. For the physical picture of the CPI, see Ref. [25].

Recall that hydrodynamics is an effective theory valid at a length scale larger than the mean free path. As the mean free path for the U(1) electromagnetic plasma is given by $l_{\rm mfp} \sim (e^4 T)^{-1}$ up to logarithmic corrections, it is necessary to meet the following condition for the use of hydrodynamics: $l_{\rm mfp} \ll l_{\rm CPI}$ or

$$\mu_5(t, \boldsymbol{x}) \ll e^2 T. \tag{13}$$

(Otherwise, the ChMHD would have an unstable mode which is beyond its applicability, and the theory would not be well defined.) Under this condition, we have

$$n_5 \approx \frac{\mu_5 T^2}{6}. \tag{14}$$

Then, the transport coefficients ν and η can be regarded as constants for static and homogeneous T.

It should be remarked that, even in the Maxwell-Chern-Simons equations (or anomalous Maxwell equations), which correspond to the limit $\boldsymbol{v} \to \boldsymbol{0}$ of the ChMHD, the condition (13) must be satisfied to use the notion of the conductivity σ itself. This is because σ is well defined only at the long length scale $l \gg l_{\rm mfp}$. A related point was emphasized in Ref. [24] from the viewpoint of the microscopic kinetic theory (see also Ref. [25]).

If one is interested in the physics beyond this regime, $\mu_5 \gtrsim T$, one needs to use the chiral kinetic theory [15–17] instead of the ChMHD, as was done in Ref. [24]. This is beyond the scope of the present paper.

C. Conservation laws

In the usual MHD, the magnetic helicity (or the Chern-Simons number),

$$\mathcal{H}_B = \int d^3x \boldsymbol{A} \cdot \boldsymbol{B}, \tag{15}$$

can be shown to be an approximate conserved quantity for sufficiently large Reynolds numbers [26,32]. On the other hand, one expects that $\mathcal{H}_B$ is not a conserved quantity in the ChMHD due to the CPI. In the following, we consider the modifications to the conventional conservation laws.

Using the ChMHD equations above, we obtain the time derivative of the energy E and the magnetic helicity $\mathcal{H}_B$ as

$$\dot{E} = -\int d^3\boldsymbol{x}[-\kappa_B \eta \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) + \eta(\boldsymbol{\nabla}\times\boldsymbol{B})^2 + \nu(\epsilon + P)(\boldsymbol{\nabla}\times\boldsymbol{v})^2], \tag{16}$$

$$\dot{\mathcal{H}}_B = 2\eta \int d^3\boldsymbol{x}[\kappa_B \boldsymbol{B}^2 - \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})]. \tag{17}$$

Here the terms with the coefficient κ_B on the right-hand sides of Eqs. (16) and (17) are the modifications to the usual MHD. As the other terms contain one more derivative compared with the κ_B terms for $B \sim v$, the latter terms are dominant at large length scale, $l \gtrsim \kappa_B^{-1} \sim l_{\rm CPI}$. Recalling that $\eta \geq 0$, we have $\dot{\mathcal{H}}_B \geq 0$ for $\mu_5 > 0$ in this regime. This means that the largest change of E and $\mathcal{H}_B$ occurs at the scale of the CPI, and so we have $\dot{\mathcal{H}}_B/\dot{E} \sim l_{\rm CPI}$. Assuming that the integral scale is also $l_{\rm CPI}$, we have $\mathcal{H}_B/E \sim l_{\rm CPI}$. We thus have $\dot{\mathcal{H}}_B/\mathcal{H}_B \sim \dot{E}/E$, and so the magnetic helicity itself is not a good conserved quantity except for $\eta = 0$.

This should be contrasted with the conventional MHD, where the change of E and $\mathcal{H}_B$ occurs only at the scale of dissipation, $l_{\rm mfp}$. In that case, $(\dot{\mathcal{H}}_B/\mathcal{H}_B)/(\dot{E}/E) \sim l_{\rm mfp}/l \ll 1$, and $\mathcal{H}_B$ is approximately conserved [26]. Here l is the scale of turbulence that is much larger than $l_{\rm mfp}$ for large Reynolds numbers.

Although the magnetic helicity alone is not conserved, one can show that the total helicity, including the helicity of fermions and the helicity of fluids, is conserved. Indeed, from Eqs. (11) and (17) [or directly from Eq. (3)], we obtain the conservation of total helicity [13],

$$\partial_t \mathcal{H}_{\rm tot} = 0, \qquad \mathcal{H}_{\rm tot} \equiv \frac{C}{2}\mathcal{H}_B + \mathcal{H}_v + N_5, \tag{18}$$

where

$$N_5 \equiv \int d^3\boldsymbol{x} n_5, \qquad \mathcal{H}_v \equiv \int d^3\boldsymbol{x} \kappa \boldsymbol{v}\cdot\boldsymbol{\omega} \tag{19}$$

are the helicity (or chiral charge) of fermions and the fluid helicity, respectively. Note here that the cross helicity $\propto \int \mu \boldsymbol{v}\cdot\boldsymbol{B}$ [13] is absent under the local charge neutrality, $\mu = 0$.

D. Scaling symmetry

We now turn to the scaling symmetry of the ChMHD. Let us first recall the scaling symmetry of the usual MHD. For $n_5 = \kappa_B = 0$, ignoring the C terms, Eqs. (9) and (10) are invariant under the scaling [33]

$$\boldsymbol{x} \to l\boldsymbol{x}, \qquad t \to l^{1-h}t, \qquad \boldsymbol{v} \to l^h \boldsymbol{v}, \qquad \boldsymbol{B} \to l^h \boldsymbol{B}, \qquad \nu \to l^{1+h}\nu, \qquad \eta \to l^{1+h}\eta, \tag{20}$$

where l is the positive scaling factor and h is any real parameter. The transformation laws for other variables follow from Maxwell's equations (4) as, e.g., $\boldsymbol{E} \to l^{2h}\boldsymbol{E}$. Imposing the condition that the coefficients ν and η are nonzero constants, h is fixed as a unique value, $h = -1$. On the other hand, in the inertial range where the ν and η terms are negligible, the MHD has generic scaling symmetries with *any* h [33].

It is easy to check that ChMHD equations (9)–(11) have the same scaling symmetry in the absence of the local fluid helicity ($\boldsymbol{v}\cdot\boldsymbol{\omega} = 0$) if we further impose the following scaling for μ_5 and n_5 at the same time[4]:

$$\mu_5 \to l^{-1}\mu_5, \qquad n_5 \to l^{1+2h}n_5. \tag{21}$$

As μ_5 and n_5 are related by $n_5 \propto \mu_5$ for $\mu_5 \ll T$ as shown in Eq. (14), h is fixed as

$$h = -1. \tag{22}$$

Coincidentally, this is the same value as the one required by constant ν and η.

It should be remarked that the local charge neutrality and the absence of the local fluid helicity are essential for this scaling symmetry; if $\mu \neq 0$, we would have the CVE of the form $\boldsymbol{j} \propto \mu\mu_5\boldsymbol{\omega}$ in Eq. (6), which would violate the scaling symmetry above. The presence of the local fluid helicity $\kappa\boldsymbol{v}\cdot\boldsymbol{\omega}$ would also break down the scaling symmetry.

In the inertial range, the chiral anomaly with the coefficient C and the CME with the coefficient κ_B do not contribute at all in Eqs. (9) and (10), while the local fluid helicity $\kappa\boldsymbol{v}\cdot\boldsymbol{\omega}$ can. In this case, the ChMHD has generic scaling symmetries with *any* h, even in the presence of the fluid helicity, if we impose the following scaling for n_5:

$$n_5 \to l^{-1+2h}n_5. \tag{23}$$

E. Physical consequences

Let us explore the physical consequences of the scaling symmetry (21) with $h = -1$ in the turbulent regime. Our argument here is analogous to the one in Refs. [33,34]. We will first leave h unspecified for later purposes and will set $h = -1$ later.

We first define the average chiral density,

$$\bar{n}_5(t) \equiv \frac{1}{V}\int d^3\boldsymbol{x}\, n_5(\boldsymbol{x}, t), \tag{24}$$

where $V = \int_{2\pi/K}^{L} d^3\boldsymbol{x}$, with $2\pi/L$ and K being the infrared and ultraviolet momentum cutoffs, respectively. In the following, we will consider the formal limit as $L \to \infty$ and $K \to \infty$. From Eq. (3), and assuming that $\boldsymbol{j}_5$ vanishes at sufficiently large distances, the time evolution of $\bar{n}_5(t)$ is given by

$$\partial_t \bar{n}_5 = \frac{C}{V}\int d^3\boldsymbol{x}\boldsymbol{E}\cdot\boldsymbol{B} = \int_0^\infty dk\mathcal{N}(k,t), \tag{25}$$

where

$$\mathcal{N}(k,t) = \frac{4\pi C}{V}k^2\langle \boldsymbol{E}(\boldsymbol{k},t)\cdot\boldsymbol{B}^*(\boldsymbol{k},t)\rangle, \tag{26}$$

for an isotropic turbulence.

We also consider the magnetic and kinetic energy densities in $k \equiv |\boldsymbol{k}|$ space:

$$\mathcal{E}_B(k,t) = \frac{2\pi k^2}{(2\pi)^3}\int d^3\boldsymbol{y} e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\langle \boldsymbol{B}(\boldsymbol{x},t)\cdot\boldsymbol{B}(\boldsymbol{x}+\boldsymbol{y},t)\rangle, \tag{27}$$

$$\mathcal{E}_v(k,t) = \frac{2\pi k^2}{(2\pi)^3}\int d^3\boldsymbol{y} e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\langle \boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{y},t)\rangle, \tag{28}$$

and the magnetic and kinetic correlation lengths defined by

$$\xi_B(t) = 2\pi\frac{\int_0^\infty dk k^{-1}\mathcal{E}_B(k,t)}{\int_0^\infty dk\mathcal{E}_B(k,t)}, \tag{29}$$

$$\xi_v(t) = 2\pi\frac{\int_0^\infty dk k^{-1}\mathcal{E}_v(k,t)}{\int_0^\infty dk\mathcal{E}_v(k,t)}, \tag{30}$$

respectively.

Let us now look into the scaling symmetries of $\mathcal{N}(k,t)$, $\mathcal{E}_B(k,t)$, and $\mathcal{E}_v(k,t)$. From Eq. (20), they satisfy

$$\mathcal{N}(l^{-1}k, l^{1-h}t) = l^{1+3h}\mathcal{N}(k,t), \tag{31}$$

$$\mathcal{E}_B(l^{-1}k, l^{1-h}t) = l^{1+2h}\mathcal{E}_B(k,t), \tag{32}$$

$$\mathcal{E}_v(l^{-1}k, l^{1-h}t) = l^{1+2h}\mathcal{E}_v(k,t). \tag{33}$$

We introduce the functions $\psi_n(k,t) \equiv k^{1+3h}\mathcal{N}(k,t)$, $\psi_B(k,t) \equiv k^{1+2h}\mathcal{E}_B(k,t)$ and $\psi_v(k,t) \equiv k^{1+2h}\mathcal{E}_v(k,t)$, such that

$$\psi_n(l^{-1}k, l^{1-h}t) = \psi_n(k,t), \tag{34}$$

$$\psi_B(l^{-1}k, l^{1-h}t) = \psi_B(k,t), \tag{35}$$

$$\psi_v(l^{-1}k, l^{1-h}t) = \psi_v(k,t). \tag{36}$$

These relations mean that ψ_n, ψ_B, and ψ_v are functions of $x \equiv k^{1-h}t$ alone: $\psi_n(k,t) = \psi_n(k^{1-h}t)$, $\psi_B(k,t) = \psi_B(k^{1-h}t)$, and $\psi_v(k,t) = \psi_v(k^{1-h}t)$. Hence, $\mathcal{N}(k,t)$, $\mathcal{E}_B(k,t)$, and $\mathcal{E}_v(k,t)$ can be expressed as

$$\mathcal{N}(k,t) = k^{-1-3h}\psi_n(k^{1-h}t), \tag{37}$$

[4] A partial transformation law (20), which does not take into account the anomaly relation (11) and the scaling (21), was previously given in Ref. [18].

$$\mathcal{E}_B(k,t) = k^{-1-2h}\psi_B(k^{1-h}t), \tag{38}$$

$$\mathcal{E}_v(k,t) = k^{-1-2h}\psi_v(k^{1-h}t). \tag{39}$$

Substituting Eqs. (37), (38), and (39) into Eqs. (25), (29), and (30), respectively, and performing the integral over t with assuming $\bar{n}_5(\infty) = 0$ in the first,[5] we have

$$\bar{n}_5(t) = \bar{n}_5(t_s)\left(\frac{t}{t_s}\right)^{\frac{1+2h}{1-h}}, \tag{40}$$

$$\xi_B(t) = \xi_B(t_s)\left(\frac{t}{t_s}\right)^{\frac{1}{1-h}}, \tag{41}$$

$$\xi_v(t) = \xi_v(t_s)\left(\frac{t}{t_s}\right)^{\frac{1}{1-h}}, \tag{42}$$

where t_s is some parameter and

$$\bar{n}_5(t_s) = \frac{1}{1+2h} t_s^{\frac{1+2h}{1-h}} \int_0^\infty dx x^{-\frac{1+2h}{1-h}}\psi_n(x), \tag{43}$$

$$\xi_B(t_s) = 2\pi t_s^{\frac{1}{1-h}} \frac{\int_0^\infty dx x^{-\frac{2+h}{1-h}}\psi_B(x)}{\int_0^\infty dx x^{-\frac{1+h}{1-h}}\psi_B(x)}, \tag{44}$$

$$\xi_v(t_s) = 2\pi t_s^{\frac{1}{1-h}} \frac{\int_0^\infty dx x^{-\frac{2+h}{1-h}}\psi_v(x)}{\int_0^\infty dx x^{-\frac{1+h}{1-h}}\psi_v(x)}. \tag{45}$$

Inserting $h = -1$, corresponding to the unique scaling symmetry (22) in the ChMHD, we obtain

$$\bar{n}_5(t) = \bar{n}_5(t_s)\left(\frac{t_s}{t}\right)^{\frac{1}{2}}, \tag{46}$$

$$\xi_B(t) = \xi_B(t_s)\left(\frac{t}{t_s}\right)^{\frac{1}{2}}, \tag{47}$$

$$\xi_v(t) = \xi_v(t_s)\left(\frac{t}{t_s}\right)^{\frac{1}{2}}. \tag{48}$$

We expect that the solutions of the ChMHD asymptotically approach these behaviors regardless of the initial conditions. In particular, Eq. (47) and (48) show that both $\xi_B(t)$ and $\xi_v(t)$ grow with time, meaning that both the magnetic and kinetic energies are transferred from a small scale to a larger scale: the inverse energy cascade.

Equation (37) with $h = -1$ exhibits the same self-similar inverse cascade of magnetic helicity observed in the Maxwell-Chern-Simons theory [35]. Our result here provides its generalization to the ChMHD, together with the new result (48), even in the presence of the fluid velocity $\boldsymbol{v}$.

[5]As seen from Eq. (40), this assumption can be satisfied when $h < -\frac{1}{2}$ or $h > 1$.

This argument shows that the self-similar behaviors in Eqs. (37)–(39) with $h = -1$ can be seen as a consequence of the scaling symmetry (20) and (21) in the ChMHD. However, it would break down away from the charge neutrality or in the presence of fluid helicity, as we have seen above.

III. CHIRAL HYDRODYNAMICS FOR NEUTRAL MATTER

We then consider the chiral hydrodynamics for neutral matter of a *single* chiral fermion at finite chemical potential $\mu \neq 0$. Our primary interest here is the application to the neutrino hydrodynamics considered in Ref. [13].

A. Hydrodynamic equations

As neutral matter does not couple to electromagnetic fields, the hydrodynamic equation in this case is

$$(\epsilon + P)(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}P + \nu\boldsymbol{\nabla}^2\boldsymbol{v}. \tag{49}$$

This is the usual relativistic hydrodynamics to the leading order in v [36]. We here include the contribution $-\boldsymbol{\nabla}P$ unlike Eq. (9) in the ChMHD, because we can consider not only the regime $\mu \ll T$, but also $\mu \gg T$ in the neutral chiral hydrodynamics, where μ is generally inhomogeneous (see below). On the other hand, the current conservation is modified by the CVE as [13]

$$\partial_t(n + \kappa\boldsymbol{v}\cdot\boldsymbol{\omega}) + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0, \tag{50}$$

$$\boldsymbol{j} = n\boldsymbol{v} + \kappa\boldsymbol{\omega}, \tag{51}$$

where $\kappa = \tilde{\kappa}_1\mu^2 + \tilde{\kappa}_2T^2$ with some constants $\tilde{\kappa}_{1,2}$ (see Ref. [27] for the detailed expressions).

Note that the neutral chiral matter does not have the CPI, unlike the charged chiral plasmas in Sec. II. Hence, we do not have the constraint like Eq. (13) in the present case.

B. Scaling symmetry

Let us now consider the scaling symmetry of the chiral hydrodynamics for neutral matter above. First, when ϵ and P are constants, Eq. (49) has the following scaling symmetry:

$$\boldsymbol{x} \to l\boldsymbol{x}, \quad t \to l^{1-h}t, \quad \boldsymbol{v} \to l^h\boldsymbol{v}, \quad \nu \to l^{1+h}\nu, \tag{52}$$

for any h. However, once the conservation law (50) is taken into account, this scaling symmetry seems not to hold for any h, even in the inertial range, at first sight.

In fact, there is a regime where the hydrodynamic equations above have a scaling symmetry. The point here is that, despite the absence of the CPI, the number density n can vary due to the CVE in Eq. (50) [13], so that n must be regarded as a dynamical variable, $n(t,\boldsymbol{x})$. We thus impose the scaling for μ as

$$\mu \to l^p \mu, \tag{53}$$

with some real parameter p.

We now show that the chiral hydrodynamics has a unique scaling symmetry,

$$h = 0, \qquad p = -1, \tag{54}$$

both for $\mu \ll T$ and $\mu \gg T$ in the inertial range where the ν term can be ignored.

When $\mu \ll T$, the thermodynamic quantities and the transport coefficient κ depend on T and μ as $\epsilon \propto T^4$, $P \propto T^4$, $n \propto \mu T^2$, and $\kappa \propto T^2$ to the leading order in $\mu/T \ll 1$. For Eqs. (50) and (51) to possess a scaling symmetry, $n \sim \kappa \boldsymbol{v} \cdot \boldsymbol{\omega}$ and $n\boldsymbol{v} \sim \kappa \boldsymbol{\omega}$ (where "∼" stands for the same scaling exponent), we must have

$$p = 2h - 1, \qquad h + p = h - 1. \tag{55}$$

The solution of these equations is given by Eq. (54). Then it is easy to check that Eq. (49) satisfies this scaling symmetry in the inertial range where the dissipative term ν can be ignored.

When $\mu \gg T$, on the other hand, we have $\epsilon \propto \mu^4$, $P \propto \mu^4$, $n \propto \mu^3$, and $\kappa \propto \mu^2$ to the leading order in $T/\mu \ll 1$. Imposing a scaling symmetry in Eqs. (50) and (51), we must have

$$3p = 2h + 2p - 1, \qquad h + 3p = h + 2p - 1, \tag{56}$$

leading to Eq. (54) again. Similarly to above, Eq. (49) satisfies this scaling symmetry in the inertial range.

In summary, the neutral chiral hydrodynamics in the inertial range has the same scaling symmetry (52) and (53) with h and p given by Eq. (54) both when $\mu \ll T$ and when $\mu \gg T$. The exponent h in this case is uniquely determined by the presence of the CVE, but it is different from Eq. (22) in the ChMHD. This uniqueness should be contrasted with the generic scaling symmetries of the usual hydrodynamics with any h in the inertial range. Note, however, that this unique scaling symmetry is lost outside the inertial range.

C. Physical consequences

Let us study the physical consequences of the scaling symmetry (52) and (53) in the chiral hydrodynamics for neutral matter in the turbulent regime where the kinetic Reynolds number is sufficiently large. We consider the kinetic energy density defined in Eq. (28) and the kinetic correlation length in Eq. (30).

From the scaling symmetry (52), it follows that

$$\mathcal{E}_v(l^{-1}k, l^{1-h}t) = l^{1+2h}\mathcal{E}_v(k,t). \tag{57}$$

Then, we can use the same argument in Sec. II E, leading to Eq. (39) for the kinetic energy density and Eq. (42) for the kinetic correlation length.

Inserting $h = 0$ as required by Eq. (54) in the neutral chiral hydrodynamics, we arrive at

$$\mathcal{E}_v(k,t) = k^{-1}\psi_v(kt), \tag{58}$$

$$\xi_v(t) = \xi_v(t_s)\left(\frac{t}{t_s}\right), \tag{59}$$

which we expect to hold universally at late times. Equation (59) shows the inverse energy cascade. Note here that the time dependence of $\xi_v(t)$ in Eq. (59) is different from that of $\xi_v(t)$ in Eq. (48) in the ChMHD; $\xi_v(t)$ in the neutral chiral hydrodynamics grows faster than $\xi_v(t)$ in the ChMHD because of the different scaling symmetries between Eqs. (22) and (54).

IV. SUMMARY AND DISCUSSION

In this paper, we found that the chiral (magneto)hydrodynamic equations for charged and neutral matter in the turbulent regime have *unique* scaling symmetry under certain conditions. These scaling symmetries dictate the behaviors of the chiral charge and magnetic and kinetic correlation lengths: $n_5(t) \sim t^{-1/2}$ and $\xi_B(t) \sim \xi_v(t) \sim t^{1/2}$ in charged chiral matter and $\xi_v(t) \sim t$ in neutral chiral matter [see Eqs. (46)–(48) and (59)]. These scaling laws suggest the inverse energy cascade in both charged and neutral chiral matter.

Among others, our results may have potential relevance in core-collapse supernovae, where the chiral transport of neutrinos are expected to play key roles [13]. Since their dynamical evolution is described by the coupled transport equations for neutrinos, electrons, and baryons, the simple scaling symmetries and scaling laws derived here may not be directly applicable. Nonetheless, the fact that the inverse energy cascade occurs both in the ChMHD and in neutral chiral hydrodynamics suggests the tendency toward the inverse energy cascade in the presence of the chiral transport of neutrinos. If this is the case, it should work favorably for supernova explosions compared with the direct energy cascade observed in the conventional neutrino transport without the effects of chirality or helicity [23]. It should be important to check the possible inverse cascade numerically by the future three-dimensional *chiral* neutrino radiation hydrodynamics.

For quark-gluon plasmas created in heavy ion collisions, the bulk fluid motion is relativistic. In this case, because of the γ factor in relativistic hydrodynamics, $\gamma = 1/\sqrt{1-\boldsymbol{v}^2}$, there is no scaling symmetry like Eqs. (20) and (21), and the self-similar behaviors and scaling laws like Eqs. (46), (47), and (48) are lost. The fate of the ChMHD turbulence in these ultrarelativistic systems would be an interesting question to be investigated in the future.

While we have concentrated on the self-similarity of the chiral (magneto)hydrodynamics in this paper, it would be interesting to study the possible self-similarity at the level of the chiral kinetic theory. Finally, one can also ask the possible effects of finite fermion masses and nonlinear chiral transport phenomena [37,38] on the scaling laws in the turbulent regime.

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