Neutron stars in Horndeski gravity

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Horndeski's theory of gravity is the most general scalar-tensor theory with a single scalar whose equations of motion contain at most second-order derivatives. A subsector of Horndeski's theory known as "Fab Four" gravity allows for dynamical self-tuning of the quantum vacuum energy, and therefore it has received particular attention in cosmology as a possible alternative to the ΛCDM model. Here we study compact stars in Fab Four gravity, which includes as special cases general relativity ("George"), Einstein-dilaton-Gauss-Bonnet gravity ("Ringo"), theories with a nonminimal coupling with the Einstein tensor ("John"), and theories involving the double-dual of the Riemann tensor ("Paul"). We generalize and extend previous results in theories of the John class and were not able to find realistic compact stars in theories involving the Paul class.

I. INTRODUCTION

The most recent cosmological observations are consistent with standard cosmological models built on general relativity (GR), but they imply the presence of a mysterious late-time acceleration phase. The late-time acceleration can be interpreted as due to the existence of new particle sectors beyond the Standard Model, or explained by assuming that GR itself is modified on cosmological scales. Modified gravity models differ widely in their physical motivations, but many of them can be reformulated in terms of scalar-tensor theories of gravitation; i.e., they are mathematically equivalent to a gravitational theory whose degrees of freedom are the metric $g_{\mu\nu}$ and one or more scalar fields ϕ. Many of the simplest dark energy or modified gravity models—including the standard ΛCDM model—are plagued by the cosmological constant problem (i.e., the problem of fine-tuning the potentially huge quantum vacuum energy against the small value of the observed cosmological constant). However some scalar-tensor theories allow for a "dynamical self-tuning mechanism" in which the effects of the cosmological constant may be compensated within the scalar field sector, so that they do not appear in the metric, by relaxing the assumptions of Weinberg's no-go theorem [1]. Here we will focus on one such model, called "Fab Four" gravity in the literature, which is a special case of Horndeski's theory.

A. Horndeski's theory

Realistic models of dark energy or modified gravity must at the very least pass the stringent experimental constraints on deviations from GR [2,3] and be theoretically viable. In particular, they must be free of the so-called "Ostrogradski ghost" [4]. Several studies led to the conclusion that the most general models with a single additional scalar degree of freedom compatible with these requirements correspond to the scalar-tensor theory formulated by Horndeski about 40 years ago, whose equations of motion contain at most second-order derivatives [5]. It was shown [6] that Horndeski's theory is equivalent to the generalization of a scalar field theory with Galilean shift symmetry in flat spacetime to curved spacetime [7], whose action reads

$$S = \sum_{i=2}^{5} \int d^4x \sqrt{-g} \mathcal{L}_i, \tag{1}$$

where

$$\mathcal{L}_2 = G_2, \tag{2a}$$

$$\mathcal{L}_3 = -G_3 \Box \phi, \tag{2b}$$

$$\mathcal{L}_4 = G_4 R + G_{4X}[(\Box \phi)^2 - \phi_{\mu\nu}^2], \tag{2c}$$

$$\mathcal{L}_5 = G_5 G_{\mu\nu} \phi^{\mu\nu} - \frac{G_{5X}}{6}[(\Box \phi)^3 + 2\phi_{\mu\nu}^3 - 3\phi_{\mu\nu}^2 \Box \phi]. \tag{2d}$$

Here $g_{\mu\nu}$ is the metric tensor and $g \equiv \det(g_{\mu\nu})$ its determinant. The Ricci scalar and Einstein tensor associated with $g_{\mu\nu}$ are denoted by R and $G_{\mu\nu}$, respectively. The functions $G_i = G_i(\phi, X)$ depend only on the scalar field ϕ and its

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kinetic energy $X = -\partial_\mu \phi \partial^\mu \phi/2$. We also introduced the shorthand notations $\phi_{\mu...\nu} \equiv \nabla_\mu ... \nabla_\nu \phi$, $\phi_{\mu\nu}^2 \equiv \phi_{\mu\nu}\phi^{\mu\nu}$, $\phi_{\mu\nu}^3 \equiv \phi_{\mu\nu}\phi^{\nu\alpha}\phi^\mu{}_\alpha$, and $\Box\phi \equiv g^{\mu\nu}\phi_{\mu\nu}$.

Special cases of Horndeski's theory correspond to well-studied models of dark energy and modified gravity, including quintessence [8,9], k-essence [10], the Dvali-Gabadadze-Porrati (DGP) model [11,12], and $f(R)$ gravity [13–16]. However it is desirable to restrict the large number of functional degrees of freedom of the action (1) by additional theoretical or phenomenological requirements. For example, it is desirable to restrict the Horndeski action to models that allow for dynamical self-tuning of the quantum vacuum energy. This requirement leads to the Fab Four theory.

B. Fab Four theory

Starting from the Horndeski action (1), Charmousis *et al.* [17,18] considered homogeneous isotropic cosmological models satisfying the following requirements:

(1) The theory admits the Minkowski vacuum for any value of the vacuum energy.

(2) The Minkowski vacuum persists across any phase transition where the vacuum energy changes instantaneously by a finite amount.

(3) The theory admits nontrivial cosmological evolution in the presence of matter.

These requirements lead to the Fab Four action

$$S = \int d^4x \sqrt{-g}(\mathcal{L}_G[g_{\mu\nu}, \phi] + \mathcal{L}_M[g_{\mu\nu}, \Psi]), \qquad (3)$$

where $\mathcal{L}_M[g_{\mu\nu}, \Psi]$ is the Lagrangian for matter fields, collectively represented by Ψ, and

$$\mathcal{L}_G[g_{\mu\nu}, \phi] = \mathcal{L}_{george} + \mathcal{L}_{ringo} + \mathcal{L}_{john} + \mathcal{L}_{paul}, \qquad (4)$$

where

$$\mathcal{L}_{george} = V_{george}(\phi)R, \qquad (5a)$$

$$\mathcal{L}_{ringo} = V_{ringo}(\phi)R_{GB}, \qquad (5b)$$

$$\mathcal{L}_{john} = V_{john}(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi, \qquad (5c)$$

$$\mathcal{L}_{paul} = V_{paul}(\phi)P^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi. \qquad (5d)$$

Here

$$R_{GB} \equiv R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} - 4R^{\mu\nu}R_{\mu\nu} + R^2 \qquad (6)$$

is the Gauss-Bonnet (GB) invariant, and the four potentials $V_{george}(\phi)$, $V_{ringo}(\phi)$, $V_{john}(\phi)$, and $V_{paul}(\phi)$ are functions of the scalar field. The quantity

$$P^{\mu\nu}{}_{\alpha\beta} \equiv -\frac{1}{4}\delta^{\mu\nu\gamma\delta}_{\sigma\lambda\alpha\beta}R^{\sigma\lambda}{}_{\gamma\delta}, \qquad (7)$$

where

$$\delta^{\mu\alpha\rho\gamma}_{\nu\beta\sigma\delta} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\mu_\beta & \delta^\mu_\gamma & \delta^\mu_\delta \\ \delta^\nu_\alpha & \delta^\nu_\beta & \delta^\nu_\gamma & \delta^\nu_\delta \\ \delta^\rho_\alpha & \delta^\rho_\beta & \delta^\rho_\gamma & \delta^\rho_\delta \\ \delta^\sigma_\alpha & \delta^\sigma_\beta & \delta^\sigma_\gamma & \delta^\sigma_\delta \end{vmatrix}, \qquad (8)$$

is the double-dual of the Riemann tensor, which shares the symmetries of the Riemann tensor and satisfies $\nabla_\mu P^{\mu\alpha\nu\beta} = 0$. We assume that $g_{\mu\nu}$ is the Jordan frame metric, so that the matter fields Ψ do not couple directly to the scalar field ϕ.

"George" reduces to GR and "Ringo"—the Einstein-dilaton-Gauss-Bonnet (EdGB) term—becomes trivial in four dimensions when the respective potentials are constant. Compact objects in these theories (George and Ringo) have been studied in detail in the existing literature. The "John" and "Paul" terms are more crucial for self-tuning and will be the main focus of this paper.

The correspondence between the Horndeski Lagrangians (2) and the Fab Four Lagrangians (5) was presented in [18], and we report it here for completeness:

$$G_2 = 2V''_{john}(\phi)X^2 - V^{(3)}_{Paul}(\phi)X^3 + 6V''_{george}(\phi)X$$
$$+ 8V^{(3)}_{ringo}(\phi)X^2(3 - \ln(|X|)), \qquad (9a)$$

$$G_3 = 3V'_{john}(\phi)X - \frac{5}{2}V''_{paul}(\phi)X^2 + 3V'_{george}(\phi)$$
$$+ 4V^{(3)}_{ringo}(\phi)X(7 - 3\ln(|X|)), \qquad (9b)$$

$$G_4 = V_{john}(\phi)X - V'_{paul}(\phi)X^2 + V_{george}(\phi)$$
$$+ 4V''_{ringo}(\phi)X(2 - \ln(|X|)), \qquad (9c)$$

$$G_5 = -3V_{paul}(\phi)X - 4V'_{ringo}(\phi)\ln(|X|). \qquad (9d)$$

C. Cosmology and black holes in Fab Four theory

Cosmological evolution in Fab Four gravity in the presence of ordinary matter and radiation has been exhaustively investigated by Copeland *et al.* [19]. They demonstrated that for a specific choice of the Fab Four potentials in Eq. (5), even if the source is dominated by the vacuum energy and there is no explicit matter fluid, the cosmological evolution toward the self-tuned Minkowski attractor can mimic the matter-dominated evolution corresponding to dark matter. Moreover, Refs. [20,21] demonstrated the existence of a self-tuned de Sitter (dS) attractor for a certain nonlinear combination of the canonical kinetic term to the Fab Four. References [22,23] presented a systematic

derivation of the most general subclass of Horndeski's theory that can allow for a spatially flat self-tuned dS vacuum. This new subclass of Horndeski's theory is expected to have a deep connection to the Fab Four theory, but it was derived in an independent way and their relation remains unclear. A specific form of John and Paul also appears in a proxy theory to nonlinear massive gravity [24], but a close inspection of cosmological dynamics revealed that there is no de Sitter attractor in this model [25].

A challenge to the Fab Four model is that self-tuning has been verified only for homogeneous, isotropic cosmological backgrounds. The presence of stars and black holes (BHs) in the Universe implies that self-tuning should still occur in the presence of local inhomogeneities of the spacetime, such as point masses or extended self-gravitating bodies. Whether self-tuning occurs in inhomogeneous spacetimes is a nontrivial question.

A first step towards answering this question is the investigation of BH solutions in Fab Four theory. Most studies of BH solutions in Horndeski's theory and Fab Four gravity have focused on the shift-symmetric subclass of the theories. An influential work by Hui and Nicolis [26] proved a BH no-hair theorem in Horndeski gravity. The theorem makes the following assumptions: (i) the spacetime is static and spherically symmetric; (ii) the scalar field shares the same symmetries as the spacetime, i.e., $\phi = \phi(r)$, where r is the radial coordinate; (iii) the theory is shift-symmetric (i.e., it is invariant under the transformation $\phi \rightarrow \phi + c$, where c is a constant); and (iv) the spacetime is asymptotically flat.

Searches for hairy BH solutions followed two main routes: they either looked for loopholes in the Hui-Nicolis theorem, or relaxed the assumptions behind the theorem. All BH solutions found so far in Horndeski's theory have secondary hair; i.e., the scalar charge is not independent of other charges, such as the mass (see, e.g., [27] for a review of BH solutions with scalar hair).

Sotiriou and Zhou found a loophole in the Hui-Nicolis no-hair theorem [28,29]. In our language, they considered the combination George + Ringo with $V_{\text{george}} = \text{constant}$ and $V_{\text{ringo}} \propto \phi$ in Eq. (5b). Other authors relaxed assumption (iv), finding asymptotically anti–de Sitter (AdS) BH solutions for actions of the John type (non-minimal coupling to the Einstein tensor) with $V_{\text{john}} = \text{constant}$ [30–32] (see [33,34] for a stability analysis of BH solutions in theories of the John subclass). BH solutions that may be more relevant for astrophysics were found by Babichev and Charmousis [35] for theories of the George + John type, with V_{george} and V_{john} both constant, relaxing assumption (ii). Babichev and Charmousis introduced a linear time dependence in the scalar field that therefore does not possess the same symmetries as the metric. However the effective energy-momentum tensor remains static because of the shift symmetry. A particularly important asymptotically flat BH solution emerging from

this analysis is a "stealth" solution in the George + John class: a Schwarzschild BH metric supports a nontrivial, regular scalar field configuration which does not backreact on the spacetime. By adding the canonical kinetic term for the scalar field and the cosmological constant Λ, Babichev and Charmousis also obtained a Schwarzschild-(A)dS solution. Interestingly, the effective cosmological constant one can read off from the Schwarzschild-(A)dS metric does *not* depend on Λ, and the Λ dependence appears only in the scalar field. Therefore this solution may be interpreted as an extension of the self-tuned dS vacuum to an inhomogeneous spacetime.

In Ref. [36], all of the above static, spherically symmetric BH solutions were generalized to slow rotation at leading order in the Hartle-Thorne approximation [37,38]. For all of these solutions, first-order corrections due to rotation were shown to be identical to GR. The Hui-Nicolis no-hair theorem was extended to slowly rotating BHs for which the scalar field is allowed to have a linear time dependence. Moreover, all the spherically symmetric solutions obtained for the John class can be naturally extended to the shift- and reflection-symmetric subclass of Horndeski's theory, namely theories with $G_2 = G_2(X)$, $G_4 = G_4(X)$, and $G_3 = G_5 = 0$ [39].

In summary, nontrivial BH solutions in Fab Four gravity were found for the Ringo and John subclasses. In particular, the Schwarzschild-dS solution found in the case of nonminimal coupling with the Einstein tensor (John) can be seen as a self-tuned BH solution. On the other hand, to our knowledge, no analytic or numerical BH solutions have been reported for the Paul subclass. Because of the similarity between John and Paul, one may naively expect that Paul should also allow for self-tuned, inhomogeneous vacuum solutions. This question was partially addressed by Appleby [40], who claimed that self-tuned BH solutions would not exist in the Paul case. This is because in a Schwarzschild-dS spacetime the Weyl components of $P^{\mu\nu}{}_{\alpha\beta}$ and R_{GB} terms in the scalar field equation of motion contain an explicit dependence on the radial coordinate, and leave no redundancy in the scalar field equation of motion. This is in contrast to the case of "John," where the scalar field equation of motion contains no Weyl component that could make it redundant for a Schwarzschild-dS metric. This also hints at the absence of similar BH solutions in the non-reflection-symmetric subclass of the shift-symmetric Horndeski theory with nonzero $G_3(X)$ and $G_5(X)$, although there are no detailed studies of this issue.

D. Plan of the paper

The next natural step to test whether the Fab Four model is compatible with local inhomogeneities is to consider self-gravitating matter configurations, and in particular static or rotating neutron stars (NSs). The main goal of this paper is precisely to investigate the existence and

properties of slowly rotating NS solutions in Fab Four gravity.

The structure and stability of rotating NSs in GR (George) is, of course, textbook material [41–43]. In the past few years there has been significant progress in our understanding of slowly [44] and rapidly rotating [45,46] NSs in Einstein-dilaton-Gauss-Bonnet gravity (Ringo), and there are also studies of stellar stability under odd-parity (axial) perturbations in this theory [47]. Recent investigations turned to theories with nonminimal coupling to the Einstein tensor (John) [48–50]. Here we complete and extend the analysis of NSs in the John subclass, and we look for solutions in theories containing the Paul term. We were unable to obtain NS solutions in theories involving the Paul term. Apparently, Paul does not want to be a star.

This paper is organized as follows. In Sec. II we derive the stellar structure equations at first order in a slow-rotation expansion in generic shift-symmetric Horndeski theories. In Sec. III we specialize our analysis to each of the Fab Four subcases. In Sec. IV we summarize our findings and point out possible directions for future research. The appendix discusses the relation between the moment of inertia and the stellar compactness in theories of the Ringo and John subclasses. Throughout the paper, unless specified otherwise, we will use geometrical units ($G = c = 1$).

II. SLOWLY ROTATING STARS IN FAB FOUR THEORY

In this section we will consider the shift-symmetric subclass of Horndeski's theory that is invariant under the transformation

$$\phi \to \phi + c, \tag{10}$$

where c is a constant. From Eqs. (9), this assumption implies that V_{john}, V_{paul}, and V_{george} must be constant, while the Ringo (EdGB) term V_{ringo} can be a linear function of ϕ. For EdGB, a constant shift in ϕ only adds a trivial topological invariant to the action, and therefore it does not affect the field equations. Equations (4) and (5) represent the basic building blocks of our theory, which will be described by the general action

$$S = S_{\text{G}} + S_{\text{M}}, \tag{11}$$

where S_{M} is the ordinary action for fluid matter and S_{G} is a combination of the Lagrangians (5c)–(5b).

To investigate slowly rotating solutions we follow the approach described by Hartle and Thorne [37,38], in which spin corrections are considered as small perturbations on an otherwise static, spherically symmetric background. In particular, at first order in the star's angular velocity Ω the metric can be written as

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$
$$- 2[\Omega - \tilde{\omega}(r)]\sin^2\theta dt d\varphi, \tag{12}$$

where $\tilde{\omega}(r)$ is the angular velocity of the fluid as measured by a freely falling observer.

Varying the action (11) with respect to the metric and the scalar field we obtain the equations of motion for $g_{\alpha\beta}$ and ϕ, respectively:

$$\mathcal{E}_{\alpha\beta} = T_{\alpha\beta}, \qquad \mathcal{E}_\phi = 0, \tag{13}$$

where

$$T_{\alpha\beta} = (\epsilon + p)u_\alpha u_\beta + pg_{\alpha\beta} \tag{14}$$

is the energy-momentum tensor of a perfect fluid. Here ϵ and p are the energy density and pressure of a fluid element with four-velocity $u^\mu = u^0(1, 0, 0, \Omega)$. The time component u^0 follows directly from the normalization condition $u^\mu u_\mu = -1$, which leads for the metric (12) to $u^0 = 1/\sqrt{A}$. The explicit form of $\mathcal{E}_{\alpha\beta}$ and $\mathcal{E}_\phi$ can be found in the Appendix of [36] (see [34] for a particular study in the case of John).

Moreover, in the Jordan frame, the energy-momentum tensor is conserved:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{15}$$

To close the system of equations we need to specify the equation of state (EOS) for the NS, i.e., a relation between the pressure and energy density:

$$p = p(\epsilon). \tag{16}$$

Taken together, Eqs. (13), (15), and (16) provide the full description of a slowly rotating star.

In this work we will consider three realistic EOSs, namely, APR [51], SLy4 [52] and GNH3 [53] in decreasing order of stiffness. To facilitate comparisons with [48,50] we will also consider a polytropic EOS of the form $p = K\rho^\Gamma$, with $K = 123M_\odot^2$ and $\Gamma = 2$. Here ρ is the mass density, related to the energy density by

$$\epsilon = \left(\frac{p}{K}\right)^{1/\Gamma} + \frac{p}{\Gamma - 1}. \tag{17}$$

In Table I we show the radius R and compactness $\mathcal{C} \equiv M/R$ of nonrotating models, as well as the moment of inertia I, for NSs with the "canonical" mass $M = 1.4\,M_\odot$ constructed using different EOS models in GR. At fixed mass, the realistic EOSs APR, SLy4, and GNH3 (in this order) yield configurations with decreasing compactness, and therefore larger moment of inertia.

TABLE I. The radius R, compactness $\mathcal{C}$, and moment of inertia I for a canonical NS with mass $M = 1.4\,M_\odot$, in GR, using three different nuclear-physics based EOS models and a $\Gamma = 2$ polytrope.

EOS	R (km)	$\mathcal{C}$	$I(10^{45}\text{ g cm}^2)$
APR	11.33	0.182	1.31
SLy4	11.72	0.176	1.37
GNH3	14.18	0.146	1.81
Polytrope	16.48	0.125	2.28

III. FAB FOUR NEUTRON STARS

In this section we discuss NSs in the four subclasses of Fab Four gravity, starting from the simplest Lagrangians.

A. George (General relativity)

The George Lagrangian for shift-symmetric theories corresponds to GR, so we refer the reader to standard treatments of rotating stars [41–43].

B. Ringo (Einstein-dilaton-Gauss-Bonnet gravity)

Nonrotating hairy BH solutions in EdGB gravity with a dilatonic coupling of the schematic form $V_{\rm ringo} \sim \zeta e^{\gamma\phi}$ were found by Kanti *et al.* [54]. These solutions were then extended to slowly and rapidly rotating BHs [44,55]. As stated in the introduction, Sotiriou and Zhou [28,29] pointed out that hairy BH solutions exist in shift-symmetric EdGB theories, in violation of the Hui-Nicolis no-hair theorem (see [36] for an extension of these results to linear order in a slow-rotation approximation). Shift-symmetric EdGB theories can be seen as a small-field Taylor series expansion of the dilatonic coupling

$$V_{\rm ringo} \simeq \zeta + \zeta\gamma\phi, \tag{18}$$

where the constant term ζ can be neglected since it gives rise to a topological invariant at the level of the action.

NSs in EdGB gravity with a dilatonic coupling were studied in [44–46] (see also [47] for axial perturbations). As it turns out, the bulk properties of NSs depend only on the combination $\zeta\gamma$; cf. the discussion around Eq. (29) of [44]. This is because the value of the scalar field is typically very small within the star, and therefore the Taylor expansion (18) is an excellent approximation. For this reason, the analysis of NSs in Ref. [44] applies also to the shift-symmetric case of interest here, and we refer the reader to the treatment in that paper for calculations of stellar structure and observational bounds on the product $\zeta\gamma$.

C. John (Nonminimal coupling with the Einstein tensor)

A more interesting case is slowly rotating compact stars in theories with a nonminimal derivative coupling with the Einstein tensor, corresponding to the John Lagrangian (5c) [48–50]. These theories are described by the action

$$S_{\rm G} = \int d^4x \sqrt{-g}(\mathcal{L}_{\rm george} + \mathcal{L}_{\rm john} + \mathcal{L}_{\rm K})$$
$$= \int d^4x \sqrt{-g}\left[\kappa R - \frac{1}{2}(\beta g^{\mu\nu} - \eta G^{\mu\nu})\partial_\mu\phi\partial_\nu\phi\right], \tag{19}$$

where $\mathcal{L}_{\rm K} = \beta X = -(\beta\partial_\mu\phi\partial^\mu\phi)/2$ is a kinetic term for the scalar field, β and η are constants, and $\kappa = (16\pi)^{-1}$. Equation (19) can be obtained from the Horndeski Lagrangian by choosing

$$G_2 = \beta X, \qquad G_4 = \kappa + \frac{\eta}{2}X, \qquad G_3 = G_5 = 0. \tag{20}$$

We also consider a real scalar field of the form [35]

$$\phi(r,t) = qt + \psi(r), \tag{21}$$

where q is a constant scalar charge. With this choice, the field's kinetic energy is a function of r only:

$$X = \frac{1}{2}\left[\frac{q^2}{A(r)} - B(r)\psi'(r)^2\right]. \tag{22}$$

In vacuum, the theory described by the action (19) leads to asymptotically AdS black hole solutions with a non-trivial scalar field configuration [30–32,35,39]. However, it has recently been shown that it is possible to construct "stealth" NS models for which the exterior solution is given by the Schwarzschild spacetime [48].

For $\beta = 0$, the scalar field outside the star (where $T_{\mu\nu} = 0$) does not backreact on the metric, leading to "stealth solutions." However inside the star (where $T_{\mu\nu} \neq 0$) the scalar field has a nontrivial effect, and the stellar structure is different from GR.

Hereafter we will focus on these stealth solutions, fixing $\beta = 0$. We recall that the action (19) is invariant under shift symmetry ($\phi \to \phi + c$). This allows us to write the equation of motion for the scalar field in terms of a conserved current J^μ:

$$\nabla_\mu J^\mu = 0, \tag{23}$$

with nonzero components given by

$$J^t = -\frac{q\eta}{r^2\kappa A}(rB' + B - 1), \tag{24}$$

$$J^r = \frac{\eta B}{r^2\kappa A}[A(B-1) + rBA']\phi'. \tag{25}$$

We also remark that Eq. (23), using the line element (12), admits the solution

$$J^r = \sqrt{\frac{B}{A}} \frac{C_1}{r^2}, \qquad (26)$$

with C_1 constant. In the following we will set $C_1 = 0$, as it has been shown that this choice is consistent with a vanishing radial energy flux, i.e., $\mathcal{E}_{tr} = 0$ [56].

Combining Eqs. (15), (25), and the (tt) and (rr) components of Eqs. (13), we obtain a set of differential equations for the spherically symmetric background. Moreover, at linear order in the angular velocity, the $(t\varphi)$ equation $\mathcal{E}_{t\varphi} - T_{t\varphi} = 0$ yields a differential equation for $\tilde{\omega}$. In summary, a slowly rotating NS at first order in the slow-rotation approximation is described by the following set of equations:

$$A' = \frac{A}{r} \frac{1-B}{B}, \qquad (27)$$

$$B' = \frac{3q^2\eta B(B-1) - A[r^2\epsilon - 4\kappa + B(4\kappa + r^2(\epsilon + 6p))]}{r[A(4\kappa + r^2 p) - 3q^2\eta B]}, \qquad (28)$$

$$p' = -\frac{\epsilon + p}{2} \frac{A'}{A}, \qquad (29)$$

$$\tilde{\omega}'' = \frac{4q^2\eta B^2 - A[4B(4\kappa + r^2 p) - r^2(\epsilon + p)]}{rB[A(4\kappa + r^2 p) - q^2\eta B]} \tilde{\omega}'$$
$$- \frac{4A(\epsilon + p)}{B[q^2\eta B - A(4\kappa + r^2 p)]} \tilde{\omega}, \qquad (30)$$

$$(\phi')^2 = \frac{r^2 A p - q^2\eta(B-1)}{\eta A B}. \qquad (31)$$

Note that q and η always appear combined in the factor $q^2\eta$.

Expanding all variables in a power series around $r = 0$, we obtain the initial values for $(A, B, \tilde{\omega}, p, \phi)$ as

$$A = A_c - \frac{r^2 A_c^2(3p_c + \epsilon_c)}{3(3q^2\eta - 4\kappa A_c)} + \mathcal{O}(r^3), \qquad (32a)$$

$$B = 1 + \frac{2}{3} \frac{r^2 A_c(3p_c + \epsilon_c)}{(3q^2\eta - 4\kappa A_c)} + \mathcal{O}(r^3), \qquad (32b)$$

$$p = p_c + \frac{r^2 A_c(p_c + \epsilon_c)(3p_c + \epsilon_c)}{6(3q^2\eta - 4\kappa A_c)} + \mathcal{O}(r^3), \qquad (32c)$$

$$\tilde{\omega} = \tilde{\omega}_c - \frac{2}{5} \frac{A_c(\epsilon_c + p_c)r^2}{q^2\eta - 4A_c\kappa} \tilde{\omega}_c + \mathcal{O}(r^3), \qquad (32d)$$

$$(\phi')^2 = \frac{p_c}{\eta} r^2 - \frac{2q^2(3p_c + \epsilon_c)}{3(3q^2\eta - 4\kappa A_c)} r^2 + \mathcal{O}(r^3), \qquad (32e)$$

where the subscript "c" means that the various variables are evaluated at the center of the star. Following [48] we set

$A_c = 1$ and chose $\tilde{\omega}_c = 1$. Given a choice of EOS, the central pressure p_c uniquely determines a NS model.

From these expansions we can obtain constraints that must be satisfied by $q^2\eta$ to obtain physically acceptable solutions. If we demand that $p''(r) < 0$ [57], we obtain

$$q^2\eta < \frac{4\kappa}{3}, \qquad (33)$$

which is automatically satisfied when $\eta < 0$, but sets an upper bound on $q^2\eta$ when $\eta > 0$. On the other hand, the requirement that the derivative of the scalar field should be real, i.e., $(\phi')^2 > 0$, implies

$$\frac{p_c}{\eta} - \frac{2q^2(3p_c + \epsilon_c)}{3(3q^2\eta - 4\kappa)} > 0. \qquad (34)$$

For $\eta > 0$ this condition is always satisfied by virtue of Eq. (33). However, when $\eta < 0$ we obtain a lower bound on $q^2|\eta|$, namely,

$$q^2|\eta| > \frac{3}{4\pi} \frac{p_c}{2\epsilon_c - 3p_c}. \qquad (35)$$

To construct NS models we integrate the system of equations (27)–(29), supplemented by the boundary conditions (32a)–(32c), from $r = 0$ up to the star's radius $r = R$, which corresponds to the point where the pressure vanishes, i.e., $p(R) = 0$. Then we match the interior solution to the exterior Schwarzschild metric. The NS mass is obtained by solving the system

$$A(R) = A_\infty\left(1 - \frac{2M}{R}\right), \qquad A'(R) = A_\infty \frac{2M}{R^2}, \qquad (36)$$

where A_∞ is an integration constant. Then we rescale the time variable $(t \to t\sqrt{A_\infty})$ so that it represents the coordinate time measured by an observer at infinity. Because of the linear dependence of the scalar field on t, we correspondingly rescale q as

$$q_\infty = \frac{q}{\sqrt{A_\infty}}. \qquad (37)$$

The stellar structure equations depend only on the combination $q^2\eta$, so we can set $\eta = \pm 1$ without loss of generality. The scalar field is computed from Eq. (31) for families of solutions with fixed values of q_∞. To facilitate comparisons with [48], here we choose these values to be 0, 0.032, and 0.064. To obtain the solutions we apply a shooting method, adjusting the value q in each integration until we obtain the desired value of q_∞.

We also integrate Eq. (30) for a given $\tilde{\omega}_c$ and we compute the star's angular velocity Ω and its angular momentum J, requiring that at the surface

$$\tilde{\omega}(R) = \Omega - \frac{2J}{R^3}, \qquad \tilde{\omega}'(R) = \frac{6J}{R^4}. \qquad (38)$$

The moment of inertia is computed through $I = J/\Omega$. We note that rescaling $\tilde{\omega}(r)$ by a constant factor does not affect Eq. (30). Therefore, once the solution $\tilde{\omega}_{old}$ has been found for given initial conditions, yielding a value Ω_{old}, a new solution $\tilde{\omega}_{new}$ can immediately be found via $\tilde{\omega}_{new} = \tilde{\omega}_{old}\Omega_{new}/\Omega_{old}$. The moment of inertia I is independent of the star's angular velocity.

In Fig. 1 we show the mass-radius diagram for all the EOS models used in this paper. The polytropic case (bottom-right panel) matches the results in [48], except for what we believe to be a mislabeling of some curves in their Fig. 2.

As pointed out in [48], the limit $q_\infty \to 0$ does not correspond to GR, and indeed the corresponding mass-radius curves are different from those of GR (solid black lines). For any EOS and fixed q_∞, positive (negative) values of η correspond to more (less) compact configurations. At fixed η, larger values of the scalar charge q_∞

correspond to stellar models with larger radii. As a reference, the horizontal colored band corresponds to the most massive known NS, PSR J0348+0432, with $M = 2.01 \pm 0.04 \, M_\odot$ [58]. When $\eta > 0$, for all values of q_∞ and EOS models considered in this paper, such massive NSs are not supported.

In Fig. 2 we show the moment of inertia as a function of mass for the same stellar models and theory parameters as in Fig. 1. In addition, in Table II we list the values of I for a canonical NS with mass $M = 1.4 \, M_\odot$. It is interesting that some theories with $\eta > 0$ cannot support stars with this value of the mass. As expected, deviations with respect to GR grow as the scalar charge increases, yielding larger (smaller) moments of inertia for $\eta < 0$ ($\eta > 0$). The relative deviation from GR can be of order 30% for $q_\infty = 0.064$ and $\eta = -1$.

In GR, the dimensionless moment of inertia $\bar{I} \equiv I/M^3$ was recently shown to be related to the NS compactness $\mathcal{C}$ by a universal relation which is almost insensitive to the adopted EOS [59] (see [60–63] for earlier studies):
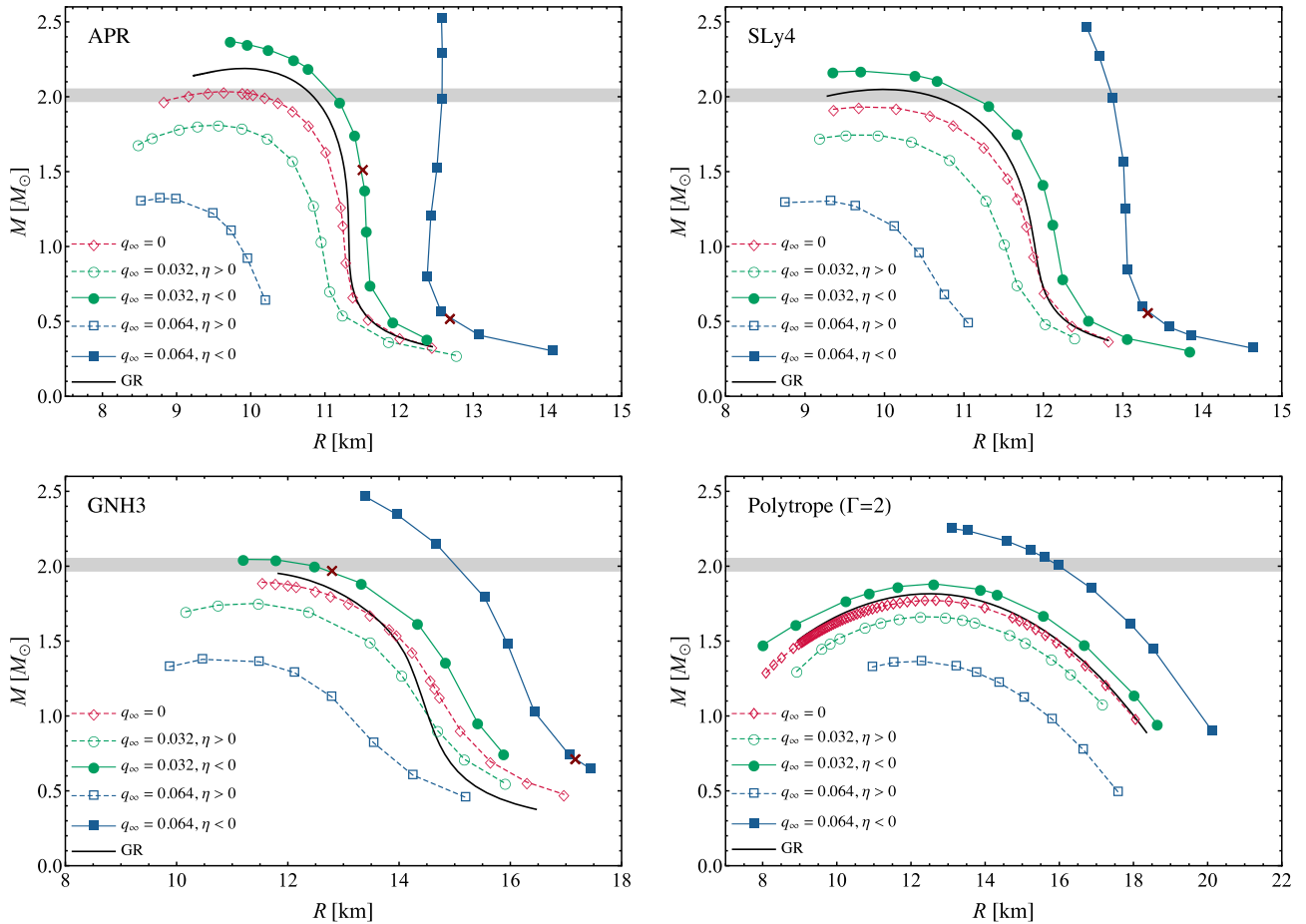

FIG. 1. Mass-radius curves for different EOS models, selected values of q_∞, and $\eta = \pm 1$. The various panels correspond to EOS APR (top left), SLy4 (top right), GNH3 (bottom left), and a polytropic (bottom right). Configurations with radii smaller than that identified by the orange cross do not satisfy the condition (34). The horizontal colored band corresponds to $M = 2.01 \pm 0.04 \, M_\odot$, the most massive NS mass known to date [58]. Note that the various panels have different x-axis ranges.
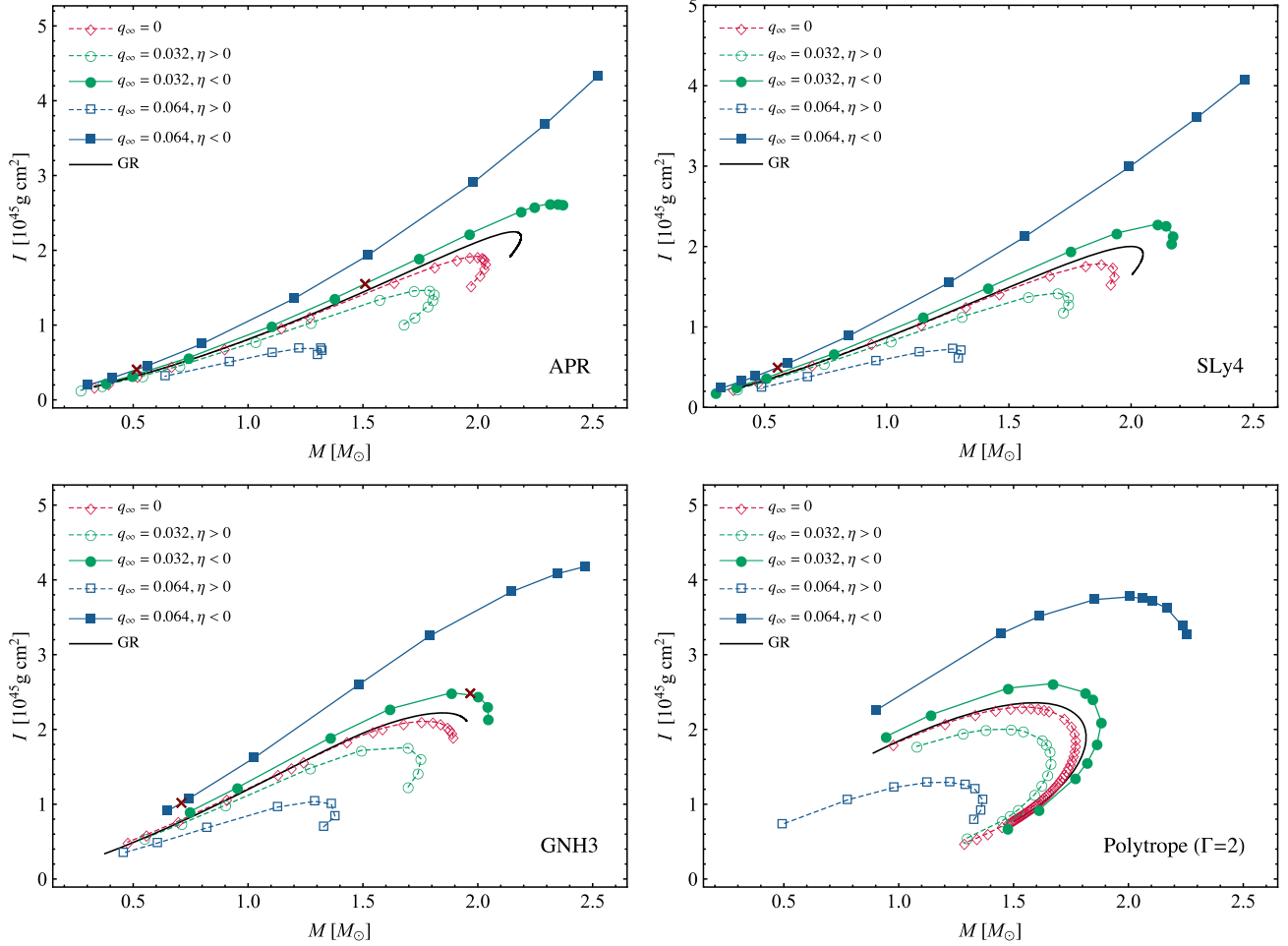
FIG. 2.   Moment of inertia I as a function of the mass M. The various panels correspond to EOS APR (top left), SLy4 (top right), GNH3 (bottom left), and a polytropic (bottom right). Configurations with masses larger than that identified by the orange cross do not satisfy the condition (34).

$$\bar{I}_{\text{fit}} = a_1 \mathcal{C}^{-1} + a_2 \mathcal{C}^{-2} + a_3 \mathcal{C}^{-3} + a_4 \mathcal{C}^{-4}, \qquad (39)$$

where the fitting coefficients a_i, $i = (1, \ldots, 4)$, are listed in Table II of [59]. This $I - \mathcal{C}$ relation reproduces numerical results with an accuracy better than 3%. The observed universality is reminiscent of the I-Love-Q relations

TABLE II.   Moment of inertia for a NS with $M = 1.4 \, M_\odot$ for selected values of q_∞ and for nuclear-physics-motivated EOS models. For $q_\infty = 0.064$ and $\eta = 1$, none of the EOS models considered here supports NSs with $M = 1.4 \, M_\odot$.

η	q_∞	$I_{\text{APR}}(10^{45}\,\text{g cm}^2)$	$I_{\text{GNH3}}(10^{45}\,\text{g cm}^2)$	$I_{\text{SLy4}}(10^{45}\,\text{g cm}^2)$
$\cdots$	GR	1.31	1.81	1.37
$\cdots$	0	1.28	1.80	1.35
-1	0.032	1.39	1.96	1.47
-1	0.064	1.70	2.42	1.81
1	0.032	1.17	1.64	1.22
1	0.064	$\cdots$	$\cdots$	$\cdots$

between the moment of inertia, tidal deformability (as encoded in the so-called Love number) and rotational quadrupole moment Q [64,65]. The extension of these near-universal relations $I - \mathcal{C}$ relations to theories of the Ringo and John subclasses is discussed in the Appendix.

It is natural to ask whether these stealth NS models are stable. Vacuum, static, spherically symmetric solutions where the scalar field has a linear time dependence were shown to be free from ghost and gradient instabilities under odd-parity gravitational perturbations as long as the following conditions are met [66]:

$$\mathcal{F} > 0, \qquad \mathcal{G} > 0, \qquad \mathcal{H} > 0, \qquad (40)$$

where

$$\mathcal{F} = 2\left(G_4 - \frac{q^2}{A}G_{4X}\right) = 2\left(\kappa + \frac{q_\infty^2 \eta}{4} - \frac{q_\infty^2 \eta A_\infty}{2A}\right), \quad (41)$$

$$\mathcal{G} = 2\left(G_4 - 2XG_{4X} + \frac{q^2}{A}G_{4X}\right)$$
$$= 2\left(\kappa - \frac{q_\infty^2\eta}{4} + \frac{q_\infty^2\eta A_\infty}{2A}\right), \qquad (42)$$

$$\mathcal{H} = 2(G_4 - 2XG_{4X}) = 2\left(\kappa - \frac{q_\infty^2\eta}{4}\right). \qquad (43)$$

Here we have used $X = q^2/(2A_\infty)$ as well as Eq. (37), which applies to the stealth BH solutions of [35]. For stealth BH solutions, $A \to 0$ in the vicinity of the event horizon; therefore the third term on the right-hand side of Eqs. (41) and (42) is the dominant one. As a consequence $\mathcal{F}\mathcal{G} < 0$, suggesting that these solutions are generically unstable [66].

A similar argument can be applied to our stealth NS solutions. In the exterior vacuum spacetime of the star, the metric function A, which satisfies $A < A_\infty$, remains positive and finite. When η is *positive*, $\mathcal{G}$ is always positive as well, and the conditions $\mathcal{F} > 0$ and $\mathcal{H} > 0$ everywhere outside the star translate into

$$q_\infty^2\eta < 4\kappa\frac{A(R)}{2A_\infty - A(R)} = 4\kappa\left(\frac{1 - 2\mathcal{C}}{1 + 2\mathcal{C}}\right), \qquad (44)$$

$$q_\infty^2\eta < 4\kappa, \qquad (45)$$

respectively, where we have used Eq. (36).

We have numerically confirmed that all NS models presented in Fig. 1 satisfy the conditions (44) and (45) for the largest value of $q_\infty = 0.064$ considered in this paper. For a typical NS the compactness is $\mathcal{C} \approx 0.2$, and the right-hand side of Eq. (44) is approximately 0.035, which is much larger than our choice $q_0^2\eta = 0.064^2 \approx 0.004$. The condition (44) will be violated only for an unrealistically compact NS with $\mathcal{C} \approx 0.45$. This suggests that hypothetical ultracompact objects—such as Lemaitre stars [65,67,68] and gravastars [69–71]—may be unstable in the presence of a stealth scalar field.

Similarly, for *negative* values of η, $\mathcal{F}$ and $\mathcal{H}$ are always positive, and the condition $\mathcal{G} > 0$ is satisfied everywhere outside the star if

$$q_\infty^2|\eta| < 4\kappa\frac{A(R)}{2A_\infty - A(R)} = 4\kappa\left(\frac{1 - 2\mathcal{C}}{1 + 2\mathcal{C}}\right). \qquad (46)$$

We have also checked that for $q_\infty = 0.064$ and $\eta = -1$, all NS models presented in Fig. 1 satisfy (46). In the Newtonian limit $\mathcal{C} \ll 1$, the stealth NS spacetime is stable for $q_\infty^2\eta < 4\kappa$ when $\eta > 0$, and for $q_\infty^2|\eta| < 4\kappa$ when $\eta < 0$. For NSs with larger values of $q_\infty^2|\eta|$ the exterior spacetime becomes unstable everywhere, including the Newtonian regime.

It is interesting to consider the nonrelativistic limit of theories of the John class. Introducing the usual mass function $m(r)$ such that $B(r) = 1 - 2m(r)/r$, we see that the pressure equation retains its standard form

$$\frac{dp}{dr} = -\frac{m\rho}{r^2}, \qquad (47)$$

where ρ is the mass density. However the mass equation is reduced to

$$\frac{dm}{dr} = \frac{4\pi r^2\rho}{1 - 12\pi q^2\eta}. \qquad (48)$$

This behavior looks reminiscent of "beyond Horndeski" theories [72,73], where a partial breakdown of the Vainshtein mechanism occurs, modifying the Newtonian limit [74]. In fact, several authors have advocated the use of this "feature" to constrain beyond Horndeski theories using Newtonian stars or white dwarfs [75–79]. While those theories modify the pressure equation (47), leaving the mass equation unaltered, theories of the John subclass seem to alter the Newtonian limit in the opposite way.

However, combining Eqs. (47)–(48) and restoring the gravitational constant G we obtain

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G_{\text{eff}}\rho, \qquad (49)$$

which is equivalent to the ordinary hydrostatic equilibrium equation in Newtonian gravity with an *effective* gravitational constant

$$G_{\text{eff}} \equiv \frac{G}{1 - 12\pi q^2\eta}. \qquad (50)$$

Therefore the nonrelativistic limit of the "John" theories considered in this section is equivalent to Newtonian gravity with an effective gravitational constant G_{eff}. Incidentally, a similar result was found by Cisterna *et al.* [50] in the context of cosmology [cf. their Eq. (38)].

D. Paul (Double-dual of the Riemann tensor)

Let us now turn to NS solutions in theories containing the Paul Lagrangian (5d). We start with the simplest model, given by the combination

$$\mathcal{L} = \mathcal{L}_{\text{george}} + \mathcal{L}_{\text{paul}}$$
$$= R - \frac{1}{3}\alpha P^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi, \qquad (51)$$

which from Eqs. (19) corresponds to the following choice of the functions G_i:

$$G_2 = G_3 = 0, \qquad G_4 = 1, \qquad G_5 = \alpha X, \qquad (52)$$

where α is a coupling parameter. As in Sec. III C, we consider a scalar field with linear time dependence of the form (21). This choice is crucial for $\phi(r)$ to have a nontrivial profile. Indeed, the nonvanishing components of the scalar current for the action (51) are

$$J^r = \frac{\alpha}{2r^2}\frac{B}{A}[q^2(B-1) + A(1-3B)B\phi'^2]\frac{A'}{A}, \quad (53)$$

$$J^t = \frac{q\alpha}{2r^2}\frac{B}{A}\left\{\phi'\left[\frac{A'}{A}(B-1) + \frac{B'}{B}(3B-1)\right]\right.$$
$$\left. + 2(B-1)\phi''\right\}. \quad (54)$$

From the first equation we conclude that in the limit $q \to 0$ the condition $J^r = 0$ implies $\phi' = 0$; i.e., the scalar field must be constant. However for $q \neq 0$ we obtain

$$(\phi')^2 = q^2\frac{1-B}{A(1-3B)B}. \quad (55)$$

Replacing this relation into the (tt) and (rr) components of Eqs. (13), we derive two first-order equations for the metric variables A and B:

$$B' = \frac{1 - B - 8\pi r^2\epsilon}{r - \frac{q^3\alpha\sqrt{1-B}[AB(1-3B)]^{3/2}}{A^3(1-3B)^3}}, \quad (56)$$

$$A' = \frac{A^3}{B}\frac{1 - B + 8\pi r^2 p}{A^2 r - \frac{q^3\alpha B\sqrt{1-B}\sqrt{AB(1-3B)}}{(1-3B)^2}}. \quad (57)$$

Equations (55)–(57), together with a choice of EOS and the energy-momentum conservation equation (15), which gives

$$p' = -\frac{\epsilon + p}{2}\frac{A'}{A}, \quad (58)$$

form a closed system of differential equations, which can be integrated by imposing suitable initial conditions at the center of the star. These conditions can be found through a Taylor expansion in r,

$$A(r) = A_c + \frac{r^2}{q^6\alpha^2}A_2(p_c, A_c) + \mathcal{O}(r^3), \quad (59a)$$

$$B(r) = 1 + \frac{r^2}{q^6\alpha^2}B_2(p_c, A_c) + \mathcal{O}(r^3), \quad (59b)$$

$$p(r) = p_c + \frac{r^2}{q^6\alpha^2}p_2(p_c, A_c) + \mathcal{O}(r^3), \quad (59c)$$

$$\phi'(r) = \pm\sqrt{\frac{B_2(p_c, A_c)}{2A_c}}\frac{r}{q^2\alpha} + \mathcal{O}(r^3), \quad (59d)$$

where A_2, B_2, and p_2 are functions of the constant parameters A_c and p_c. Unlike Eqs. (56) and (57), which reduce to GR for $\alpha \to 0$ (or $q \to 0$), the initial conditions for the metric functions, $(\phi')^2$, and the pressure are ill defined. Note that such a pathological behavior is not expected in the naive $\alpha \to 0$ limit of (56) and (57), because this is a "nonperturbative" effect such that the leading behavior $\sqrt{1-B} \propto 1/\alpha$ obtained from (59b) cancels the α terms in (56) and (57), making the deviation from GR evident.

To better understand this issue, let us reconsider the $\eta \to 0$ limit of the John action. In that case, as we see from Eqs. (32a)–(32e), the only divergent quantity as $\eta \to 0$ is the derivative of the scalar field ϕ', while all other metric and matter quantities have a finite limit. Since we work in the Jordan frame there is no direct coupling between the scalar field and matter. Furthermore the scalar field does not backreact on the spacetime in the stealth exterior, and therefore a singular behavior of the scalar field does not affect the geodesics of particles outside the star. In contrast, for the Paul case all physical quantities diverge in the limit $\alpha \to 0$, indicating a pathological behavior in the NS interior. Furthermore, at variance with the John case, we could not find a stealth exterior solution for Paul. Our results suggest that exterior stealth solutions for Paul do not exist under the ansatz (21) for the scalar field.

We observed a similar behavior for other Fab Four theories involving the Paul term. We considered the following combinations:

$$\mathcal{L}_{\text{george}} + \mathcal{L}_{\text{paul}} + \mathcal{L}_{\text{K}}, \qquad \phi(r), \quad (60)$$

$$\mathcal{L}_{\text{george}} + \mathcal{L}_{\text{paul}} + \mathcal{L}_{\text{john}} + \mathcal{L}_{\text{K}}, \qquad \phi(t,r), \quad (61)$$

$$\mathcal{L}_{\text{george}} + \mathcal{L}_{\text{paul}} + \mathcal{L}_{\text{ringo}} + \mathcal{L}_{\text{K}}, \qquad \phi(t,r), \quad (62)$$

$$\mathcal{L}_{\text{george}} + \mathcal{L}_{\text{paul}} + \mathcal{L}_{\text{john}} + \mathcal{L}_{\text{ringo}} + \mathcal{L}_{\text{K}}, \qquad \phi(t,r). \quad (63)$$

In all of these cases the physical variables suffer from the same divergence when the coupling parameter α of the Paul term vanishes.

Appleby [40] found that the self-tuning mechanism is not applicable for spherically symmetric black hole spacetimes in theories of the Paul class. Our results strengthen his conclusions, suggesting that the Paul term does not allow for physically well-behaved compact object solutions.

IV. CONCLUSIONS

We have presented an exhaustive study of slowly rotating NS solutions in the shift-symmetric class of Fab Four gravity, namely, the subclass of Horndeski's gravity that may allow for dynamical self-tuning of the quantum vacuum energy,

and for this reason has been the subject of intense scrutiny in a cosmological context. Our main goal was to investigate whether Fab Four gravity is compatible with the existence of relativistic stars, such as NSs.

Among the nonminimal couplings in Fab Four gravity listed in Eqs. (5c)–(5b), we especially focused on the John (nonminimal derivative coupling to the Einstein tensor) and Paul (nonminimal derivative coupling to the double dual of the Riemann tensor) subclasses. This is both because George (GR) and Ringo (EdGB gravity) have been extensively studied in the past and because John and Paul are the crucial terms allowing for self-tuning of the quantum vacuum energy in cosmological scenarios.

In the case of John, if we make the assumption that the scalar field has a linear time dependence of the form (21), there is a stealth solution such that the scalar field does not backreact on the metric in the exterior, while it introduces nontrivial modifications of the interior stellar structure with respect to GR in the stellar interior. Our results on spherically symmetric NSs agree with previous work [48] and extend it to slowly rotating solutions. As pointed out in [48], in the limit of vanishing scalar charge ($q_\infty \to 0$) the mass-radius curves differ from GR. Irrespective of the chosen EOS, positive (negative) values of the coupling constant η in (19) yield more (less) compact stellar configurations. For positive values of η this fact can be used to put mild (EOS-dependent) constraints on the maximum value of q_∞; cf. Ref. [50].

We have also shown that the approximately EOS-independent relations between the moment of inertia I and compactness $\mathcal{C}$ within GR break down in this theory. Therefore, in principle, future measurements of I could potentially constrain the value of q_∞ [80]. We also obtained improved $I - \mathcal{C}$ relations that depend of the value of q_∞ and are accurate within ~5%.

Based on stability studies in the context of BH solutions [66], we have argued that the NS models studied here are generically stable under odd-parity gravitational perturbations. A systematic study of stellar perturbations within theories of the John subclass is desirable, and it could follow in the footsteps of similar studies for scalar-tensor theory [81–84] and EdGB gravity [47].

Surprisingly, we also found that in all subclasses of the Fab Four and its minimal extensions that involve Paul, not only the scalar field, but also all metric functions and the pressure diverge at the center of the star in the small-coupling limit. Therefore "healthy" BH and stellar solutions do not seem to exist in the shift-symmetric Paul subclass. It will be interesting to determine whether this conclusion still holds in the absence of shift symmetry.

As a straightforward generalization of the present work, one could search for NS solutions in Fab Four theories where the potentials (5c)–(5b) have nontrivial functional forms, as well as in more general (non-shift-symmetric) versions of Horndeski's theory. The general formalism

developed in [36] can be straightforwardly applied to these cases.

Barausse and Yagi [85] have recently shown that the so-called sensitivities of compact objects [86] vanish in shift-symmetric Horndeski gravity, which includes the Fab Four class. Consequently the dynamics of binaries involving NSs is, to leading post-Newtonian order, the same as in GR. It would be interesting to determine whether these conclusions hold at higher post-Newtonian orders, and whether gravitational waves can be used at all to constrain these theories.

Note added.—Recently, a similar study of slowly rotating stars appeared on the arXiv [50]. Their work focuses on theories of the John class and deals also with their cosmological interpretation. Where our works overlap, they agree with our main conclusions.

APPENDIX: $I - \mathcal{C}$ RELATIONS

In this appendix we discuss the relation between the moment of inertia and the compactness for NSs in theories of the John and Ringo subclasses.

1. John (Nonminimal coupling with the Einstein tensor)

The behavior of I as function of q_∞ can be accurately described by a simple quadratic fit of the form

$$I = p_0 + p_1 q_\infty + p_2 q_\infty^2, \qquad (A1)$$

where (p_0, p_1, p_2) are constants. In the top panel of Fig. 3 we compare this relation with numerical data for $\eta = -1$ (note that for this figure we have computed models with additional values of q_∞ that were not displayed in Figs. 1 and 2 to avoid cluttering). The bottom panel of Fig. 3 shows that the relative errors between the numerical data and the fit are typically of order 0.1% or smaller.

To understand whether these relations hold also for theories of the John subclass, we have compared our numerical data against Eq. (39), computing the relative error $\Delta \bar{I}/\bar{I} = |1 - \bar{I}_{\text{fit}}/\bar{I}|$. The results are shown in the bottom panel of Fig. 4. Errors are always larger than in GR, and they
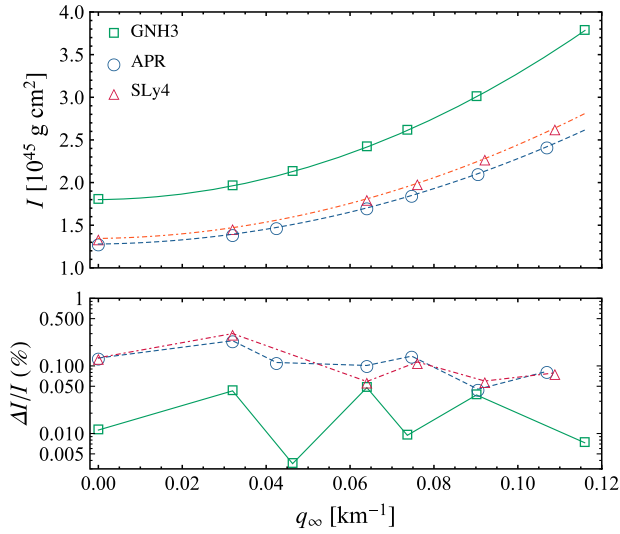
FIG. 3. Top panel: The moment of inertia I versus charge q_∞ for a canonical NS with $M = 1.4 M_\odot$ and $\eta = -1$, and the realistic EOSs APR, GNH3, and SLy4. Bottom panel: Relative percentage errors between the numerical data and the relation (A1).

FIG. 5. Relative errors between the improved fits and the numerical data. Top panel: $q_\infty = 0$. Middle panel: $q_\infty = 0.064$ and $\eta = 1$. Lower panel: $q_\infty = 0.064$ and $\eta = -1$.

can be as high as 40% for low-compactness configurations. A similar trend is observed for the I-Love-Q relations in GR in [87]. Deviations from the GR relation are due to the strong dependence of the star's bulk properties on the scalar charge q_∞, which spoils the (approximate) EOS universality of the relation proposed in [59]. Therefore we conclude that a theory-independent fit would perform poorly.
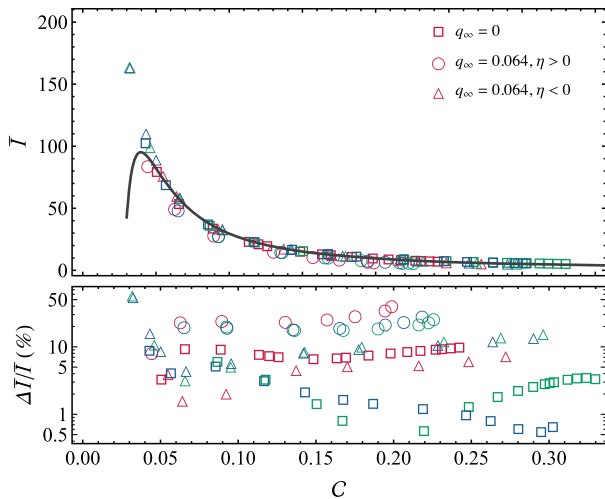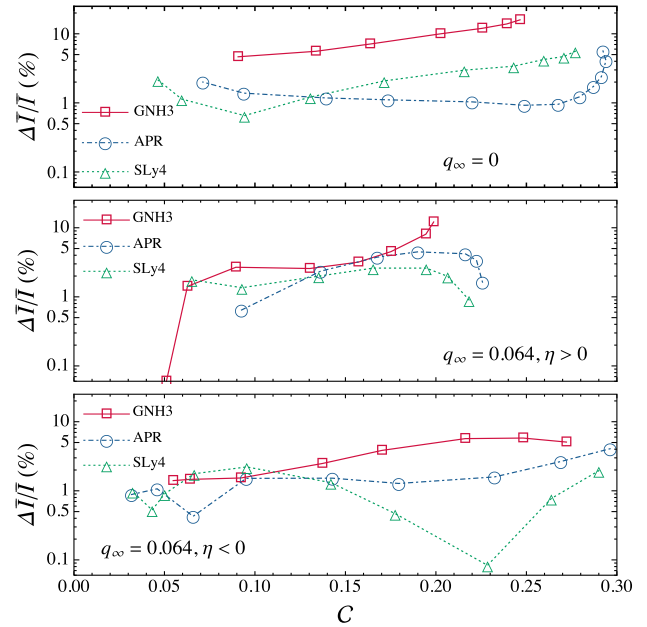

FIG. 4. Top panel: $\bar{I} - \mathcal{C}$ relation for different values of the scalar charge q_∞ and the realistic EOS APR (blue), GNH3 (red), SLy4 (green). The solid curve represents the fit given by Eq. (39), obtained in [59]. Bottom panel: Relative errors between the numerical data and the analytic relation. For illustrative purposes, we show the cases $q_\infty = 0$ and $q_\infty = 0.064$. For the latter, deviations from GR are more dramatic.

It is still possible to introduce approximately EOS-independent relations for $I - \mathcal{C}$ at fixed values of the theory parameters q_∞ and η using the functional form given in Eq. (39). The relative errors between the numerical data and these fits are shown in Fig. 5, and the corresponding fitting coefficients are listed in Table III. For almost all configurations the new relations perform better than Eq. (39), with relative errors that can be an order of magnitude smaller.

2. Ringo (Einstein-dilaton-Gauss-Bonnet gravity)

We have also investigated the $I - \mathcal{C}$ relations for theories of the Ringo subclass (EdGB gravity) using the numerical data from [44]. We found that the fit proposed in [59] works remarkably well for EdGB, with relative percentage errors $\lesssim 10\%$ for a wide range of compactness. This result is complementary to the $I - Q$ relations in EdGB obtained in [45]. We recall, however, that our calculations are limited to slow rotation. The question of whether or not rapidly rotating NSs in EdGB satisfy the same $I - \mathcal{C}$ relations of [59] could be addressed following the analysis of [45,46].

TABLE III. Numerical coefficients of the new universal $I - \mathcal{C}$ relations, for fixed values of q_∞ and η.

q_∞	η	a_1	a_2	a_3	a_4
0	$\cdots$	0.684	0.265	−0.0062	6.87×10^{-5}
0.032	1	0.666	0.240	−0.00364	-2.01×10^{-6}
0.032	−1	0.776	0.273	−0.00809	1.64×10^{-4}
0.064	1	0.0654	0.348	−0.0125	1.81×10^{-4}
0.064	−1	0.872	0.276	−0.00574	4.53×10^{-5}

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