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Minisuperspace models as infrared contributions

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A direct correspondence of quantum mechanics as a minisuperspace model for a self-interacting scalar quantum-field theory is established by computing, in several models, the infrared contributions to 1-loop effective potentials of Coleman-Weinberg type. A minisuperspace *approximation* rather than truncation is thereby obtained. By this approximation, the spatial averaging scale of minisuperspace models is identified with an infrared scale (but not a regulator or cutoff) delimiting the modes included in the minisuperspace model. Some versions of the models studied here have discrete space or modifications of the Hamiltonian expected from proposals of loop quantum gravity. They shed light on the question of how minisuperspace models of quantum cosmology can capture features of full quantum gravity. While it is shown that modifications of the Hamiltonian can be well described by minisuperspace truncations, some related phenomena such as signature change, confirmed and clarified here for modified scalar field theories, require at least a perturbative treatment of inhomogeneity beyond a strict minisuperspace model. The new methods suggest a systematic extension of minisuperspace models by a canonical effective formulation of perturbative inhomogeneity.

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I. INTRODUCTION

Quantum mechanics can be seen as a "minisuperspace model" of quantum-field theory obtained by restricting the latter to spatially constant fields. Standard derivations in these frameworks, interpreted appropriately, can therefore shed light on the question of how minisuperspace models of quantum cosmology [1] might be related to some putative full theory of quantum gravity, and what physical information they can be able to capture. Here, we initiate a detailed treatment of this form by comparing semiclassical contributions to effective potentials in quantum mechanics with different versions of the Coleman-Weinberg potential of self-interacting scalar quantum-field theories [2].

At first glance, these two potentials look very different from each other, suggesting that a relationship between the quantum-mechanical result, as our minisuperspace model, and full quantum-field theory may not be obvious. A quantum-mechanical system with Hamiltonian $\hat{H}_{\rm QM} = \frac{1}{2}\hat{p}^2 + V(\hat{q})$ has, to first order in $\hbar$, a semiclassical contribution to its effective potential given by

$$V_{\rm eff}(q) = V(q) + \frac{1}{2}\hbar\sqrt{V_{qq}(q)}, \tag{1}$$

as it can be derived by path-integral methods [3] or canonically [4]. (Subscripts indicate the order of derivatives by the argument of a function, here the potential. Note that we assume a kinetic term $\frac{1}{2}\hat{p}^2$ without mass in order to facilitate the following comparison with scalar quantum-field theory, where the mass appears in the quadratic contribution to the potential. In standard quantum mechanics, the effective potential would be $\frac{1}{2}\hbar\sqrt{V_{qq}(q)/m}$.)

The Coleman-Weinberg potential, on the other hand, can be expressed in various versions, none of which suggest a clear comparison. With a covariant cutoff and a quartic potential in the scalar field $\phi(x)$, the renormalized potential given in [2] is

$$W_{\rm renorm}(\phi_0) = \lambda\phi_0^4 + \frac{9\lambda^2}{4\pi^2}\phi_0^4(\log(\phi_0^2/M^2) - 25/6) \tag{2}$$

with the renormalization scale M and ϕ_0 the spatially constant background value chosen for $\phi(x)$. Before a cutoff and renormalization are employed, the potential is expressed as an integral over modes of the quantum field,

$$W_{\rm eff}(\phi_0) = \lambda\phi_0^4 + \frac{1}{2}i\hbar\int\frac{{\rm d}^4k}{(2\pi)^4}\log\left(1 + \frac{12\lambda\phi_0^2}{\|\mathbf{k}\|^2}\right) \tag{3}$$

or, performing the integration over the time component k^0,

$$W_{\rm eff}(\phi_0) = \lambda\phi_0^4 + \frac{1}{2}\hbar\int\frac{{\rm d}^3k}{(2\pi)^3}\left(\sqrt{|\vec{k}|^2 + 12\lambda\phi_0^2} - |\vec{k}|\right). \tag{4}$$

In this last form, the effective potential is directly obtained by canonical methods [5], briefly reviewed below. Being canonical, the derivation does not make covariance manifest, and accordingly only the spatial wave vector $\vec{k}$ appears. The formal derivation from (3) indeed shows that one cannot directly use a covariant cutoff in this setting

because k^0 must be integrated over its whole infinite range for (4) to result. The remaining integral is still divergent, so that it can only be treated by a noncovariant cutoff limiting the range of $|\vec{k}|^2$, rather than the range of $\|\mathbf{k}\|^2$. Nevertheless, the expression (4) has advantages in our present context because it makes a comparison with (1) more clear. Indeed, just as (1), the square root in (4) contains the term $12\lambda\phi_0^2 = W_{\phi\phi}(\phi_0)$, the second derivative of the self-interacting potential $W(\phi) = \lambda\phi^4$ of the Coleman-Weinberg model.

As we will demonstrate in what follows, the quantum-mechanical effective potential can be extracted as the infrared contribution to the Coleman-Weinberg potential of the corresponding quantum-field theory. In this way, a direct relation between a minisuperspace model and its full theory can be established, providing useful insights about the related question in quantum gravity, where not much is known about concrete results in a possible full theory. We will also be led to a systematic formulation of a minisuperspace approximation as an expansion of the infrared contribution by orders of $k_{\rm IR}/\sqrt{W_{\phi\phi}(\phi_0)}$ with the wave number $k_{\rm IR}$ corresponding to the infrared scale. Perturbative inhomogeneity will be shown to combine quantum mechanics as a minisuperspace model with a quantum-field theory of modes coupled to it, providing a manageable setting situated between a strict minisuperspace model and the corresponding full theory.

II. QUANTUM MECHANICS AS A MINISUPERSPACE MODEL

We first have to make the minisuperspace picture of quantum mechanics more precise. It should be a model of quantum-field theory seen as the full theory, obtained by performing a spatial averaging of the fields. The traditional construction of minisuperspace models starts at the classical level where the averaging is performed and then applies quantization methods to the resulting model. The relationship between the quantum-mechanics model constructed in this way and the full quantum-field theory has remained obscure. We will arrive at a clear link by comparing effective potentials in both settings.

A. Quantum mechanics as 0 + 1-dimensional field theory

A seemingly related question has already been studied in quite some detail, applying methods from quantum-field theory to quantum mechanics [3,6,7]. To leading order in a derivative expansion, the low-energy effective action then produces an effective potential of the form (1). However, this result presents a formal analogy and does not illuminate the question of how specific properties of quantum mechanics could be recognized in quantum-field theory or how quantum-mechanics derivations could be used to learn something about quantum-field theory. In standard situations, the last question may not be very relevant since many calculations can be done directly in quantum-field theory. But this is no longer true in some approaches to quantum gravity, where models with spatially nonconstant fields are usually hard to evaluate, owing for instance to the absence of a background space-time. The proposal [1] to use tractable minisuperspace models in order to gain information about potential full theories of quantum gravity is still one of the most actively used. The question of what such a model can tell us about a full theory then becomes crucial, but it has not been systematically analyzed yet.

A detailed look at the classical relation between a field theory and a corresponding mechanical system of homogeneous fields shows why the models studied for instance in [3,6,7] provide analogies rather than strict relationships between quantum mechanics and quantum-field theory. The dynamics of a homogeneous mode $q = \phi_0$ can be derived by inserting $\phi(x) = \phi_0$ in the field-theory Lagrangian. Taken at face value, the result is infinite because it is a homogeneous function of fields integrated over all of space. However, the fields being the same at all points in space, we may restrict the spatial integration to a finite region. If we know the dynamics in this region, we know it everywhere thanks to homogeneity. A finite Lagrangian is then obtained which can be used to derive equations of motion or the Hamiltonian of this minisuperspace model, but it differs from the usual models of classical mechanics in that it depends on a new parameter, the volume V_0 of the finite region selected for spatial integrations. This new parameter, which will be exhibited more fully in the detailed derivations of the next subsection, turns out to be crucial for a relationship between quantum mechanics and quantum-field theory. It has not appeared in papers that studied the formal relationship between effective potentials based on analogies.

The parameter V_0 has, however, appeared in quantum cosmology as a minisuperspace model of quantum gravity. Its role in this context has remained unclear. One of the main results of this paper, in addition to the relationship between minisuperspace models and quantum-field theories, is an explanation of this parameter as the infrared scale of an approximation used to implement homogeneous fields. In fact, our considerations will lead us to a minisuperspace approximation as opposed to the minisuperspace truncations used so far. We broadly define the former as a set of equations that provides some information about the magnitude of terms that are ignored in a minisuperspace truncation, together with a prescription of how to include higher-order terms. More details will be provided below once the relationship between minisuperspace models and quantum-field theory has been found.

B. Minisuperspace model

Our classical full theory is a scalar field theory on Minkowski space-time with Lagrangian

$$L = \int \mathrm{d}^3x \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}|\nabla\phi|^2 - W(\phi)\right) \tag{5}$$

with some potential $W(\phi)$. The integral is performed over all of space up to infinity. In order to obtain a finite Lagrangian after averaging the field (setting it to a spatially constant value ϕ_0), we have to choose a bounded region of coordinate volume $\int \mathrm{d}^3x = V_0$, so that the averaged, minisuperspace Lagrangian is

$$L_{\mathrm{mini}} = V_0\left(\frac{1}{2}\dot{\phi}^2 - W(\phi)\right). \tag{6}$$

(At the minisuperspace level, we drop the subscript "0" of ϕ or its momentum introduced below.) The choice of the integration region and its volume does not matter for the classical theory because we can evaluate a constant field anywhere we like. However, as we will see now, the value of V_0 does affect some quantum properties, which has been a constant source of puzzlement in recent investigations of (especially loop) quantum cosmology. (For instance, it has been suggested that one should take a limit $V_0 \to \infty$ in these models, interpreting V_0 as an infrared cutoff [8]. However, the value of V_0 does not affect the classical dynamics at all, unlike what a cutoff would do. Changing V_0 rather corresponds to a symmetry of the classical minisuperspace theory, which seems broken by quantum effects, but this would mean that quantization introduces a new kind of coordinate dependence. Moreover, if one does take the limit $V_0 \to \infty$, the effective potential is identical with the classical potential and, as claimed in [8], the states no longer obey uncertainty relations. Quantized minisuperspace models then do not seem to follow the usual rules of quantum mechanics. In our conclusions we will present an interpretation of V_0 free of such problems.)

We switch to the Hamiltonian picture by introducing the momentum $p = \partial L_{\mathrm{mini}}/\partial\dot{\phi} = V_0\dot{\phi}$. The classical Hamiltonian is

$$H_{\mathrm{mini}} = \frac{1}{2}\frac{p^2}{V_0} + V_0 W(\phi). \tag{7}$$

At this stage, we can easily quantize the minisuperspace model, giving us the Hamiltonian operator

$$\hat{H}_{\mathrm{mini}} = \frac{1}{2}\frac{\hat{p}^2}{V_0} + V_0 W(\hat{\phi}). \tag{8}$$

Our minisuperspace model differs from standard quantum mechanics only in the presence of coefficients depending on V_0. Nevertheless, the usual methods can, of course, be applied.

We use the canonical effective methods of [4,9] and derive the effective Hamiltonian by expanding the expectation value $\langle\hat{H}_{\mathrm{mini}}\rangle$ in a generic semiclassical state in terms of moments of the state. We will only use second-order moments appropriate for a semiclassical expansion to first order in $\hbar$: we have the two fluctuations $\Delta(\phi^2) = (\Delta\phi)^2$ and $\Delta(p^2) = (\Delta p)^2$, as well as the covariance $\Delta(\phi p) = \frac{1}{2}\langle\hat{\phi}\,\hat{p} + \hat{p}\,\hat{\phi}\rangle - \phi p$. Here and in what follows, we simplify the notation by dropping brackets around expectation values of basic operators, $\langle\hat{\phi}\rangle = \phi$ and $\langle\hat{p}\rangle = p$.

Following this procedure, we obtain the effective Hamiltonian

$$H_{\mathrm{eff}} = \frac{1}{2}\frac{p^2}{V_0} + V_0 W(\phi) + \frac{1}{2V_0}\Delta(p^2) + \frac{1}{2}V_0 W_{\phi\phi}(\phi)\Delta(\phi^2) + \cdots, \tag{9}$$

where the dots indicate higher-moment terms, resulting from an expansion of the potential. The effective Hamiltonian provides Hamiltonian equations of motion for expectation values and moments, which can be derived from a Poisson bracket defined using commutators: $\{\langle\hat{A}\rangle, \langle\hat{B}\rangle\} := \langle[\hat{A},\hat{B}]\rangle/i\hbar$ (extended to products of expectation values by the Leibniz rule). For second-order moments, we obtain

$$\dot{\Delta}(\phi^2) = \frac{2}{V_0}\Delta(\phi p) \tag{10}$$

$$\dot{\Delta}(\phi p) = \frac{1}{V_0}\Delta(p^2) - V_0 W_{\phi\phi}(\phi)\Delta(\phi^2) \tag{11}$$

$$\dot{\Delta}(p^2) = -2V_0 W_{\phi\phi}(\phi)\Delta(\phi p). \tag{12}$$

The effective potential is well defined if the moments behave adiabatically, corresponding to a derivative expansion in time [10]. To lowest order in an adiabatic expansion, we ignore the time derivatives on the left-hand sides of (10)–(12) and solve the resulting linear equations for the moments. We obtain $\Delta_0(\phi p) = 0$ and $\Delta_0(p^2) = V_0^2 W_{\phi\phi}(\phi)\Delta_0(\phi^2)$. Saturating the uncertainty relation $\Delta(\phi^2)\Delta(p^2) - \Delta(\phi p)^2 \geq \hbar^2/4$ in order to minimize fluctuations, we determine all the second-order moments:

$$\Delta_0(\phi^2) = \frac{1}{2}\frac{\hbar}{V_0\sqrt{W_{\phi\phi}(\phi)}},$$
$$\Delta_0(p^2) = \frac{1}{2}\hbar V_0\sqrt{W_{\phi\phi}(\phi)}. \tag{13}$$

The effective potential in (9) can then be expressed solely in terms of ϕ:

$$V_{\mathrm{eff}}(\phi) = V_0 W(\phi) + \frac{1}{2V_0}\Delta_0(p^2) + \frac{1}{2}V_0 W_{\phi\phi}(\phi)\Delta_0(\phi^2) = V_0 W(\phi) + \frac{1}{2}\hbar\sqrt{W_{\phi\phi}(\phi)}. \tag{14}$$

In order to arrive at this result of the effective potential expressed solely in terms of ϕ, eliminating the moments in (9), two assumptions were necessary: the validity of an adiabatic approximation for the moments and the saturation condition of uncertainty relations. Both assumptions have

been shown to lead to correct results for anharmonic oscillators in quantum mechanics [4,10] and, in the context of the Coleman-Weinberg potential, for quartic self-interactions of a scalar field [5]. In the rest of this paper, we work with general potentials $W(\phi)$ but assume that approximations as in the preceding brief derivation are valid both in the (full) quantum-field theory and the minisuperspace quantum-mechanics model. Our main interest here lies in relating these two frameworks, not in probing the range of semiclassical methods.

We divide by V_0 in order to extract the effective version of the original potential $W(\phi)$ of the scalar theory before averaging:

$$W_{\rm eff}(\phi) = W(\phi) + \frac{\hbar}{2V_0}\sqrt{W_{\phi\phi}(\phi)}. \tag{15}$$

As promised, while the classical potential does not depend on the averaging volume V_0, its first-order quantum correction does. The meaning of this parameter cannot be understood within the minisuperspace model. The classical reduction would suggest that the value of V_0 should not play any role in physical results because the physics of an exactly constant field should not depend on the volume of the region in which the field is constant, provided the theory is local. Indeed, results in the classical model do not depend on the value of V_0. Quantum effects, however, are known to be nonlocal, which might explain the presence of V_0 in the correction term of (15). Still, this general observation does not elucidate the physical meaning of V_0. In order to understand this issue, we now relate the minisuperspace result to a calculation in the corresponding full theory.

C. Effective potential in the infrared

If we consider the integral in (4) only for small values of k that are in the infrared, we obtain an effective potential of the form (15), up to numerical factors. Indeed, if we include modes with $|\vec{k}|$ between zero and $2\pi/R_0$, with a large R_0 for the maximum wavelength allowed, we can approximate the integral as the volume $\frac{4}{3}\pi(2\pi/R_0)^3$ times the integrand evaluated at $|\vec{k}| = 0$. The result, $\frac{2}{3}\pi\hbar\sqrt{W_{\phi\phi}(\phi_0)}/R_0^3$, agrees with (15) up to a numerical factor if we identify $\frac{4}{3}\pi R_0^3$ with the averaging volume V_0.

More accurately, we can write the infrared contribution to the effective potential as

$$W_{\rm IR}(\phi_0) = W(\phi_0) + \frac{1}{4\pi^2}\hbar\int_0^{2\pi/R_0} {\rm d}k k^2\left(\sqrt{k^2 + W_{\phi\phi}(\phi_0)} - k\right) \tag{16}$$

$$= W(\phi_0) + \frac{1}{32\pi^2}\hbar\left(\sqrt{(2\pi/R_0)^2 + W_{\phi\phi}(\phi_0)}\left(2\left(\frac{2\pi}{R_0}\right)^3 + \frac{2\pi W_{\phi\phi}(\phi_0)}{R_0}\right) - 2\left(\frac{2\pi}{R_0}\right)^4 - W_{\phi\phi}(\phi_0)^2\log\frac{\sqrt{(2\pi/R_0)^2 + W_{\phi\phi}(\phi_0)} + 2\pi/R_0}{\sqrt{W_{\phi\phi}(\phi_0)}}\right) \tag{17}$$

$$\approx W(\phi_0) + \frac{5}{2^9\pi^2}\hbar\frac{\sqrt{W_{\phi\phi}(\phi_0)}}{R_0^3}. \tag{18}$$

As before, identifying $V_0 = \frac{4}{3}\pi R_0^3$ gives us (15) up to numerical factors. In the last step, we have kept only the leading term in an expansion by $(R_0\sqrt{W_{\phi\phi}(\phi_0)})^{-1}$, so that we require the second derivative of the potential to be much larger than the largest wave number squared, $W_{\phi\phi}(\phi_0) \gg R_0^{-2}$. This approximation means that we restrict ϕ_0 to values such that the potential dominates the spatial-derivative term in the field-theory Hamiltonian: in

$$\frac{1}{2}|\nabla\phi|^2 + W(\phi) \approx \frac{1}{2}(-k^2 + W_{\phi\phi}(\phi_0))\delta\phi^2, \tag{19}$$

using an expansion $\phi(x) = \phi_0 + \delta\phi(x)$ around the minimum ϕ_0 of $W(\phi)$, the potential term is then dominant.

We are now in a position to specify our minisuperspace approximation, as opposed to a truncation. Both give the same leading corrections at least qualitatively, but the former also specifies the range of validity. In the present example, the minisuperspace *approximation* is defined as the infrared contribution to a field theory and is valid as long as $W_{\phi\phi}(\phi_0) \gg R_0^{-2}$. The minisuperspace *truncation*, on its own, could be used to conclude that R_0 should be large for homogeneous fields to capture the relevant physics at this scale, but without any known relation with a full effective potential, it would not prevent one from applying it to values ϕ_0 for which $W_{\phi\phi}(\phi_0) \gg R_0^{-2}$ is violated and the relationship with a full theory breaks down. The minisuperspace truncation would formally be valid for all ϕ_0 if the limit $R_0 \to \infty$ is taken, but then there are no quantum corrections in (15) and

the truncation does not tell us anything about quantum-field theory.

We recall that our main motivation comes from quantum gravity, where an open question is to see how minisuperspace results can find a home in some corresponding full theory. Depending on the specific application of such models, additional effects may have to be included. An example would be microscopic properties, transported to a minisuperspace model by a detailed procedure of coarse graining. Such a procedure would replace our simpler posing of an infrared scale, but it is outside the scope not only of this article but also of most approaches to quantum gravity, at least at present. Nevertheless, some of the detailed models presented in the next section show that certain microscopic features suggested by quantum gravity can leave a trace in minisuperspace models even if a simple infrared scale is posed.

III. INFRARED CONTRIBUTIONS TO ONE-DIMENSIONAL FIELD THEORIES

The results presented so far show that the low-energy effective potential in quantum mechanics agrees, up to numerical factors, with the infrared contribution to a related quantum-field theory. The averaging volume V_0 of the minisuperspace model corresponds to the infrared scale used in the quantum-field theory. In this section, we explore how different choices of full theories with the same potential, which therefore produce the same minisuperspace result, affect the infrared contribution. Although we focus on effective potentials, similar correspondences can be established between two-point functions of quantum-field theories and quantum-mechanical fluctuations.

A. One-dimensional field theory

We begin by providing a more detailed derivation of the field-theory effective potential with a one-dimensional spatial manifold, following [5]. We take this opportunity to show more details of the derivation that leads to the integral used in (16), but at the same time give an example that demonstrates the dependence of the infrared contribution on the spatial dimension.

The quantum Hamiltonian, formally expanded to second-order moments, is

$$H_{\rm Q}=\frac{1}{2}\int {\rm d}x(\pi(x)^2+\phi'(x)^2+2W(\phi(x)) + G^{2,0}(x,x)+{\rm D}^2G^{0,2}(x,x)+W_{\phi\phi}(\phi(x))G^{0,2}(x,x)). \tag{20}$$

The derivative ${\rm D}^2G^{0,2}(x,x)$ is defined as $\lim_{y\to x}{\rm d}^2G^{0,2}(x,y)/{\rm d}x{\rm d}y$. For the moments, we have equations of motion

$$\dot{G}^{0,2}(y,z)=G^{1,1}(y,z)+G^{1,1}(z,y) \tag{21}$$

$$\dot{G}^{1,1}(y,z)=G^{2,0}(y,z)-\left(W_{\phi\phi}(\phi)-\frac{{\rm d}^2}{{\rm d}y^2}\right)G^{0,2}(y,z) \tag{22}$$

$$\dot{G}^{2,0}(y,z)=-\left(W_{\phi\phi}(\phi)-\frac{{\rm d}^2}{{\rm d}z^2}\right)G^{1,1}(y,z) - \left(W_{\phi\phi}(\phi)-\frac{{\rm d}^2}{{\rm d}y^2}\right)G^{1,1}(z,y). \tag{23}$$

To leading order in an adiabatic expansion, the first equation implies $G^{1,1}(z,y)=-G^{1,1}(y,z)$, from which the last one implies that $G^{1,1}=0$ using standard boundary conditions.

We solve (22) by using a Fourier decomposition

$$G^{0,2}(x,y)=\int {\rm d}k_x{\rm d}k_y f(k_x,k_y)e^{i(k_xx+k_yy)}. \tag{24}$$

For a translation-invariant theory, $f(k_y,k_y)$ must be of the form $g(k)\delta(k_x+k_y)$, so that

$$G^{0,2}(x,y)=\int {\rm d}k g(k)e^{ik(x-y)}. \tag{25}$$

By (22), we then have

$$G^{2,0}(x,y)=\int {\rm d}k(W_{\phi\phi}(\phi)+k^2)g(k)e^{ik(x-y)}. \tag{26}$$

The moments $G^{0,2}$ and $G^{2,0}$ (with $G^{1,1}=0$) appear in the uncertainty relation

$$G^{0,2}(x_1,x_2)G^{2,0}(y_1,y_2)\geq\frac{\hbar^2}{8}(\delta(x_1-y_1)\delta(x_2-y_2) + \delta(x_1-y_2)\delta(x_2-y_1)). \tag{27}$$

The saturation condition is singular for moments such as (25) and (26), but it can nevertheless be evaluated in order to restrict the values. If we set $x_1=x_2$ in the saturated uncertainty relation and integrate over this value, we can see that the free function $g(k)$ must be

$$g(k)=\frac{\hbar}{2\pi}\frac{1}{2\sqrt{W_{\phi\phi}(\phi)+k^2}} \tag{28}$$

if the theory is reflection symmetric, $g(-k)=g(k)$. Therefore,

$$G^{0,2}(x,y)=\frac{\hbar}{2\pi}\int {\rm d}k\frac{1}{2\sqrt{W_{\phi\phi}(\phi)+k^2}}e^{ik(x-y)} \tag{29}$$

$$G^{2,0}(x,y) = \frac{\hbar}{2\pi}\int \mathrm{d}k \frac{1}{2}\sqrt{W_{\phi\phi}(\phi)+k^2}e^{ik(x-y)}. \quad (30)$$

We insert these moments in the effective Hamiltonian and arrive at the Coleman-Weinberg potential

$$\begin{aligned} W_{\mathrm{eff}}(\phi_0) &= W(\phi_0) + \frac{\hbar}{4\pi}\int \mathrm{d}k\left(\sqrt{W_{\phi\phi}(\phi_0)+k^2}-|k|\right) \\ &= W(\phi_0) + \frac{\hbar}{2\pi}\int_0^k \mathrm{d}k\left(\sqrt{W_{\phi\phi}(\phi_0)+k^2}-k\right) \end{aligned} \quad (31)$$

with a spatially constant ϕ_0. [In the integrand, we subtract $|k|$ in order to ensure that there is no contribution in the free case when $W=0$. This term amounts to an infinite subtraction of the diverging moments $G^{a,b}(x,x)$.]

The integral in W_{eff} can be evaluated explicitly, but we are interested only in the infrared contribution given by modes up to some scale $2\pi/L_0$. Using the additional assumption of potential dominance at the chosen value of ϕ_0, $W_{\phi\phi}(\phi_0) \gg (2\pi/L_0)^2$, we obtain

$$W_{\mathrm{IR}}(\phi_0) = W(\phi_0) + \hbar\frac{\sqrt{W_{\phi\phi}(\phi_0)}}{L_0}. \quad (32)$$

The result differs from the minisuperspace model by a factor of 2.

B. Discrete space

We now modify our one-dimensional field theory by putting it on a discrete space. The classical Hamiltonian is

$$H = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\pi_n^2 + \frac{1}{8\ell_0^2}(\phi_{n+1}-\phi_{n-1})^2 + W(\phi_n)\right). \quad (33)$$

We have chosen a symmetric discretization of the spatial derivative and denoted the discreteness scale by ℓ_0. Accordingly, there will be a maximal wave number $k_{\max} = 2\pi/\ell_0$ in Fourier decompositions.

We obtain the quantum Hamiltonian, as before to second-order moments, given by

$$\begin{aligned} H_{\mathrm{Q}} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\Bigg(&\pi_n^2 + \frac{1}{4\ell_0^2}(\phi_{n+1}-\phi_{n-1})^2 + 2W(\phi_n) \\ &+ G^{2,0}_{n,n} + \frac{1}{4\ell_0^2}(G^{0,2}_{n+1,n+1} - 2G^{0,2}_{n+1,n-1} \\ &+ G^{0,2}_{n-1,n-1}) + W_{\phi\phi}(\phi_n)G^{0,2}_{n,n}\Bigg). \end{aligned} \quad (34)$$

It is slightly easier to compute equations of motion for the moments if we first rearrange the spatial difference term by a suitable shift of the summation labels:

$$\begin{aligned} H_{\mathrm{Q}} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\Bigg(&\pi_n^2 + \frac{1}{4\ell_0^2}(\phi_{n+1}-\phi_{n-1})^2 + 2W(\phi_n) \\ &+ G^{2,0}_{n,n} + \frac{1}{2\ell_0^2}(G^{0,2}_{n,n} - G^{0,2}_{n+1,n-1}) + W_{\phi\phi}(\phi_n)G^{0,2}_{n,n}\Bigg). \end{aligned} \quad (35)$$

Equations of motion are

$$\dot{G}^{0,2}_{n,m} = G^{1,1}_{n,m} + G^{1,1}_{m,n} \quad (36)$$

$$\begin{aligned} \dot{G}^{1,1}_{n,m} = &G^{2,0}_{n,m} - W_{\phi\phi}(\phi_n)G^{0,2}_{n,m} \\ &- \frac{1}{4\ell_0^2}(2G^{0,2}_{n,m} - G^{0,2}_{n,m-2} - G^{0,2}_{n,m+2}) \end{aligned} \quad (37)$$

$$\begin{aligned} \dot{G}^{2,0}_{n,m} = &-\frac{1}{4\ell_0^2}(2G^{1,1}_{n,m} - G^{1,1}_{n-2,m} - G^{1,1}_{n+2,m} + 2G^{1,1}_{m,n} \\ &- G^{1,1}_{m-2,n} - G^{1,1}_{m+2,n}). \end{aligned} \quad (38)$$

In an adiabatic approximation, the first equation implies $G^{1,1}_{n,m} = -G^{1,1}_{m,n}$, from which the last equation gives

$$G^{1,1}_{n-2,m} + G^{1,1}_{n+2,m} = G^{1,1}_{n,m-2} + G^{1,1}_{n,m+2}. \quad (39)$$

Boundary conditions are now more complicated, but we may use $G^{1,1}_{n,m} = 0$ in order to realize the correct continuum limit of the preceding subsection.

In order to solve (37), we write

$$G^{0,2}_{n,m} = \int_0^{k_{\max}} \mathrm{d}k g(k) e^{ik\ell_0(n-m)} \quad (40)$$

as a Fourier decomposition in the required k-range. Equation (37) then implies

$$G^{2,0}_{n,m} = \left(W_{\phi\phi}(\phi_n) + \frac{1}{2\ell_0^2}\right)G^{0,2}_{n,m} - \frac{1}{4\ell_0^2}(G^{0,2}_{n,m-2} + G^{0,2}_{n,m+2}) \quad (41)$$

$$\begin{aligned} &= \int_0^{k_{\max}} \mathrm{d}k g(k)\Bigg(W_{\phi\phi}(\phi_n) - \frac{1}{4\ell_0^2}(e^{2ik\ell_0} \\ &\quad + e^{-2ik\ell_0} - 2)\Bigg)e^{ik\ell_0(n-m)} \end{aligned} \quad (42)$$

$$= \int_0^{k_{\max}} \mathrm{d}k g(k)\left(W_{\phi\phi}(\phi_n) + \frac{\sin^2(\ell_0 k)}{\ell_0^2}\right)e^{ik\ell_0(n-m)}. \quad (43)$$

We saturate the uncertainty relation in the form $\sum_n G^{0,2}_{n,m}G^{2,0}_{n,m'} = \frac{1}{4}\hbar^2\delta_{m,m'}$ and obtain the effective potential

$$W_{\rm eff}(\phi_0) = W(\phi_0) + \frac{\hbar}{2\pi}\int_0^{k_{\rm max}} {\rm d}k\left(\sqrt{W_{\phi\phi}(\phi_0) + \sin^2(\ell_0 k)/\ell_0^2} - \sin(\ell_0 k)/\ell_0\right). \quad (44)$$

The integral can again be evaluated. For an infrared scale of $2\pi/L_0 \ll 2\pi/\ell_0$ and potential domination, $W_{\phi\phi}(\phi_0) \gg (2\pi/L_0)^2$, we have

$$W_{\rm IR}(\phi_0) = W(\phi_0) + \hbar\left(\frac{\sqrt{W_{\phi\phi}(\phi_0)}}{L_0} + \frac{8}{3}\pi^2\frac{1}{\sqrt{W_{\phi\phi}(\phi_0)}L_0^3} - \frac{2}{15}\pi^4\frac{4W_{\phi\phi}(\phi_0)\ell_0^2+3}{W_{\phi\phi}(\phi_0)^{3/2}L_0^5} + W_{\phi\phi}(\phi_0)O(L_0^{-6}W_{\phi\phi}(\phi_0)^{-3})\right). \quad (45)$$

As the expansion to higher orders in $(L_0\sqrt{W_{\phi\phi}(\phi_0)})^{-1}$ shows, one has to go well beyond the leading result of the infrared expansion in order to see the discreteness scale ℓ_0. Moreover, the fifth-order term depends significantly on ℓ_0 only if $W_{\phi\phi}(\phi_0)\ell_0^2 \approx 1$ or larger, so that the second derivative of the potential must be huge for a small discreteness scale.

C. Periodic space

In our next model, we assume that the spatial manifold is a circle with coordinate radius $\mathcal{L}$. The Hamiltonian is

$$H = \int_0^{\mathcal{L}} {\rm d}x\left(\frac{1}{2}\pi(x)^2 + \frac{1}{2}\phi'(x)^2 + W(\phi)\right), \quad (46)$$

and we impose periodic boundary conditions $\phi(\mathcal{L}) = \phi(0)$, $\phi'(\mathcal{L}) = \phi'(0)$. All our local equations of motion are as in the continuum model we started with, but Fourier decompositions are discrete. We therefore write

$$G^{0,2}(y,z) = \sum_{j=-\infty}^{\infty} g_j e^{2\pi i j(y-z)/\mathcal{L}} \quad (47)$$

and the leading-order adiabatic equations imply that

$$G^{2,0}(y,z) = \sum_{j=-\infty}^{\infty} g_j\left(W_{\phi\phi}(\phi) + \left(\frac{2\pi}{\mathcal{L}}\right)^2 j^2\right)e^{2\pi i j(y-z)/\mathcal{L}}. \quad (48)$$

Upon saturating the uncertainty relation, the Fourier coefficients are determined by

$$g_j = \frac{\hbar}{\mathcal{L}}\frac{1}{2\sqrt{W_{\phi\phi} + (2\pi/\mathcal{L})^2 j^2}}. \quad (49)$$

The effective potential is

$$W_{\rm eff}(\phi_0) = W(\phi_0) + \frac{\hbar}{2\mathcal{L}}\sum_{j=-\infty}^{\infty}\left(\sqrt{W_{\phi\phi}(\phi_0) + \left(\frac{2\pi}{\mathcal{L}}\right)^2 j^2} - \frac{2\pi j}{\mathcal{L}}\right) = W(\phi_0) + \frac{1}{2}\hbar\frac{\sqrt{W_{\phi\phi}(\phi_0)}}{\mathcal{L}} + \frac{\hbar}{\mathcal{L}}\sum_{j=1}^{\infty}\left(\sqrt{W_{\phi\phi}(\phi_0) + \left(\frac{2\pi}{\mathcal{L}}\right)^2 j^2} - \frac{2\pi j}{\mathcal{L}}\right). \quad (50)$$

We choose a length $L_0 \le \mathcal{L}$ as the infrared scale that determines the infrared contribution

$$W_{\rm IR} = W(\phi_0) + \frac{1}{2}\hbar\frac{\sqrt{W_{\phi\phi}(\phi_0)}}{\mathcal{L}} + \frac{\hbar}{\mathcal{L}}\sum_{j=1}^{\mathcal{L}/L_0-1}\left(\sqrt{W_{\phi\phi}(\phi_0) + \left(\frac{2\pi}{\mathcal{L}}\right)^2 j^2} - \frac{2\pi j}{\mathcal{L}}\right). \quad (51)$$

We have assumed that $\mathcal{L}/L_0 = N$ is an integer, so that averaging over the infrared scale eliminates all modes with $j = N$ or higher. The value $N = 1$ is allowed, in which case the averaging is complete and only homogeneous modes are left. There remains an infrared contribution to the classical potential, given by

$$W_{\rm IR}^{L_0=\mathcal{L}}(\phi_0) = W(\phi_0) + \frac{1}{2}\hbar\frac{\sqrt{W_{\phi\phi}(\phi_0)}}{L_0}. \quad (52)$$

In this case, the infrared contribution agrees exactly with the quantum-mechanical effective potential (15), even in its numerical factor.

We are not restricted to $N = 1$ as long as N is a positive integer. For $N > 1$, the infrared contribution (51) is a finite sum with contributions in addition to the quantum-mechanical one. If we assume potential domination, as before, we have $W_{\phi\phi}(\phi_0) \gg (2\pi/L_0)^2 \ge (2\pi/\mathcal{L})^2$, and we can expand the square root in order to evaluate the sums:

$$W_{\rm IR}(\phi_0) = W(\phi_0) + \frac{1}{2}\hbar\frac{\sqrt{W_{\phi\phi}(\phi_0)}}{\mathcal{L}} + \sum_{j=1}^{\mathcal{L}/L_0-1}\left(W_{\phi\phi}(\phi_0)\left(1 + \frac{1}{2}\left(\frac{2\pi}{\mathcal{L}}\right)^2 \times \frac{1}{W_{\phi\phi}(\phi_0)}j^2 + \cdots\right) - \frac{2\pi j}{\mathcal{L}}\right) \quad (53)$$

$$= W(\phi_0) + \frac{1}{2}\hbar\frac{\sqrt{W_{\phi\phi}(\phi_0)}}{\mathcal{L}} - \pi\frac{\hbar}{\mathcal{L}^2}\frac{\mathcal{L}}{L_0}\left(\frac{\mathcal{L}}{L_0}-1\right) + \frac{\pi^2}{3}\frac{\hbar}{\sqrt{W_{\phi\phi}(\phi_0)}\mathcal{L}^3}\frac{\mathcal{L}}{L_0}\left(\frac{\mathcal{L}}{L_0}-1\right)\left(2\frac{\mathcal{L}}{L_0}-1\right)+\cdots. \tag{54}$$

For large $N = \mathcal{L}/L_0 \gg 1$,

$$W_{\rm IR}(\phi_0) = W(\phi_0) + \frac{1}{2N}\hbar\frac{\sqrt{W_{\phi\phi}(\phi_0)}}{L_0} - \pi\frac{\hbar}{L_0^2} + \frac{\pi^2}{3}\frac{\hbar}{\sqrt{W_{\phi\phi}(\phi_0)}L_0^3}. \tag{55}$$

D. Scalar field on a homogeneous metric background

If the background space-time of the scalar field theory is not Minkowskian, the spatial metric q and lapse function N (distinct from the mode number in the preceding subsection) appear in the Hamiltonian

$$H = \int {\rm d}x N(x)\sqrt{q}\left(\frac{1}{2}\frac{\pi(x)^2}{q} + \frac{\phi'(x)^2}{q} + W(\phi)\right). \tag{56}$$

The momentum is now related to the time derivative of ϕ by $\pi = \sqrt{q}\dot{\phi}$. In one dimension the metric is just a function, given by $q(x) = a^2$ for a homogeneous background with the cosmological scale factor a. For our purposes, the lapse function can be ignored, and we may absorb the factor of $\sqrt{q} = a$ in the spatial coordinate x. The only effect of the spatial metric is then to write the entire Hamiltonian, including the spatial derivative of ϕ, in terms of ax instead of x. Accordingly, mode expansions are formulated with respect to the physical wave number k/a instead of the comoving one, k. Similarly, if we use the discrete model, the discreteness parameter in the spatial difference operator is the geometrical distance $a\ell_0$ rather than the coordinate distance ℓ_0.

In the corresponding minisuperspace model the coordinate value L_0 of the infrared scale is replaced by the geometrical distance aL_0 in all equations, including the resulting effective potential. The dependence of the effective potential on the infrared scale is therefore coordinate invariant. It only depends on the physical scale used to average fields.

E. Quantum-geometry modifications

Models of loop quantum gravity have been used to suggest two types of modifications which turn out to deform the classical notion of covariance. Accordingly, the kinetic term of a scalar Hamiltonian is modified by inserting different types of functions. These functions, in canonical form, may depend on the spatial metric or extrinsic curvature, which in our context would be treated as background fields, but they can also depend on the scalar field or its momentum and then change the quadratic form of kinetic terms. There are two important forms of modifications, inverse-triad corrections first introduced in loop quantum cosmology in [11], and holonomy corrections introduced in [12–14].

1. Inverse-triad corrections

Inverse-triad corrections result from a quantization of the inverse metric factors in a Hamiltonian such as (56), in which the kinetic and spatial-derivative term are divided by metric components. Loop quantizations (using a densitized triad instead of the spatial metric) lead to discrete metric operators with zero in their discrete spectra, so that no direct inverse exists. Nevertheless, as proposed in [15,16], an inverse can be quantized in an indirect way, schematically writing $q^{-1/2} = 2\{q^{1/2}, p_q\}$ and replacing the Poisson bracket by a commutator divided by $i\hbar$. A densely defined operator results, but its spectrum differs from the expected $q_n^{-1/2}$ for small values of q_n [17,18].

One can model this effect of discrete quantum geometry by changing the q-dependence of a Hamiltonian (56) on a classical background. With two given positive functions ν and σ, a scalar Hamiltonian modified by inverse-triad corrections has the form

$$H = \int {\rm d}x\left(\frac{1}{2}\nu\pi(x)^2 + \frac{1}{2}\sigma\phi'(x)^2 + W(\phi)\right). \tag{57}$$

For simplicity, we will assume ν and σ to depend only on time, perhaps implicitly, but not on x. The time dependence of the modification functions could result from an expanding cosmological background, but we omit corresponding factors of $q = a^2$. Such modified Hamiltonians have been derived for instance via inhomogeneous (but still tractable) quantum-gravity models such as [19,20], or by perturbative treatments of inhomogeneity in lattice models [21].

We can eliminate the function σ by employing a canonical transformation $\tilde{\phi} := \sqrt{\sigma}\phi$ and $\tilde{\pi} := \pi/\sqrt{\sigma}$. The only modification in the kinetic term is then a factor of $\beta := \nu\sigma$ multiplying the momentum squared, and there is a substitution of $\tilde{\phi}/\sqrt{\sigma}$ for ϕ in the potential. We will therefore assume the Hamiltonian to be of the form

$$H = \int {\rm d}x\left(\frac{1}{2}\beta\tilde{\pi}(x)^2 + \frac{1}{2}\tilde{\phi}'(x)^2 + W(\tilde{\phi}/\sqrt{\sigma})\right) \tag{58}$$

without changing our notation for the fields.

With our new Hamiltonian, the equation of motion for $G^{2,0}$, after transitioning to an expanded quantum Hamiltonian, is unchanged, while

$$\dot{G}^{0,2}(y,z) = \beta(G^{1,1}(y,z) + G^{1,1}(z,y)). \tag{59}$$

To leading adiabatic order, the conclusion that $G^{1,1}=0$ is therefore unchanged. For $G^{1,1}$, we have the equation of motion

$$\dot{G}^{1,1}(y,z)=\beta G^{2,0}(y,z)-\left(\frac{W_{\phi\phi}(\tilde{\phi}/\sqrt{\sigma})}{\sigma}-\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right)G^{0,2}(y,z). \tag{60}$$

Upon Fourier decomposition and saturating the uncertainty relation, we obtain coefficients

$$g(k)=\frac{\hbar}{2\pi}\frac{\sqrt{\beta}}{2\sqrt{W_{\phi\phi}(\tilde{\phi}/\sqrt{\sigma})/\sigma+k^2}}. \tag{61}$$

For potential domination, the infrared contribution to the effective potential now reads

$$\begin{aligned}W_{\mathrm{IR}}(\phi_0)&=W(\tilde{\phi}_0/\sqrt{\sigma})+\hbar\sqrt{\frac{\beta}{\sigma}}\frac{\sqrt{W_{\phi\phi}(\tilde{\phi}_0/\sqrt{\sigma})}}{L_0}\\&=W(\phi_0)+\hbar\sqrt{\frac{\beta}{\sigma}}\frac{\sqrt{W_{\phi\phi}(\phi_0)}}{L_0},\end{aligned} \tag{62}$$

which differs from the quantum-mechanical result by an additional factor $\sqrt{\beta/\sigma}=\sqrt{\nu}$ depending on the modification functions. The same factor is obtained if one implements the modification in the quantum-mechanical Hamiltonian

$$\hat{H}=\frac{1}{2}\nu\hat{p}^2+W(\hat{q}) \tag{63}$$

and follows the derivations leading up to (1). The modification by ν follows from a direct reduction of (57) to homogeneity.

2. Holonomy corrections

Holonomy modifications are motivated by the use of holonomies of a gravitational connection as basic operators in loop quantum gravity, while connection components cannot be quantized directly. Any object, such as a Hamiltonian, which depends on the connection in its classical expression, must therefore be modified for it to be quantized on the kinematical Hilbert space of loop quantum gravity. The form of holonomies (or their matrix elements) of compact groups suggests that polynomial dependences on the connection are replaced by bounded and periodic functions.

a.Modified spatial-derivative term.—Similarly, a scalar field is not represented directly on the kinematical Hilbert space of loop quantum gravity, but only through its exponentials $h_x(\phi):=\exp(i\phi(x))$. In our context, this modification would be similar to inverse-triad corrections with $\nu=1$ and σ related to a derivative of the modification function. However, in this case we can no longer assume that the modification function depends only on time, so that a more detailed analysis is required. (The final result will nevertheless turn out to be closely related to the one for inverse-triad corrections because the effective potentials to orders considered here depend on the field expectation value only via its spatially constant average.)

In the presence of scalar holonomy modifications, the discrete model used in Sec. III B is more suitable. We now use a classical Hamiltonian

$$\begin{aligned}H=\sum_{n=-\infty}^{\infty}\Big(&\frac{1}{2}\pi_n^2+\frac{1}{4\ell_0^2}(g(\phi_n)^2\\&-g(\phi_{n+1})g(\phi_{n-1}))+W(g(\phi_n))\Big)\end{aligned} \tag{64}$$

with some local function $g(\phi_n)$, which is nonlinear in the presence of scalar holonomy modifications. We compute the quantum Hamiltonian

$$\begin{aligned}H_{\mathrm{Q}}=\frac{1}{2}\sum_{n=-\infty}^{\infty}\Big(&\pi_n^2+\frac{1}{2\ell_0^2}(g(\phi_n)^2-g(\phi_{n+1})g(\phi_{n-1}))\\&+2W(g(\phi_n))\\&+G^{2,0}_{n,n}+\frac{1}{2\ell_0^2}(\Delta_1 g[\phi_n]G^{0,2}_{n,n}-\Delta_2 g[\phi_n]G^{0,2}_{n+1,n-1})\\&+W_{\phi_n\phi_n}(g(\phi_n))G^{0,2}_{n,n}\Big),\end{aligned} \tag{65}$$

where we introduced the nonlocal functions

$$\begin{aligned}\Delta_1 g[\phi_n]:=g_{\phi_n}(\phi_n)^2-\frac{1}{2}g_{\phi_n\phi_n}(\phi_n)(&g(\phi_{n+2})\\&-2g(\phi_n)+g(\phi_{n-1}))\end{aligned} \tag{66}$$

$$\Delta_2 g[\phi_n]:=g_{\phi_{n+1}}(\phi_{n+1})g_{\phi_{n-1}}(\phi_{n-1}). \tag{67}$$

Both functions are equal to 1 in the unmodified case.

The equation of motion for $G^{0,2}_{n,m}$ is unchanged compared to the case with $g(\phi_n)=\phi_n$, and while the equation for $G^{2,0}_{n,m}$ has coefficients depending on $\Delta_1 g[\phi_n]$ and $\Delta_2 g[\phi_n]$, the leading adiabatic solutions are still consistent with $G^{1,1}_{n,m}=0$. The remaining equation is

$$\begin{aligned}\dot{G}^{1,1}_{n,m}=G^{2,0}_{n,m}&-\left(W_{\phi_n\phi_n}(g(\phi_n))+\frac{1}{2\ell_0^2}\Delta_1 g[\phi_n]\right)G^{0,2}_{n,m}\\&-\frac{1}{4\ell_0^2}\Delta_2 g[\phi_n](G^{0,2}_{n,m-2}+G^{0,2}_{n,m+2}).\end{aligned} \tag{68}$$

Using the same Fourier decomposition as before, we solve this equation by

$$G^{2,0}_{n,m} = \int_0^{k_{\max}} \mathrm{d}k g(k) \left(W_{\phi_n\phi_n}(g(\phi_n)) + \frac{1}{2\ell_0^2}(\Delta_1 g[\phi_n] - \Delta_2 g[\phi_n]) - \Delta_2 g[\phi_n] \frac{\sin^2(\ell_0 k)}{\ell_0^2} \right) e^{ik\ell_0(n-m)} \tag{69}$$

where $g(k)$ appears in

$$G^{0,2}_{n,m} = \int_0^{k_{\max}} \mathrm{d}k g(k) e^{ik\ell_0(n-m)}. \tag{70}$$

Saturating the uncertainty relation leads to the effective potential

$$W_{\mathrm{eff}}(\phi_0) = W(g(\phi_0)) + \frac{\hbar}{2\pi} \int_0^{k_{\max}} \mathrm{d}k \left(\sqrt{W_{\phi\phi}(g(\phi_0)) + g_\phi(\phi_0)^2 \sin^2(\ell_0 k)/\ell_0^2} - \sin(\ell_0 k)/\ell_0 \right). \tag{71}$$

Here, we have used the fact that for constant $\phi_n = \phi_0$ (slightly abusing the subscript notation), as assumed in effective potentials of Coleman-Weinberg type, $\Delta_1 g[\phi_n] = g_\phi(\phi_0)^2 = \Delta_2 g[\phi_n]$. The subtraction is chosen in such a way that it removes the contribution only when $W(\phi) = 0$ and $g_\phi(\phi) = 1$, because nonlinear $g(\phi)$ implies nonquadratic derivative terms in the Hamiltonian and therefore interactions even if the classical potential vanishes. The leading-order infrared contribution depends on $g(\phi)$ only through the potential. The same modification is obtained in the minisuperspace model of (64).

b.Modified momentum dependence.—Alternatively, in order to model the formal dependence of gravitational Hamiltonians on connection components, it is instructive to modify the quadratic dependence of our scalar Hamiltonian on $\pi(x)$. The momentum of the scalar field would then be represented by its exponentials of integrated fields, $h_{x_0,\delta}(\pi) := \exp\left(\int_{x_0}^{x_0+\delta} \mathrm{d}x \pi(x)\right)$. (Written on a metric background, the momentum has a density weight and can therefore be integrated directly without any additional measure factors.) Using $h_{x_0,\delta}(\pi)$ instead of π in the Hamiltonian leads to a nonlocal expression, but as is often done, here we model the effect by replacing $\pi(x)^2$ by a local function $f(\pi(x))$:

$$H = \int \mathrm{d}x \left(\frac{1}{2} f(\pi(x)) + \frac{1}{2}\phi'(x)^2 + W(\phi) \right). \tag{72}$$

In this form, the scalar model resembles the three-dimensional version proposed in [22,23] and further evaluated in [24–27].

The quantum Hamiltonian

$$H_{\mathrm{Q}} = \frac{1}{2} \int \mathrm{d}x (f(\pi(x)) + \phi'(x)^2 + 2W(\phi) + f_{\pi\pi}(\pi) G^{2,0}(x,x) + \mathrm{D}^2 G^{0,2}(x,x) + W_{\phi\phi}(\phi) G^{0,2}(x,x)), \tag{73}$$

expanded to second order in moments, implies equations of motion

$$\dot{G}^{0,2}(y,z) = \frac{1}{2} f_{\pi\pi}(\pi)(G^{1,1}(y,z) + G^{1,1}(z,y)) \tag{74}$$

and an unchanged equation for $\dot{G}^{2,0}$, so that we conclude $G^{1,1} = 0$ as before, using the leading adiabatic approximation. The remaining equation to be solved is

$$\dot{G}^{1,1}(y,z) = \frac{1}{2} f_{\pi\pi}(\pi) G^{2,0}(y,z) - \left(W_{\phi\phi}(\phi) - \frac{\mathrm{d}^2}{\mathrm{d}y^2} \right) G^{0,2}(y,z). \tag{75}$$

A Fourier decomposition can be done as before, just with additional factors of $f_{\pi\pi}(\pi)/2$ in some solutions. We arrive at the effective potential

$$W_{\mathrm{eff}}(\phi_0) = W(\phi_0) + \frac{\hbar}{4\pi} \sqrt{\frac{f_{\pi\pi}(\pi_0)}{2}} \times \int \mathrm{d}k \left(\sqrt{W_{\phi\phi}(\phi_0) + k^2} - |k| \right) \tag{76}$$

with a spatially constant π_0, and the infrared contribution is

$$W_{\mathrm{IR}}(\phi_0) = W(\phi_0) + \hbar \sqrt{\frac{f_{\pi\pi}(\pi_0)}{2}} \frac{\sqrt{W_{\phi\phi}(\phi_0)}}{L_0}. \tag{77}$$

The same modification by a factor of $\sqrt{f_{\pi\pi}(\pi_0)/2}$ is obtained in the minisuperspace model if its quadratic dependence on the momentum p is replaced using the same function $f(p)$.

In holonomy-modified models of gravity, in which the quadratic momentum dependence is replaced by a bounded function f just as in the model used here, signature change is an interesting consequence [28–30]. This phenomenon was first noticed by an analysis of space-time structures in the presence of a modified kinetic term, as well as in equations of motion derived from such Hamiltonians [31].

In these models, a signature change to four-dimensional Euclidean space happens whenever the second derivative of the modification function is negative, or around local maxima of the function. Instead of a discrete classical signature parameter $\epsilon = \pm 1$, the modified space(-time) structures have signatures given by $\beta(p) = \frac{1}{2} f_{pp}(p)$, which appears in the new coefficient in (77). When $\beta < 0$, the infrared contribution to the effective potential is imaginary, indicating an instability. An instability is, in fact, the main consequence of modified space-time structures with signature change, as analyzed in the cosmological [30,32] and black-hole context [33].

While the full space-time structure cannot be seen in homogeneous minisuperspace models, it is interesting to note that a minisuperspace model of holonomy corrections is able to reproduce the correct factor in (77). However, the imaginary contribution to the potential means that we have reached the limits of the adiabatic approximation, making it difficult to interpret the complex value within the minisuperspace setting. Nevertheless, the same conclusion of instabilities is then confirmed by an analysis of equations of motion in the corresponding field theory, which are elliptic rather than hyperbolic when $f_{pp}(p)/2 < 0$. We note that the minisuperspace Hamiltonian is positive definite for any positive function $f(p)$, even around a local maximum where $f_{pp}(p) < 0$. We will discuss signature change in more detail after we have developed further techniques in the next section.

IV. BETWEEN MINISUPERSPACE AND THE FULL THEORY

The controlled setting of scalar (field) theories has allowed us to introduce a minisuperspace approximation instead of just a truncation of degrees of freedom: the quantum-mechanical effective potential can, up to a numerical factor, be obtained from the field-theoretical Coleman-Weinberg potential by restricting the latter to its infrared contribution and expanding the resulting integral by $(\sqrt{W_{\phi\phi}(\phi_0)} L_0)^{-1}$, corresponding to a potential term dominating the spatial-derivative term in the Hamiltonian. One could use higher-order terms of such an expansion, as for instance in (55), in order to go beyond the minisuperspace result. However, one would have to begin with the full field-theory calculation, so that this kind of approximation does not seem to lead to strong simplifications. In this section, we explore the possibility of introducing models that are simpler than the full theory but still capture quantum terms that cannot be seen in a strict minisuperspace setting.

A. Perturbative inhomogeneity

Instead of truncating the canonical fields $\phi(x)$ and $\pi(x)$ to their average values, as the starting point of a minisuperspace quantization, we decompose the fields into their average values $\bar{\phi}$ and $\bar{\pi}$, as well as their zero-average variations $\delta\phi$ and $\delta\pi$. (The average values $\bar{\phi}$ and $\bar{\pi}$ will appear in some expressions in a form very similar to ϕ_0 and π_0 previously. We use a different notation for them in order to highlight their different origin in a perturbative expansion, rather than just constant field values.) Such decompositions have been used in canonical cosmological models going back to [34], and more recently in [35,36]; see also [32]. Formally, this decomposition can be introduced by defining

$$\bar{\phi} := \frac{1}{L_0} \int_0^{L_0} \mathrm{d}x \phi(x), \qquad \bar{\pi} := \int_0^{L_0} \mathrm{d}x \pi(x), \tag{78}$$

with integrations over the averaging volume of length L_0, and the variations

$$\delta\phi(x) := \phi(x) - \bar{\phi}, \qquad \delta\pi(x) := \pi(x) - \frac{1}{L_0}\bar{\pi}. \tag{79}$$

(Anticipating the Poisson brackets derived below, we absorb a factor of L_0 in $\bar{\pi}$ in order to have canonical fields. Absorbing L_0 in $\bar{\pi}$, rather than $\bar{\phi}$, mimics the density weight of the corresponding field.) By definition, the variations satisfy the conditions

$$\int_0^{L_0} \mathrm{d}x \delta\phi(x) = 0 = \int_0^{L_0} \mathrm{d}x \delta\pi(x). \tag{80}$$

They ensure that we do not double-count the average degrees of freedom, which have been separated off as $\bar{\phi}$ and $\bar{\pi}$.

The canonical structure of decomposed fields follows from the original one, for instance by pulling back the symplectic or Liouville form. For the average values, computing

$$\int_0^{L_0} \mathrm{d}x \dot{\phi}(x)\pi(x) = \dot{\bar{\phi}}\,\bar{\pi} + \int_0^{L_0} \mathrm{d}x \delta\dot{\phi}(x)\delta\pi(x) \tag{81}$$

using (80), one sees that $\bar{\pi}$ is indeed the momentum of $\bar{\phi}$, so that

$$\{\bar{\phi}, \bar{\pi}\} = 1 \tag{82}$$

for the average values. The variations are canonically conjugate, but also subject to the second-class constraints (80). Therefore, we switch from the direct result of (81) to their Dirac brackets

$$\{\delta\phi(x), \delta\pi(y)\} = \delta(x,y) - L_0^{-1}. \tag{83}$$

[With two second-class constraints C_1 and C_2 given in (80), the Dirac bracket is obtained by subtracting from the original Poisson bracket the term $1/\{C_1, C_2\} = 1/\int \mathrm{d}x \int \mathrm{d}y \delta(x,y) = 1/L_0$.]

The Poisson brackets can be used to compute Hamiltonian equations of motion from the decomposed Hamiltonian

$$H = \int_0^{L_0} \mathrm{d}x \left(\frac{1}{2}\pi(x)^2 + \frac{1}{2}\phi'(x)^2 + W(\phi)\right) = \frac{\bar{\pi}^2}{2L_0} + L_0 W(\bar{\phi}) + \frac{1}{2}\int_0^{L_0} \mathrm{d}x(\delta\pi(x)^2 + \delta\phi'(x)^2 + W_{\phi\phi}(\bar{\phi})\delta\phi(x)^2 + \cdots), \tag{84}$$

where the dots now indicate terms of higher than second order in the variations. By the decomposition of canonical fields and the expansion of H to the given order, the original interacting field theory is converted into a nonharmonic mechanical system coupled to a free field theory. Extending the minisuperspace quantization, we can now quantize this system and obtain nonharmonic quantum mechanics coupled to a free quantum-field theory. Such a system is simpler than the full interacting quantum-field theory. Even if one expands the Hamiltonian to higher orders in the variations, there may be simplifications because a possibly nonpolynomial field-theory potential would be converted into a polynomial expansion.

Effective potentials of the decomposed theory can be computed using moments of the two subsystems, $G^{c,d}$ for the quantum-mechanics part and $G^{a,b}(x_1,\ldots,x_b;y_1,\ldots,y_a)$ for the field-theory part. As a consequence of (80), the field-theory moments are subject to the conditions

$$\int_0^{L_0} \mathrm{d}x G^{a,b}(\ldots,x,\ldots) = 0. \tag{85}$$

Their Poisson brackets follow using the Dirac bracket for variations, for instance,

$$\begin{aligned}\{G^{0,2}(x_1,x_2), G^{2,0}(y_1,y_2)\} &= G^{1,1}(x_2,y_2)(\delta(x_1,y_1) - L_0^{-1}) \\ &\quad + G^{1,1}(x_1,y_1)(\delta(x_2,y_2) - L_0^{-1}) \\ &\quad + G^{1,1}(x_2,y_1)(\delta(x_1,y_2) - L_0^{-1}) \\ &\quad + G^{1,1}(x_1,y_2)(\delta(x_2,y_1) - L_0^{-1}).\end{aligned} \tag{86}$$

Equations of motion for second-order moments take the same form as those obtained for the full field theory. Therefore, the effective potential is the same, except that the restricted space of the averaging volume requires us to limit the wave numbers to be larger than $k_{\min} = 2\pi/L_0$. Wavelengths larger than the averaging volume are therefore excluded from the remaining Coleman-Weinberg potential for the quantum field theory of variations, but they are included in the effective potential of the quantum-mechanics part. Up to numerical factors, the full Coleman-Weinberg potential is therefore split correctly into the infrared contribution as in (1), plus the remainder.

As before, the numerical factors can be matched exactly in a periodic model with its clearer separation of discrete modes. We have already seen that the minisuperspace model produces exactly the infrared contribution to the Coleman-Weinberg potential if we choose the averaging volume to equal the length of the periodic space. The same term is now produced by the quantum-mechanics part of our extended model. The field-theory part is almost identical to the previous model, except that the conditions (80) for variations remove the zero mode from the Fourier sum in (50). But the zero mode, split off explicitly in (51), is just the quantum-mechanics contribution, so that the infrared contribution of our decomposed model is identical with the infrared contribution in the full quantum-field theory, to the orders considered.

B. Application: Signature change

As a simple application of the perturbed model in the context of effective potentials, we now revisit the question of signature change in holonomy-modified theories. The quantum-mechanical minisuperspace model gives a hint of signature change because the $\hbar$-correction to its effective potential, with a factor of $\sqrt{f_{pp}(p)/2}$ if the quadratic momentum dependence of the classical Hamiltonian is replaced by some function $f(p)$, is imaginary around a local maximum of f, where $f_{pp} < 0$. However, the adiabatic approximation used to derive the effective potential breaks down in this regime, so that the conclusion of unstable behavior is not reliable in this setting. The minisuperspace model has a nonstandard but positive Hamiltonian and is well defined. Moreover, the instability by itself does not directly indicate signature change, even though field theories on Euclidean space lead to instabilities of initial-value problems.

The perturbed model allows us to address the question without having to go to the full field theory. We write the holonomy-modified Hamiltonian as

$$H = \int_0^{L_0} \mathrm{d}x \left(f(\pi(x)) + \frac{1}{2}\phi'(x)^2 + W(\phi)\right) = \frac{\bar{\pi}^2}{2L_0} + L_0 W(\bar{\phi}) + \frac{1}{2}\int_0^{L_0} \mathrm{d}x(f_{\bar{\pi}\bar{\pi}}(\bar{\pi})\delta\pi(x)^2 + \delta\phi'(x)^2 + W_{\phi\phi}(\bar{\phi})\delta\phi(x)^2 + \cdots). \tag{87}$$

It is now clear that the field-theory Hamiltonian of perturbative inhomogeneity is not positive definite when $f_{\bar{\pi}\bar{\pi}}(\bar{\pi}) < 0$. Already at the classical level, the Hamiltonian equations

$$\delta\dot{\phi} = f_{\bar{\pi}\bar{\pi}}(\bar{\pi})\delta\pi, \qquad \delta\dot{\pi} = \delta\phi'' - W_{\phi\phi}(\bar{\phi})\delta\phi \tag{88}$$

imply as a second-order field equation the mixed-type partial differential equation

$$-\delta\ddot{\phi}+f_{\bar{\pi}\bar{\pi}}(\bar{\pi})\delta\phi''=f_{\bar{\pi}\bar{\pi}}(\bar{\pi})W_{\phi\phi}(\bar{\phi})\delta\phi-\frac{f_{\bar{\pi}\bar{\pi}\bar{\pi}}(\bar{\pi})}{f_{\bar{\pi}\bar{\pi}}(\bar{\pi})}\dot{\bar{\pi}}\delta\dot{\phi}. \tag{89}$$

For $f_{\bar{\pi}\bar{\pi}}(\bar{\pi})<0$, the equation is elliptic and is well posed with a two-dimensional boundary-value problem instead of an initial-value problem.

It is not necessary to include moment terms or consider a quantum-field theory in order to see signature change, in contrast to the minisuperspace model. If we include moments and compute the Coleman-Weinberg potential of the modified (perturbed or full) theory, we obtain (76). For $f_{\bar{\pi}\bar{\pi}}(\bar{\pi})<0$, the $\hbar$-correction to the classical potential has an imaginary factor, as it does for a Euclidean quantum-field theory with the same potential. Regarding signature change, the classical perturbed model and the quantized perturbed or full theory therefore agree. This observation supports the conclusions about signature change in models of loop quantum gravity [28–30], which have been made by considering spherically symmetric or perturbed cosmological models without including moment terms [31,37].

1. Avoiding an adiabatic approximation

In order to discuss signature change in more detail, we go back to our derivation of effective potentials in holonomy-modified models. In particular, we should revisit the adiabatic assumption in this case because it is tied to an evolution picture or initial-value problem, which is available only in the Lorentzian regime but not in the Euclidean one if there is signature change. It turns out that our previous Lorentzian solutions can be recovered without using an adiabatic approximation, but they still rely on the presence of a well-posed initial-value problem.

The three equations of motion for the second-order moments are given by (25), (75), and (23). Instead of applying the adiabatic approximation, we eliminate $G^{1,1}$ to get the two equations

$$\ddot{G}^{0,2}(y,z,t)=\frac{1}{2}f_{\pi\pi}\left[f_{\pi\pi}G^{0,2}(y,z,t)-\left(2W_{\phi\phi}-\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)\right]G^{0,2}(y,z,t) \tag{90}$$

$$\ddot{G}^{2,0}(y,z,t)=-\left(W_{\phi\phi}-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)\left[\frac{f_{\pi\pi}}{2}G^{2,0}(y,z,t)-\left(W_{\phi\phi}-\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right)G^{0,2}(y,z,y)\right]+(y\leftrightarrow z). \tag{91}$$

We further assume that π_0 and ϕ_0 in the coefficients $f_{\pi\pi}(\pi_0)$ and $W_{\phi\phi}(\phi_0)$ are spatially constant. Then eliminating $G^{2,0}$ from (91), we obtain a fourth-order differential equation for $G^{0,2}$:

$$\left[\frac{\mathrm{d}^4}{\mathrm{d}t^4}-f_{\pi\pi}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{\mathrm{d}^2}{\mathrm{d}z^2}-2W_{\phi\phi}\right)+\frac{1}{4}f_{\pi\pi}^2\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2}-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)^2\right]G^{0,2}(y,z,t)=0. \tag{92}$$

For $f_{\pi\pi}<0$, the principal symbol of this equation is positive semidefinite, and therefore the equation is elliptic, while the equation has characteristics for $f_{\pi\pi}>0$. In the latter case, it is therefore meaningful to look for solutions determined by initial values at some fixed t, which by Fourier decomposition can be written as

$$G^{0,2}(y,z,t)=\int \mathrm{d}k_y\int \mathrm{d}k_z[g_1(k_y,k_z)e^{i(k_y y+k_z z-\omega_k t)}+g_1^*(k_y,k_z)e^{-i(k_y y+k_z z-\omega_k t)}], \tag{93}$$

with $\omega_k=\sqrt{f_{\pi\pi}/2}(\omega_y-\omega_z)$, where $\omega_i=\sqrt{W_{\phi\phi}+k_i^2}$. Here we are choosing the minus sign so that we get the correct limit for the usual Minkowski case, on which we will comment further below. (Solutions to the fourth-order differential equation would be consistent with a sum $\omega_y+\omega_z$ as well.) Since we assume $f_{\pi\pi}>0$ for now, ω_k is real. For $f_{\pi\pi}<0$, only imaginary solutions for ω_k exist, consistent with the absence of characteristics in this elliptic case.

The arbitrariness of the state about which these functions are evaluated is captured by the function $g_1(k_y,k_z)$ and its conjugate. These can be fixed by specifying the initial conditions for the moments through the relations

$$g_1(k_y,k_z)=\frac{1}{2}\left[G^{0,2}(k_y,k_z)+\frac{i}{\omega_k}\dot{G}^{0,2}(k_y,k_z)\right] \tag{94}$$

$$g_1^*(k_y,k_z)=\frac{1}{2}\left[G^{0,2}(-k_y,-k_z)-\frac{i}{\omega_k}\dot{G}^{0,2}(-k_y,-k_z)\right], \tag{95}$$

where the newly introduced function $G^{0,2}(k_y,k_z)$ and its time derivative are defined as

$$G^{0,2}(k_y,k_z)\coloneqq\int \mathrm{d}y\int \mathrm{d}z\, G^{0,2}(y,z,t=0)e^{-i[k_y y+k_z z]} \tag{96}$$

$$\dot{G}^{0,2}(k_y,k_z)\coloneqq\int \mathrm{d}y\int \mathrm{d}z\, \dot{G}^{0,2}(y,z,t=0)e^{-i[k_y y+k_z z]}. \tag{97}$$

Further, if the theory is assumed to have translational invariance in space, we can use $g_1(k_y,k_z)=\tilde{g}(k_y)\delta(k_y+k_z)$ to get

$$G^{0,2}(y,z,t) = \int \mathrm{d}k[\tilde{g}(k)e^{ik(y-z)} + \tilde{g}^*(k)e^{-ik(y-z)}]. \quad (98)$$

We can immediately see that these moments do not depend on time explicitly, even without using an adiabatic approximation. Having a (spatially) translationally invariant theory is sufficient to imply time translation invariance in the Lorentzian case. Physically, translation invariance in space implies the absence of propagating modes in initial values, so that time translation invariance of solutions is implied on a static background. While this heuristic interpretation justifies the conclusion about symmetric solutions, it follows only because ω_k has the form given after Eq. (93), which is a direct consequence of the form of the equations of motion, *together with* the sign choice commented on above. One could therefore invert these arguments and fix the sign choice $\omega_k = \sqrt{f_{\pi\pi}/2}(\omega_y - \omega_z)$, as opposed to $\omega_k = \sqrt{f_{\pi\pi}/2}(\omega_y + \omega_z)$, avoiding a comparison with the usual Minkowski solutions.

If we further demand that the theory have reflection symmetry, that is, $\tilde{g}(-k) = \tilde{g}(k)$, it is evident that we can write the solution of $G^{0,2}(y,z)$ as

$$G^{0,2}(y,z) = \int \mathrm{d}k g(k) e^{ik(y-z)}, \quad (99)$$

where we have absorbed a factor of 2 in the function $g(k)$ and have hence dropped the tilde. Thus we have the same solution for the moments, even without the adiabatic approximation, as long as we have spatial translational invariance and reflection symmetry. Both of these properties are expected to be realized even with modified momentum dependence (motivated by some kind of spatial quantum geometry) in a continuum effective theory.

For $f_{\pi\pi} < 0$, the initial-value formulation based on (93) with $g_1(k_y, k_z)$ related to the values of $G^{0,2}$ at some fixed time should be replaced by a boundary-value problem in the t-direction. We then have to prescribe $G^{0,2}(y,z,t)$ at two fixed values of $t = t_1$ and $t = t_2$. It is then clear that time translation invariance can be respected by solutions only in the special case in which we choose $G^{0,2}(y,z,t_1) = G^{0,2}(y,z,t_2)$. There seems to be no independent physical condition that justifies such a choice, which is consistent with the formal result that the adiabatic approximation breaks down for $f_{\pi\pi} < 0$: in this case, using the adiabatic approximation results in imaginary contributions to the effective potential. While a complex potential, or complex moments obtained in the derivation of such a potential, are not meaningful, imaginary contributions can be interpreted as indications of instabilities that occur when one attempts to solve elliptic partial differential equations by initial-value problems.

2. Euclidean theories

In our discussion of holonomy modifications, we have seen that a change of sign in $f_{\pi\pi}$ implies that second-order moments and the effective potential acquire an imaginary factor. However, $G^{0,2}$ and $G^{2,0}$ are fluctuations and should never be negative, let alone imaginary. We can resolve this apparent inconsistency by comparing the effective holonomy-modified theory with standard Euclidean field theory, motivated by the observations made elsewhere that a change of sign in $f_{\pi\pi}$ indicates a signature change. Such a comparison will allow us to elucidate the role of signature change further.

We begin by recalling that the Coleman-Weinberg potential takes the same form in Lorentzian and Euclidean quantum-field theory, as stated already in [2]. Some details leading to this result will be relevant for our further comments on signature change: the Euclidean action (with one spatial dimension as before) is traditionally defined as

$$S_{\mathrm{E}} = \int \mathrm{d}t\mathrm{d}x\left(\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 + W(\phi)\right) \quad (100)$$

with a positive potential term. With this choice, the Euclidean Hamiltonian

$$H_{\mathrm{E}} = \int \mathrm{d}x\left(\frac{1}{2}\pi^2 - \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 - W(\phi)\right) \quad (101)$$

is transformed to the positive-definite energy functional

$$-H_{\mathrm{EW}} = \int \mathrm{d}x\left(\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^2 + W(\phi)\right) \quad (102)$$

after a Wick rotation from t to $\tau = it$ (so that $\pi = \mathrm{d}\phi/\mathrm{d}t = i\mathrm{d}\phi/\mathrm{d}\tau$). The weight $\exp(iS)$ of a Lorentzian path integral then becomes the correct weight $\exp(\int \mathrm{d}\tau H_{\mathrm{EW}}) = \exp(-\int \mathrm{d}\beta E)$ of the partition function of statistical mechanics, with the energy E (and a periodic range for β). Since H_{EW} is, up to a total minus sign and the use of τ instead of t, the same as the Hamiltonian of the Lorentzian theory, our canonical methods imply the same Coleman-Weinberg potential independently of the signature, in agreement with [2].

Signature change in models of loop quantum gravity leads to different results because it is not accompanied by a Wick rotation. The momentum term in the Hamiltonian changes sign, which leads to elliptic field equations as in Euclidean space, but all coordinates remain real. The Hamiltonian obtained with holonomy modifications has a nonquadratic kinetic term $f(\pi(x))$, which remains positive so that in this form we do not see directly what role the sign of $f_{\pi\pi}$ should play. We can make this coefficient appear explicitly if we consider a Hamiltonian expanded for small variations $\delta\phi$ and $\delta\pi$ around some background fields ϕ_0 and π_0, which would be constant when used in the Coleman-Weinberg potential:

$$H = \int \mathrm{d}x \left(f(\pi_0) + \frac{1}{2}(\phi_0')^2 + W(\phi_0) \right) + \int \mathrm{d}x \left(\frac{1}{2} f_{\pi\pi}(\pi_0)\delta\pi(x)^2 + \frac{1}{2}\delta\phi'(x)^2 + \frac{1}{2} W_{\phi\phi}(\phi_0)\delta\phi(x)^2 + \cdots \right). \quad (103)$$

Up to a total minus sign, the Hamiltonian for perturbations, with $f_{\pi\pi}(\pi_0) = -1 < 0$, is identical to the Euclidean Hamiltonian H_{E}, but without any Wick rotation the Hamiltonian remains of indefinite sign and is unbounded from above and below.

Perturbative modes evolve according to a Hamiltonian in which the sign of the kinetic term is given by the sign of $f_{\pi\pi}$. Signature change is implied because the kinetic and spatial-derivative terms have different relative signs depending on the sign of $f_{\pi\pi}$, and correspondingly, second-order field equations for $\delta\phi$ are hyperbolic partial differential equations for a positive $f_{\pi\pi}$, but elliptic ones for a negative $f_{\pi\pi}$. The relative signs of these two terms therefore agree with what one obtains for field theories on Lorentzian (positive $f_{\pi\pi}$) and Euclidean space-times (negative $f_{\pi\pi}$), respectively, provided these space-times are kept real and not modified by Wick rotations. The nonpositive Hamiltonian then leads to instabilities, as shown by imaginary quantum corrections to effective potentials.

V. CONCLUSIONS

In a variety of scalar field theories, we have found good qualitative agreement between infrared contributions to their Coleman-Weinberg-type potentials and effective potentials in quantum-mechanical systems. With simple periodic spaces, quantitative agreement was found. These models provide several instructive conclusions about the question of how minisuperspace truncations can be related to field theories, and how to go beyond the minisuperspace setting in a controlled way.

A. Minisuperspace truncation vs approximation

In quantum cosmology, minisuperspace models have been studied for decades, after their introduction in [1]; see also [38]. While their relation to some putative full theory remains weak, it has been known for some time that certain dynamical properties of the full theory cannot be captured in this setting [39]. More recently, signature change has joined the list of concrete phenomena which, at high density or large momenta, can have important implications not seen in minisuperspace truncations.

Loop quantum cosmology [40,41] was initially motivated by the hope that the controlled kinematical setting of loop quantum gravity might make it possible to derive some aspects of reduced models [42]. However, also in this context, not much progress on strict derivations of the dynamics has been made, even setting aside the problem that the dynamics of full loop quantum gravity remains poorly controlled, owing to quantization ambiguities and possible anomalies. (Some progress on the latter question has recently been made [43–45], but so far without indications on the minisuperspace question.) The kinematical side of a possible reduction of loop quantum cosmology from loop quantum gravity is still being analyzed [46–53], and possible dynamical relations between minisuperspace models and the full theory are tentative. A recent idea uses condensate states in order to describe homogeneous space-times, either as an approximation [54] or as a reduction from a full theory defined by group-field theories [55–61]. Symmetry reduction of standard quantum-field theories has been used in [62] in order to understand minisuperspace truncations in a controlled setting, but so far only for free theories in which the conclusions of [62] appear to be restricted. In the present paper, we have provided results for interacting quantum-field theories, focusing on effective potentials in order to overcome the more complicated questions of how to relate states.

In our models, a quantum-mechanical system does not just appear as a minisuperspace truncation of a field theory with a reduced number of degrees of freedom, but can be embedded in a controlled approximation. The infrared contribution to a Coleman-Weinberg potential is seen to agree, up to a numerical factor, with the quantum-mechanical effective potential to leading order in an expansion by $(\sqrt{W_{\phi\phi}(\phi)}L_0)^{-1} = (2\pi)^{-1}k_{\mathrm{IR}}/\sqrt{W_{\phi\phi}(\phi)}$ with the infrared scale L_0 or wave number k_{IR}. (For a periodic space with discrete wave numbers, we have found exact agreement with the quantum-mechanics result.)

Taken to higher orders, this expansion corresponds to an approximation with controlled correction terms, valid as long as the potential dominates the spatial-derivative term in the Hamiltonian. There is an important difference between traditional minisuperspace truncations and the minisuperspace approximation provided here: while truncations are usually performed at the kinematical level, introducing the dynamics by formulating a quantum Hamiltonian on the state space of the reduced model, our minisuperspace approximation is dynamical in a crucial way. We need to solve some of the field-theory equations of motion for moments in order to obtain the effective potential whose infrared contribution we expand. The leading order of this expansion can be derived in a truncated model, just using the classical potential and moment equations of quantum mechanics. Going to the next orders is then not just a matter of computing a higher-order term, but requires more detailed information about the full dynamics. In this sense, there is a big leap between minisuperspace truncations and the approximations developed here.

It might seem pointless to compute a minisuperspace approximation if it requires one to solve the full dynamical equations first. At this stage, it is important that, at least to

first order in $\hbar$, we can derive the expanded infrared contribution to the effective potential by combining the semiclassical approximation with an expansion in terms of perturbative inhomogeneity as shown in Sec. IV. The resulting models between minisuperspaces and the full theory take the form of quantum mechanics coupled to a free quantum-field theory (again, to first order in $\hbar$), in which semiclassical equations of motion for moments are manageable. The formal setting is closely related to canonical cosmological perturbation theory.

B. Infrared scale

The minisuperspace correspondence found here explains the meaning of the dependence of quantum corrections on the averaging volume V_0, or the averaging distance L_0 in our one-dimensional models. The situation within minisuperspace models has been very unclear in the context of quantum cosmology, in which the physical meaning of L_0 is hard to see. Moreover, while physical results in the classical theory do not depend on the value of L_0, quantum corrections usually do. In quantum cosmology, it has sometimes been suggested that one can make quantum corrections arbitrarily small by choosing a large averaging scale L_0, which indeed appears with a negative power in our effective potential. But while it is true that quantum corrections in our minisuperspace model can be made arbitrarily small in this way, at the same time the minisuperspace approximation becomes worse: fewer modes are then included in the infrared contribution of the field-theory effective potential corresponding to the minisuperspace effective potential. Averaging over a larger L_0 eliminates more quantum corrections, which are no longer being considered in the minisuperspace model but remain a significant part of corrections in the field theory. If quantum corrections are small only by virtue of choosing a large averaging distance, this property would be a minisuperspace artifact, lacking physical meaning.

We have explained the discrepancy between the classical independence and quantum dependence on the averaging scale by the fact that classical theories are local, while quantum theories have nonlocal features so that they can be sensitive to averaging volumes via the number of modes included in quantum corrections. Moreover, we have given the parameter L_0 a direct physical meaning as the infrared scale of quantum-field theory. Contributions to effective potentials depend on the number of modes included in their derivation, so that a dependence of quantum corrections on L_0 is nothing but the well-known running of coupling constants in quantum-field theory.

Our results also show that L_0 is not an infrared cutoff or a regulator, as sometimes suggested in loop quantum cosmology; see for instance [8]. The limit $L_0 \to \infty$ is not required in order to have qualitative agreement of minisuperspace and quantum-field theory results; this limit would just eliminate all modes considered for the effective potential and therefore lead to a vanishing contribution without any qualitative comparison. In fact, in the models considered, no infrared cutoff is required because the effective potentials are infrared finite. If one computes the Coleman-Weinberg potential by summing up 1-loop Feynman diagrams with a free number of external lines, as introduced in [2], one may encounter infrared divergences: for the original $\lambda\phi^4$-potential,

$$W_{\rm eff}(\phi) = \lambda\phi^4 + \frac{\hbar}{(2\pi)^4}\int {\rm d}^4k \sum_{n=1}^{\infty} \frac{1}{2n}\left(\frac{12\lambda\phi^2}{k^2+i\epsilon}\right)^n \qquad (104)$$

has infinite contributions if each term in the sum (for $n > 1$) is integrated individually at small k. However, these infrared divergences disappear in the final effective potential if the sum is performed before integrating, giving rise to (3). While the ultraviolet divergence remains, the infrared one is replaced by a logarithmic divergence of the effective potential at $\phi_0 = 0$. The infrared contributions used here do not encounter this last divergence because it is outside of the range of the expansion by $(\sqrt{W_{\phi\phi}(\phi_0)}L_0)^{-1}$. Therefore, a minisuperspace approximation of scalar quantum-field theory does not require an infrared cutoff.

Unlike an ultraviolet cutoff, the infrared scale used here does not depend on unknown physics. It has a clear physical meaning as a selection of modes included in the averaged effective potential. Therefore, it is sufficient to work with a given scale L_0, once selected, and no renormalization is required. One could try to absorb L_0 completely in renormalized coupling constants, as done for renormalizable theories with the ultraviolet cutoff. However, it is easy to see that this can be possible only for a few special potentials. To first order in $\hbar$, $\sqrt{W_{\phi\phi}(\phi_0)}$ would have to be of the same functional form as $W(\phi_0)$ if a combined coupling constant could be used in $W(\phi_0) + \frac{1}{2}\hbar\sqrt{W_{\phi\phi}(\phi_0)}/L_0$ that absorbs L_0. The potential would have to satisfy the differential equation $W_{\phi\phi}(\phi) = AW(\phi)^2$ with an arbitrary constant A. This nonlinear second-order equation can be transformed to two coupled first-order ones by introducing $U := W_\phi$, so that $U{\rm d}U/{\rm d}W = W_\phi {\rm d}U/{\rm d}W = U_\phi = W_{\phi\phi} = AW^2$ or, solving $U{\rm d}U = AW^2{\rm d}W$, $W_\phi = \sqrt{\frac{2}{3}AW^3 + c}$ with another constant c. The remaining equation can be integrated by separation, but the resulting expression for W has a closed inverse only in the case $c = 0$, for which $W(\phi) = a\phi^{-2}$ with a constant a. In this case, the effective potential can be written as

$$W_{\rm eff}(\phi_0) = \frac{a + \sqrt{6a}\hbar/L_0}{\phi_0^2} =: \frac{a_{\rm ren}}{\phi_0^2} \qquad (105)$$

with a renormalized $a_{\rm ren}$. If the bare coupling constant a is allowed to depend on L_0, $a_{\rm ren}$ may be assumed to be independent of L_0 and a scale-free theory is obtained.

However, potentials not obeying the equation $W_{\phi\phi} \propto W^2$ do not give rise to scale-free effective potentials.

C. Quantum-geometry modifications

Although we did not consider quantum gravity, our models can be used to show that modifications suggested by quantum geometry, mainly loop quantum gravity in the canonical setting used here, can be captured by corresponding modifications in the minisuperspace models. The field-theory modifications modeling discrete quantum space usually depend on microscopic parameters, for instance the discreteness scale ℓ_0. The minisuperspace result matches the infrared contribution to the field-theory effective potential if the same scale is used in minisuperspace modifications (rather than the only scale, L_0, present in the minisuperspace model). In loop quantum cosmology, such a dependence has been recognized in the context of lattice refinement [63,64].

We have even seen hints of signature change in a minisuperspace model by way of an instability implied by a complex effective potential. However, the phenomenon itself lies outside of the range of validity of the adiabatic approximations used. In order to make the effect reliable, one has to go beyond the minisuperspace setting, as done in Sec. IV by introducing perturbative inhomogeneity as it has been used in cosmological models of loop quantum gravity as well. We have further elucidated signature change in models of loop quantum gravity by contrasting the Euclidean versions implied by this phenomenon with the usual notion of Euclidean quantum-field theory. A crucial difference is the absence of a Wick rotation in the former case, so that instabilities cannot be fully removed.

The scalar field theories studied here can model several aspects of features expected for quantum gravity and cosmology. Since our new minisuperspace approximation makes use of infrared contributions to effective potentials, the nonrenormalizability of perturbative quantum gravity is not an issue. However, new ingredients would have to be included for a direct application to quantum cosmology: first, gravitational models are necessarily constrained systems, which requires an extension of the canonical effective framework used here. Methods for effective constraints have been developed in [65,66]. Secondly, in the absence of an absolute time parameter, it is not clear how the adiabatic expansion can be formulated. This question can be circumvented by considering moment couplings to expectation values without trying to solve for the moments separately. The emphasis would then not be on effective potentials but on other suitable properties referring more directly to the dynamics of moments. Finally, one must address the question of how an infrared scale can meaningfully be fixed in a diffeomorphism-covariant theory.

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