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Mass decomposition of SLACS lens galaxies in Weyl conformal gravity

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We study here, using the Mannheim-Kazanas solution of Weyl conformal theory, the mass decomposition in the representative subsample of 57 early-type elliptical lens galaxies of the Sloan Lens Advanced Camera for Surveys (SLACS) on board the Hubble Space Telescope. We begin by showing that the solution need not be an exclusive solution of conformal gravity but can also be viewed as a solution of a class of $f(R)$ gravity theories coupled to nonlinear electrodynamics thereby rendering the ensuing results more universal. Since lensing involves light bending, we shall first show that the solution adds to Schwarzschild light bending caused by the luminous mass (M_*) a *positive* contribution $+\gamma R$ contrary to the previous results in the literature, thereby resolving a long-standing problem. The cause of the error is critically examined. Next, applying the expressions for light bending together with an input equating Einstein and Weyl angles, we develop a novel algorithm for separating the luminous component from the total lens mass (luminous + dark) within the Einstein radius. Our results indicate that the luminous mass estimates differ from the observed total lens masses by a linear proportionality factor across the subsample, which qualitatively agrees with the common conclusion from a number of different simulations in the literature. In quantitative detail, we observe that the ratios of luminous over total lens mass (f^*) within the Einstein radius of individual galaxies take on values near unity, many of which remarkably fall inside or just marginally outside the specified error bars obtained from a simulation based on the Bruzual-Charlot stellar population synthesis model together with the Salpeter initial mass function favored on the ground of metallicity [Grillo *et al.*, Astron. Astrophys. **501**, 461 (2009)]. We shall also calculate the average dark matter density $\langle\rho\rangle_{\mathrm{av}}$ of individual galaxies within their respective Einstein spheres. To our knowledge, the present approach, being truly analytic, seems to be the first of its kind attempting to provide a new decomposition scheme distinct from the simulational ones.

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I. INTRODUCTION

Unlike for spiral galaxies, which appear to be embedded in large dark matter "halos," we cannot in general measure rotation curves for elliptical galaxies, a vast majority of which act as strong gravitational lenses. Only very rarely have strong lenses been identified with spiral galaxies since the observed lens properties suggest that they are produced not by the galactic disk but solely by the elliptical galaxy bulge. Thus, using gravitational lensing, one can study the mass-to-light ratio of elliptical galaxies, which shows that there is no sign of a large amount of dark matter surrounding these galaxies. If dark matter is present in these galaxies, it has to be mixed in with the luminous matter giving a total enclosed lens mass. We shall be concerned in this work with the decomposition of this lens mass into dark and luminous parts within the Einstein radius of elliptical galaxies.

A few lines about the dark matter hypothesis seems to be in order. Early observations [1,2] on rotational data of spiral galaxies, now reconfirmed by several observations extending well beyond the optical disk [3–14], indicate that they do not conform to Newtonian gravity predictions. Doppler emissions from stable circular orbits of neutral hydrogen clouds in the halo allow measurement of tangential velocity v_{tg} of the clouds treated as probe particles. Contrary to Newton's laws, where v_{tg}^2 should decay with radius r, observations indicate that it approximately levels off with r in the galactic halo region, which in turn calls for the presence of additional nonluminous mass, the so-called "dark matter." Several well known theoretical models for dark matter exist in the literature, and it is impossible to list all of them here (only some are mentioned in [15–39]). The most recent model, to our knowledge, seems to be the so-called Eddington-inspired Born-Infeld theory, succinctly called the "gravitational avatar of non-linear electrodynamics" [40], developed by Bañados and Ferreira [41]. This new, and more general, theory has led to many interesting observable predictions about dark matter including the

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possibility of nonsingular cosmological models alternative to inflation [42–49].

However, there is yet another variety of theories that do not at all require dark matter for the interpretation of the observed rotation curves associated with spirals. This class of theories include, e.g., modified Newtonian dynamics developed by Milgrom [50–52] and Bekenstein and Milgrom [53]. [The theory is very widely discussed; see e.g., [54,55], and other modified gravities such as developed in [56], $f(R)$ gravity theories [57,58], and Weyl conformal gravity[1] [61].]

We shall consider below what we call the Mannheim-Kazanas (MK) vacuum (meaning matter-free) solution [61,62] of Weyl conformal gravity applying it to elliptical lens galaxies.[2] The solution has three universal constant parameters, γ^*, γ_0, and k, that are associated with potentials of cosmic origin. This implies that the Weyl vacuum is not really empty but is an arena for the energetic interplay of these potentials.[3] To be specific, the constant γ_0 is associated with a universal linear potential term $V_{\gamma_0}(r) = \gamma_0 c^2 r/2$ that is induced by the cosmic background and k is associated with a de Sitter–like potential term $V_k(r) = -kc^2r^2/2$ that is induced by inhomogeneities in the cosmic background. The value γ^* is associated with the linear potential of the Sun and is so small that there are no modifications to standard solar system phenomenology. The constants were used to successfully fit the rotation data of individual spiral galaxies including possible noncircular motions in the halo [65,66] and also to determine their halo sizes from the condition of stability of circular orbits [67]. However, we are not considering rotation curves in this paper but nevertheless using those universal constants for the mass decomposition in lens elliptical galaxies.

The purpose of the present paper is twofold: First, we show that the light bending in the MK solution enhances the Schwarzschild bending by an amount $+\gamma R$ contrary to the previous result of $-\gamma R$ in the literature. We shall point out the causes for this discrepancy. The relative contributions to bending from different terms will also be worked out. Second, we shall find an application of this light bending. Using it in the lens equation together with a certain logical input (explained below), we shall investigate how far the MK solution can account for the mass decomposition in the 57 Sloan Lens Advanced Camera for Surveys (SLACS) early-type elliptical lens galaxies [68], which provide an unbiased subsample representative of the complete sample of early-type galaxies in the Sloan Digital Sky Survey (SDSS) database of over 10^5 galaxies. This part of the task means that we shall be trying to quantify the effect of the cosmology induced potentials V_{γ_0} and V_k on the galactic matter distribution within the Einstein radius observed in lensing measurements.

To reach our goals, we shall calculate, following Ishak *et al.* [69], different contributions to light deflection in the MK solution using the artifact of a vacuole that is assumed to enclose the galaxy (lens) at its center. The predecessor of the vacuole method is the Rindler-Ishak method [70], developed for the asymptotically nonflat SdS metric of general relativity (GR), that exposed the effect of cosmological constant Λ on light bending thereby debunking a prevailing belief to the contrary [71]. Then we shall use a logical input, already employed in the literature [48,49,72], viz., that the observed value of the Einstein angle θ_{E} (caused by the luminous + dark matter) should be equal to the Weyl angle θ_{W} (caused by the luminous matter + potentials), i.e., $\theta_{\mathrm{Ein}} = \theta_{\mathrm{Weyl}}$ for a light ray having the *same* impact parameter.[4] Using the input, we shall develop a new algorithm that would lead to the decomposition of the total lens mass $M_{\mathrm{tot}}^{\mathrm{lens}}$ within the observed Einstein (or Weyl) radius into dark (M_{dm}) and luminous (M_*) matter parts and compare the mass ratios with known simulational predictions.

[1]There has been much debate for and against the Weyl conformal gravity. For instance, Flanagan [59] argues that if the source has associated with it a macroscopic long range scalar field breaking conformal symmetry, the theory does not reproduce attractive gravity in the solar system. However, subsequently, Mannheim [60] has counterargued that Schwarzschild tests of solar gravity could still be recovered even in the presence of such macroscopic fields.

[2]The solution is called here the MK solution for easy reference. Note that it is distinguished from the Schwarzschild–de Sitter (SdS) solution by an extra *linear* term γr contributed exclusively by conformal gravity and reminding us of Mach's principle (see [63]). However, we shall soon show that the MK solution can occur in other theories such as in $f(R)$ gravity as well (see Sec. II C). The solution reduces to the familiar de Sitter form $B(r) = 1 - kr^2$ in the limit $M = 0$, $\gamma = 0$ and to the SdS form under $\gamma = 0$.

[3]The approach involving potentials, though not necessarily of cosmic origin, is not new. A potential determined by rotation curves that led to strong constraints on dark matter, to our knowledge, was first envisaged by Lake [23]. Conversely, a modified gravitational potential Φ fitting rotation curves has been suggested by Capozziello *et al.* [64] in the context of $f(R)$ gravity. The most commonly known potential is Newtonian, whose strength M_* dictates the deflection of light rays grazing the sun in the environment of the Schwarzschild vacuum. Likewise, the Weyl vacuum has two potentials $V_{\gamma_0}(r)$ and $V_k(r)$ making up the MK metric, and the associated potential strengths γ_0 and k dictate the light deflection there.

[4]We wish to recall that Einstein and Weyl theories of gravity are both metric theories but otherwise very different. Thus, by the equality $\theta_{\mathrm{Ein}} = \theta_{\mathrm{Weyl}}$, we are not saying that the two theories are merging into one another as a whole but only saying that logically the deflection must have a unique value for a unique impact parameter b, and that all competing theories must predict the same numerical value. The true justification for the adopted equality, however, has to come from other observable *predictions* that the equality would possibly lead to. In fact, one such prediction is the galactic mass decomposition that can be compared with those obtained by independent simulations available in the literature [73].

The paper is organized as follows: To get a reasonable view of what Weyl conformal gravity looks like, we start in Sec. II from the action discussing certain pertinent issues related to conformal symmetry. We further show that the MK metric can occur in the $f(R)$ gravity theory too, thus making the metric more universal than thought heretofore.[5] In Sec. III, we provide critical reappraisals of some steps used previously in the literature for the calculation of light deflection in the MK metric. Next, in Sec. IV, we derive the light trajectory in the MK metric by perturbatively solving the null geodesic equation and calculate in Sec. V the light deflection by using the vacuole method of Ishak *et al.* [69]. In Sec. VI, the algorithm for mass decomposition in the Sloan Lens Advanced Camera for Surveys (SLACS) lens galaxies is developed and applied to the considered subsample. The numerical results for light bending and mass decomposition are shown in Tables I and II, respectively. Section VII concludes the paper. We shall choose units in which $G = 1$, $\hbar = 1$, the vacuum speed of light $c = 1$ unless specifically restored, and the signature chosen is $(+,-,-,-)$.

II. WEYL CONFORMAL GRAVITY

In recent times, it has increasingly been realized that conformal gravity might hold the key to resolving many outstanding problems of astrophysics. This possibility has been advocated in a very recent article by 't Hooft [74] in which he specifically refers to the works of MK including their 1989 paper that brought to focus anew the conformal gravity.[6] Further, it is well known that all the homogeneous and isotropic space-times described by the Robertson-Walker line element have zero Weyl tensor and are thus solutions to the vacuum conformal gravity. Such a highly symmetric empty space-time is a good candidate for the creation of the universe "from nothing"—a possibility first proposed by Vilenkin [76]. The idea that the initial state of the universe should be conformally invariant was advocated also by Penrose [77] and 't Hooft [74]. However, this empty universe might seem unrealistic unless it could be filled with particles, but how could this happen? Spontaneous breaking of conformal symmetry in the early hot universe

[5] We thank an anonymous referee for suggesting this interesting possibility and for advising some other major points relating to conformal symmetry to be addressed in all the detail. Section II is entirely devoted to these issues.

[6] A bit of curious history can be found in a recent article by Berezin *et al.* [63]: Conformal gravity was invented by Weyl in 1918 with a motivation to combine the gravitational and electromagnetic fields into one unified theory. However, the theory was rejected by Einstein and Weyl because it was recognized that the conformal symmetry at most allows only massless particles to exist. But this obstacle can be overcome today by means of the Higgs mechanism for generating particle masses (for the latest account, see [75]).

TABLE I. Different contributions to Schwarzschild light bending t_{Sch} with the associated impact parameter b. The last two columns show that the positive contribution t_γ is overtaken by the negative contribution t_k in all the cases. However, their combined effect is 1 or 2 orders of magnitude less than the contribution t_{Sch}. The overall bending is thus positive and toward the Galactic center. The average dark matter density $\langle\rho\rangle_{\mathrm{av}}^{\mathrm{MK}}$ over the Einstein sphere of radius b is seen to be $\sim 10^6 M_\odot (\mathrm{kpc})^{-3}$.

Galaxy	$\langle\rho\rangle_{\mathrm{av}}^{\mathrm{MK}}$	b	t_{Sch}	t_γ	t_k
…	$\times 10^6 M_\odot (\mathrm{kpc})^{-3}$	(kpc)	$\times 10^{-6}$	$\times 10^{-7}$	$\times 10^{-7}$
J0008 − 0004	5.77	6.59	9.98	4.40	−5.05
J0029 − 0055	0.93	3.49	6.58	1.03	−1.87
J0037 − 0942	5.51	4.97	11.09	2.85	−3.57
J0044 + 0113	0.17	1.71	10.11	0.42	−0.83
J0109 + 1500	2.15	3.03	8.21	0.94	−1.66
J0157 − 0056	6.70	4.88	10.07	2.55	−3.38
J0216 − 0813	17.84	5.53	16.54	4.93	−4.73
J0252 + 0039	2.48	4.41	7.78	1.73	−2.70
J0330 − 0020	3.88	5.44	8.72	2.76	−3.72
J0405 − 0455	0.00	1.14	5.05	0.16	−0.38
J0728 + 3835	3.27	4.22	9.04	1.80	−2.68
J0737 + 3216	8.16	4.67	11.76	2.68	−3.36
J0822 + 2652	5.02	4.45	10.26	2.19	−3.00
J0903 + 4116	7.93	7.23	11.59	5.97	−6.02
J0912 + 0029	12.70	4.58	16.51	3.46	−3.67
J0935 − 0003	25.07	4.27	18.04	3.27	−3.44
J0936 + 0913	2.15	3.45	8.31	1.19	−1.99
J0946 + 1006	5.94	4.93	11.15	2.83	−3.54
J0956 + 5100	10.26	5.04	13.86	3.55	−3.93
J0959 + 4416	3.21	3.61	9.01	1.36	−2.17
J0959 + 0410	0.00	2.23	6.87	0.51	−1.04
J1016 + 3859	2.57	3.13	9.18	1.07	−1.80
J1020 + 1122	8.80	5.12	12.56	3.35	−3.88
J1023 + 4230	3.96	4.48	9.77	2.13	−2.98
J1029 + 0420	0.00	1.93	5.97	0.37	−0.82
J1100 + 5329	8.39	7.03	12.49	6.04	−5.93
J1106 + 5228	0.08	2.18	7.89	0.53	−1.06
J1112 + 0826	9.89	6.21	13.59	5.15	−5.16
J1134 + 6027	1.71	2.92	8.52	0.91	−1.60
J1142 + 1001	3.36	3.50	9.27	1.32	−2.11
J1143 − 0144	3.80	3.26	11.14	1.34	−2.03
J1153 + 4612	0.65	3.18	6.62	0.88	−1.65
J1204 + 0358	2.55	3.68	8.82	1.39	−2.21
J1205 + 4910	5.86	4.25	11.18	2.16	−2.91
J1213 + 6708	1.72	3.13	8.54	1.02	−1.76
J1218 + 0830	2.25	3.47	8.82	1.25	−2.04
J1250 + 0523	2.62	4.18	8.22	1.64	−2.56
J1402 + 6321	7.05	4.54	12.13	2.61	−3.26
J1403 + 0006	0.53	2.62	7.32	0.68	−1.32
J1416 + 5136	7.03	6.08	11.45	4.27	−4.74
J1420 + 6019	0.00	1.26	6.08	0.20	−0.46
J1430 + 4105	1.31	6.53	15.41	6.34	−5.77
J1436 − 0000	3.97	4.81	9.09	2.29	−3.20
J1443 + 0304	0.00	1.92	5.98	0.37	−0.81
J1451 − 0239	0.00	2.33	6.58	0.53	−1.09
J1525 + 3327	11.08	6.56	13.64	5.74	−5.58
J1531 − 0105	4.84	4.71	10.89	2.55	−3.31
J1538 + 5817	0.07	2.51	6.87	0.61	−1.22

(Table continued)

TABLE I. *(Continued)*

Galaxy	$\langle\rho\rangle_{\rm av}^{\rm MK}$	b	$t_{\rm Sch}$	t_γ	t_k
J1621 + 3931	6.09	4.97	11.07	2.85	−3.57
J1627 − 0053	4.88	4.18	10.47	1.99	−2.78
J1630 + 4520	8.66	6.91	13.25	6.18	−5.92
J1636 + 4707	2.95	3.98	8.64	1.56	−2.43
J2238 − 0754	1.44	3.07	8.09	0.95	−1.69
J2300 + 0022	8.06	4.54	12.54	2.68	−3.30
J2303 + 1422	5.78	4.35	11.81	2.36	−3.05
J2321 − 0939	1.31	2.47	9.31	0.72	−1.32
J2341 + 0000	3.48	4.48	9.35	2.06	−2.94

giving rise to a generation of particle masses is the answer (see the classic works in [78–80]).

A. The action and spontaneous breaking of conformal symmetry

We start from the vacuum conformal gravity action

$$I_{\rm W} = -\alpha_g \int d^4x\sqrt{-g}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}, \qquad (1)$$

where α_g is the coupling constant and $C^{\alpha\beta\gamma\delta}$ is the Weyl tensor. This action is fully covariant with an additional symmetry of conformal invariance under transformations of the metric $g_{\mu\nu}(x) \to \Omega^2(x)g_{\mu\nu}(x)$. Conformal gravity thus possesses no fundamental scale (no intrinsic G or fundamental Λ) at all, leading to an intrinsically scale-free cosmology at sufficiently high enough temperatures [81]. Newton's constant G_N might be generated as a "macroscopic/low energy" limit (like the Fermi constant G_F generated in the electroweak theory) to be measured by a Cavendish experiment in a universe decoupled from the hot early stage.

We consider here conformal gravity exactly in this low energy limit; that is, after spontaneous violation of symmetry has actually happened, particle masses have been generated and galaxies formed as we see today. Conformal gravity does not require elusive dark matter or dark energy for interpreting (*albeit* in a different way) the astrophysical observations, as well as provides singularity and a ghost-free solution to some of the known problems plaguing standard cosmology including the cosmological constant [61,82] and the age problem [83] (that is, of course, not to say that conformal gravity has no problems of its own, though some seem to have been well answered; see, e.g., [60,84]).

Variation of $g_{\mu\nu}$ in (1) leads to the field equations [61]

$$4\alpha_g W_{\mu\nu} = T_{\mu\nu}, \qquad (2)$$

where

$$W_{\mu\nu} \equiv W_{\mu\nu}^{(2)} - \frac{1}{3}W_{\mu\nu}^{(1)}, \qquad (3)$$

$$W_{\mu\nu}^{(2)} = \frac{g_{\mu\nu}}{2}\nabla^\beta\nabla_\beta(R_\alpha^\alpha) + \nabla^\beta\nabla_\beta R_{\mu\nu} - \nabla_\beta\nabla_\nu R_\mu^\beta - \nabla_\beta\nabla_\mu R_\nu^\beta - 2R_{\mu\beta}R_\nu^\beta + \frac{g_{\mu\nu}}{2}R_{\alpha\beta}R^{\alpha\beta}, \qquad (4)$$

$$W_{\mu\nu}^{(1)} = 2g_{\mu\nu}\nabla^\beta\nabla_\beta(R_\alpha^\alpha) - 2\nabla_\mu\nabla_\nu R_\alpha^\alpha - 2R_\alpha^\alpha R_{\mu\nu} + \frac{g_{\mu\nu}}{2}R^2. \qquad (5)$$

∇_β is the covariant derivative operator and $T_{\mu\nu}$ is the conformally invariant energy momentum tensor to be supplied. Solving this intimidating system of equations, MK [62] computed the solution exterior ($T_{\mu\nu} = 0$) to a static, spherically symmetric gravitating source, which is

$$d\tau^2 = B(r)dt^2 - \frac{1}{B(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (6)$$

$$B(r) = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2, \qquad (7)$$

where β, γ, k are dimensionful integration constants. Defining the Schwarzschild mass $M = \frac{\beta(2-3\beta\gamma)}{2}$, we can rewrite the exact metric as

$$B(r)^{\rm exact} = \alpha - \frac{2M}{r} + \gamma r - kr^2, \qquad (8)$$

where $\alpha \equiv (1-6M\gamma)^{1/2}$. From the observed galaxy (treated as lens) masses M and the rotation curve fitted universal value of γ, the constant factor $6M\gamma \approx 10^{-15}$ and can easily be neglected for simplicity. Thus, for the galactic mass decomposition, we shall take the MK metric relevant on the scales of the Einstein radius of lens mass M and beyond to be[7]

$$B(r)^{\rm galactic} = 1 - \frac{2M}{r} + \gamma r - kr^2, \qquad (9)$$

which has also been used for predicting flat rotation curves in the galactic halo with universal values of γ and k in the galactic halo region [65]. The emergence of the Schwarzschild mass together with other dimensionful

[7]The reason why the $B(r)$ are separately designated as exact and galactic is that the metric (9) does not follow from (8) at the exact value $\alpha = 1$ as it would then require either M or γ or both to be exactly zero. However, one could technically say that (9) follows from (8) only in the *limit* $\alpha \to 1$. Physically, it implies that the metric (9) can apply to a distance scale, where effects of both M and γ are nonzero, with the combined effect $M\gamma$ to be negligibly small. Such a scale is naturally provided by the galactic halo radius [67], which is intermediate between the Schwarzschild radius of M and the cosmological de Sitter radius.

TABLE II. Lens mass decomposition by the algorithm Eq. (85) and the vacuole radius r_b for individual galaxies.

Galaxy	$M_*^{\rm MK}$	$M_{\rm dm}^{\rm MK}$	$M_{\rm tot}^{\rm lens}$	r_b	$f_{*{\rm Sal,BC}}^{\rm Grillo}$	$f_*^{\rm MK}$
…	$(\times 10^{11} M_\odot)$	$(\times 10^{8} M_\odot)$	$(\times 10^{11} M_\odot)$	(kpc)		
J0008 − 0004	3.43	69.12	3.50	842.98	$0.54^{+0.10}_{-0.33}$	0.98
J0029 − 0055	1.19	1.65	1.20	590.01	$0.76^{+0.35}_{-0.23}$	0.99
J0037 − 0942	2.87	28.34	2.90	791.76	$0.74^{+0.17}_{-0.28}$	0.99
J0044 + 0113	0.89	0.03	0.90	536.05	$0.55^{+0.27}_{-0.15}$	0.99
J0109 + 1500	1.29	2.51	1.3	605.96	$1.08^{+0.29}_{-0.22}$	0.99
J0157 − 0056	2.56	32.85	2.6	763.46	$1.21^{+0.20}_{-0.52}$	0.98
J0216 − 0813	4.77	126.99	4.9	943.08	$0.71^{+0.19}_{-0.28}$	0.97
J0252 + 0039	1.79	8.97	1.8	675.39	$0.52^{+0.07}_{-0.24}$	0.99
J0330 − 0020	2.47	26.32	2.5	753.54	$0.99^{+0.11}_{-0.27}$	0.99
J0405 − 0455	0.30	0.00	0.3	371.68	$0.73^{+0.43}_{-0.23}$	1.00
J0728 + 3835	1.99	10.33	2.0	699.53	$0.50^{+0.25}_{-0.08}$	0.99
J0737 + 3216	2.86	34.99	2.90	791.76	$0.77^{+0.09}_{-0.17}$	0.98
J0822 + 2652	2.38	18.59	2.4	743.36	$0.93^{+0.10}_{-0.16}$	0.99
J0903 + 4116	4.37	126.14	4.5	916.64	$0.87^{+0.09}_{-0.31}$	0.97
J0912 + 0029	3.95	51.46	4	881.35	$0.51^{+0.11}_{-0.09}$	0.98
J0935 − 0003	4.01	82.12	4.1	888.64	$0.52^{+0.09}_{-0.12}$	0.98
J0936 + 0913	1.49	3.72	1.5	635.56	$0.71^{+0.21}_{-0.17}$	0.99
J0946 + 1006	2.87	30.02	2.9	791.76	$0.51^{+0.06}_{-0.11}$	0.99
J0956 + 5100	3.64	55.33	3.7	858.74	$0.74^{+0.11}_{-0.33}$	0.98
J0959 + 4416	1.69	6.33	1.7	662.64	$0.82^{+0.19}_{-0.20}$	0.99
J0959 + 0410	0.80	0.00	0.8	515.42	$0.65^{+0.13}_{-0.15}$	1.00
J1016 + 3859	1.49	3.31	1.5	635.56	$0.63^{+0.16}_{-0.18}$	0.99
J1020 + 1122	3.35	49.56	3.4	834.88	$0.46^{+0.19}_{-0.12}$	0.98
J1023 + 4230	2.28	15.01	2.3	732.89	$0.71^{+0.09}_{-0.15}$	0.99
J1029 + 0420	0.60	.0.00	0.6	468.29	$0.81^{+0.32}_{-0.27}$	1.00
J1100 + 5329	4.57	122.47	4.7	930.03	$0.49^{+0.32}_{-0.15}$	0.97
J1106 + 5228	0.89	0.04	0.9	536.05	$1.01^{+0.38}_{-0.29}$	0.99
J1112 + 0826	4.40	99.48	4.5	916.64	$0.70^{+0.10}_{-0.10}$	0.97
J1134 + 6027	1.29	1.79	1.3	605.96	$0.65^{+0.28}_{-0.25}$	0.99
J1142 + 1001	1.69	6.08	1.7	662.64	$0.59^{+0.24}_{-0.13}$	0.99
J1143 − 0144	1.89	5.53	1.9	687.67	$0.46^{+0.09}_{-0.06}$	0.99
J1153 + 4612	1.09	0.89	1.1	573.14	$0.51^{+0.34}_{-0.07}$	0.99
J1204 + 0358	1.69	5.37	1.7	662.64	$0.43^{+0.14}_{-0.12}$	0.99
J1205 + 4910	2.48	18.99	2.5	753.54	$0.61^{+0.19}_{-0.19}$	0.99
J1213 + 6708	1.39	2.23	1.4	621.12	$0.71^{+0.21}_{-0.18}$	0.99
J1218 + 0830	1.59	3.94	1.6	649.38	$0.64^{+0.21}_{-0.14}$	0.99
J1250 + 0523	1.79	8.03	1.8	675.39	$1.04^{+0.27}_{-0.33}$	0.99
J1402 + 6321	2.87	27.71	2.9	791.76	$0.73^{+0.13}_{-0.12}$	0.99
J1403 + 0006	0.99	0.40	1	555.22	$0.84^{+0.27}_{-0.30}$	0.99
J1416 + 5136	3.63	66.60	3.7	858.74	$0.75^{+0.09}_{-0.17}$	0.98
J1420 + 6019	0.40	0.00	0.4	409.08	$1.01^{+0.43}_{-0.36}$	1.00
J1430 + 4105	5.24	153.07	5.4	974.08	$0.38^{+0.07}_{-0.13}$	0.97
J1436 − 0000	2.28	18.57	2.3	732.89	$0.77^{+0.24}_{-0.30}$	0.99
J1443 + 0304	0.60	0.00	0.6	468.29	$1.00^{+0.32}_{-0.44}$	1.00
J1451 − 0239	0.80	0.00	0.8	515.42	$0.97^{+0.22}_{-0.48}$	1.00
J1525 + 3327	4.66	131.70	4.8	936.57	$0.68^{+0.09}_{-0.21}$	0.97
J1531 − 0105	2.68	21.34	2.7	773.13	$0.70^{+0.15}_{-0.14}$	0.99
J1538 + 5817	0.89	0.05	0.9	536.05	$0.84^{+0.08}_{-0.19}$	0.99

(Table continued)

TABLE II. *(Continued)*

Galaxy	$M_*^{\rm MK}$	$M_{\rm dm}^{\rm MK}$	$M_{\rm tot}^{\rm lens}$	r_b	$f_{*{\rm Sal,BC}}^{\rm Grillo}$	$f_*^{\rm MK}$
J1621 + 3931	2.87	31.39	2.9	791.76	$0.75_{-0.25}^{+0.12}$	0.98
J1627 − 0053	2.28	15.03	2.3	732.89	$0.61_{-0.13}^{+0.12}$	0.99
J1630 + 4520	4.78	120.36	4.9	943.03	$0.69_{-0.10}^{+0.07}$	0.97
J1636 + 4707	1.79	7.80	1.8	675.39	$0.59_{-0.12}^{+0.18}$	0.99
J2238 − 0754	1.29	1.77	1.3	605.96	$0.64_{-0.25}^{+0.25}$	0.99
J2300 + 0022	2.97	31.68	3	800.76	$0.60_{-0.11}^{+0.07}$	0.98
J2303 + 1422	2.68	20.01	2.7	773.13	$0.63_{-0.09}^{+0.13}$	0.99
J2321 − 0939	1.19	0.83	1.2	590.01	$0.90_{-0.18}^{+0.26}$	0.99
J2341 + 0000	2.18	13.22	2.2	722.11	$0.73_{-0.28}^{+0.13}$	0.99

constants in the metric (8) already indicates the first instance of local symmetry violation in the vacuum Weyl gravity at the solution level. As a second instance of symmetry violation, we consider conformal cosmology.

B. Conformal cosmology

Consider the action of the conformally coupled matter [81]

$$I_{\rm M} = -\hbar \int d^4x\sqrt{-g}\Big[\underbrace{\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{12}\phi^2 R + \lambda\phi^4}_{scalar} + \underbrace{i\bar{\psi}\gamma^\mu(x)[\partial_\mu + \Gamma_\mu(x)]\psi}_{fermion} - \underbrace{g\phi\bar{\psi}\psi}_{interaction}\Big], \tag{10}$$

where $\Gamma_\mu(x)$ is the fermion spin connection, λ and g are the dimensionless coupling constants, $\phi(x)$ is the symmetry breaking scalar field, and ψ is a fermionic field. The matter energy momentum tensor following from the action (10) is

$$T^{\mu\nu} = \hbar\Big[i\bar{\psi}\gamma^\mu(\partial_\mu + \Gamma_\mu)\psi + \frac{2}{3}\nabla^\mu\phi\nabla^\nu\phi - \frac{g^{\mu\nu}}{6}\nabla^\alpha\phi\nabla_\alpha\phi - \frac{\phi}{3}\nabla^\mu\nabla^\nu\phi + \frac{g^{\mu\nu}\phi\nabla^\alpha\nabla_\alpha\phi}{3} - \frac{\phi^2}{6}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) - g^{\mu\nu}\lambda\phi^4\Big]. \tag{11}$$

Defining the "density" ρ of a perfect fluid (with u^μ being the timelike four-velocity satisfying $u^\mu u_\mu = 1$)

$$\rho u^\mu u^\nu = i\hbar\bar{\psi}\gamma^\mu(\partial^\nu + \Gamma^\nu)\psi + \frac{\hbar}{2}\nabla^\mu\phi\nabla^\nu\phi, \tag{12}$$

and isotropic "pressure" p

$$p u^\mu u^\nu = -\frac{\hbar}{3}\phi\nabla^\mu\nabla^\nu\phi + \frac{\hbar}{6}\nabla^\mu\phi\nabla^\nu\phi, \tag{13}$$

the energy momentum tensor may be rewritten in a more elegant form as

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + p g_{\mu\nu} - \frac{\phi^2}{6}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - g_{\mu\nu}\lambda\phi^4. \tag{14}$$

This provides for the right hand side of Eq. (2). In an isotropic and homogeneous universe, the left hand side of Eq. (2) is identically zero (empty universe), thus leading to $T_{\mu\nu} = 0$. When the scalar field $\phi(x)$ in $I_{\rm M}$ obtains a nonzero mass (which we are free to rotate to some "space-time constant" ϕ_0 due to conformal freedom), we get

$$\frac{\phi_0^2}{6}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = (p+\rho)u_\mu u_\nu + p g_{\mu\nu} - g_{\mu\nu}\lambda\phi_0^4. \tag{15}$$

Thus conformal cosmology looks like the standard cosmology with a "perfect fluid" source and a nonzero cosmological constant $\Lambda = \lambda\phi_0^4$ with the important exception that Newton's constant G_N has been replaced by an "effective" constant of the form

$$G_{\rm eff} = -\frac{3}{4\pi\phi_0^2}, \tag{16}$$

just as has been advocated by 't Hooft [74]. This is not the low energy Newton's constant G_N that Cavendish measured, but instead a term which we identify to be the negative gravitational constant $G_{\rm eff}$ providing *repulsion* at cosmological distances. Thus, in the isotropic and homogeneous case, we end up breaking conformal symmetry, again recovering standard cosmology with a nonzero cosmological constant Λ. Local gravity is fixed by small, local variations in the background scalar field $\phi(x)$, with such variations being completely decoupled from the homogeneous, constant, cosmological background field ϕ_0 itself. It is the distinction between inhomogeneity on the local scale and homogeneity on the global scale that provides the demarcation between local and global gravity, respectively. Hence conformal gravity is attractive on local galactic scales, while it is repulsive on cosmological scales, a fact that has been explicitly demonstrated very recently by Phillips [85].

Finally, we wish to point out that the MK metric (8) or (9) differs from the SdS black hole by an important linear term γr, which is a specific contribution from the fourth order vacuum conformal gravity. Nonetheless, the metric need not correspond exclusively to vacuum conformal gravity. An alternative theory of which the MK metric may again be a solution could be a similar fourth order theory. The best candidates are the $f(R)$ gravity theories that have been very widely discussed in the literature (see the review [86], [39,87–94]), notably in connection with modeling dark matter and dark energy as curvature effects [64], instability, and antievaporation of black holes [95,96]. For a full account of these and other effects, we refer the reader to the excellent treatise [97]. We shall now show that the MK metrics (8) and (9) could indeed be viewed as a solution of $f(R)$ gravity coupled to nonlinear electrodynamics (NED), so that the results of the present paper are actually more universal than thought heretofore, these now being valid in a wider class of $f(R)$ theories as well.

C. MK metric as $f(R)$ gravity solution

The $f(R)$ gravity is more general than GR, the latter recovered only when $f(R) = R$. To reach our goal stated above, we shall follow an algorithm developed by Rodrigues *et al.* [98]. A similar algorithm was developed much earlier by Capozziello *et al.* [99]. They have recently obtained solutions of $f(R)$ gravity coupled to NED that may be viewed as generalizations of known solutions within the GR theory. Here we are considering a known solution of conformal gravity instead of GR. The action for $f(R)$ gravity with sources is given by [99]

$$I_{f(R)} = \int d^4x\sqrt{-g}[f(R) + 2\kappa^2\mathcal{L}_m], \qquad (17)$$

where $\kappa^2 = 8\pi G/c^4$ and $\mathcal{L}_m$ is the matter Lagrangian. Varying the action with respect to the metric yields the field equations

$$f_R R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu f + (\delta^\mu_\nu\Box - g^{\mu\beta}\nabla_\beta\nabla_\nu)f_R = \kappa^2\Theta^\mu_\nu, \qquad (18)$$

where $f_R = df(R)/dR$, $\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the D'Alembertian and Θ^μ_ν is the source stress tensor. With $\mathcal{L}_m = \mathcal{L}_{\rm NED}(F)$, where $F = \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor [98], it follows that $\Theta^\mu_\nu = \delta^\mu_\nu\mathcal{L}_{\rm NED} - \frac{\partial\mathcal{L}_{\rm NED}(F)}{\partial F}F^{\mu\alpha}F_{\nu\alpha}$. Varying A_μ in (17), one has $\nabla_\mu[F^{\mu\nu}\mathcal{L}_{\rm F}] = 0$, where the two Lagrangian densities are related by $\mathcal{L}_{\rm F} = \frac{\partial\mathcal{L}_{\rm NED}(F)}{\partial F} = \frac{\partial\mathcal{L}_{\rm NED}(F)}{\partial r}(\frac{\partial F}{\partial r})^{-1}$. Because of radial symmetry of the metrics, the only nonzero component of the Faraday tensor is $F^{10}(r) = \frac{q}{r^2}\mathcal{L}_{\rm F}^{-1}$ and $F = -\frac{1}{2}(F^{10})^2$. The field equations (18) can be rewritten in terms of the effective stress tensor of GR ("curvature fluid" [64]),

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = f_R^{-1}\Big[\kappa^2\Theta_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(R - Rf_R) - (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R\Big] \equiv \kappa^2 T^{\rm eff}_{\mu\nu}. \qquad (19)$$

Rodrigues *et al.* [98] start from a radial mass function $M(r)$ which, for the metric (9), is obtained by rewriting $B(r) = 1 - \frac{2M(r)}{r}$,

$$M(r) = M\left(1 - \frac{\gamma r^2}{2M} + \frac{kr^3}{2M}\right), \qquad (20)$$

where M is the constant Schwarzschild mass from the metric (9). The Ricci scalar $R(r)$ is independent of M,

$$R = \frac{6}{r}(\gamma - 2kr) \Rightarrow r(R) = \frac{6\gamma}{R + 12k}. \qquad (21)$$

We give below only the final results for the galactic scale metric (9). Following the algorithm in [98], it follows that the class of $f(R)$ gravity theories

$$f(R)^{\rm galactic} = c_0R + c_1\int r(R)dR = c_0R + 6c_1\gamma\ln(R + 12k), \qquad (22)$$

where c_0 and c_1 are arbitrary constants, would also produce the MK solution (9). The NED sector gives

$$F^{10} = -\frac{2c_0\gamma r + c_1\{r(2 + 3\gamma r) - 6M\}}{2\kappa^2 q}, \qquad (23)$$

$$\mathcal{L}_{\rm F} = -\frac{2\kappa^2q^2}{r^2[2c_0\gamma r + c_1\{r(2 + 3\gamma r) - 6M\}]},$$
$$\mathcal{L}_{\rm NED} = -\frac{c_0\gamma + c_1(1 - 3\gamma r\ln r)}{\kappa^2 r}. \qquad (24)$$

We see that $f(R)^{\rm galactic}$ is a nice logarithmic function that can be analyzed by choosing values of c_0 and c_1 and/or solution constants γ and k. For instance, $\gamma = 0$ gives the SdS case with $f(R) = R = -12k$. For $k = 0$, one obtains $f(R) = c_0R + 6c_1\gamma\ln R$, which vividly shows the role of $f(R)$ beyond GR leading to the emergence of an extra linear term γr in the metric (9), that has both Cauchy and event horizons (see Sec. III), as a deviation from the SdS black hole. The influence of γr on the antievaporation phenomenon will be a challenging and interesting future task. Exactly the same algorithm applies also to the exact form (8) yielding

$$f(R)^{\rm exact} = c_0 R + 2c_1\Bigg[\sqrt{9\gamma^2 + 2(R+12k)(\alpha-1)} + 3\gamma \ln\left(\sqrt{9\gamma^2 + 2(R+12k)(\alpha-1)} - 3\gamma\right)\Bigg]. \tag{25}$$

The energy conditions, except the strong energy condition, are satisfied by the $T_{\mu\nu}^{\rm eff}$ for both the metrics (8) and (9), now posed as solutions of $f(R)$ gravity. The Dolgov-Kawasaki [100] stability condition $\frac{d^2 f(R)}{dR^2} \geq 0$ is satisfied only when $c_1 \leq 0$ in both (22) and (25). It is interesting to note that logarithmic $f(R) = \ln(\lambda R)$, investigated in [95], follow as special cases Eq. (22) under $c_0 = 0$, $k = 0$. A fuller analysis for both the $f(R)$ gravities equivalent to conformal solutions will be given elsewhere as they will take us outside the scope of the present paper.

III. CRITICAL REAPPRAISALS

We shall here critically reexamine three issues that relate to light deflection in the MK metric of Weyl gravity. Some years ago, Edery and Paranjape [72] calculated the light deflection in the asymptotically nonflat MK metric (9) to be $\Delta\phi = \frac{4M}{R} - \gamma R$, where M is the luminous mass of a galaxy, γ is a constant parameter, and R is identified with the closest approach distance r_0. However, Edery and Paranjape had already recognized that the negative sign in $-\gamma R$ was discrepant requiring further investigation because, only for $\gamma < 0$, the contribution becomes positive imitating the effect of attractive dark matter, but then the problem is that the rotation curve fit requires $\gamma > 0$, an exactly opposite sign.

We argue here that the discrepant sign is a result of an illegitimate range of integration. To derive the above deflection, Edery and Paranjape considered integration over the radial coordinate r from R to ∞ arguing that the incoming light followed a "straight line" path at infinity associated with the metric function $B_\infty(r) = 1 + \gamma r - kr^2 > 0$, $k > 0$ [neglecting M/r in metric (9)]. It is this reduced metric that, under transformation $r \to \rho$ [see Eq. (28) below], is conformal to cosmological Robertson-Walker metric with negative space curvature ($K = -k - \frac{\gamma^2}{4}$) [62]. They further argued that the straight line path was justified by the limit $\frac{d\varphi}{dr} \to 0$ as $r \to \infty$. We wish to point out that the limit $r \to \infty$ does not make sense because there is a finite horizon radius in the metric, which limits the motion of light *inside* this radius. Outside the horizon, at $r \to \infty$, where the light ray has been assumed to pass, the metric function $B_\infty(r)$ changes sign leading to violation of the metric signature, which forbids a meaningful integration from R to ∞.

To be more specific, the metric function $B_\infty(r) = 0$ gives the horizon radius

$$r = r_{\rm hor} = \frac{\gamma + \sqrt{4k + \gamma^2}}{2k}. \tag{26}$$

For the special case $\gamma = 0$ and $k = \frac{\Lambda}{3} > 0$, one simply retrieves the de Sitter horizon radius $r_{\rm hor}^{\rm dS} = \sqrt{\frac{3}{\Lambda}}$. Outside the horizon $r > r_{\rm hor}$, say at $r = 2r_{\rm hor}$, $B_\infty(2r_{\rm hor})$ becomes generically negative no matter what the sign of γ is.[8] This can be seen from

$$B_\infty(2r_{\rm hor}) = -\frac{3k + \gamma(\gamma + \sqrt{4k + \gamma^2})}{k} < 0. \tag{27}$$

On the other hand, inside the horizon $r < r_{\rm hor}$, say at $r = r_{\rm hor}/2$, $B_\infty(r_{\rm hor}/2) > 0$, which is consistent with the required signature protection, thereby limiting the light motion to only within $r < r_{\rm hor}$. If one changes from static r to comoving radial coordinate ρ by [62]

$$\rho(r) = \frac{4r}{2(1 + \gamma r - kr^2)^{1/2} + 2 + \gamma r}, \tag{28}$$

one obtains

$$\rho_{\rm hor}(r_{\rm hor}) = \frac{4(\gamma + \sqrt{4k + \gamma^2})}{4k + \gamma(\gamma + \sqrt{4k + \gamma^2})}, \tag{29}$$

which is just the redefined horizon radius. However, it can easily be verified that $\rho(r = 2r_{\rm hor})$ is imaginary. Also, as $r \to \infty$, $\rho \to \frac{4}{2\sqrt{-k} + \gamma}$, which too is imaginary for $k > 0$ and so is physically meaningless.

Exactly the same arguments hold for the full MK solution $B(r) = 1 - \frac{2M}{r} + \gamma r - kr^2$ as well because there now appear two horizons for $k > 0$ at radii given by (for details, see Ref. [101])

$$r_{\rm hor}^1 = 2\sqrt{\frac{3k + \gamma^2}{9k^2}} \cos\left(\frac{\theta + 4\pi}{3}\right) + \frac{\gamma}{3k}, \tag{30}$$

$$r_{\rm hor}^2 = 2\sqrt{\frac{3k + \gamma^2}{9k^2}} \cos\left(\frac{\theta}{3}\right) + \frac{\gamma}{3k}, \tag{31}$$

$$r_{\rm hor}^2 > r_{\rm hor}^1, \qquad \theta = \theta(M, \gamma, k), \tag{32}$$

where $r_{\rm hor}^1$ and $r_{\rm hor}^2$ are the radii of the inner Cauchy horizon and the outer event horizon, respectively, and away outward from the event horizon, the metric function $B(r)$ becomes negative. Thus, because of the presence of horizons, the usual integration from R to ∞ does not make sense either in the mutilated version

[8] We understand that straight line on a curved geometry is distinguished here from the Euclidean straight line on flat geometry. However, the issue is not about this distinction but about the validity of the limit $r \to \infty$. The situation here is exactly opposite to what one finds, e.g., in the Schwarzschild geometry, where $B(r) = 1 - 2M/r > 0$ for $r > r_{\rm hor} = 2M$.

($M = 0$) or in the full MK solution ($M \neq 0$). Hence, the resultant deflection $\Delta\phi = \frac{4M}{R} - \gamma R$ cannot be accepted as valid. To find the valid light deflection, it is necessary to find a new method that does not rely on such integration. That very new method has been developed by Rindler and Ishak [70], which is based on the geometric invariant angle and is most suited to the asymptotically nonflat space-times. We shall soon see that the bending expression, apart from the term $+\gamma R$, yields also other terms in which γ couples with M, k, and R in different combinations.

The second issue relates to two previous papers [102,103] that unfortunately overlooked a practical condition, viz., that the closest approach distance R must be far greater than the Schwarzschild radius of the galaxy, i.e., $R \gg 2M$. This condition technically induces a certain function $|A|$ to assume a *positive* value, which is crucial for obtaining the known expression for bending. For instance, with $r = 1/u$, Rindler and Ishak [70] defined the function $A(r,\varphi) \equiv \frac{dr}{d\varphi} = (-r^2)\frac{du}{d\varphi}$ but used only the positive numerical value $|A| = r^2\frac{du}{d\varphi} = \frac{R^3}{4M^2}$ in their prescription for bending that led to the correct Schwarzschild deflection. The importance of their positivity prescription is that without it even the known Schwarzschild deflection would not follow. Since this positivity was not accommodated in [102,103], an erroneous two-way negative contribution $-\gamma R$ appeared there, supporting the existing result of Edery and Paranjape [72]. We shall respect this prescription in the present paper, which will show that the two-way contribution in Weyl gravity actually is $+\gamma R$, thus enhancing the Schwarzschild bending and imitating the effects of attractive dark matter, as should be the case.

The third and final issue concerns the appropriate setup needed for calculating the light deflection. Note that the Rindler-Ishak method originally proposed in [70] was based on a source and an observer located in a static SdS background. On the other hand, given the environment of Mannheim-O'Brien [65] cosmological potentials, it should be more appropriate to derive the corresponding light bending equation in a cosmological setup, that is, in the Friedmann-Lemaître-Robertson-Walker (FLRW) background. To do that, Ishak *et al.* [69] extended the original method in which the galaxy (lens) is now placed at the center of a SdS vacuole exactly embedded into the FLRW space-time using the Einstein-Strauss [104] prescription and appropriate junction conditions. We shall follow this extended method in this paper.

IV. LIGHT TRAJECTORY IN THE MK SPACE-TIME

We restate the asymptotically nonflat spherically symmetric galactic MK solution as used in [65,72],

$$d\tau^2 = B(r)dt^2 - \frac{1}{B(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$B(r) = 1 - \frac{2M}{r} + \gamma r - kr^2, \tag{33}$$

where M is the luminous central mass, and k and γ are constants. Denoting $u = 1/r$, we derive the path equation for a test particle of mass m_0 on the equatorial plane $\theta = \pi/2$ as follows:

$$\frac{d^2u}{d\varphi^2} + u = 3Mu^2 - \frac{\gamma}{2} + \frac{M}{h^2} + \frac{1}{2h^2u^2}\left(\gamma - \frac{2k}{u}\right), \tag{34}$$

where $h = \frac{J}{m_0}$, the conserved angular momentum per unit test mass. For photon, $m_0 = 0 \Rightarrow h \to \infty$, and one ends up with the null geodesic equation,

$$\frac{d^2u}{d\varphi^2} + u = 3Mu^2 - \frac{\gamma}{2}. \tag{35}$$

We shall perturbatively solve this equation. To zeroth order, Eq. (35) gives

$$\frac{d^2u_0}{d\varphi^2} + u_0 = -\frac{\gamma}{2}, \tag{36}$$

and its exact solution is

$$u_0 = \frac{\cos\varphi}{R} - \frac{\gamma}{2}, \tag{37}$$

where R is a constant related to the distance of closest approach r_0 to the origin. For transparency, we shall consider only first order perturbation in M. Thus, following the usual method of small perturbations, we want to derive the solution as

$$u = u_0 + u_1, \tag{38}$$

where u_1 satisfies

$$\frac{d^2u_1}{d\varphi^2} + u_1 = 3Mu_0^2. \tag{39}$$

The perturbative expansion holds only for small u or large r. Thus we are considering galactic parameters M, R and solution parameter γ such that the nondimensional quantities $\frac{2M}{R} \ll 1$ and $\gamma R \ll 1$. The exact solution of Eq. (39) is

$$u_1 = \frac{M}{4R^2}[6 + 3R^2\gamma^2 - 6R\gamma\cos\varphi - 2\cos 2\varphi - 6R\gamma\varphi\sin\varphi]. \tag{40}$$

When $\gamma = 0$, it may be verified that one recovers the equation for the light trajectory in first order in the

Schwarzschild metric. Formally changing $\varphi \to \frac{\pi}{2} - \varphi$, the final solution for the light trajectory up to first order in M can be written as

$$u \equiv \frac{1}{r} = \frac{\sin\varphi}{R} - \frac{\gamma}{2} + \frac{M}{4R^2}[6 + 3R^2\gamma^2 - 3R\gamma(\pi - 2\varphi)\cos\varphi + 2\cos 2\varphi - 6R\gamma\sin\varphi]. \quad (41)$$

The closest approach distance r_0 is obtained from Eq. (41) by putting $\varphi = \pi/2$ and is given by

$$\frac{1}{r_0} = \frac{1}{R} + \left[\frac{M(4 - 6R\gamma + 3R^2\gamma^2)}{4R^2} - \frac{\gamma}{2}\right] \simeq \frac{1}{R}, \quad (42)$$

because for typical observed galactic values of M, R, and γ, which we shall soon see, it follows that $\frac{1}{R} \approx 10^{-22}$ cm^{-1}, while the piece in the square bracket $\approx 10^{-30}$ cm^{-1}; hence it can be ignored.[9] Thus, R can be identified with the closest approach distance

$$R \simeq r_0. \quad (43)$$

Equation (41) is the desired equation to be used in the sequel.

V. LIGHT DEFLECTION: ISHAK *et al.* VACUOLE METHOD

The original method proposed by Rindler and Ishak [70] did not require the concept of a vacuole because the source and observer were assumed to be located in a static SdS background. The work nonetheless contained the essential ingredient for its vacuole extension: It combined the standard perturbative solution with an invariant geometric definition of the bending angle that took into account the explicit effect of the metric $B(r)$ that contains k, which is formally similar to but not numerically exactly the same as Λ.

The invariant geometric formula for the cosine of the angle ψ between two coordinate directions d and δ is given by

$$\cos\psi = \frac{g_{ij}d^i\delta^j}{(g_{ij}d^i d^j)^{1/2}(g_{ij}\delta^i\delta^j)^{1/2}}. \quad (44)$$

Differentiating u with respect to φ, denoting $\frac{dr}{d\varphi} = (-r^2)\frac{du}{d\varphi} \equiv A(r,\varphi)$, and imposing the Rindler-Ishak positivity condition on $A(r,\varphi)$ [70], we get

$$|A(r,\varphi)| = r^2\frac{du}{d\varphi}. \quad (45)$$

[9]The symbol "≈" means "of the order of."

For the light path Eq. (41), the function $|A|$ reads

$$|A(r,\varphi)| = (r^2)\left[\frac{3MR\gamma(\varphi - 2\pi)\sin\varphi + 4(R - 2M\sin\varphi)\cos\varphi}{4R^2}\right] > 0. \quad (46)$$

The method of Ishak *et al.* [69] treats the light bending in a cosmological setting that requires the concept of a vacuole, which is not present in nature by itself but is devised here only as an artifact of the investigative procedure. The vacuole is assumed to be a large hypothetical sphere that houses the lens galaxy at its center and is exactly embedded into the FLRW space-time. It is further assumed that all the light bending occurs inside the vacuole and that once the light transitions out of the vacuole and into FLRW space-time, all bending stops. The cutoff point, where the transition occurs, is tailored to each individual lens, namely the edge of the vacuole defined by its radius r_b. The vacuole concept as such is not used except for this one purpose, viz., to obtain a realistic order-of-magnitude estimate of the range of influence of the lens. Once we are able to obtain the deflection angle, we can dismiss the vacuole as redundant.

Assuming a small light entry angle φ_b at the vacuole boundary radius $r = r_b$ such that $\sin\varphi_b \simeq \varphi_b$, and $\cos\varphi_b \simeq 1$, Eq. (41) gives

$$\frac{1}{r_b} = \frac{\varphi_b}{R} + \frac{2M}{R^2} + \left[M\left(\frac{3\gamma^2}{4} - \frac{3\pi\gamma}{4R}\right) - \frac{\gamma}{2}\right], \quad (47)$$

or, equivalently,

$$\varphi_b = \left(\frac{R}{r_b} - \frac{2M}{R}\right) + \left[\frac{3\pi M\gamma}{4} + \frac{\gamma R}{2} - \frac{3MR\gamma^2}{4}\right]. \quad (48)$$

Note that we have only one equation (47) connecting two unknowns φ_b and r_b. Hence we need to specify any one of them from independent considerations. Ishak *et al.* [69] employed the Einstein-Strauss prescription [104,105] to determine the boundary r_b assuming that the SdS vacuole has been matched to an expanding FLRW universe via the Sen-Lanczos-Darmois-Israel junction conditions [106–109]. In general, the vacuole radius r_b would also change due to cosmic expansion, but Ishak *et al.* [69] considered r_b at that particular instant t_0 of the cosmic epoch, when the light ray just happens to pass the point of closest approach to the lens. The Einstein-Strauss prescription adapted to the MK solution reads

$$r_{b\,\mathrm{in\,MK}} = a(t)r_{b\,\mathrm{in\,FLRW}},$$
$$M_{\mathrm{MK}} = \frac{4\pi}{3}r^3_{b\,\mathrm{in\,MK}} \times \rho_{\mathrm{in\,FLRW}}, \quad (49)$$

where M_{MK} is just the same luminous lens mass M appearing in the metric (9).

We shall take the WMAP estimate of the observed density of the universe that is the critical density ρ_c implying that the universe is spatially flat. Thus, for our computation, we take $\rho_{\mathrm{in\,FLRW}} = \rho_c = \frac{3H_0^2}{8\pi} = 9.47 \times 10^{-30}\ \mathrm{gm\,cm^{-3}} = 7.03 \times 10^{-58}\ \mathrm{cm^{-2}}$ [80]. Normalizing the scale factor to $a(t_0) = 1$ and dropping suffixes, the above prescription translates to

$$r_b = \left(\frac{3M}{4\pi\rho_c}\right)^{1/3}, \tag{50}$$

where the luminous mass M is often expressed in units of the Sun's mass $M_\odot = 1.989 \times 10^{33}\ \mathrm{gm} = 1.48 \times 10^5\ \mathrm{cm}$. Equation (25) provides a vacuole boundary radius r_b, where the space-time transitions from a MK space-time to an FLRW background. Evidently, by the prescription (50), r_b depends explicitly only on the observable cosmological Hubble parameter H_0 and on the galactic parameter M, none of which depends on the constant γ. So, for consistency, the γ terms in Eq. (47) [or, in Eq. (48)] should be discarded. To verify what it entails, let us once again note that, for typical galactic values from rotation curve fit [65]

$$M \approx 10^{16}\ \mathrm{cm}, \quad R \approx 10^{22}\ \mathrm{cm}, \quad \gamma \approx 10^{-30}\ \mathrm{cm}, \tag{51}$$

and for $r_b \approx 10^{24}$ cm coming from the prescription (50), the first piece in Eq. (48) turns out to be $(\frac{R}{r_b} - \frac{2M}{R}) \approx 10^{-2}$, while the last piece in the square bracket is $\approx 10^{-8}$ and hence can easily be ignored in comparison. Thus, to the leading order, $\varphi_b \approx 10^{-2}$, which is consistent with the small angle approximation. Therefore, without any loss of rigor in what follows, we can take

$$\varphi_b \simeq \frac{R}{r_b} - \frac{2M}{R}, \tag{52}$$

or, equivalently,

$$r_b \simeq \left(\frac{\varphi_b}{R} + \frac{2M}{R^2}\right)^{-1}. \tag{53}$$

Returning to Eq. (46), we get

$$\begin{aligned} |A_b| &\equiv |A(r_b, \varphi_b)| \\ &= (r_b^2) \times \left[\frac{3MR\gamma(\varphi_b - 2\pi)\varphi_b + 4(R - 2M\varphi_b)}{4R^2}\right] \\ &= \frac{r_b^2}{R}\left[1 - \frac{2M\varphi_b}{R} + \frac{3M\pi\gamma\varphi_b}{4} - \frac{3M\gamma\varphi_b^2}{2}\right] > 0. \end{aligned} \tag{54}$$

This positivity is possible as the observed galactic data M, R and the small values of φ_b from Eq. (52) render the quantity in the square bracket positive.

The formula of Ishak *et al.* [69] for the bending angle ψ is

$$\tan\psi = \frac{r_b\sqrt{B(r_b)}}{|A_b|}, \tag{55}$$

where

$$\sqrt{B(r_b)} \simeq 1 - \frac{M}{r_b} + \frac{1}{2}\gamma r_b - \frac{1}{2}kr_b^2, \tag{56}$$

since, for galactic values of M, rotation curve-fitted values of γ, k and the values of r_b, the last three terms add to $\approx 10^{-6}$, which is too small compared to unity justifying that the higher power expansion terms in $\sqrt{B(r_b)}$ be ignored. The main thing to note is that k appears in the bending only through $\sqrt{B(r_b)}$.

For small ψ, $\tan\psi \simeq \psi$, and for small entry angle, $\tan\varphi_b \simeq \varphi_b$, so that the one-way deflection ϵ for nonzero φ_b is, by definition [70],

$$\epsilon = \tan(\psi - \varphi_b) \simeq \psi - \varphi_b, \tag{57}$$

where

$$\psi = \frac{2R^2[2M + r_b(kr_b^2 - \gamma r_b - 2)]}{r_b^2[8M\varphi_b + R\{3M\gamma\varphi_b(2\varphi_b - \pi) - 4\}]}. \tag{58}$$

Equation (57) is the exact one-way expression but is rather unilluminating, so we shall expand it in the first powers to see what it yields. Expanding Eq. (57) in the first power of γ, we get, with $\psi \equiv \epsilon_0 + \epsilon_1$,

$$\begin{aligned} \epsilon &= (\epsilon_0 - \varphi_b) + \epsilon_1 \\ &= \left[\frac{2R^2(2M - 2Rr_b + kRr_b^3)}{r_b^2(8M\varphi_b - 4R)} - \varphi_b\right] + \epsilon_1, \end{aligned} \tag{59}$$

$$\epsilon_1 \equiv \frac{2\gamma R^2}{r_b^2}\left[\frac{r_b^2}{4R - 8M\varphi_b} + \frac{3MR(2M - 2r_b + kRr_b^3)(\pi - 2\varphi_b)\varphi_b}{16(R - 2M\varphi_b)^2}\right]. \tag{60}$$

Expanding ϵ_1 in the first power of M, we get

$$\begin{aligned} \epsilon_1 \equiv \frac{\gamma R}{2} + \frac{M}{8r_b}[&8\gamma r_b\varphi_b + 3\pi kR\gamma r_b^2\varphi_b \\ &+ 12R\gamma\varphi_b^2 - 6kR\gamma r_b^2\varphi_b^2 - 6\pi R\gamma\varphi_b] + O(M^2) \end{aligned} \tag{61}$$

$$\simeq \frac{\gamma R}{2}, \tag{62}$$

since, for typical galactic values of M, R, γ, k, r_b, and φ_b, it follows that $\frac{\gamma R}{2} \approx 10^{-7}$, while the second term in Eq. (61) leads to a value $\approx 10^{-16}$ and so is ignored here by comparison. Expressing r_b in terms of φ_b using Eq. (52) in the square bracket of the right hand side of Eq. (59), we get

$$\epsilon_0 - \varphi_b \equiv \frac{2(\frac{2M}{R}+\varphi_b)^2[2M - \frac{2R^2}{2M+R\varphi_b} + \frac{kR^6}{(2M+R\varphi_b)^3}]}{8M\varphi_b - 4R} - \varphi_b. \tag{63}$$

Expanding it in the first power of M, whence φ_b on the right hand side cancels out, we find

$$\epsilon_0 - \varphi_b = \frac{2M}{R} - \frac{kR^2}{2\varphi_b} + M\left(\frac{\varphi_b^2}{R} + \frac{kR}{\varphi_b^2} - kR\right) + O(M^2). \tag{64}$$

For typical galactic values mentioned above and for $k \approx 10^{-54}\ \mathrm{cm}^{-2}$, the relative strength of the terms in Eq. (64) are as follows:

$$\frac{2M}{R} \approx 10^{-6}, \qquad -\frac{kR^2}{2\varphi_b} \approx -10^{-7},$$
$$M\left(\frac{\varphi_b^2}{R} + \frac{kR}{\varphi_b^2} - kR\right) \approx 10^{-11}. \tag{65}$$

Hence, we can ignore the third term on the right hand side of Eq. (64). Collecting the leading order terms from Eqs. (61) and (65), the result is

$$\delta = 2\epsilon = 2[(\epsilon_0 - \varphi_b) + \epsilon_1] = \frac{4M}{R} + \gamma R - \frac{kR^2}{\varphi_b}. \tag{66}$$

In the limit, $\gamma = 0$, one recovers the known bending expression for the SdS metric [69]. The last cosmological term is expectedly repulsive and looks familiar if one formally chooses $k = \Lambda/3$ so that, using Eq. (52), one finds

$$-\frac{kR^2}{\varphi_b} \simeq -\frac{\Lambda R r_b}{3}, \tag{67}$$

which is exactly the contribution obtained by Ishak *et al.* [69]. Further notice that the term $+\gamma R$ in Eq. (66) enhances the Schwarzschild bending $\frac{4M}{R}$, contrary to previous results, which is what we wanted to prove.

Expressing R in terms of the impact parameter b, and using Eq. (43), we get[10]

$$\frac{1}{R} \simeq \frac{1}{r_0} \simeq \frac{1}{b} + \frac{\Lambda b}{6}, \tag{68}$$

so we obtain, to leading order, from Eq. (66)

$$\delta = \frac{4M}{b} + \frac{2M\Lambda b}{3} + \gamma b - \frac{\Lambda b r_b}{3} \tag{69}$$

$$\equiv t_{\mathrm{Sch}} + t_{\mathrm{Sereno}} + t_\gamma + t_\Lambda. \tag{70}$$

The term $\frac{2M\Lambda b}{3}$ has been obtained by Sereno [111], and he called it a local coupling term. There is also another term proportional to M^2 adding to δ, viz., $\frac{15M^2\gamma}{b}$ derived previously [112] but not shown here. However, for galactic lenses their values are too minute to be of any interest, e.g., $t_{\mathrm{Sereno}} \approx 10^{-15}$, as well as the still smaller other term, $\frac{15M^2\gamma}{b} \approx 10^{-20}$, so we ignore them here. But the remaining terms are nearly of comparable magnitude, so we preserve them. Restoring k, we therefore have

$$\delta = \frac{4M}{b} + \gamma b - kbr_b \equiv t_{\mathrm{Sch}} + t_\gamma + t_k. \tag{71}$$

This is our final expression for light bending in the MK solution to be used in our analysis of the galactic mass decomposition.

VI. ALGORITHM FOR MASS DECOMPOSITION

The luminous mass M in the metric (9) will hereafter be denoted by M_* for more notational clarity. Thus, the MK light deflection δ, Eq. (71), can be rewritten as

$$\delta = \frac{4M_*}{b} + \tilde{\gamma} b, \tag{72}$$

$$= t_{\mathrm{Sch}} + t_\gamma + t_k, \tag{73}$$

where

$$t_{\mathrm{Sch}} = \frac{4M_*}{b}, \qquad t_\gamma = \gamma b = (N^*\gamma^* + \gamma_0)b,$$
$$t_k = -kbr_b, \qquad M_* = N^*\beta^*, \tag{74}$$

$$\gamma = N^*\gamma^* + \gamma_0, \tag{75}$$

$$\tilde{\gamma} \equiv \gamma - kr_b, \tag{76}$$

where the solar mass is $\beta^* = 1.48 \times 10^5$ cm, and the universal constants in the MK solution are $\gamma^* = 5.42 \times 10^{-41}\ \mathrm{cm}^{-1}$, $\gamma_0 = 3.06 \times 10^{-30}\ \mathrm{cm}^{-1}$, and $k = 9.54 \times 10^{-54}\ \mathrm{cm}^{-2}$. Table I below shows the values of different contributions to δ in the case of 57 lens galaxies as well as the corresponding values of b and r_b.

On the other hand, the Schwarzschild deflection α in Einstein theory is

[10] The impact parameter b for the metric (9) is defined by [110]: $b = (\frac{1}{u_{\max}})[\frac{1}{B(u_{\max})}]^{1/2}$, where $u_{\max} = 1/r_{\min} = 1/r_0$. Hence, $b = r_0[\frac{1}{B(r_0)}]^{1/2} \simeq r_0 + M - \frac{\gamma r_0^2}{2} + \frac{\Lambda r_0^3}{6}$. Ignoring M compared to r_0 (as $M/r_0 \approx 10^{-6}$) and inverting, one obtains $\frac{1}{b} \simeq \frac{1}{r_0} + \frac{\gamma}{2} - \frac{\Lambda r_0}{6}$. Solving for $\frac{1}{r_0}$, we get $\frac{1}{r_0} = \frac{(6-3b\gamma)+\sqrt{((6-3b\gamma)^2+24b^2\Lambda}}{12b}$. Since $3b\gamma \ll 6$, we can write $\frac{1}{r_0} \simeq \frac{6+\sqrt{36+24b^2\Lambda}}{12b} = \frac{1}{b} + \frac{\Lambda b}{6} + O(b^2)$, which is just our Eq. (68). It also follows by expanding $R \simeq r_0 = (\frac{1}{b} + \frac{\Lambda b}{6})^{-1}$ in powers of b that $R = b + O(b^2)$.

$$\alpha = \frac{4M_{\rm tot}^{\rm lens}}{b}, \tag{77}$$

where $M_{\rm tot}^{\rm lens}$ is the total projected lens mass enclosed within the Einstein radius $R_{\rm Ein}$ defined by $R_{\rm Ein} = d_{\rm ol}\theta_{\rm Ein}$, which is nothing but the impact parameter b. We shall imbed the deflection expressions into the lens equation [113], which is

$$\theta d_{\rm os} = \beta d_{\rm os} + \alpha d_{\rm ls}, \tag{78}$$

where $d_{\rm os}$, $d_{\rm ls}$, $d_{\rm ol}$ are the angular diameter distances between observer-source, lens-source, and observer-lens. The Einstein angle $\theta = \theta_{\rm Ein}$ is defined by the case when the source, lens, and observer stay in a line, that is, when $\beta = 0$. Thus

$$\theta = \alpha\left(\frac{d_{\rm ls}}{d_{\rm os}}\right) = \frac{4M_{\rm tot}^{\rm lens}}{b}\left(\frac{d_{\rm ls}}{d_{\rm os}}\right) = \frac{4M_{\rm tot}^{\rm lens}}{\theta d_{\rm ol}}\left(\frac{d_{\rm ls}}{d_{\rm os}}\right) \tag{79}$$

$$\Rightarrow \theta = \theta_{\rm Ein} = \sqrt{\frac{4M_{\rm tot}^{\rm lens}}{D}}, \tag{80}$$

$$D \equiv \frac{d_{\rm ol}d_{\rm os}}{d_{\rm ls}}. \tag{81}$$

Note that $\theta_{\rm Ein}$ is caused by the total mass (luminous $M_* +$ dark) enclosed within the Einstein radius b.

The Weyl angle, for the *same* impact parameter $b = \theta d_{\rm ol}$ is

$$\theta = \delta\left(\frac{d_{\rm ls}}{d_{\rm os}}\right) = \left[\frac{4M_*}{\theta d_{\rm ol}} + \tilde{\gamma}(\theta d_{\rm ol})\right]\left(\frac{d_{\rm ls}}{d_{\rm os}}\right) \tag{82}$$

$$\Rightarrow \theta = \theta_{\rm Weyl} = \sqrt{\frac{4M_*}{D - \tilde{\gamma} d_{\rm ol}^2}}. \tag{83}$$

Note that $\theta_{\rm Weyl}$ is caused by the luminous mass M_* plus cosmology induced potentials within the same radius b (i.e., same lensing geometry, same impact parameter). Following Edery and Paranjape [72], and later works [48,49], we use the input $\theta_{\rm Ein} = \theta_{\rm Weyl}$, which yields

$$\tilde{\gamma} = \frac{d_{\rm os}}{d_{\rm ls}d_{\rm ol}}\left(1 - \frac{M_*}{M_{\rm tot}^{\rm lens}}\right). \tag{84}$$

Using Eq. (50) for r_b in Eq. (76), Eq. (84) can be explicitly written as

$$N^*\gamma^* + \gamma_0 = \frac{d_{\rm os}}{d_{\rm ls}d_{\rm ol}}\left(1 - \frac{N^*\beta^*}{M_{\rm tot}^{\rm lens}}\right) + \frac{k(\frac{3}{\pi})^{1/3}}{2^{5/3}}\left(\frac{N^*\beta^*}{\rho_{\rm c}}\right)^{1/3}, \tag{85}$$

where [114]

$$\rho_{\rm c} = \frac{3H_0^2}{8\pi} = 9.47 \times 10^{-30}\ {\rm gm\,cm^{-3}} = 7.03 \times 10^{-58}\ {\rm cm^{-2}}.$$

Equation (85) *is a cubic equation in* N^* *and is central to our mass decomposition scheme.*

Our algorithm is the following: In the above Eq. (85), the galaxy independent universal MK constants (γ_0, γ^*, k) are known [62,65], the distances $d_{\rm ls}$, $d_{\rm ol}$, $d_{\rm os}$ ($\equiv d_{\rm ol} + d_{\rm ls}$) and the total mass $M_{\rm tot}^{\rm lens}$ are provided by the observed SLACS data for each individual galaxy [73,115]. Plugging these values into Eq. (85), we first have to find the numerical value of N^* specific to each galaxy. The resultant cubic equation in N^* fortunately yields only one positive root, which then enables us to find the value of the luminous component $M_* = N^*\beta^*$. We henceforth call it $M_*^{\rm MK}$ to distinguish it from the luminous mass values obtained from other independent simulations. Subtracting $M_*^{\rm MK}$ from the observed total mass of the lens $M_{\rm tot}^{\rm lens}$, we obtain the dark matter component $M_{\rm dm}$ as well as the mass ratios $f_*^{\rm MK} = (M_*^{\rm MK}/M_{\rm tot}^{\rm lens})|_{\le R_{\rm Ein}}$.

As to the existing mass ratios in the literature, we note that simulations depending on different stellar-population models and initial mass functions (IMFs) have thrown up rather widely different values with error bars. This can be seen from the work of Grillo *et al.* [73], who fitted the lens spectral energy distributions with a three-parameter grid of Bruzual and Charlot's (indexed BC) [116] and Maraston's (indexed M) [117] composite stellar-population models, computed by adopting solar metallicity and various IMFs. They obtained the mass decomposition within the Einstein radius and found that a Salpeter IMF (indexed Sal) [118] was better suited than either a Chabrier (Cha) [119] or Kroupa (Kro) IMF [120] for describing the considered subsample of 57 lenses. It was concluded that in all the models, the observed total mass $M_{\rm tot}^{\rm lens}$ is linearly proportional to the estimated luminous mass of the lenses denoted by $M_*^{\rm Sal,BC}$, $M_*^{\rm Sal,M}$, $M_*^{\rm Cha,BC}$, and $M_*^{\rm Kro,M}$. However, the dark matter component was found to be considerably higher for the two models [(Cha,BC), (Kro,M)] than for the models [(Sal,BC), (Sal,M)].

Our results on mass decomposition are shown in Table II below. We find that the mass ratios generated from the better suited model (Sal,BC), viz., $f_*^{\text{Grillo}\,et\,al.} = (M_*^{\rm Sal,BC}/M_{\rm tot}^{\rm lens})|_{\le R_{\rm Ein}}$ [73], come closer to our computed ratios than those from other models. For the 57 galaxies, the values of $f_*^{\text{Grillo}\,et\,al.}$ range from 0.43 to 1.21, which can be compared with the ratios $f_*^{\rm MK}$ that are seen to take on values very near unity ($f_*^{\rm MK} \sim 0.98$ on the average) for all these galaxies (Table II). These values show that the matter content within the Einstein radius of the lens is not dominated by dark matter. However, it is evident from Table II that $M_{\rm tot}^{\rm lens}$ is linearly proportional to $M_*^{\rm MK}$ across the entire subsample, which is in complete agreement with the common prediction of average linearity by other models

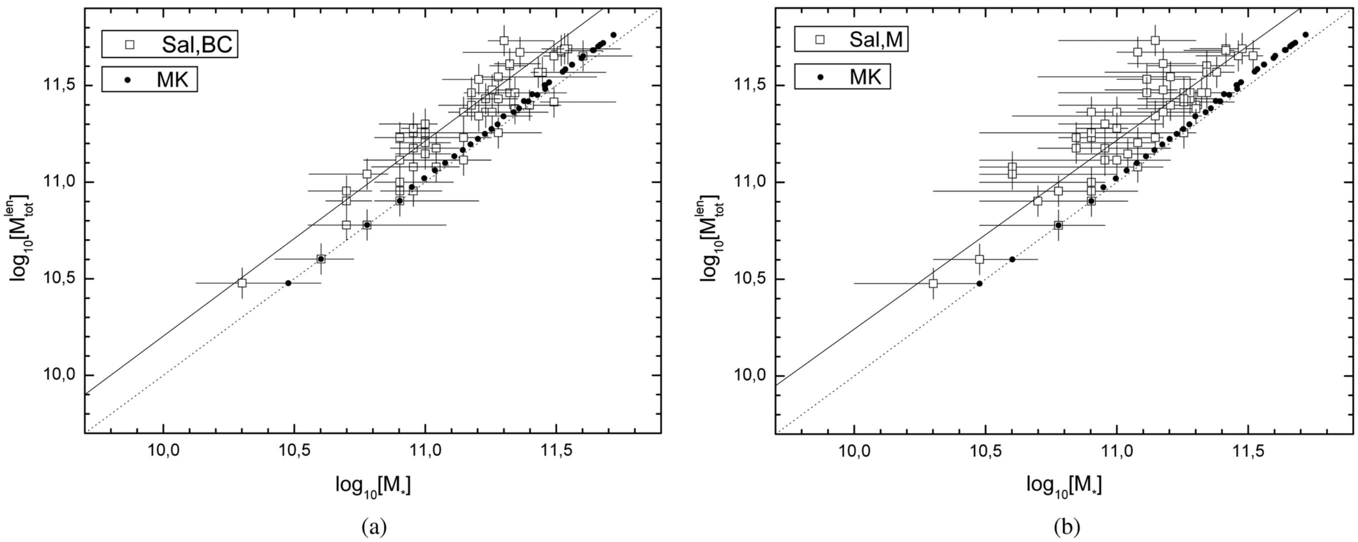

FIG. 1. The linear plot (black dots) of observed total lens mass $M_{\rm tot}^{\rm lens}(\leq R_{\rm Ein})$ for the 57 lens galaxies of the SLACS survey versus the luminous mass $M_*^{\rm MK}(\leq R_{\rm Ein})$ within the Einstein radius $R_{\rm Ein}$ obtained from our algorithm (Table II). The best-fit correlation line from different composite stellar-population models [Bruzual and Charlot (BC)–panel (a) and Maraston (M)–panel (b)] and IMFs [Salpeter (Sal)] and the one-to-one relation line $M_*/M_{\rm tot}^{\rm lens}=1$ are shown by solid and dotted lines, respectively (taken from [73]). Our linear plot and the best-fit correlation line are parallel and can be merged into one another by a small constant numerical shift.

(Figs. 1 and 2) illustrated by a line parallel to the one-to-one line defined by $M_*/M_{\rm tot}^{\rm lens}=1$, where M_* could be $M_*^{\rm MK}$, $M_*^{\rm Sal,BC}$, or $M_*^{\rm Cha,BC}$, etc.[11] Figure 1 compares the average linear profile of the model $f_*^{{\rm Grillo}\,et\,al.}$ with the more exact linearity of $f_*^{\rm MK}$ denoted by dots sitting just above the one-to-one line.

In detail, we find that 17 galaxies show mass ratios that fall within the (Sal,BC) projected error bars shown in [73]. Interestingly, out of these 17, we find that 5 galaxies exhibit very little dark matter, 4 of which are supported by the (Sal, BC) model, which yield $f_*^{\rm MK}=f_*^{{\rm Grillo}\,et\,al.}\approx 1$, marked as the coincident points in Fig. 1(a). The remaining one galaxy ($J0959+0410$) falls outside the one-to-one line by the margin of an additive factor 0.28. The ratios for the 22 galaxies fall marginally outside the error bar, while for the remaining 18 galaxies the ratios fall outside by a maximum margin 0.52. Figure 2 shows linearity profiles for the two other models [(Cha,BC), (Kro,M)] that throw up higher amounts of dark matter within the Einstein radius (the higher is the average line over the one-to-one line, and the more is the dark matter). The profiles can be compared by noting that the linear fit by Grillo [121], his Eq. (7) is

$$\mathrm{Log}_{10}[M_{\rm tot}^{\rm lens}(R_{\rm Ein})]=-0.58+1.09*\mathrm{Log}_{10}[M_*(R_{\rm Ein})]. \tag{86}$$

There is an average difference of about 0.36 with our line, which fits to

$$\mathrm{Log}_{10}[M_{\rm tot}^{\rm lens}(R_{\rm Ein})]=-0.94+1.09*\mathrm{Log}_{10}[M_*(R_{\rm Ein})]. \tag{87}$$

In view of the above, our prediction of nondominance of dark matter within the Einstein radius suggests that the present analytic approach is more akin to models [(Sal,BC), (Sal,M)] that provide relatively low dark matter inside the Einstein radius though all of their predicted ratios do not exactly coincide with, but do not stray far away from either, those from our approach. Given the model-dependent varying mass decompositions in the literature and lack of any direct experimental support yet, we can regard our model-independent analytic approach as an alternative scheme for deriving mass decompositions.

The mean density $\langle\rho\rangle_{\rm av}^{\rm MK}$ is obtained by averaging the dark matter mass $M_{\rm dm}^{\rm MK}=M_{\rm tot}^{\rm lens}(\leq R_{\rm Ein})-M_*^{\rm MK}(\leq R_{\rm Ein})$ over the Einstein sphere of radius $R_{\rm Ein}$ centered at the galactic origin,

$$\langle\rho\rangle_{\rm av}^{\rm MK}=\frac{3M_{\rm dm}^{\rm MK}}{4\pi b^3}. \tag{88}$$

Table I includes the values of $\langle\rho\rangle_{\rm av}^{\rm MK}$ together with the values of the impact parameter and the relevant deflection components. Table II gives the estimates of

[11] It should be mentioned that the *exact* equality $M_*/M_{\rm tot}^{\rm lens}=1$ is inconsistent with SDSS data. For the subsample under study, photometric and spectroscopic data are available. By using the SDSS multicolor photometry and lens modeling, Grillo *et al.* [73] studied the luminous and dark matter composition in the sample. It is possible for the data to be consistent with $M_*/M_{\rm tot}^{\rm lens}\approx 1$ allowing for dark matter however little, as is the case with a few galaxies [Fig. 1(a)] lying almost on the one-to-one line.

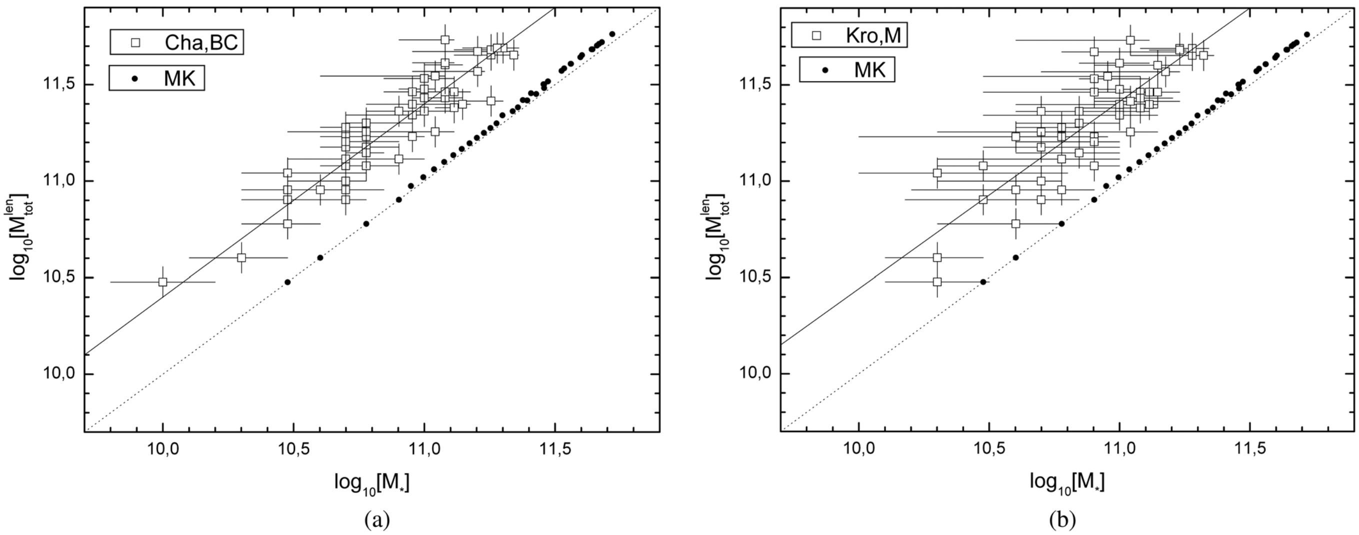

FIG. 2. The linear plot (black dots) of observed total lens mass $M_{\rm tot}^{\rm lens}(\leq R_{\rm Ein})$ for the 57 lens galaxies of the SLACS survey versus the luminous mass $M_*^{\rm MK}(\leq R_{\rm Ein})$ within the Einstein radius $R_{\rm Ein}$ obtained from our algorithm (Table II). The best-fit correlation line from different composite stellar-population models [Bruzual and Charlot (BC)–panel (a) and Maraston (M)–panel (b)] and IMFs [Chabrier (Cha), Kroupa (Kro)] and the one-to-one relation line $M_*/M_{\rm tot}^{\rm lens}=1$ are shown by solid and dotted lines, respectively (taken from [73]). Our linear plot and the best-fit correlation line are parallel and can be merged into one another by a constant but a bit larger numerical shift than in Fig. 1, which indicates the presence of comparatively more dark matter.

stellar and dark matter masses together with $f_*^{\rm MK}$. Data for distances $d_{\rm os}$, $d_{\rm ol}$, and $M_{\rm tot}^{\rm lens}$ are taken from [115]. The conversions used are the following: 1 arcsec $=$ $(1/206265)$ rad, $1\,{\rm Mpc}=3.085\times10^{24}\,{\rm cm}$, $1\,{\rm cm}^{-2}=1.98\times10^{59}M_\odot({\rm kpc})^{-3}$.

VII. CONCLUSIONS

We started with an outline of Weyl conformal gravity focusing in particular on the local and global spontaneous breakdown of conformal symmetry. Next, we showed that the MK metric need not be an exclusive solution of conformal gravity but can alternatively be viewed as a solution of a class of $f(R)$ gravity theories coupled to a nonlinear electrodynamic source. This possibility endows our galactic mass decomposition scheme, and the derived results, with more universality than thought heretofore.

We achieved two other goals in the foregoing work. First, we calculated light deflection in the MK solution of Weyl conformal theory explicitly bringing out the effect of the parameters γ and k appearing in the metric (9). The calculation based on the vacuole method reveals that the effect of γ is to enhance two-way Schwarzschild bending $(4M_*/b)$ by an amount $+\gamma b$ (noting $R\simeq b$), while the effect of k is to reduce it by an amount $-kbr_b$. The positive contribution $+\gamma b$ is contrary to the previously obtained result $-\gamma b$ in the literature [72,102,103]. Only for $\gamma<0$, the latter becomes positive and truly imitates the effect of attractive dark matter, but then the problem is that the negative sign before γ is *opposite* to that required to fit the rotation curves [72]. This long-standing problem with the MK solution has now been solved with the contribution $+\gamma b$ that is positive for $\gamma>0$, the sign required to fit the rotation curves. Notably, we required the same $\gamma>0$ for our mass decomposition as well [see Eqs. (75) and (85)]. We argued in Sec. II that the cause leading to erroneous $-\gamma b$ lay in the illegitimate range of integration leading to a metric signature change and an incomplete use of the Rindler-Ishak method not respecting their prescribed positivity condition on $|A|$.

The contribution $-kbr_b$ completely agrees with the expression obtained by Ishak *et al.* [69]. Let us for the moment notationally identify $k\equiv\Lambda/3$ so that $t_\Lambda=-\frac{\Lambda br_b}{3}$ and recall a bit of curious history here. There has been a prevailing belief that light deflection in the SdS space-time is uninfluenced by the cosmological constant Λ appearing in the metric. The reason is that Λ cancels out of the second order null geodesic equation as probably first shown by Islam [71]—naturally, the light bending expression also does not contain it. This fact can be seen in the second order null geodesic Eq. (35) from where the constant k has disappeared, even though k does influence the non-null geodesic as seen in the exact deflection Eq. (59). Eddington [122] was the first to examine one possible manifestation of the latter in the perihelion advance of the planets and derived a limit $\Lambda\leq10^{-42}\ {\rm cm}^{-2}$. On the other hand, Λ appears in the first order null geodesic equation, only its further differentiation removes it from the second order equation. Obviously, for the sake of logical consistency, the perturbative solution of the second order equation must also satisfy the first order equation, which would then yield a relation among the involved constants, one of which is the impact parameter b. An explicit calculation should in turn imply that the removed Λ would reappear in the light bending as well. This is exactly what we have found happening—a reflection of which is t_Λ and the Sereno term t_{Sereno} [see Eq. (70) and the expression for b derived in footnote 10].

Table I shows contributions to light bending coming from different factors as well as the estimates of the average dark matter density $\langle\rho\rangle_{\mathrm{av}}^{\mathrm{MK}}$ over the Einstein sphere. We see from the last two columns that the positive contribution t_γ is overtaken by the negative contribution t_k in all the cases. However, the combined effect is still 1 or 2 orders of magnitude less than the contribution t_{Sch}. Only future measurement of higher order corrections to t_{Sch} could detect this combined effect, if any. For the moment, the contributions in Eq. (71) to light deflection directly impact the mass decomposition calculated in this paper.

Second, we applied the light deflection Eq. (71) together with a logical input $\theta_{\mathrm{Ein}} = \theta_{\mathrm{Weyl}}$ to obtain the mass decomposition into luminous and dark components of the lens galaxy within its Einstein radius. The idea behind this input is that Weyl theory without dark matter and Einstein theory with dark matter both should logically predict exactly the *same* numerical value for the angle of the observed ring image of a background source if the former theory has to be in the reckoning at all as a competing theory. This input automatically implies that Weyl vacuum need not truly be a vacumm but an arena of cosmic potentials V_{γ_0} and V_k bringing about quantitatively the same lensing effect as would the dark matter in Einstein's theory. In our opinion, an estimate of the dark matter component thus provides an observable *quantification* of the strength of such potentials symbolized by the associated constants.

Table II shows mass decomposition in a representative subsample of 57 lens galaxies and that the observed total lens mass $M_{\mathrm{tot}}^{\mathrm{lens}}$ (luminous + dark) is linearly proportional to our derived luminous mass M_*^{MK} across the subsample, which qualitatively agrees with the existing conclusion in the literature as shown in Figs. 1 and 2. Table II provides the exact ratios from our analysis showing that those 57 lens galaxies are low in dark matter content within their Einstein radii. Therefore, the ratios appear to be more akin to the simulation based on (Sal,BC) than to others. Our ratios fall within the (Sal,BC) simulational error bars for many individual galaxies, and for a minority of cases the ratios fall slightly outside the error bars in varying degrees. Since in the literature [73] these ratios are also seen to vary rather significantly depending on the choice of stellar population models and the IMFs, one could equally regard the present analytic approach as yet another addition to the existing scheme for decompositions.

Finally, it should be mentioned that there is some controversy about the presence of dark matter halos around elliptical galaxies [123,124]. However, massive ellipticals are generally considered as the result of fusion of spiral galaxies. It is thus hard to understand how dark matter halos would be present around spirals and absent after their fusion. The aim of the present paper was, however, to focus on the inside of the Einstein radius and not on the distant halo region.

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