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Relativistic stars in scalar-tensor theories with disformal coupling

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We present a general formulation to analyze the structure of slowly rotating relativistic stars in a broad class of scalar-tensor theories with disformal coupling to matter. Our approach includes theories with generalized kinetic terms, generic scalar field potentials and contains theories with conformal coupling as particular limits. In order to investigate how the disformal coupling affects the structure of relativistic stars, we propose a minimal model of a massless scalar-tensor theory and investigate in detail how the disformal coupling affects the spontaneous scalarization of slowly rotating neutron stars. We show that for negative values of the disformal coupling parameter between the scalar field and matter, scalarization can be suppressed, while for large positive values of the disformal coupling parameter stellar models cannot be obtained. This allows us to put a mild upper bound on this parameter. We also show that these properties can be qualitatively understood by linearizing the scalar field equation of motion in the background of a general-relativistic incompressible star. To address the intrinsic degeneracy between uncertainties in the equation of state of neutron stars and gravitational theory, we also show the existence of universal equation-of-state-independent relations between the moment of inertia and compactness of neutron stars in this theory. We show that in a certain range of the theory's parameter space the universal relation largely deviates from that of general relativity, allowing, in principle, to probe the existence of spontaneous scalarization with future observations.

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I. INTRODUCTION

Although Einstein's general relativity (GR) has passed all the experimental tests of gravity in the weak-field/slow-motion regimes with flying colors [1], it remains fairly unconstrained in the strong-gravity regime [2] and on the cosmological scales [3]. The recent observation of gravitational waves generated during the merger of two black holes (BHs) by the LIGO/Virgo Collaboration, in accordance with general-relativistic predictions [4,5], has offered us a first glimpse of gravity in a fully nonlinear and highly dynamical regime whose theoretical implications are still being explored [6]. Nevertheless, the pressing issues on understanding the nature of dark matter and dark energy, the inflationary evolution of the early Universe and the quest for an ultraviolet completion of GR have served as driving forces in the exploration of modifications to GR [2,3].

In general modifications of GR introduce new gravitational degree(s) of freedom in addition to the metric tensor and can be described by a scalar-tensor theory of gravity [7]. On the theoretical side, scalar-tensor theories should not contain Ostrogradski ghosts [8], i.e. the equations of motion should be written in terms of the second-order differential equations despite the possible existence of the higher-order derivative interactions at the action level. On the experimental/observational side, any extension of GR must pass all the current weak-field tests which GR has successfully passed. Therefore realistic modifications of gravity should contain a mechanism to suppress scalar interactions at small scales [9,10] or (to be interesting) satisfy weak-field tests, but deviate from GR at some energy scale. Some models satisfying these requirements belongs to the so-called Horndeski theory [11–14], the most general scalar-tensor theory with second-order equations of motion.

In scalar-tensor theories, the scalar field may directly couple to matter, and hence matter does not follow geodesics associated with the metric $g_{\mu\nu}$ but with another $\tilde{g}_{\mu\nu}$. In the simplest case these two metrics are related as

$$\tilde{g}_{\mu\nu} = A^2(\varphi) g_{\mu\nu}, \tag{1}$$

which is known as the conformal coupling [3]. The two frames described by $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are often referred to as the Einstein and Jordan frames, respectively.

A. Spontaneous scalarization

For relativistic stars, such as neutron stars (NSs), the conformal coupling to matter can trigger a tachyonic instability (due to a negative effective mass) of the scalar field when the star has a compactness above a certain

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threshold. This instability *spontaneously scalarizes* the NS, whereupon it harbors a nontrivial scalar field configuration which smoothly decays outside the star. In its simplest realization, scalarization occurs when the conformal factor in Eq. (1) is chosen as $A(\varphi) = \exp(\beta_1 \varphi^2/2)$, where β_1 is a free parameter of the theory and φ is a massless scalar field. This theory passes all weak-field tests, but the presence of the scalar field can significantly modify the bulk properties of NSs, such as masses and radii, in comparison with GR. This effect was first analyzed for isolated NSs by Damour and Esposito-Farèse [15,16]. The properties and observational consequences of this phenomenon were studied in a number of situations, including stability [17,18], asteroseismology [19–22], slow (and rapidly) rotating NS solutions [16,23–26], its influence on geodesic motion of particles around NSs [27,28], tidal interactions [26] and the multipolar structure of the spacetime [29,30]. Moreover, the dynamical process of scalarization was studied in Ref. [31] and stellar collapse (including the associated process of scalar radiation emission) was investigated in Refs. [32–34]. We refer the reader to Ref. [35] for an extensive literature review.

Additionally, a semiclassical version of this effect [36] (cf. also Refs. [37–40] and Ref. [41] for a connection with the Damour-Esposito-Farèse model [15,16]) has been shown to *awaken the vacuum* state of a quantum field leading to an exponential growth of its vacuum energy density in the background of a relativistic star.

These nontrivial excitations of scalar fields induced by relativistic stars are a consequence of the generic absence of a "no-hair theorem" for these objects (see Refs. [42–44] for counterexamples), in contrast to the case of BHs, and can potentially be an important source for signatures of the presence of fundamental gravitational scalar degrees of freedom through astronomical observations [2,45], including the measurements of gravitational and scalar radiation signals [46].

The phenomenological implications of spontaneous scalarization have also been explored in binary NS mergers [47–50] and in BHs surrounded by matter [51,52]. In the former situation, a *dynamical* scalarization allows binary members to scalarize under conditions where this would not happen if they were isolated. This effect can dramatically change the dynamics of the system in the final cycles before the merger with potentially observable consequences. In the latter case, the presence of matter can cause the appearance of a nontrivial scalar field configuration, growing "hair" on the BH.

On the experimental side, binary-pulsar observations [53] have set stringent bounds on β_1, whose value is presently constrained to be $\beta_1 \gtrsim -4.5$. This tightly constrains the effects of spontaneous scalarization in isolated NSs, for it has been shown that independently of the choice of the equation of state (EOS) scalarization can occur only if $\beta_1 \lesssim -4.35$ for NSs modeled by a perfect fluid [31,54,55]. These two results confine β_1 to a very limited range, in which, even if it exists in nature, the effects of scalarization on isolated NSs are bound to be small; see Refs. [24,55] for examples where the threshold value of β_1 can be increased and Refs. [56–58] for recent work exploring the large positive β_1 region of the theory.

B. Disformal coupling

It was recently understood that modern scalar-tensor theories of gravity, under the umbrella of Horndeski gravity [11,59], offer a more general class of coupling [60,61] between the scalar field and matter through the so-called *disformal coupling* [62]

$$\tilde{g}_{\mu\nu} = A^2(\varphi)[g_{\mu\nu} + \Lambda B^2(\varphi)\varphi_\mu \varphi_\nu], \tag{2}$$

where $\varphi_\mu = \nabla_\mu \varphi$ is the covariant derivative of the scalar field associated with the gravity frame metric $g_{\mu\nu}$, and Λ is a constant with dimensions of $(\mathrm{length})^2$. For $\Lambda = 0$ we recover the purely conformal case of Eq. (1). Disformal transformations were originally introduced by Bekenstein and consist of the most general coupling constructed from the metric $g_{\mu\nu}$ and the scalar field φ that respects causality and the weak equivalence principle [62]. Disformal couplings have been investigated so far mainly in the context of cosmology [63–65]. They also arise in higher-dimensional gravitational theories with moving branes [66,67] in relativistic extensions of modified Newtonian theories, the tensor-vector-scalar theories [68,69], and in the decoupling limit of the nonlinear massive gravity [70–73]. Moreover, in Ref. [60] it was shown that the mathematical structure of Horndeski theory is preserved under the transformation (2), namely if the scalar-tensor theory written in terms of $g_{\mu\nu}$ belongs to a class of the Horndeski theory the same theory rewritten in terms of $\tilde{g}_{\mu\nu}$ belongs to another class of the Horndeski theory. Thus disformal transformations provide a natural generalization of conformal transformations.

Disformal coupling was also considered in models of a varying speed of light [74] and inflation [75,76]. The invariance of cosmological observables in the frames related by the disformal relation (2) was verified in Refs. [77–82]. Although applications to early Universe models are still limited, disformal couplings have been extensively applied to late-time cosmology [63,65,66, 83–89]. A new screening mechanism of the scalar force in the high-density region was proposed in Ref. [86], where in the presence of disformal coupling the nonrelativistic limit of the scalar field equation seemed to be independent of the local energy density. However, a reanalysis suggested that no new screening mechanism from disformal coupling could work [83,90]. It was also argued that disformal coupling could not contribute to a chameleon screening mechanism around a nonrelativistic source [91]. Experimental and observational constraints on disformal coupling to particular matter sectors have also been investigated. Disformal couplings to baryons and photons have been severely constrained in terms of the nondetection of new physics in collider experiments [75,92–96], the absence of spectral distortion of the cosmic microwave background and the violation of distance reciprocal relations [94,97–99], respectively. On the other hand, disformal coupling to the dark sector has been proposed in Refs. [84,100] and is presently less constrained in comparison with coupling to visible matter sectors.

When conformal and disformal couplings are *universal* to all the matter species, they can only be constrained through experimental tests of gravity. A detailed study of scalar-tensor theory with the pure disformal coupling $A(\varphi) = 1$ and $B(\varphi) = 1$ in the weak-field limit was presented in Ref. [83] and the post-Newtonian (PN) corrections due to the presence of pure disformal coupling were computed [90]. In these papers [83,90], in contrast to the claim of Refs. [66,86], it was shown that no screening mechanism which could suppress the scalar force in the vicinity of the source exists and the difference of the parametrized post-Newtonian (PPN) parameters from GR are of order $|\Lambda| H_0^2$, where $H_0(\sim 10^{-28}\ \mathrm{cm}^{-1})$ is the present-day Hubble scale. The strongest bound on $|\Lambda|$ comes from the constraints on the PPN preferred frame parameter α_2. The near perfect alignment between the Sun's spin axis and the orbital angular momenta of the planets provides the constraint $\alpha_2 < 4 \times 10^{-5}$ (see Ref. [101] for a discussion), which implies that $|\Lambda| \lesssim 10^{-6} H_0^{-2} (\sim 10^{40}\ \mathrm{km}^2)$. With the inclusion of the conformal factor, i.e. $A(\varphi) \neq 1$, the authors of Ref. [90] argued that the Cassini bound $|\gamma - 1| < 2.1 \times 10^{-5}$ [102] imposes a constraint on $\alpha(\varphi_0)$, where φ_0 is the cosmological background value of the scalar field and

$$\alpha(\varphi) \coloneqq \frac{d \log A(\varphi)}{d\varphi}, \qquad \beta(\varphi) \coloneqq \frac{d \log B(\varphi)}{d\varphi}. \tag{3}$$

On the other hand the disformal part of the coupling $\beta(\varphi_0)$ remains unconstrained, because corrections to the PPN parameters which include $\beta(\varphi)$ are subdominant compared to the conformal part. These weaker constraints on the disformal coupling parameters are due to the fact that in the nonrelativistic regime with negligible pressure and a slowly evolving scalar field the disformal coupling becomes negligible. We also point out that in the weak-field regime such as in the Solar System, typical densities are small therefore preventing the appearance of ghosts in the theory for negative values of Λ.

In the strong-gravity regime such as that found in the interior of NSs, the pressure cannot be neglected and the disformal coupling is expected to be as important as the conformal one. This would affect the spontaneous scalarization mechanism and consequently influence the structure (and stability) of relativistic stars, or have a significant impact on gravitational-wave astronomy [2]. The influence of disformal coupling on the stability of matter configurations around BHs was analyzed in Ref. [103]. The authors of Ref. [103] derived the stability conditions of the system by generalizing the case of pure conformal coupling [51,52]. They also generalized these works to scalar-tensor theories with noncanonical kinetic terms and disformal coupling, finding that the disformal coupling could make matter configurations more unstable, triggering spontaneous scalarization. In the present work within the same class of scalar-tensor theory considered in Ref. [103], we will study relativistic stars and investigate the influence of disformal coupling on the scalarization of NSs.

C. Organization of this work

This paper is organized as follows. In Sec. II we review the fundamentals of scalar-tensor theories with a generalized kinetic term and disformal coupling. In Sec. III we present a general formulation to analyze the structure of slowly rotating stars in theories with disformal coupling. In Sec. IV, as a case study, we consider a canonical scalar field with a generic scalar field potential. We particularize the stellar structure equations to this model and discuss how to solve them numerically. In Sec. V we explore the consequences of the disformal coupling by studying small scalar perturbations to an incompressible relativistic star in GR. In particular we investigate the conditions for which spontaneous scalarization happens. In Sec. VI we present our numerical studies about the influence of disformal coupling on the spontaneous scalarization by solving the full stellar structure equations. In Sec. VII as an application of our numerical integrations, we examine the EOS independence between the moment of inertia and compactness of NSs in scalar-tensor theory comparing it against the results obtained in GR. Finally, in Sec. VIII we summarize our main findings and point out possible future avenues of research.

II. SCALAR-TENSOR THEORY WITH THE DISFORMAL COUPLING

We consider scalar-tensor theories in which matter is disformally coupled to the scalar field. The action in the Einstein frame reads

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [R + 2P(X, \varphi)] + \int d^4x \sqrt{-\tilde{g}(\varphi, \varphi_\mu)} \mathcal{L}_{\mathrm{m}} [\tilde{g}_{\mu\nu}(\varphi, \varphi_\mu), \Psi], \tag{4}$$

where x^μ ($\mu = 0, 1, 2, 3$) represents the coordinate system of the spacetime, $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are respectively the Einstein and Jordan frame metrics disformally related by (2), $g \coloneqq \det(g_{\mu\nu})$ and $\tilde{g} \coloneqq \det(\tilde{g}_{\mu\nu})$, R is the Ricci scalar curvature associated with $g_{\mu\nu}$, $\kappa \coloneqq (8\pi G)/c^4$, where G is the gravitational constant defined in the Einstein frame and c is the speed of light in vacuum. $P(X,\varphi)$ is an arbitrary function of the scalar field φ and $X \coloneqq -\frac{1}{2}g^{\mu\nu}\varphi_\mu\varphi_\nu$, and $\mathcal{L}_{\rm m}$ represents the Lagrangian density of matter fields Ψ. We note that the canonical scalar field corresponds to the case of $P(X,\varphi) = 2X - V(\varphi)$, but we will not restrict the form of $P(X,\varphi)$ at this stage. In this paper we will not omit G and c.

Varying the action (4) with respect to the Einstein frame metric $g_{\mu\nu}$, we obtain the Einstein field equations

$$G^{\mu\nu} = \kappa(T^{\mu\nu}_{(\rm m)} + T^{\mu\nu}_{(\varphi)}), \tag{5}$$

where the energy-momentum tensors of the matter fields Ψ and scalar field φ are given by

$$T^{\mu\nu}_{(\rm m)} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-\tilde{g}}\mathcal{L}_{\rm m}[\tilde{g}(\varphi),\Psi])}{\delta g_{\mu\nu}}, \tag{6}$$

and

$$\begin{aligned} T^{\mu\nu}_{(\varphi)} &\coloneqq \frac{1}{\kappa}\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}P(X,\varphi))}{\delta g_{\mu\nu}} \\ &= \frac{1}{\kappa}(P_X\varphi^\mu\varphi^\nu + Pg^{\mu\nu}), \end{aligned} \tag{7}$$

respectively, where $P_X \coloneqq \partial_X P$ and $\varphi^\mu \coloneqq g^{\mu\nu}\varphi_\nu$. From Eq. (2), the inverse Jordan frame metric $\tilde{g}^{\mu\nu}$ is related to the inverse Einstein frame metric $g^{\mu\nu}$ by

$$\tilde{g}^{\mu\nu} = A^{-2}(\varphi)\left[g^{\mu\nu} - \frac{\Lambda B^2(\varphi)}{\chi(X,\varphi)}\varphi^\mu\varphi^\nu\right], \tag{8}$$

where we have defined

$$\chi(X,\varphi) \coloneqq 1 - 2\Lambda B^2(\varphi)X. \tag{9}$$

The volume element in the Jordan frame $\sqrt{-\tilde{g}}$ is given by $\sqrt{-\tilde{g}} = A^4(\varphi)\sqrt{-g}\sqrt{\chi(X,\varphi)}$. In order to keep the Lorentzian signature of the Jordan frame metric $\tilde{g}_{\mu\nu}$, χ must be non-negative. We note that in the purely conformal coupling limit $\Lambda = 0$ and $\chi = 1$.

The contravariant energy-momentum tensor in the Jordan frame $\tilde{T}^{\mu\nu}_{(\rm m)}$ is related to that in the Einstein frame by

$$\begin{aligned} \tilde{T}^{\mu\nu}_{(\rm m)} &\coloneqq \frac{2}{\sqrt{-\tilde{g}}}\frac{\delta(\sqrt{-\tilde{g}}\mathcal{L}_{\rm m}[\tilde{g},\Psi])}{\delta\tilde{g}_{\mu\nu}}, \\ &= \sqrt{\frac{g}{\tilde{g}}}\frac{\delta g_{\alpha\beta}}{\delta\tilde{g}_{\mu\nu}}T^{\alpha\beta}_{(\rm m)} = \frac{A^{-6}(\varphi)}{\sqrt{\chi(X,\varphi)}}T^{\mu\nu}_{(\rm m)}. \end{aligned} \tag{10}$$

The mixed and covariant energy-momentum tensors in the Jordan frame are respectively given by

$$\tilde{T}_{(\rm m)\mu}{}^{\nu} = \frac{A^{-4}(\varphi)}{\sqrt{\chi(X,\varphi)}}(\delta^\alpha_\mu + \Lambda B^2(\varphi)\varphi_\mu\varphi^\alpha)T_{(m)\alpha}{}^{\nu}, \tag{11a}$$

$$\begin{aligned} \tilde{T}_{(\rm m)\mu\nu} = \frac{A^{-2}(\varphi)}{\sqrt{\chi(X,\varphi)}}&(\delta^\alpha_\mu + \Lambda B^2(\varphi)\varphi_\mu\varphi^\alpha) \\ &\times(\delta^\beta_\nu + \Lambda B^2(\varphi)\varphi_\nu\varphi^\beta)T_{(\rm m)\alpha\beta}, \end{aligned} \tag{11b}$$

and

$$T^{\mu\nu}_{(\rm m)} = A^6(\varphi)\sqrt{\chi(X,\varphi)}\tilde{T}^{\mu\nu}_{(\rm m)}, \tag{12a}$$

$$T_{(\rm m)\nu}{}^{\mu} = A^4(\varphi)\sqrt{\chi(X,\varphi)}\left(\delta^\rho_\nu - \frac{\Lambda B^2(\varphi)\varphi^\rho\varphi_\nu}{\chi(X,\varphi)}\right)\tilde{T}_{(\rm m)\rho}{}^{\mu}, \tag{12b}$$

$$\begin{aligned} T_{(\rm m)\mu\nu} = A^2(\varphi)\sqrt{\chi(X,\varphi)}&\left(\delta^\rho_\mu - \frac{\Lambda B^2(\varphi)\varphi^\rho\varphi_\mu}{\chi(X,\varphi)}\right) \\ &\times\left(\delta^\sigma_\nu - \frac{\Lambda B^2(\varphi)\varphi^\sigma\varphi_\nu}{\chi(X,\varphi)}\right)\tilde{T}_{(\rm m)\rho\sigma}. \end{aligned} \tag{12c}$$

In terms of the covariant tensors, the Einstein equations in the Einstein frame (5) can be recast as

$$\begin{aligned} G_{\mu\nu} = \kappa A^2(\varphi)\sqrt{\chi(X,\varphi)}&\left(\delta^\rho_\mu - \frac{\Lambda B^2(\varphi)\varphi^\rho\varphi_\mu}{\chi(X,\varphi)}\right) \\ &\times\left(\delta^\sigma_\nu - \frac{\Lambda B^2(\varphi)\varphi^\sigma\varphi_\nu}{\chi(X,\varphi)}\right)\tilde{T}_{(\rm m)\rho\sigma} + P_X\varphi_\mu\varphi_\nu \\ &+ g_{\mu\nu}P. \end{aligned} \tag{13}$$

Varying the action (4) with respect to the scalar field φ, we obtain the scalar field equation of motion

$$P_X\Box\varphi + P_\varphi - P_{XX}\varphi^\rho\varphi^\sigma\varphi_{\rho\sigma} - 2XP_{X\varphi} = \kappa\mathcal{Q}, \tag{14}$$

where the function $\mathcal{Q}$ characterizes the strength of the coupling of matter to the scalar field

$$\begin{aligned} \mathcal{Q} \coloneqq \Lambda\nabla_\rho(B^2(\varphi)T^{\rho\sigma}_{(\rm m)}\varphi_\sigma) &- \alpha(\varphi)T_{(\rm m)} \\ &- \Lambda B^2(\varphi)[\alpha(\varphi) + \beta(\varphi)]T^{\rho\sigma}_{(\rm m)}\varphi_\rho\varphi_\sigma, \end{aligned} \tag{15}$$

where $T_{(\rm m)} \coloneqq g^{\rho\sigma}T_{(\rm m)\rho\sigma}$ is the trace of $T_{(m)\rho\sigma}$, and $\alpha(\varphi)$ and $\beta(\varphi)$ were defined in Eq. (3). Taking the divergence of Eq. (5), employing the contracted Bianchi identity $\nabla_\rho G^{\rho\sigma} = 0$, and using the scalar field equation of motion (14), we obtain

$$\nabla_\rho T^{\rho\sigma}_{(\rm m)} = -\nabla_\rho T^{\rho\sigma}_{(\varphi)} = -\mathcal{Q}\varphi^\sigma, \tag{16}$$

and the coupling strength $\mathcal{Q}$ can be rewritten as

$$\mathcal{Q} = \Lambda B^2(\varphi)(\nabla_\rho T^{\rho\sigma}_{(\mathrm{m})})\varphi_\sigma + \mathcal{Y}, \tag{17}$$

where we have introduced

$$\mathcal{Y} \coloneqq \Lambda B^2(\varphi)\{[\beta(\varphi) - \alpha(\varphi)]T^{\rho\sigma}_{(\mathrm{m})}\varphi_\rho\varphi_\sigma + T^{\rho\sigma}_{(\mathrm{m})}\varphi_{\rho\sigma}\} - \alpha(\varphi)T_{(\mathrm{m})}. \tag{18}$$

Multiplying Eq. (16) by φ_σ and solving it with respect to $(\nabla_\rho T^{\rho\sigma}_{(\mathrm{m})})\varphi_\sigma$, we obtain

$$\chi(\nabla_\rho T^{\rho\sigma}_{(\mathrm{m})})\varphi_\sigma = 2X\mathcal{Y}. \tag{19}$$

Then, substituting it in Eq. (17), using $\mathcal{Q} = \mathcal{Y}/\chi$, and finally eliminating $\mathcal{Q}$ from Eq. (14), we obtain the reduced scalar field equation of motion

$$\begin{aligned} &P_X\Box\varphi + P_\varphi - P_{XX}\varphi^\rho\varphi^\sigma\varphi_{\rho\sigma} - 2XP_{X\varphi} \\ &\quad = \frac{\kappa}{\chi(X,\varphi)} \times \{\Lambda B^2(\varphi)[(\beta(\varphi) - \alpha(\varphi))T^{\rho\sigma}_{(\mathrm{m})}\varphi_\rho\varphi_\sigma \\ &\qquad + T^{\rho\sigma}_{(\mathrm{m})}\varphi_{\rho\sigma}] - \alpha(\varphi)T_{(\mathrm{m})}\}. \end{aligned} \tag{20}$$

III. THE EQUATIONS OF STELLAR STRUCTURE

A. Equations of motion

In this section, we consider a static and spherically symmetric spacetime with line element

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -e^{\nu(r)}c^2dt^2 + e^{\lambda(r)}dr^2 + r^2\gamma_{ij}d\theta^i d\theta^j, \end{aligned} \tag{21}$$

where $\nu(r)$ and $\lambda(r)$ are functions of the radial coordinate r only, γ_{ij} is the metric of the unit 2-sphere, and the coordinates θ^i $(i = 1, 2)$ run over the directions of the unit 2-sphere, such that $\gamma_{ij}d\theta^i d\theta^j = d\theta^2 + \sin^2\theta d\phi^2$. We also assume by symmetry that the scalar field is only a function of r, $\varphi = \varphi(r)$. Hence the coupling functions $A(\varphi)$ and $B(\varphi)$ are also only functions of r through $\varphi(r)$.

We assume that in the Jordan frame only the diagonal components of the energy-momentum tensor of matter are nonvanishing

$$\tilde{T}_{(\mathrm{m})}{}^t{}_t = -\tilde{\rho}c^2, \qquad \tilde{T}_{(\mathrm{m})}{}^r{}_r = \tilde{p}_r, \qquad \tilde{T}_{(\mathrm{m})}{}^i{}_j = \tilde{p}_t\delta^i{}_j, \tag{22}$$

where $\tilde{\rho}$, $\tilde{p}_r$ and $\tilde{p}_t$ are respectively the energy density, radial and tangential pressures of an anisotropic fluid in the Jordan frame [104]. Using Eq. (12b), they are related to the components of the energy-momentum tensor of matter in the Einstein frame, which are represented by the quantities without a tilde, by

$$\begin{aligned} &\rho = A^4(\varphi)\sqrt{\chi}\tilde{\rho}, \qquad p_r = \frac{A^4(\varphi)}{\sqrt{\chi}}\tilde{p}_r, \\ &p_t = A^4(\varphi)\sqrt{\chi}\tilde{p}_t, \end{aligned} \tag{23}$$

where in the background given by Eq. (21), the quantity χ defined in Eq. (9) reduces to

$$\chi = 1 + e^{-\lambda}\Lambda B^2(\varphi)(\varphi')^2. \tag{24}$$

We note that even if the fluid in the Jordan frame has an isotropic pressure, $\tilde{p}_r = \tilde{p}_t$, it is transformed into an anisotropic one in the Einstein frame i.e. $p_r \neq p_t$ in the presence of disformal coupling $\chi \neq 1$.

The (t,t), (r,r) and the trace of (i,j) components of the Einstein equations (13) are given by

$$\frac{1}{r^2}[1 - e^{-\lambda}(1 - r\lambda')] = -P + A^4(\varphi)\sqrt{\chi}\kappa\tilde{\rho}c^2, \tag{25}$$

$$\frac{e^\lambda}{r^2}[1 - e^{-\lambda}(1 + r\nu')] = -(\varphi')^2P_X - e^\lambda\left[P + \frac{A^4(\varphi)}{\sqrt{\chi}}(\kappa\tilde{p}_r)\right], \tag{26}$$

$$\frac{1}{2}\left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r}\right)(\nu' - \lambda')\right] = e^\lambda[P + A^4(\varphi)\sqrt{\chi}(\kappa\tilde{p}_t)]. \tag{27}$$

On the other hand, the scalar field equation of motion (20) reduces to

$$\begin{aligned} &\chi\left\{P_X e^{-\lambda}\left[\varphi'' + \left(\frac{\nu'}{2} - \frac{\lambda'}{2} + \frac{2}{r}\right)\varphi'\right] + P_\varphi - P_{XX}e^{-2\lambda}(\varphi')^2\left(\varphi'' - \frac{\lambda'}{2}\varphi'\right) + e^{-\lambda}(\varphi')^2P_{X\varphi}\right\} \\ &\quad = \kappa\frac{A^4(\varphi)}{\varphi'}\left\{\frac{\tilde{p}_r}{\sqrt{\chi}}\left[-\alpha(\varphi)\varphi' + \Lambda B^2(\varphi)e^{-\lambda}\varphi'\left(\varphi'' + \left(\beta(\varphi)\varphi' - \alpha(\varphi)\varphi' - \frac{\lambda'}{2}\right)\varphi'\right)\right]\right. \\ &\qquad \left. - \sqrt{\chi}\left[\alpha(\varphi)\varphi'(-\tilde{\rho}c^2 + 2\tilde{p}_t) + \Lambda B^2(\varphi)e^{-\lambda}\left(\frac{\nu'}{2}\tilde{\rho}c^2 - \frac{2}{r}\tilde{p}_t\right)(\varphi')^2\right]\right\}. \end{aligned} \tag{28}$$

The nontrivial radial component of the energy-momentum conservation law in the Einstein frame (16) gives us

$$\frac{d\tilde{p}_r}{dr} = -\left[\frac{\nu'}{2} + \alpha(\varphi)\varphi'\right](\tilde{\rho}c^2 + \tilde{p}_r) - 2\left[\frac{1}{r} + \alpha(\varphi)\varphi'\right]\tilde{\sigma}, \tag{29}$$

where we have defined $\tilde{\sigma} := \tilde{p}_r - \tilde{p}_t$, which measures the degree of anisotropy of the fluid [104]. The same result can be obtained from the conservation law in the Jordan frame $\tilde{\nabla}_\rho \tilde{T}^{\rho r}_{(\mathrm{m})} = 0$, where $\tilde{\nabla}_\rho$ represents the covariant derivative associated with the Jordan frame metric $\tilde{g}_{\mu\nu}$. The conservation law (29) depends implicitly on $B(\varphi)$ and its derivative through ν' [cf. Eq. (26)].

B. The reduced equations of motion

We then reduce the set of equations (25)–(27), (28) and (29) into a form more convenient for a numerical integration. We introduce the mass function $\mu(r)$ through

$$e^{-\lambda(r)} := 1 - \frac{2\mu(r)}{r}, \tag{30}$$

and replace all $\lambda(r)$ dependence with $\mu(r)$. We also introduce the first-order derivative of the scalar field $\psi(r)$, i.e.

$$\psi := \frac{d\varphi}{dr}. \tag{31}$$

We can write the kinetic energy as

$$X = -\frac{r-2\mu}{2r}\psi^2 \tag{32}$$

and χ can then be expressed as

$$\chi = 1 + \frac{r-2\mu}{r}\Lambda B^2(\varphi)\psi^2. \tag{33}$$

The (t,t) component of the Einstein equations [cf. Eq. (25)] determines the gradient of μ

$$\frac{d\mu}{dr} = \frac{r^2}{2}[A^4(\varphi)\sqrt{\chi}\kappa\tilde{\rho}c^2 - P]. \tag{34}$$

Similarly, the (r,r) component of the Einstein equations (26) reduces to

$$\frac{d\nu}{dr} = \frac{2\mu}{r(r-2\mu)} + r\left\{\psi^2 P_X + \frac{r}{r-2\mu}\left[P + \frac{A^4(\varphi)}{\sqrt{\chi}}(\kappa\tilde{p}_r)\right]\right\}. \tag{35}$$

The conservation law (29) combined with Eq. (35) leads to

$$\frac{d\tilde{p}_r}{dr} = -\left\{\alpha(\varphi)\psi + \frac{\mu}{r(r-2\mu)} + \frac{r}{2}\left[\psi^2 P_X + \frac{r}{r-2\mu}\left(P + \frac{A^4(\varphi)}{\sqrt{\chi}}(\kappa\tilde{p}_r)\right)\right]\right\}(\tilde{\rho}c^2 + \tilde{p}_r) - 2\left[\frac{1}{r} + \alpha(\varphi)\psi\right]\tilde{\sigma}. \tag{36}$$

Finally, the scalar field equation of motion (28) reduces to

$$\begin{aligned}
&\left[\chi(P_X - e^{-\lambda}\psi^2 P_{XX}) - \kappa\Lambda A^4(\varphi)B^2(\varphi)\frac{\tilde{p}_r}{\sqrt{\chi}}\right]\psi' + \left\{\chi\left[\left(\frac{\nu'}{2} - \frac{\lambda'}{2} + \frac{2}{r}\right)P_X + \frac{\lambda'}{2}e^{-\lambda}\psi^2 P_{XX} + \psi P_{X\varphi}\right]\right.\\
&\quad\left. - \kappa\Lambda A^4(\varphi)B^2(\varphi)\left[\sqrt{\chi}\left(-\frac{\nu'}{2}\tilde{\rho}c^2 + \frac{2}{r}\tilde{p}_t\right) + \frac{\tilde{p}_r}{\sqrt{\chi}}\left(\beta(\varphi)\psi - \alpha(\varphi)\psi - \frac{\lambda'}{2}\right)\right]\right\}\psi\\
&= -e^{\lambda}\chi P_\varphi + \kappa A^4(\varphi)\alpha(\varphi)e^{\lambda}\left[-\frac{\tilde{p}_r}{\sqrt{\chi}} + \sqrt{\chi}(\tilde{\rho}c^2 - 2\tilde{p}_t)\right].
\end{aligned} \tag{37}$$

Eliminating λ' and ν' from Eq. (37), and using Eqs. (25)–(26), the scalar field equation of motion (37) can be rewritten as

$$C_2\frac{d\psi}{dr} = -C_1\psi + \frac{r}{r-2\mu}\left\{-\chi P_\varphi + \kappa A^4(\varphi)\alpha(\varphi)\left[-\frac{\tilde{p}_r}{\sqrt{\chi}} - \sqrt{\chi}(-\tilde{\rho}c^2 + 2\tilde{p}_t)\right]\right\}, \tag{38}$$

where we introduced

$$
C_2 = \chi\left[P_X - \left(1 - \frac{2\mu}{r}\right)\psi^2 P_{XX}\right] - \kappa\Lambda A^4(\varphi)B^2(\varphi)\frac{\tilde{p}_r}{\sqrt{\chi}},
$$

$$
\begin{aligned}
C_1 = \chi\Bigg\{ & P_X\left[\frac{2(r-\mu)}{r(r-2\mu)} + \frac{r}{2}\psi^2 P_X + \frac{r^2}{r-2\mu}\left(P - \frac{\kappa}{2}A^4(\varphi)\left(\sqrt{\chi}\tilde{\rho}c^2 - \frac{\tilde{p}_r}{\sqrt{\chi}}\right)\right)\right] \\
& + \frac{1}{2}\left[-\frac{2\mu}{r^2} + r(-P + A^4(\varphi)\sqrt{\chi}(\kappa\tilde{\rho}c^2))\right]\psi^2 P_{XX} + \psi P_{X\varphi}\Bigg\} \\
& - \kappa\Lambda A^4(\varphi)B^2(\varphi)\Bigg\{-\frac{1}{r-2\mu}\left(\frac{\mu}{r} + \frac{r^2 P}{2}\right)\left(\sqrt{\chi}\tilde{\rho}c^2 - \frac{\tilde{p}_r}{\sqrt{\chi}}\right) - \frac{\tilde{\rho}c^2\sqrt{\chi}}{2}\psi^2 P_X r \\
& - \frac{\kappa r^2}{r-2\mu}(\tilde{\rho}c^2\tilde{p}_r)A^4(\varphi) + \frac{2\sqrt{\chi}}{r}\tilde{p}_t + \frac{\psi}{\sqrt{\chi}}(\beta(\varphi) - \alpha(\varphi))\tilde{p}_r\Bigg\}. \qquad (39)
\end{aligned}
$$

The set of Eqs. (31), (34), (35), (36) and (38) together with a given EOS

$$
\tilde{p}_r = \tilde{p}_r(\tilde{\rho}), \qquad \tilde{p}_t = \tilde{p}_t(\tilde{\rho}), \qquad (40)
$$

form a closed system of equations to analyze the structure of relativistic stars in the scalar-tensor theory (4).

C. Slowly rotating stars

In this subsection, we extend our calculation to the case of slowly rotating stars. Once the set of the equations of motion for a static and spherically symmetric star is given, it is simple to take first-order corrections due to rotation into consideration using the Hartle-Thorne scheme [105,106]. At first order in the Hartle-Thorne perturbative expansion, we derive our results in a manner as general as possible, similarly to the previous section.

In the Einstein frame, the line element including the first-order correction due to rotation is given by

$$
\begin{aligned}
ds^2 = & -e^{\nu(r)}c^2dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\
& + 2(\omega - \Omega)r^2\sin^2\theta dt d\phi, \qquad (41)
\end{aligned}
$$

where $\omega(r)$ is a function of r, which is of the same order as the star's angular velocity Ω. We can construct the Jordan frame line element using Eqs. (2) and (8). The construction of the energy-momentum tensor for the anisotropic fluid in the Jordan frame is similar to what was done before, except that now, the normalization of the four-velocity, demands that

$$
\tilde{u}^t = [-(\tilde{g}_{tt} + 2\tilde{\Omega}\tilde{g}_{t\phi} + \tilde{\Omega}^2\tilde{g}_{\phi\phi})]^{-1/2}, \qquad (42a)
$$

$$
\tilde{u}^r = \tilde{u}^\theta = 0, \qquad \tilde{u}^\phi = \tilde{\Omega}\tilde{u}^t, \qquad (42b)
$$

where $\tilde{\Omega}$ is the star's angular velocity in the Jordan frame [measured in the coordinates of $x^\mu = (t, r, \theta, \phi)$],

$$
\tilde{g}_{tt} = A^2 g_{tt}, \qquad \tilde{g}_{rr} = A^2[g_{rr} + \Lambda B^2(\varphi')^2], \qquad (43a)
$$

$$
\tilde{g}_{ij} = A^2 g_{ij}, \qquad (i, j = \theta, \phi) \qquad (43b)
$$

$$
\tilde{g}_{t\phi} = A^2 g_{t\phi}, \qquad (43c)
$$

and we must expand all expressions, keeping only terms of order $\mathcal{O}(\Omega)$. As shown in the Appendix, the star's angular velocity is disformally invariant, $\tilde{\Omega} = \Omega$. We also note that rotation can induce a dependence of the scalar field on θ, which appears however only at more than second order in rotation, $\mathcal{O}(\Omega^2)$ [26]. Thus in our case, the scalar field configuration remains the same as in the nonrotating situation.

At the first order in rotation, the diagonal components of the Einstein equations and the scalar field equation of motion remain the same as Eqs. (31), (34), (35), (36) and (38). A new equation comes however from the (t, ϕ) component of the Einstein equation:

$$
\begin{aligned}
& \frac{d^2\omega}{dr^2} - \left(\frac{4}{r} - \frac{\lambda' + \nu'}{2}\right)\frac{d\omega}{dr} \\
& + 2\kappa A^4(\varphi)r\sqrt{\chi}\frac{(\tilde{\rho}c^2 + \tilde{p}_r - \tilde{\sigma})}{(r - 2\mu)}\omega(r) = 0. \qquad (44)
\end{aligned}
$$

By eliminating ν' and λ' with the use of Eqs. (34) and (35), we obtain the frame-dragging equation

$$
\begin{aligned}
& \frac{d^2\omega}{dr^2} + \left[\frac{1}{2}rP_X\psi^2 + \frac{\kappa r^2 A^4(\varphi)}{2\sqrt{\chi}(r - 2\mu)}(\tilde{p}_r + \chi\tilde{\rho}c^2) - \frac{4}{r}\right]\frac{d\omega}{dr} \\
& + 2\kappa A^4(\varphi)r\sqrt{\chi}\frac{(\tilde{\rho}c^2 + \tilde{p}_r - \tilde{\sigma})}{(r - 2\mu)}\omega(r) = 0. \qquad (45)
\end{aligned}
$$

Equation (45) can be solved together with Eqs. (31), (34), (35), (36) and (38). Together these equations fully describe a slowly rotating anisotropic relativistic star in the theory described by the action (4).

D. Particular limits

The equations obtained in the previous section represent the most general set of stellar structure equations for a broad class of scalar-tensor theories with a single scalar degree of freedom with a disformal coupling between the scalar field and a spherically symmetric slowly rotating anisotropic fluid distribution. Because of its generality, we can recover many particular cases previously studied in the literature:

(1) In the limit of the pure conformal coupling, $\Lambda \to 0$ (thus $\chi \to 1$), we recover the case studied in Ref. [55].

(2) If we additionally assume isotropic pressure $\tilde{p}_r = \tilde{p}_t = \tilde{p}$, we recover the standard equations given in Refs. [15,16].

(3) If we assume a kinetic term of the form $P(X,\varphi) = 2X - V(\varphi)$, where $V(\varphi)$ is a mass term $m^2\varphi^2$, isotropic pressure and purely conformal coupling we recover the massive scalar-tensor theory studied in Refs. [107,108] and the asymmetron scenario proposed in Ref. [109] by appropriately choosing $A(\varphi)$.

IV. SCALAR-TENSOR THEORY WITH A CANONICAL SCALAR FIELD

A. Stellar structure equations

Now let us apply the general formulation developed in the previous section to the canonical scalar field with the potential $V(\varphi)$, i.e. $P = 2X - V(\varphi)$. The stellar structure equations (31), (34), (35), (36) and (38) reduce to

$$\frac{d\mu}{dr} = \frac{r(r-2\mu)}{2}\psi^2 + \frac{r^2}{2}V(\varphi) + A^4(\varphi)\sqrt{\chi}\left(\frac{\kappa}{2}\tilde{\rho}c^2 r^2\right), \tag{46a}$$

$$\frac{d\nu}{dr} = \frac{2\mu}{r(r-2\mu)} + r\psi^2 - \frac{r^2}{r-2\mu}V(\varphi) + \frac{r^2}{r-2\mu}\frac{A^4(\varphi)}{\sqrt{\chi}}(\kappa\tilde{p}_r), \tag{46b}$$

$$\frac{d\tilde{p}_r}{dr} = -\left[\alpha(\varphi)\psi + \frac{\mu}{r(r-2\mu)} + \frac{r}{2}\psi^2 - \frac{r^2}{2(r-2\mu)}V(\varphi) + \frac{r^2}{r-2\mu}\frac{A^4(\varphi)}{\sqrt{\chi}}\left(\frac{\kappa}{2}\tilde{p}_r\right)\right](\tilde{\rho}c^2 + \tilde{p}_r) - 2\left(\frac{1}{r} + \alpha(\varphi)\psi\right)\tilde{\sigma}, \tag{46c}$$

$$\frac{d\varphi}{dr} = \psi, \tag{46d}$$

$$C_2\frac{d\psi}{dr} = -C_1\psi + \frac{r\chi V_\varphi(\varphi)}{r-2\mu} + \frac{\kappa r}{r-2\mu}A^4(\varphi)\alpha(\varphi) \times\left[-\frac{\tilde{p}_r}{\sqrt{\chi}} + \sqrt{\chi}(\tilde{\rho}c^2 - 2\tilde{p}_r) + 2\sqrt{\chi}\tilde{\sigma}\right], \tag{46e}$$

where

$$\begin{aligned} C_1 = \frac{2\chi}{r-2\mu}\left[\frac{2(r-\mu)}{r} - r^2V(\varphi) - \frac{\kappa}{2}A^4(\varphi)r^2\left(\sqrt{\chi}\tilde{\rho}c^2 - \frac{\tilde{p}_r}{\sqrt{\chi}}\right)\right] - \kappa\Lambda A^4(\varphi)B^2(\varphi)\Bigg[-\frac{\mu}{r(r-2\mu)}\left(\sqrt{\chi}\tilde{\rho}c^2 - \frac{\tilde{p}_r}{\sqrt{\chi}}\right) \\ - \frac{r\psi^2}{2}\left(\sqrt{\chi}\tilde{\rho}c^2 + \frac{\tilde{p}_r}{\sqrt{\chi}}\right) + \frac{r^2V(\varphi)}{2(r-2\mu)}\left(\sqrt{\chi}\tilde{\rho}c^2 - \frac{\tilde{p}_r}{\sqrt{\chi}}\right) - \kappa A^4(\varphi)\frac{r^2}{r-2\mu}\tilde{p}_r\tilde{\rho}c^2 \\ + \frac{2\sqrt{\chi}}{r}(\tilde{p}_r - \tilde{\sigma}) + \frac{\psi}{\sqrt{\chi}}(\beta(\varphi) - \alpha(\varphi))\tilde{p}_r\Bigg] \end{aligned} \tag{47}$$

and

$$C_2 = 2\chi - \kappa\Lambda A(\varphi)^4 B(\varphi)^2\frac{\tilde{p}_r}{\sqrt{\chi}}. \tag{48}$$

In the case of a slowly rotating star, the frame-dragging equation (45) becomes

$$\frac{d^2\omega}{dr^2} - \left[r\psi^2 + \frac{\kappa r^2 A^4(\varphi)}{2(r-2\mu)}\left(\frac{\tilde{\rho}c^2}{\sqrt{\chi}} + \sqrt{\chi}\tilde{p}_r\right) - \frac{4}{r}\right]\frac{d\omega}{dr} - 2\kappa A^4(\varphi)r\sqrt{\chi}\frac{(\tilde{\rho}c^2 + \tilde{p}_r - \tilde{\sigma})}{(r-2\mu)}\omega(r) = 0. \tag{49}$$

Through the Einstein equation (26), we find that if $\Lambda > 0$ the second term of C_2 in Eq. (48) is of order $\mathcal{O}(\Lambda B^2/r^2)$, from which we can estimate the radius within which the contributions of disformal coupling to the gradient terms become comparable to the standard ones in the scalar-tensor theory as $R_\mathrm{D} \coloneqq \sqrt{\Lambda}B(\varphi)$. If $R_\mathrm{D} > R$, where $r = R$ is the star's radius, the contributions of disformal coupling to the gradient terms become important throughout the star, while if $R_\mathrm{D} < R$ they could be important only in a portion of the star's interior $r < R_\mathrm{D}$. When $B \to 1$, $R_\mathrm{D} \approx \sqrt{\Lambda}$ and therefore $\sqrt{\Lambda}$ characterizes the length scale for which the disformal coupling effects become apparent. As the radius of a typical NS is about 10 km, the effects of disformal coupling of the star become apparent when $\Lambda > \mathcal{O}(100\ \mathrm{km}^2)$.

We note that in the presence of the disformal coupling, when integrating the scalar field equation (38), the coefficient C_2 in the $d\psi/dr$ equation may vanish at some $r = R_*$, i.e. $C_2(R_*) = 0$. This could happen when both $\Lambda > 0$ and the pressure at the center of the star is large enough such that $C_2 < 0$ in the vicinity of $r = 0$. In such a case, as we integrate the equations outwards, since the radial pressure $\tilde{p}_r$ decreases and vanishes at the surface of the star, there must be a point R_* where C_2 vanishes. This point represents a singularity of our equations and a regular stellar model cannot be constructed. The nonexistence of a regular relativistic star for a large positive Λ is one of the most important consequences due to the disformal coupling. The appearance of the singularity is due to the fact that the gradient term in the scalar field equation of motion (46e) picks a wrong sign (i.e., negative speed of sound) and is an illustration of the gradient instability pointed out in Refs. [72,86,110].

B. Interior solutions

From this section onwards, we focus on the case of isotropic pressure $\tilde{p} = \tilde{p}_r = \tilde{p}_t$. We then derive the boundary conditions at the center of the star, $r = 0$, which have to be specified when integrating Eqs. (46) and (49). We assume that at $r = 0$, $\tilde{\rho}(0) = \tilde{\rho}_c$. The remaining metric and matter variables can be expanded as

$$\mu(r) = \frac{1}{6}[\kappa\tilde{\rho}_c c^2 A^4(\varphi_c) + V(\varphi_c)]r^3 + \mathcal{O}(r^5), \tag{50a}$$

$$\nu(r) = \frac{1}{6}[\kappa(\tilde{\rho}_c c^2 + 3\tilde{p}_c)A^4(\varphi_c) - 2V(\varphi_c)]r^2 + \mathcal{O}(r^4), \tag{50b}$$

$$\varphi(r) = \varphi_c + \frac{\kappa A^4(\varphi_c)\alpha(\varphi_c)(\tilde{\rho}_c c^2 - 3\tilde{p}_c) + V_\varphi(\varphi_c)}{12[2 - \kappa\Lambda\tilde{p}_c A^4(\varphi_c)B^2(\varphi_c)]} r^2 + \mathcal{O}(r^4), \tag{50c}$$

$$\begin{aligned}\tilde{p}(r) = \tilde{p}_c - \frac{1}{12}(\tilde{\rho}_c c^2 + \tilde{p}_c)\Bigg\{&\kappa A^4(\varphi_c)\left[\tilde{\rho}_c c^2 + 3\tilde{p}_c + \alpha(\varphi_c)^2 \frac{\tilde{\rho}_c c^2 - 3\tilde{p}_c}{1 - \frac{\kappa}{2}\Lambda\tilde{p}_c A^4(\varphi_c)B^2(\varphi_c)}\right] \\ &- 2V(\varphi_c)\left[1 - \frac{\alpha(\varphi_c)V_\varphi(\varphi_c)}{2V(\varphi_c)} \frac{1}{1 - \frac{\kappa}{2}\Lambda\tilde{p}_c A^4(\varphi_c)B^2(\varphi_c)}\right]\Bigg\} r^2 + \mathcal{O}(r^4),\end{aligned} \tag{50d}$$

where $\tilde{p}_c$ is fixed by $\tilde{\rho}_c$ through the EOS, i.e. $\tilde{p}_c = \tilde{p}(\tilde{\rho}_c)$. The central value of the scalar field φ_c is fixed by demanding that outside the star the scalar field approaches a given cosmological value φ_0 as $r \to \infty$, which is consistent with observational constraints. We will come back to this in Sec. IV C.

As a well-behaved stellar model requires $\tilde{p}''(0) < 0$, we impose

$$\begin{aligned}&\kappa A^4(\varphi_c)\left[\tilde{\rho}_c c^2 + 3\tilde{p}_c + \alpha(\varphi_c)^2 \frac{\tilde{\rho}_c c^2 - 3\tilde{p}_c}{1 - \frac{\kappa}{2}\Lambda\tilde{p}_c A^4(\varphi_c)B^2(\varphi_c)}\right] \\ &\quad - 2V(\varphi_c)\left[1 - \frac{\alpha(\varphi_c)V_\varphi(\varphi_c)}{2V(\varphi_c)} \frac{1}{1 - \frac{\kappa}{2}\Lambda\tilde{p}_c A^4(\varphi_c)B^2(\varphi_c)}\right] > 0.\end{aligned} \tag{51}$$

For a large positive disformal coupling parameter $\Lambda > 0$ and a large pressure at the center $\tilde{p}_c$ such that $|1 - \frac{\kappa\Lambda}{2}\tilde{p}_c A^4(\varphi_c)B^2(\varphi_c)| \ll 1$, the r^2 terms of the scalar field and pressure diverge and the Taylor series solution (50) breaks down. Such a property is a direct consequence of the appearance of the singularity inside the star which was mentioned in the previous subsection. Assuming that $A(\varphi_c) \approx 1$ and $B(\varphi_c) \approx 1$, the maximal positive value of $\Lambda_{\max}$ can be roughly estimated as

$$\Lambda_{\max} \approx \frac{2}{\kappa\tilde{p}_c} = \frac{c^4}{4\pi G\tilde{p}_c} \approx 10^2\ \mathrm{km}^2, \tag{52}$$

for $\tilde{p}_c = 10^{36}\ \mathrm{dyne/cm}^2$, which agrees with the numerical analysis done in Sec. VI. On the other hand, for a large negative value of the disformal coupling $\Lambda < 0$, no singularity appears, from Eq. (50c) the r^2 correction to the scalar field amplitude is suppressed, and $\varphi(r) \to \varphi_c$ everywhere inside the star. This indicates that $A(\varphi_c) \approx$ constant, and for a vanishing potential $V(\varphi) = 0$ the stellar configuration approaches that in GR.

In the case of slowly rotating stars, the boundary condition for ω near the origin reads

$$\omega = \omega_c\left[1 + \frac{\kappa}{5}A^4(\varphi_c)(\tilde{\rho}_c c^2 + \tilde{p}_c)r^2\right] + \mathcal{O}(r^4). \tag{53}$$

1. Stellar models in purely disformal theories

It is interesting to analyze the stellar structure equations in the purely disformal coupling limit, when $A(\varphi) = 1$. In this case we find that the expansions near the origin are

$$\mu(r) = \frac{1}{6}[\kappa\tilde{\rho}_c c^2 + V(\varphi_c)]r^3 + \mathcal{O}(r^5),$$
$$\nu(r) = \frac{1}{6}[\kappa(\tilde{\rho}_c c^2 + 3\tilde{p}_c) - 2V(\varphi_c)]r^2 + \mathcal{O}(r^4),$$
$$\varphi(r) = \varphi_c + \frac{V_\varphi(\varphi_c)}{12[1 - \frac{\kappa}{2}\Lambda\tilde{p}_c B^2(\varphi_c)]}r^2 + \mathcal{O}(r^4),$$
$$\tilde{p}(r) = \tilde{p}_c - \frac{1}{12}(\tilde{p}_c + \tilde{\rho}_c c^2)[\kappa(\tilde{\rho}_c c^2 + 3\tilde{p}_c) - 2V(\varphi_c)]r^2 + \mathcal{O}(r^4). \tag{54}$$

Thus for $V(\varphi) = 0$, $\varphi = \varphi_c$ everywhere, and the disformal coupling term does not modify the stellar structure with respect to GR. In other words, to obtain a nontrivial scalar field configuration inside the NS we must have a potential $V(\varphi) \neq$ constant. It was argued in Ref. [83] that for a simple mass term potential $V_\varphi \sim m^2\varphi$, where m is the mass of the scalar field, disformal contributions can be neglected and the NS solution is the same as in GR.

2. Metric functions in the Jordan frame

Finally, we mention the behaviors of the metric functions in the Jordan frame. In the Appendix we derive the relationship of the physical quantities defined in the two frames. The boundary conditions (50) indicate that in the singular stellar solution of the Einstein frame the metric functions μ and ν remain regular. Using Eqs. (A3) and (A9), the metric functions in the Jordan frame behave as

$$\bar{\nu}(r) = \ln A(\varphi_c)^2 + \frac{1}{6}\left[(\tilde{\rho}_c c^2 + 3\tilde{p}_c) - 2V(\varphi_c) + \alpha(\varphi_c)\frac{\kappa A^4(\varphi_c)\alpha(\tilde{\rho}_c c^2 - 3\tilde{p}_c) + V'(\varphi_c)}{1 - \frac{\kappa\Lambda}{2}A^4(\varphi_c)B^2(\varphi_c)\tilde{p}_c}\right]r^2 + O(r^4), \tag{55}$$

$$\bar{\mu}(r) = \frac{A(\varphi_c)}{18}\left[3(A^4(\varphi_c)\tilde{\rho}c^2 + V(\varphi_c)) + 3\alpha(\varphi_c)\frac{\kappa A^4(\varphi_c)\alpha(\tilde{\rho}_c c^2 - 3\tilde{p}_c) + V'(\varphi_c)}{1 - \frac{\kappa\Lambda}{2}A^4(\varphi_c)B^2(\varphi_c)\tilde{p}_c} + \Lambda B^2(\varphi_c)\frac{(\kappa A^4(\varphi_c)\alpha(\tilde{\rho}_c c^2 - 3\tilde{p}_c) + V'(\varphi_c))^2}{4(1 - \frac{\kappa\Lambda}{2}A^4(\varphi_c)B^2(\varphi_c)\tilde{p}_c)^2}\right]r^3 + O(r^5). \tag{56}$$

Therefore, for $|1 - \frac{\kappa\Lambda}{2}A^4(\varphi_c)B^2(\varphi_c)\tilde{p}_c| \ll 1$, the Taylor series solutions for $\bar{\mu}(r)$ and $\bar{\nu}(r)$ break down, which indicates that the metric functions in the Jordan frame $\bar{\mu}$ and $\bar{\nu}$ diverge at some finite radius and a curvature singularity appears there.

C. Exterior solution

In the vacuum region outside the star $r > R$, the fluid variables $\tilde{\rho}$, $\tilde{p}_r$ and $\tilde{p}_t$ vanish. The exterior solution should be the vacuum solution of GR coupled to the massless canonical scalar field. The following exact solution can be obtained [15,111]:

$$ds^2 = -e^{\nu(\rho)}c^2dt^2 + e^{-\nu(\rho)} \times \left[d\rho^2 + \left(\rho^2 - \frac{2Gs}{c^2}\rho\right)\gamma_{ij}d\theta^i d\theta^j\right], \tag{57}$$

$$\nu(\rho) = \nu_0 + \ln\left(1 - \frac{2Gs}{c^2\rho}\right)^{\frac{M}{s}}, \tag{58}$$

$$\varphi(\rho) = \varphi_0 - \frac{Q}{2M}\ln\left(1 - \frac{2Gs}{c^2\rho}\right)^{\frac{M}{s}}, \tag{59}$$

where ν_0 represents the freedom of the rescaling of the time coordinate, φ_0 is the cosmological value of the scalar field at $r \to \infty$, M and Q are the integration constants and $s := \sqrt{M^2 + Q^2}$. The metric (57) can be rewritten in terms of the Schwarzschild-like coordinate r by the transformations

$$r(\rho) = \rho\left(1 - \frac{2Gs}{c^2\rho}\right)^{\frac{s-M}{2s}}, \tag{60}$$

$$\mu(\rho) = M\left[1 - \frac{G(s-M)^2}{2M\rho c^2(1 - \frac{2Gs}{c^2\rho})}\right]\left(1 - \frac{2Gs}{c^2\rho}\right)^{\frac{s-M}{2s}}. \tag{61}$$

As $r \to \infty$, the solution (57) behaves as

$$\mu(r) = \frac{GM}{c^2} - \frac{G^2Q^2}{2c^4 r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{62a}$$

$$\nu(r) = \nu_0 - \frac{2GM}{c^2 r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{62b}$$

$$\varphi(r) = \varphi_0 + \frac{GQ}{c^2 r} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{62c}$$

Thus the integration constants M and Q correspond to the Arnowitt-Deser-Misner (ADM) mass and the scalar charge in the Einstein frame, respectively. For later convenience we also define the fractional binding energy

$$\mathcal{E}_\mathrm{b} \coloneqq \frac{M_\mathrm{b}}{M} - 1, \tag{63}$$

which is positive for bound (but not necessarily stable) configurations. We note that for the vanishing scalar field at asymptotic infinity the ADM mass is disformally invariant, $\bar{M} = M$ [see Eq. (A10)].

In the slowly rotating case, the integration of Eq. (44) in vacuum $\tilde{\rho} = \tilde{p}_t = 0$ gives

$$\omega' = \frac{6G}{c^2 r^4} e^{\frac{\lambda+\nu}{2}} J, \tag{64}$$

where J is the integration constant. In the vacuum case, we can find the exact exterior solution at the first order in rotation [16]. Expanding it in the vicinity of $r \to \infty$ gives

$$\omega = \Omega - \frac{2GJ}{c^2 r^3} + O\left(\frac{1}{r^5}\right). \tag{65}$$

Thus J corresponds to the angular momentum in the exterior spacetime.

D. Matching

At the surface of the star, the interior solution is matched to the exterior solution (57). Then the cosmological value of the scalar field φ_0, the ADM mass M and the scalar charge Q are evaluated as

$$\varphi_0 = \varphi_\mathrm{s} + \ln\left(\frac{x_1 + x_2}{x_1 - x_2}\right)^{\frac{\psi_\mathrm{s}}{x_2}}, \tag{66a}$$

$$M = \frac{c^2 R^2 \nu'_\mathrm{s}}{2G}\left(1 - \frac{2\mu_\mathrm{s}}{R}\right)^{\frac{1}{2}}\left(\frac{x_1 + x_2}{x_1 - x_2}\right)^{-\frac{\nu'_\mathrm{s}}{2x_2}}, \tag{66b}$$

$$q \coloneqq \frac{Q}{M} = -\frac{2\psi_\mathrm{s}}{\nu'_\mathrm{s}} \tag{66c}$$

where we introduced $x_1 = \nu'_s + 2/R$ and $x_2 \coloneqq \sqrt{\nu'^2_\mathrm{s} + 4\psi^2_\mathrm{s}}$. We also defined $\mu_\mathrm{s} \coloneqq \mu(R)$ and $\nu_\mathrm{s} \coloneqq \nu(R)$.

In the case of a slowly rotating star, the angular velocity and angular momentum of the star, Ω and J, are evaluated as

$$\Omega = \omega_s - \frac{3c^4 J}{4G^2 M^3 (3 - \alpha(\varphi_\mathrm{s})^2)}\left[\frac{4}{x_1^2 - x_2^2}\left(\frac{x_1 - x_2}{x_1 + x_2}\right)^{\frac{2\nu'_\mathrm{s}}{x_2}} \times \left(\frac{3\nu'_\mathrm{s}}{R} + \frac{1}{R^2} + 3\nu'^2_\mathrm{s} - \psi^2_\mathrm{s}\right) - 1\right], \tag{67}$$

$$J = \frac{c^2 R^4}{6G}\sqrt{1 - \frac{2\mu_\mathrm{s}}{R}} e^{-\frac{\nu'_\mathrm{s}}{2}} \omega'_\mathrm{s}. \tag{68}$$

The moment of inertia can be obtained by

$$I \coloneqq \frac{J}{\Omega}, \tag{69}$$

or equivalently by integrating Eq. (44), using Eqs. (34) and (35)

$$I = \frac{8\pi}{3c^2}\int_0^R dr A^4(\varphi)\sqrt{\chi} r^4 e^{-\frac{\nu-\lambda}{2}}(\tilde{\rho}c^2 + \tilde{p}_t)\left(\frac{\omega}{\Omega}\right). \tag{70}$$

We observe that this relation for the moment of inertia holds for any choice of $P(X, \varphi)$, $A(\varphi)$ and $B(\varphi)$. In the purely conformal theory we obtain the result of Ref. [55].

For a given EOS the equations of motion (46) and (49) are numerically integrated from $r = 0$ up to the surface of the star $r = R$, where the pressure vanishes $\tilde{p}(R) = 0$. With the values of various variables at the surface at hand, we can compute φ_0, M, q and I using the matching conditions.

From the Einstein frame radius R, we can calculate the physical Jordan frame radius $\tilde{R}$ through [cf. Eq. (2)]

$$\tilde{R} \coloneqq \sqrt{A^2(\varphi_\mathrm{s})[R^2 + \Lambda B^2(\varphi_\mathrm{s})\psi^2_\mathrm{s}]} \tag{71}$$

where we introduced $\varphi_\mathrm{s} \coloneqq \varphi(R)$ and $\psi_\mathrm{s} \coloneqq \psi(R)$. For a vanishing scalar field we have $\tilde{R} = R$.

The total baryonic mass of the star M_b can be obtained by integrating

$$M_\mathrm{b} = \int_0^R dr A^3(\varphi)\sqrt{\chi}\frac{4\pi \tilde{m}_\mathrm{b} r^2}{\sqrt{1 - \frac{2\mu}{r}}}\tilde{n}(r), \tag{72}$$

where $\tilde{m}_\mathrm{b} = 1.66 \times 10^{-24}$ g is the atomic mass unit and $\tilde{n}$ is the baryonic number density.

In the Appendix we show that the physical quantities related to the rotation of fluid and spacetime, namely I and J as well as ω and Ω, are invariant under the disformal transformation (2).

V. A TOY MODEL OF SPONTANEOUS SCALARIZATION WITH AN INCOMPRESSIBLE FLUID

Before carrying out the full numerical integrations of the stellar structure equations it is illuminating to study under which conditions scalarization can occur in our model. This can be accomplished by studying a simple toy model where a scalar field lives on the background of an incompressible fluid star. The results obtained in this section will be validated in Sec. VI.

Let us start by assuming that the star has a constant density ρ (incompressible) and an isotropic pressure $p = p_r = p_t$. The scalar field φ is massless, and has a canonical kinetic term and small amplitude, such that we can linearize the equations of motion. The conformal and disformal coupling functions can be expanded as

$$A(\varphi) = 1 + \frac{1}{2}\beta_1\varphi^2 + \mathcal{O}(\varphi^3),$$
$$B(\varphi) = 1 + \frac{1}{2}\beta_2\varphi^2 + \mathcal{O}(\varphi^3), \tag{73}$$

where we have defined $\beta_1 \coloneqq A_{\varphi\varphi}(0)$ and $\beta_2 \coloneqq B_{\varphi\varphi}(0)$. As at the background level the scalar field is trivial $\varphi = 0$, the Jordan and Einstein frames coincide, and $\tilde{\rho} = \rho$ and $\tilde{p} = p$. For an incompressible star, the Einstein field equations admit an exact solution of the form (21) given by [112]

$$e^{\lambda(r)} = \left(1 - \frac{2GMr^2}{c^2R^3}\right)^{-1}, \tag{74a}$$

$$e^{\nu(r)} = \left[\frac{3}{2}\left(1 - \frac{2GM}{c^2R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2GMr^2}{c^2R^3}\right)^{1/2}\right]^2, \tag{74b}$$

$$p(r) = \rho c^2 \frac{(1 - \frac{2GMr^2}{c^2R^3})^{1/2} - (1 - \frac{2GM}{c^2R})^{1/2}}{3(1 - \frac{2GM}{c^2R})^{1/2} - (1 - \frac{2GMr^2}{c^2R^3})^{1/2}}, \tag{74c}$$

where $r = R$ is the surface of the star, at which $p(R) = 0$. Here, M and $\mathcal{C}$ are the total mass and compactness of the star:

$$M = \frac{4\pi R^3}{3}\rho, \qquad \mathcal{C} = \frac{GM}{c^2R}. \tag{75}$$

We then consider the perturbations to the background (74) induced by the fluctuations of φ. Since the corrections to the Einstein equations appear in $\mathcal{O}(\varphi^2, {\varphi_\mu}^2)$, at the leading order of φ only the scalar field equation of motion becomes nontrivial. In the linearized approximation, $\chi = 1 + \mathcal{O}({\varphi_\mu}^2)$, $\alpha = \beta_1\varphi + \mathcal{O}(\varphi^2)$ and $\beta = \beta_2\varphi + \mathcal{O}(\varphi^2)$, and the scalar field equation of motion (20) for the massless and minimally coupled scalar field $P = 2X$ reduces to

$$\left(g^{\rho\sigma} - \frac{\kappa\Lambda}{2}T_{(\mathrm{m})}^{\rho\sigma}\right)\varphi_{\rho\sigma} = -\frac{\kappa\beta_1}{2}T_{(\mathrm{m})}{}^{\rho}{}_{\rho}\varphi + \mathcal{O}(\varphi^2, \varphi_\mu^2). \tag{76}$$

Thus, as expected, in the Einstein frame the corrections from disformal coupling appear as the modification of the kinetic term via the coupling to the energy-momentum tensor.

Taking the s-wave configuration for a stationary field, $\dot{\varphi} = \ddot{\varphi} = 0$, we get

$$\varphi'' + \frac{\frac{\nu'-\lambda'}{2} + \frac{2}{r} - \frac{\kappa\Lambda}{2}[-\frac{\nu'}{2}\rho c^2 + (-\frac{\lambda'}{2} + \frac{2}{r})p(r)]}{1 - \frac{\kappa\Lambda}{2}p(r)}\varphi'$$
$$- \frac{\kappa\beta_1}{2}e^{\lambda(r)}\frac{\rho c^2 - 3p(r)}{1 - \frac{\kappa\Lambda}{2}p(r)}\varphi + \mathcal{O}(\varphi^2, \varphi'^2) = 0. \tag{77}$$

Inside the star, the scalar field equation of motion in the stationary background (77) can be expanded as

$$\varphi'' + \frac{2}{r}\left[1 + \mathcal{O}\left(\mathcal{C}\frac{r^2}{R^2}\right)\right]\varphi' + u\left[1 + \mathcal{O}\left(\mathcal{C}\frac{r^2}{R^2}\right)\right]\varphi = 0, \tag{78}$$

where we have defined

$$u \coloneqq \frac{6(3\sqrt{1-2\mathcal{C}} - 2)\mathcal{C}}{(3\sqrt{1-2\mathcal{C}} - 1)R^2 + 3\mathcal{C}(\sqrt{1-2\mathcal{C}} - 1)\Lambda}|\beta_1|. \tag{79}$$

By neglecting the correction terms of order $\mathcal{O}(\mathcal{C}\frac{r^2}{R^2})$ in Eq. (78), the approximated solution inside the star satisfying the regularity boundary condition at the center, $\varphi(0) = \varphi_{\mathrm{c}}$ and $\varphi'(0) = 0$, is given by

$$\varphi(r) \approx \varphi_{\mathrm{c}}\frac{\sin(\sqrt{u}r)}{\sqrt{u}r}. \tag{80}$$

We note that at the surface of the star, $r = R$, the corrections to this approximate solution (80) would be of $\mathcal{O}(\mathcal{C})$, which is negligible for $\mathcal{C} \ll 1$ and gives at most a 10% error even for $\mathcal{C} \simeq 0.1$. Thus the solution (80) provides a good approximation to the precise interior solution of Eq. (77), up to corrections of $\mathcal{O}(10\%)$ for typical NSs.

Outside the star, where $\tilde{\rho} = \tilde{p} = 0$, the scalar field equation of motion (77) reduces to

$$\varphi'' + \left(\frac{1}{r} + \frac{1}{r - \frac{2GM}{c^2}}\right)\varphi' = 0. \tag{81}$$

The exterior solution of the scalar field is given by

$$\varphi(r) = \varphi_0 + \frac{Q}{2M}\ln\left(1 - \frac{2GM}{c^2r}\right), \tag{82}$$

which can be expanded as

$$\varphi(r) = \varphi_0 - \frac{GQ}{c^2r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{83}$$

where Q denotes scalar charge. Matching at the surface $r = R$ gives

$$\frac{GQ}{c^2R\varphi_0} = -\frac{2\mathcal{C}(1-2\mathcal{C})(\sqrt{u}R - \tan(\sqrt{u}R))}{\Xi}, \tag{84}$$

$$\frac{\varphi_c}{\varphi_0} = -\frac{2\mathcal{C}\sqrt{u}R}{\cos(\sqrt{u}R)}\frac{1}{\Xi} \tag{85}$$

where we introduced

$$\Xi = (1-2\mathcal{C})\sqrt{u}R\ln(1-2\mathcal{C}) - [2\mathcal{C} + (1-2\mathcal{C})\ln(1-2\mathcal{C})]\tan(R\sqrt{u}). \tag{86}$$

The scalar charge Q and the central value of the scalar field φ_c blow up when

$$\frac{\tan(\sqrt{u}R)}{\sqrt{u}R} = \frac{(1-2\mathcal{C})\ln(1-2\mathcal{C})}{2\mathcal{C} + (1-2\mathcal{C})\ln(1-2\mathcal{C})}. \tag{87}$$

Thus, inside the star, the scalar field can be enhanced and the scalarization takes place when

$$\sqrt{u}R \approx \frac{\pi}{2}\left(1 + \frac{4}{\pi^2}\mathcal{C}\right). \tag{88}$$

The condition (88) can be rewritten as

$$|\beta_1^{\mathrm{crit}}| \approx \frac{\pi^2}{24\mathcal{C}}\frac{3\sqrt{1-2\mathcal{C}} - 1 + 3\mathcal{C}(\sqrt{1-2\mathcal{C}} - 1)\frac{\Lambda}{R^2}}{3\sqrt{1-2\mathcal{C}} - 2} \times \left(1 + \frac{4}{\pi^2}\mathcal{C}\right)^2, \tag{89}$$

where β_1^{crit} is the critical value of β_1 for which scalarization can be triggered.

For small compactness $\mathcal{C} \ll 1$, we find at leading order

$$|\beta_1^{\mathrm{crit}}| \approx \frac{\pi^2}{12\mathcal{C}}\left(1 - \frac{3\mathcal{C}^2}{2R^2}\Lambda\right). \tag{90}$$

For a typical NS, the compactness parameter $\mathcal{C} \simeq 0.2$, and if Λ is negligibly small $|\beta_1^{\mathrm{crit}}| = \pi^2/(12\mathcal{C}) \simeq 4.1$, which agrees with the ordinary scalarization threshold [15,17]. On the other hand, disformal coupling becomes important when $\Lambda \simeq (R/\mathcal{C})^2$, which for $R \sim 10$ km and $\mathcal{C} \simeq 0.2$, corresponds to $\Lambda \simeq 2500\ \mathrm{km}^2$.

In the other limit, for sufficiently large negative disformal coupling parameters $|\Lambda| \gg (R/\mathcal{C})^2$, as $uR^2 \simeq 2R^2/(|\Lambda|\mathcal{C}^2) \ll 1$, from Eqs. (84) and (85) we have

$$\frac{GQ}{c^2 R\varphi_0} \simeq -\frac{1-2\mathcal{C}}{3}uR^2 \ll 1 \quad \text{and} \quad \varphi_c \simeq \varphi_0, \tag{91}$$

and the scalar field excitation is suppressed inside the star: the stellar configuration is that of GR.

In the next section, we will show explicit examples of the numerical integrations of the stellar structure and scalar field equations (46) and (49), and explore how the disformal coupling affects the standard scalarization mechanism in the models proposed in Refs. [15,16]. We will confirm our main conclusions from the perturbative calculations presented here.

VI. NUMERICAL RESULTS

Having gained analytical insight into the effect of the disformal coupling on spontaneous scalarization, we now will perform full numerical integrations of the stellar structure equations.

For simplicity, we will focus on the simple case of a canonical scalar field without a potential $V(\varphi) = 0$ and we will assume the special form of the coupling functions that enter Eq. (2)

$$A(\varphi) = e^{\frac{1}{2}\beta_1\varphi^2}, \qquad B(\varphi) = e^{\frac{1}{2}\beta_2\varphi^2}, \tag{92}$$

as a *minimal* model to include the disformal coupling in our problem. In the absence of the disformal coupling function ($\Lambda = 0$), this model reduces to that studied originally by Damour and Esposito-Farèse [15,16]. Another input from the theory is the cosmological value of the scalar field φ_0, which for simplicity we take to be zero throughout this section. We also studied the case $\varphi_0 = 10^{-3}$, which does not alter our conclusions.

Under these assumptions our model is invariant under the transformation $\varphi \to -\varphi$ (reflection symmetry). Therefore for each scalarized NS with scalar field configuration φ, there exists a reflection-symmetric counterpart with $\varphi \to -\varphi$. For both families of solutions the bulk properties (such as masses, radii and moment of inertia) are the same, while the scalar charges Q have opposite sign, but the same magnitudes. Moreover, $\varphi = 0$ is a trivial solution of the stellar structure equations. These solutions are equivalent to NSs in GR.

In this section we sample the $(\beta_1, \beta_2, \Lambda)$ parameter space of the theory, analyzing each parameter's influence on NS models and on spontaneous scalarization. As mentioned in Sec. I, binary-pulsar observations have set a constraint of $\beta_1 \gtrsim -4.5$ in what corresponds to the purely conformal coupling ($\Lambda = 0$) limit of our model. This lower bound on β_1 is not expected to apply for our more general model and therefore, so far, the set of parameters $(\beta_1, \beta_2, \Lambda)$ are largely unconstrained.

A. Equation of state

To numerically integrate the stellar structure equations we must complement them with a choice of EOS. Here we consider three realistic EOSs, namely APR [113], SLy4 [114] and FPS [115], in decreasing order of stiffness. The first two support NSs with masses larger than the $M = 2.01 \pm 0.04\ M_\odot$ lower bound from the pulsar PSR J0348 + 0432 in GR [116]. On the other hand, EOS FPS has a maximum mass of $\sim 1.8\ M_\odot$ in GR and is in principle ruled out by Ref. [116]. Nevertheless, as we will see this EOS can support NSs with $M \gtrsim 2\,M_\odot$, albeit scalarized, for certain values of the theory's parameters.

With this set of EOSs we validated our numerical code by reproducing the results of Refs. [28,55] in the purely conformal coupling limit. Our results including the presence of the disformal coupling are presented next.

B. Stellar models in the minimal scalar-tensor theory with disformal coupling

In Sec. V we found that β_1 always needs to be sufficiently negative for scalarization to be triggered. For this reason, let us first analyze how Λ and β_2 affect scalarized nonrotating NSs assuming a fixed value of β_1.

In Fig. 1, we consider what happens when we change the value of Λ while maintaining β_1 and β_2 fixed. We observe that for sufficiently negative values of Λ the effects of scalarization become suppressed. This can be qualitatively understood from Eq. (90): as $\Lambda/R^2 \to -\infty$ we need $|\beta_1^{\rm crit}| \to \infty$ for scalarization to happen. For fixed values of β_1 and $\mathcal{C}$, there will be a sufficiently negative value of Λ, for which $\beta_1^{\rm crit} > \beta_1$ and scalarization ceases to occur. Although in Fig. 1 we show $\Lambda = -3000\ {\rm km}^2$, we have confirmed this by constructing stellar models for even smaller values of Λ. Also, in agreement with Sec. V, we see that Λ alters the threshold for scalarization. This is most clearly seen in the right panel of Fig. 1, where for different values of Λ scalarization starts (evidenced by a nonzero scalar charge q) when different values of compactness $\mathcal{C}$ are reached.[1] In particular, for $\Lambda > 0$, because of the minus sign in the disformal term in Eq. (90), NSs can scalarize for smaller values of $\mathcal{C}$, while the opposite happens when $\Lambda < 0$. We remark that for large positive Λ the structure equations become singular at the origin as discussed in Sec. IV. This prevents nonrelativistic stars, for which $\mathcal{C} \to 0$, from scalarizing.

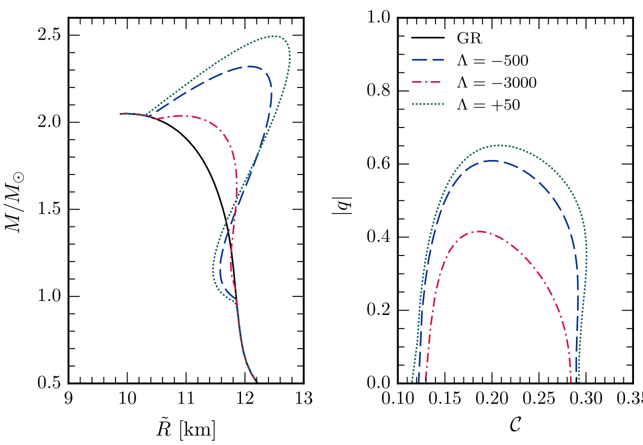

FIG. 1. We show the role of Λ in spontaneous scalarization. In both panels we consider stellar models using EOS SLy4 with $\beta_1 = -6.0$, $\beta_2 = 0$ and for $\Lambda = (-500, -3000, 50)\ {\rm km}^2$. For reference the solid line corresponds to GR. Left panel: The mass-radius relation. Right panel: The dimensionless scalar charge $q \coloneqq -Q/M$ [15] as a function of the compactness $\mathcal{C}$. We see that $\Lambda > 0$ slightly increases scalarization with respect to the purely conformal theory (cf. Fig. 2). On the other hand, $\Lambda < 0$ can dramatically suppress scalarization. Note also that unlike β_2, Λ can change the compactness threshold above which scalarization can happen, as predicted by the analysis of Sec. V. These results are qualitatively independent of the choice of EOS.

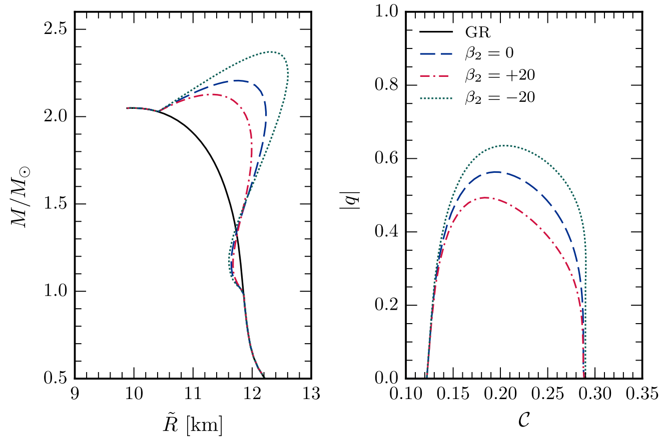

FIG. 2. We show the role of β_2 in spontaneous scalarization. As in Fig. 1, in both panels we consider stellar models using SLy4 EOS but with $\beta_1 = -6.0$ and $\Lambda = -1000\ {\rm km}^2$ for $\beta_2 = (-20, 0, 20)$. For reference the solid line corresponds to GR. Left panel: The mass-radius relation. Right panel: The dimensionless scalar charge $q = -Q/M$ as a function of the compactness $\mathcal{C}$. We see that β_2 affects highly scalarized stellar models making scalarization stronger (in the sense of increasing the value of q) when $\beta_2 < 0$, or weaker for $\beta_2 > 0$. Observe that β_2 has a negligible effect on weakly scalarized models ($|q| \lesssim 0.35$). This is in agreement with its absence from the perturbative analysis of Sec. V. Note that the range of $\mathcal{C}$ for which scalarization occurs is the same, irrespective of the choice of β_2. Again, these results are qualitatively independent of the choice of EOS.

In Fig. 2, we consider what happens when we change the value of β_2 while maintaining β_1 and Λ fixed. We see that in agreement with Eq. (90), the parameter β_2 does not affect the threshold for scalarization. Moreover, we observe that $\beta_2 < 0$ ($\beta_2 > 0$) makes scalarization more (less) evident with respect to $\beta_2 = 0$. In fact, in Eqs. (48) and (47), we see that β_1 and β_2 contribute to the scalar field equation through the factors $\Lambda A^4 B^2$ and $\beta - \alpha$, which have competing effects in sourcing the scalar field for $\beta_1 < 0$ and $\beta_2 \neq 0$. Our numerical integrations indicate that the former is dominant and that $\beta_2 \neq 0$ affects only very compact NSs ($\mathcal{C} \gtrsim 0.15$ in the example of Fig. 2).

[1] In the preceding section, because of the weak (scalar) field approximation the Jordan and Einstein frame radii are approximately the same, i.e $\tilde{R} = R$. This is not the case in this section and hereafter the compactness uses the Jordan frame radius, i.e. $\mathcal{C} = GM/(c^2\tilde{R})$.

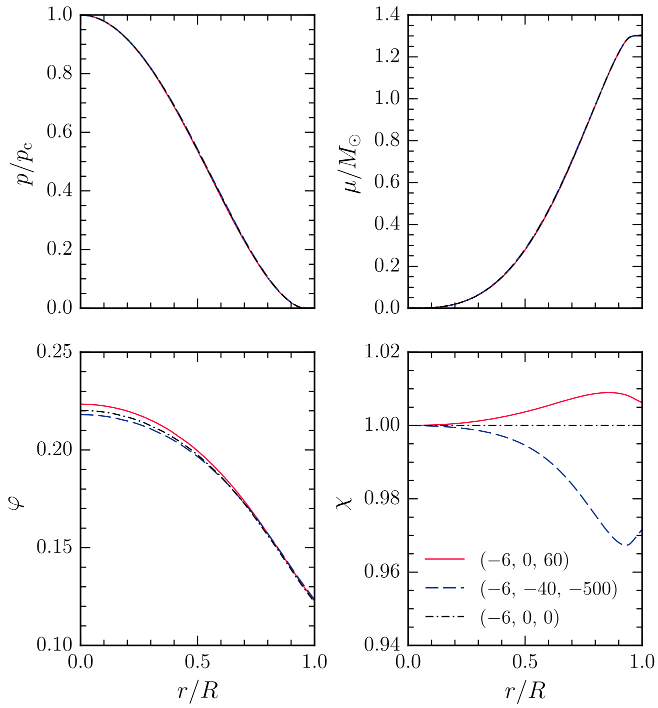

FIG. 3. We show the normalized pressure profile p/p_c (top left), dimensionless mass function $\mu/M_\odot$ (top right), scalar field φ (bottom left) and the disformal factor χ (bottom right) in the stellar interior. The radial coordinate was normalized by the Einstein frame radius R. The radial profiles above correspond to three stellar configurations using SLy4 EOS, with fixed baryonic mass $M_b/M_\odot = 1.5$ and theory parameters $(\beta_1, \beta_2, \Lambda) = (-6, 0, 60)$, $(-6, -40, -500)$ and $(-6, 0, 0)$, the latter corresponding to a stellar model in the Damour-Esposito-Farèse theory [15,111]. While the fluid variables are not dramatically affected, models with $\Lambda > 0$ ($\Lambda < 0$) become more (less) scalarized due to the disformal coupling. The bulk properties of these models are summarized in Table I.

It is also of interest to see how scalarization affects the interior of NSs. In Fig. 3, we show the normalized pressure profile p/p_c (top left), the dimensionless mass function $\mu/M_\odot$ (top right), the scalar field φ (bottom left) and the disformal factor χ (bottom right) in the stellar interior. The radial coordinate was normalized by the Einstein frame radius R. The quantities correspond to three stellar configurations using SLy4 EOS with fixed baryonic mass $M_b/M_\odot = 1.5$, which in GR yields a canonical NS with mass $M \approx 1.4\ M_\odot$, for the sample values of $(\beta_1, \beta_2, \Lambda)$ indicated in Table I. In agreement with our previous discussion we see that NSs with $\Lambda > 0$ ($\Lambda < 0$) support a larger (smaller) value of φ_c, which translates to a larger (smaller) value of q. It is particularly important to observe that χ is non-negative for all NS models, guaranteeing the Lorentzian signature of spacetime [cf. Eq. (9)].

In Fig. 4 we show the mass-radius curves (top panels) and moment of inertia-mass (lower panels) for increasing values of β_1 (from left to right), for three realistic EOSs,

TABLE I. The properties of NSs in GR and scalar-tensor theory using EOS SLy4 and fixed baryonic mass $M_b/M_\odot = 1.5$. The radial profiles of some of the physical variables involved in the integration of the stellar model are shown in Fig. 3.

$(\beta_1, \beta_2, \Lambda)$	$\tilde{R}$ [km]	M [$M_\odot$]	I [10^{45} g cm^2]	φ_c	q
GR	11.72	1.363	1.319	–	–
$(-6, 0, 0)$	11.60	1.354	1.431	0.220	0.613
$(-6, -40, -500)$	11.64	1.354	1.438	0.218	0.622
$(-6, 0, 60)$	11.59	1.354	1.430	0.223	0.615

keeping $\beta_2 = 0$, but using different values of Λ. As we anticipated in Fig. 1, negative values of Λ reduce the effects of scalarization, while positive values increase them. The case $\Lambda = 0$ corresponds to the purely conformal theory of Ref. [15]. We observe that scalarized NS models branch from the GR family at different points for different values of Λ (when β_1 is fixed). In agreement with our previous discussion, sufficiently negative values of Λ can completely suppress scalarization. Indeed for $\beta = -4.5$ the solutions with $\Lambda = -1000\ \mathrm{km}^2$ are identical to GR, while scalarized solutions exist when $\Lambda = 0$.

Additionally, we observe degeneracy between families of solutions in theories with different parameters. For instance, the maximum mass for a NS assuming EOS APR is approximately the same, $M/M_\odot \approx 2.38$, for both $\beta_1 = -5.5$, $\Lambda = 0$ and $\beta_1 = -6.0$, $\Lambda = -1000\ \mathrm{km}^2$. We also point out the degeneracy between the choice of EOS and of the parameters of the theory. For instance, the maximum mass predicted by EOS FPS in the theory with $\beta_1 = -5.5$ and $\Lambda = 50\ \mathrm{km}^2$ is approximately the same as that predicted by GR, but for EOS SLy4, i.e. $M/M_\odot \approx 2.05$. We emphasize that these two types of degeneracies are not exclusive to the theory we are considering, but are generic to *any* modification to GR [117].

In Fig. 5, we exhibit the mass-radius (top panels) and moment of inertia-mass (lower panels) for increasing values of β_1 (from left to right), but now keeping $\Lambda = -1000\ \mathrm{km}^2$ and changing the value of β_2. Once more, sufficiently negative values of Λ can completely suppress scalarization. This is clearly seen in the panels for $\beta_1 = -4.5$, where $\Lambda = -1000\ \mathrm{km}^2$, suppresses scalarization for all values of β_2 considered. We observe that independently of the choice of EOS, $\beta_2 > 0$ ($\beta_2 < 0$) yields smaller (larger) deviations from GR.

C. Stability of the solutions

Let us briefly comment on the stability of the scalarized solutions obtained in this section. In general, for a given set of parameters $(\beta_1, \beta_2, \Lambda)$ and fixed values of M_b and φ_0, we have more than one stellar configuration with different values of the mass M. Following the arguments of Refs. [15,17,35], we take the solution of smallest mass M i.e. larger fractional binding energy $\mathcal{E}_b$ defined in Eq. (63), to

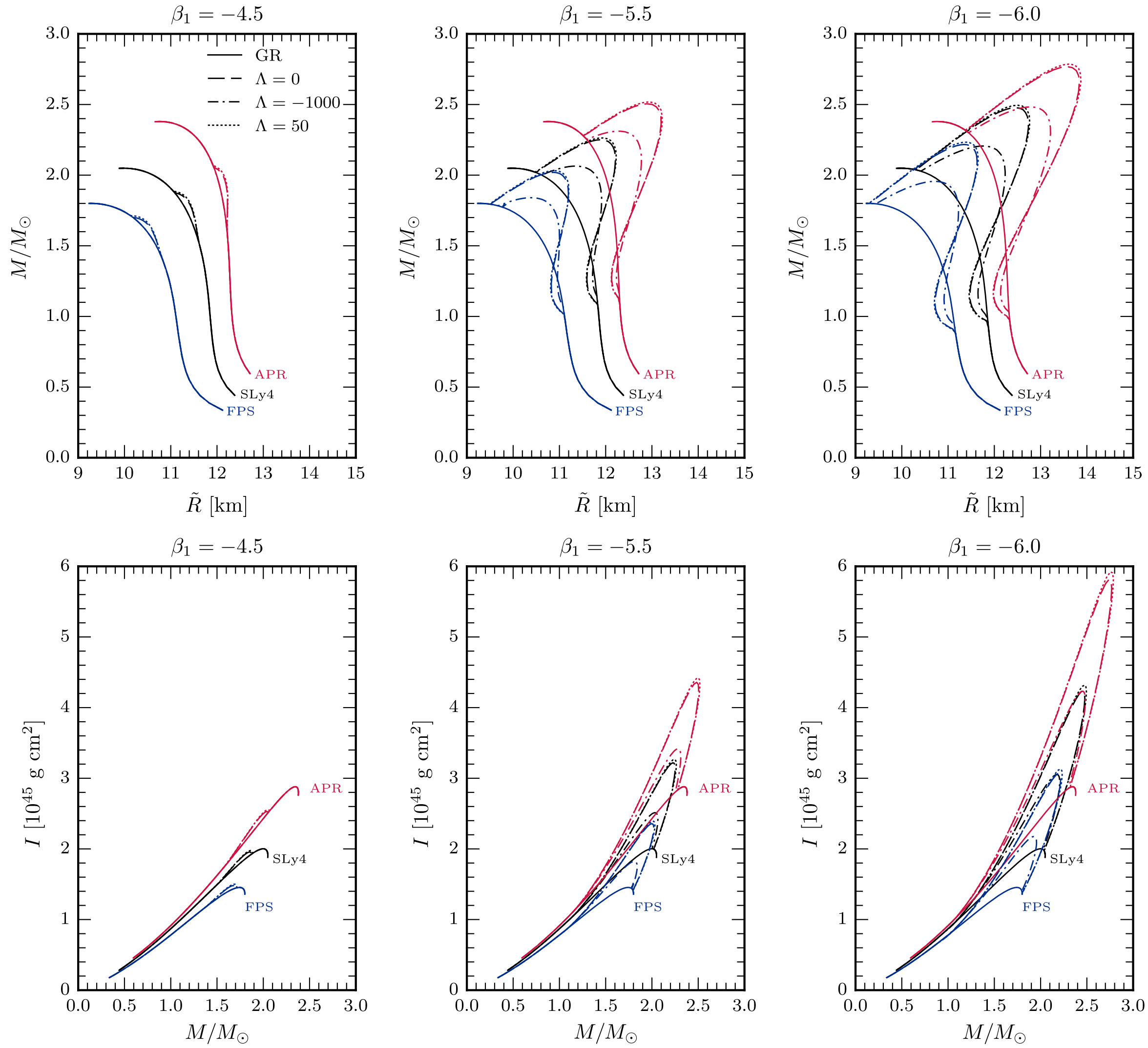

FIG. 4. We show NS models in scalar-tensor theories with disformal coupling for three choices of realistic EOSs, namely APR, SLy4 and FPS, in decreasing order of stiffness. We illustrate the effect of varying the values of β_1 and Λ, while keeping β_2 fixed ($\beta_2 = 0$) for simplicity. The curves corresponding to $\Lambda = 0$, represent stellar models in purely conformal theory [15,16]. Top panels: Mass-radius relations. Bottom panels: Moment of inertia versus mass. As seen in Fig. 1 already, while $\Lambda < 0$ weakens scalarization, $\Lambda > 0$ strengthens the effect. For $\beta_2 = 0$, this latter effect is very mild, being more evident by $\beta_2 < 0$ (cf. Fig. 5).

be the one which is energetically favorable to be realized in nature. In Fig. 6, we show $\mathcal{E}_\mathrm{b}$ as a function of M_b for the two families of solutions in a theory with $(\beta_1, \beta_2, \Lambda) = (-6, 0, 50)$ and $\varphi_0 = 0$. The dashed line corresponds to solutions which are indistinguishable from the ones obtained in GR, while the solid line (which branches off from the former around $M_\mathrm{b}/M_\odot \approx 1.1$) corresponds to scalarized solutions. We see that scalarized stellar configurations in our model are energetically favorable, as happens in the case of purely conformal coupling theory [15,17,35].

VII. AN APPLICATION: EOS-INDEPENDENT $I - \mathcal{C}$ RELATIONS

As we have seen in the previous sections the presence of the disformal coupling modifies the structure of NSs making scalar-tensor theories generically predict different bulk properties with respect to GR. However, as we discussed based on Figs. 4 and 5, modifications caused by scalarization are usually degenerate with the choice of EOS, severely limiting our ability to constrain the parameters of the theory using current NS observations (see e.g. Ref. [118]). Moreover, different theory parameters can yield similar stellar models for a fixed EOS.

An interesting possibility to circumvent these problems is to search for EOS-independent (or at least weakly EOS-dependent) properties of NSs. Accumulating evidence favoring the existence of such EOS independence between certain properties of NSs, culminated with the discovery of the I-Love-Q relations [119,120] connecting the moment of inertia, the tidal Love number and the rotational

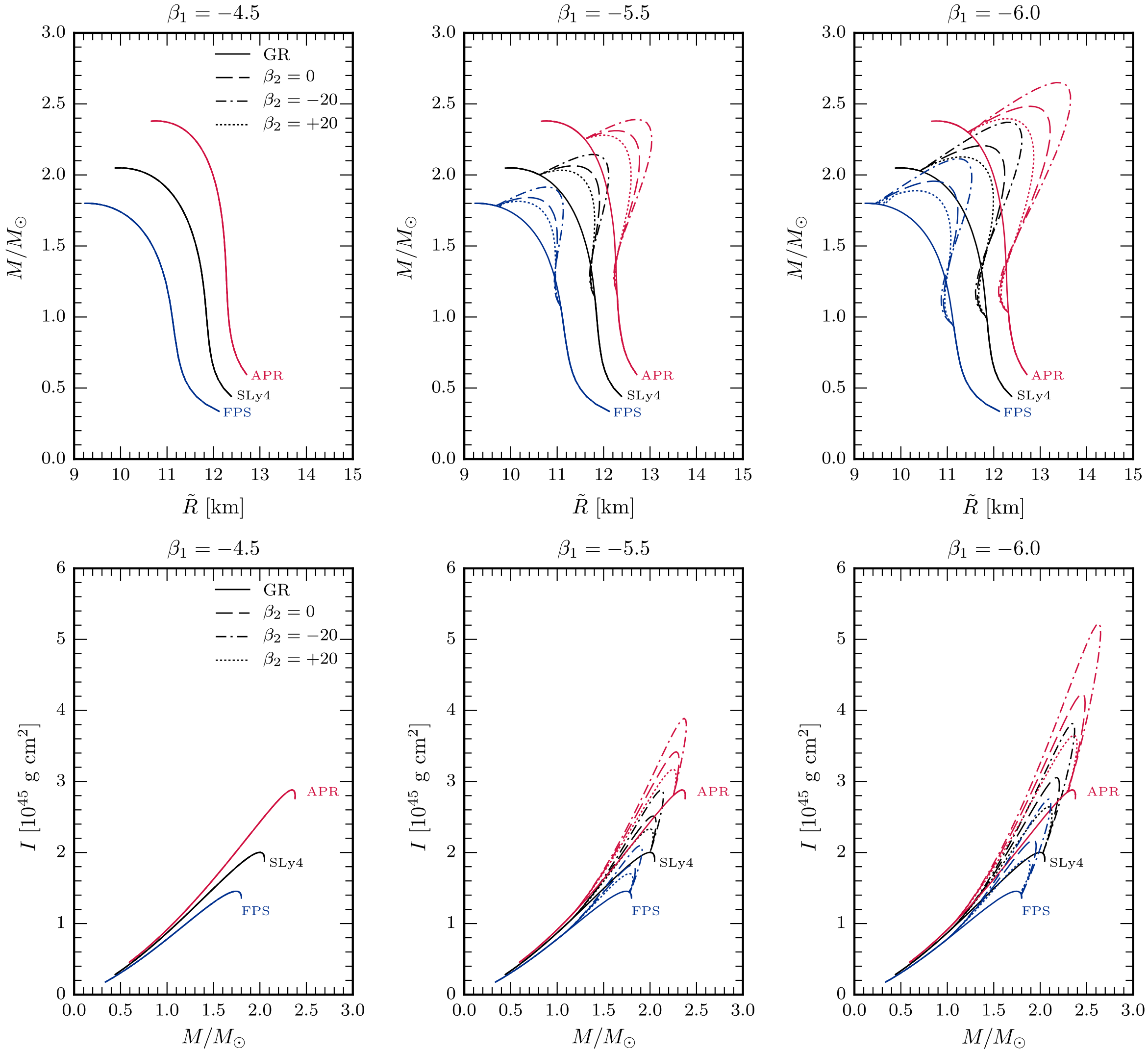

FIG. 5. In comparison to Fig. 4, here we show the influence of β_2 on spontaneous scalarization while keeping $\Lambda = -1000$ km^2. As we have seen in Fig. 1 (and by the analytic treatment of Sec. V), negative values of Λ suppress scalarization. This effect is such that for $\beta_1 = -4.5$, scalarization is suppressed altogether (top left panel). For smaller values of β_1, this value of Λ weakens scalarization and we clearly see that β_2 affects the most scalarized stellar models in the conformal coupling theory. Note that the range covered by the axis here and in Fig. 4 is the same, making it clear that scalarization is less strong for the values of β_2 adopted.

quadrupole moment (all made dimensionless by certain multiplicative factors) of NSs in GR.

If such relations hold in modified theories of gravitation they can potentially be combined with future NS measurements to constrain competing theories of gravity. This attractive idea was explored in the context of dynamical Chern-Simons theory [120], Eddington-inspired Born-Infeld gravity [121], Einstein-dilaton-Gauss-Bonnet (EdGB) gravity [122,123], $f(R)$ theories [124] and the Damour-Esposito-Farèse model of scalar-tensor gravity [24,26].

Within the present framework we cannot compute the I-Love-Q relations, since while on one hand we can compute I, the tidal Love number requires an analysis of tidal interactions, and the rotational quadrupole moment Q requires pushing the Hartle-Thorne perturbative expansion up to order $\mathcal{O}(\Omega^2)$. Nevertheless, we can investigate whether the recently proposed I-$\mathcal{C}$ relations [125] between the moment of inertia I and the compactness $\mathcal{C}$ remain valid in our theory. For a recent study in the Damour-Esposito-Farèse and R^2 theories, see Ref. [126]. This relation was also studied for EdGB and the subclass of Horndeski gravity with nonminimal coupling between the scalar field and the Einstein tensor in Ref. [127].

The relation proposed in Ref. [125] for the moment of inertia $\bar{I} := I/M^3$ and the compactness $\mathcal{C}$ is

$$\bar{I} = a_1\mathcal{C}^{-1} + a_2\mathcal{C}^{-2} + a_3\mathcal{C}^{-3} + a_4\mathcal{C}^{-4}, \qquad (93)$$

where the coefficients a_i ($i = 1, \ldots, 4$) are given by $a_1 = 8.134 \times 10^{-1}$, $a_2 = 2.101 \times 10^{-1}$, $a_3 = 3.175 \times 10^{-3}$ and

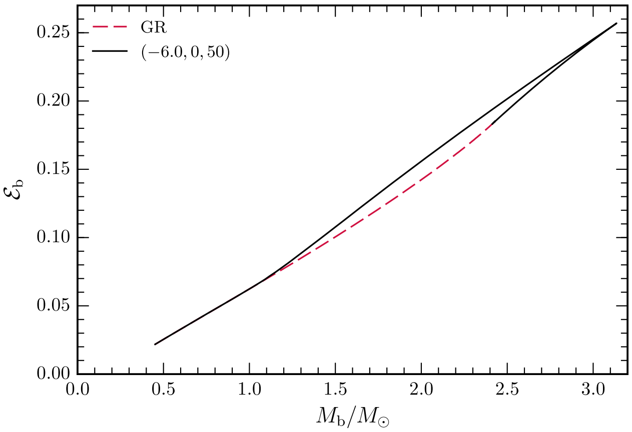

FIG. 6. We show the fractional binding energy $\mathcal{E}_b$ as a function of the baryonic mass for stellar models using EOS SLy4 and for theory with $(\beta_1, \beta_2, \Lambda) = (-6, 0, 50)$. Solutions in this theory branch around $M_b/M_\odot \approx 1.1$ with scalarized solutions (solid line) being energetically favorable over the general-relativistic ones (dashed line). The turning point at the solid curve corresponds to the maximum in the M-R relation, cf. Fig. 1.

$a_4 = -2.717 \times 10^{-4}$. This result is valid for slowly rotating NSs in GR, although it can easily be adapted for rapidly rotating NSs [125]. The coefficients in Eq. (93) are obtained by fitting the equation to a large sample of EOSs. For earlier work considering a different normalization for $\bar{I}$, namely $I/(MR^2)$, see e.g. Refs. [128–132].

We confront this fit against stellar models in two scalar-tensor theories with the parameters $(\beta_1, \beta_2, \Lambda)$ having the values $(-6, -20, -500)$ and $(-7, -20, -500)$ that support highly scalarized solutions. As seen in Fig. 7, the deviations from GR can be quite large, up to 40% for the theory with $\beta_1 = -7$ in the range of compactness for which spontaneous scalarization happens (cf. Fig. 7, bottom panel). Nevertheless, the EOS independence between $\bar{I}$ and $\mathcal{C}$ remains even when scalarization occurs (cf. Fig. 7, top panel).

Since our model is largely unconstrained observationally, measurements of the moment of inertia and compactness of NSs could in principle be used to constrain it or, more optimistically, indicate the occurrence of spontaneous scalarization in NSs. This is in contrast with the standard Damour-Esposito-Farèse model, for which the theory's parameters are so tightly constrained by binary pulsar observations [133], that spontaneous scalarization (if it exists) is bound to have a negligible influence on the I-$\mathcal{C}$ relation [126]. We stress however that in general it will be difficult to constrain the parameter space $(\beta_1, \beta_2, \Lambda)$ only through the I-$\mathcal{C}$ relation. The reason is in the degeneracy of stellar models for different values of the parameters; see the discussion in Sec. VI B.

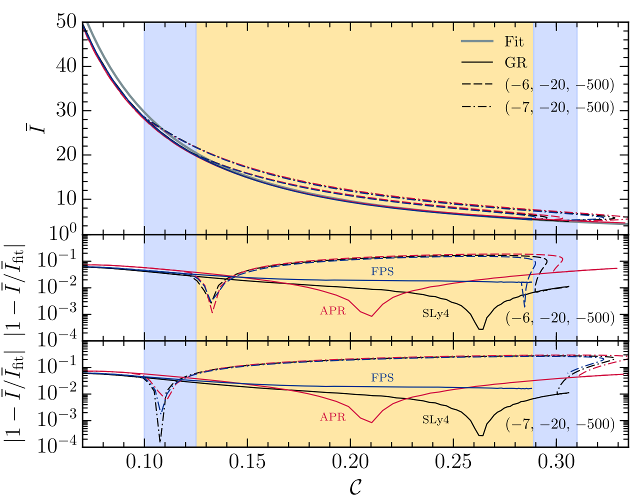

FIG. 7. We consider the I-$\mathcal{C}$ relation in scalar-tensor gravity. Top panel: The fit (93) obtained in the context of GR (thick solid line) is confronted against stellar models obtained in GR (solid line); and scalar-tensor theories with parameters $(\beta_2, \Lambda) = (-20, -500)$, but with $\beta_1 = -6$ (dashed lines) and $\beta_2 = -7$ (dash-dotted lines), using EOSs APR, SLy4 and FPS. Middle panel: Relative error between the fit for GR against scalar-tensor theory with $\beta_1 = -6$. Bottom panel: Similarly, but for $\beta_1 = -7$. In all panels the shaded regions correspond to approximately the domain of compactness for which spontaneous scalarization occurs in each theory. While for GR, the errors are typically below 6%, scalarized models can deviate from GR by 20% (for $\beta_1 = -6$) and up to 40% (for $\beta_1 = -7$).

VIII. CONCLUSIONS AND OUTLOOK

In this paper we have presented a general formulation to analyze the structure of relativistic stars in scalar-tensor theories with disformal coupling, including the leading-order corrections due to slow rotation. The disformal coupling is negligibly small in comparison with conformal coupling in the weak-gravity or slow-motion regimes, where the scalar field is slowly evolving and typical pressures are much smaller than the energy density scales, but it may be comparable to the ordinary conformal coupling in the strong-gravity regime found inside relativistic stars. Our calculation covers a variety of scalar-tensor models, especially, conformal and disformal couplings to matter, nonstandard scalar kinetic terms and generic scalar potential terms.

After obtaining the stellar structure equations, we have particularly focused on the case of a canonical scalar field with a generic scalar potential. We showed that in the absence of both conformal coupling and a scalar potential, the disformal coupling does not modify the stellar structure with respect to GR. On the other hand, this result shows us that inside relativistic stars the effects of disformal coupling always appear *only* when there is conformal coupling to matter and/or a nontrivial potential term. The strength of disformal coupling crucially depends on the coupling strength Λ in Eq. (2) with dimensions of $(\text{length})^2$. For a canonical scalar field, Λ has to be of $\mathcal{O}(10^3)$ km^2 to significantly influence the structure of NSs.

In our numerical analyses, we have investigated the effects of the disformal coupling on the spontaneous scalarization of NSs in the scalar-tensor theory with purely conformal coupling. We found that the effects of disformal coupling depend on the sign of Λ. We showed that for negative values of Λ the mass and moment of inertia of NSs decrease, approaching the values in GR for sufficiently large negative values of Λ. We speculate that this is the consequence of a mechanism similar to the disformal screening proposed in Ref. [86] where in a high-density or a large disformal coupling limit the response of the scalar field becomes insensitive to the local matter density, exemplified here by studying relativistic stars. On the other hand, for positive values of Λ, we showed that the mass and moment of inertia increase but for too large positive values of Λ the stellar structure equation becomes singular and a regular NS solution cannot be found. This allowed us to derive a mild upper bound of $\Lambda \lesssim 100\ \mathrm{km}^2$, that does not depend on the choice of the EOS.

We have also tested the applicability of a recently proposed EOS-independent relation between the dimensionless moment of inertia I/M^3 and the compactness $\mathcal{C}$ for NSs in GR. We found that for a certain domain of the theory's parameter space, the deviations from GR can be as large as ∼40%, suggesting that future measurements of the NS moment of inertia might be used to test scalar-tensor theories with disformal coupling. Because of the large dimensionality of the parameter space, modifications with respect to GR are generically degenerate between different choices of β_1, β_2 and Λ. Thereby, even though deviations from GR can be larger, it seems unlikely that constraints can be put on the theory's parameters using exclusively the I-$\mathcal{C}$ relation. In this regard, it would be worth extending our work and studying how the I-Love-Q relations are affected by the disformal coupling, generalizing the works of Refs. [24–26] for scalar-tensor theories with disformal coupling.

Still in this direction, one could investigate whether the "three-hair" relations—EOS-independent relations connecting higher-order multipole moments of rotating NSs in terms of the first three multipole moments in GR [134–136]—remain valid in scalar-tensor theory, including those with disformal coupling. This could be accomplished by combining the formalism developed in Ref. [29] with numerical solutions for rotating NSs such as those obtained in Ref. [24].

Although the main subject of this paper was to investigate the hydrostatic equilibrium configurations in scalar-tensor theories with disformal coupling, let us briefly comment on the gravitational (core) collapse resulting in the formation of a NS (see e.g. Ref. [34]). A fully numerical analysis of dynamical collapse in this theory is beyond the scope of our paper, but an important issue in this dynamical process may be the possible appearance of ghost instabilities for negative values of Λ [72,75,86,110]. During collapse, matter density at a given position increases, and if at some instant it reaches the threshold value where the effective kinetic term in the scalar field equation vanishes, the time evolution afterwards cannot be determined. For a canonical scalar field $P = 2X$, in a linearized approximation where $\chi \simeq 1$ and $B(\varphi) \simeq 1$, the effective kinetic term of the equation of motion (20) is roughly given by

$$-\left(1 - \frac{\kappa|\Lambda|}{2}\tilde{\rho}c^2\right)\ddot{\varphi}, \tag{94}$$

where the dot represents a time derivative. The sign of the kinetic term may change in the region of a critical density higher than $\tilde{\rho}_{\rm crit} = 2/(\kappa c^2|\Lambda|)$. The choice of $\Lambda = -100\ \mathrm{km}^2$ gives $\tilde{\rho}_{\rm crit} \simeq 10^{15}\ \mathrm{g/cm}^3$, which is a typical central density of NSs. Thus for $|\Lambda| \lesssim 100\ \mathrm{km}^2$ a NS is not expected to suffer an instability while for other values it might occur in the interior of the star. Of course, for a more precise estimation, nonlinear interactions between the dynamical scalar field, spacetime and matter must be taken into consideration. A detailed study of time-dependent processes in our theory is definitely important, but is left for future work.

Another interesting prospect for future work would be to study compact binaries within our model. The most stringent test of scalar-tensor gravity comes from the measurement of the orbital decay of binaries with asymmetric masses, which constrains the emission of dipolar scalar radiation by the system [53]. We expect that the disformal coupling parameters β_2 and Λ should play a role in the orbital evolution of a binary system by influencing the emission of scalar radiation from the system. In fact, both parameters are expected to modify the so-called sensitivities [137,138] that enter at the lowest PN orders sourcing the emission of dipolar scalar radiation. An investigation of compact binaries within our model could, combined with current observational data, yield tight constraints on disformal coupling. Moreover one could study NS solutions for other classes of scalar-tensor theories not considered here. This task is facilitated by the generality of our calculations presented in Sec. III. Work in this direction is currently underway and we hope to report it soon.

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APPENDIX: DISFORMAL INVARIANCE

In this appendix, we study how the physical quantities associated with the stellar properties transform under the disformal transformation (2). We write the metrics with slow rotation of spacetimes in the Einstein and Jordan frames as

$$ds^2 = -e^{\nu(r)}c^2dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + 2(\omega - \Omega)r^2\sin^2\theta dt d\phi, \tag{A1}$$

and

$$d\bar{s}^2 = -e^{\bar{\nu}(\bar{r})}c^2dt^2 + e^{\bar{\lambda}(\bar{r})}d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2) + 2(\bar{\omega} - \bar{\Omega})\bar{r}^2\sin^2\theta dt d\phi. \tag{A2}$$

We can relate Eqs. (A1) and (A2) using the disformal relation (2) as

$$e^{\bar{\nu}} = A^2(\varphi)e^{\nu}, \tag{A3}$$

$$e^{\frac{\bar{\lambda}}{2}}d\bar{r} = A(\varphi)\sqrt{\chi}e^{\frac{\lambda}{2}}dr, \tag{A4}$$

$$\bar{r} = rA(\varphi), \tag{A5}$$

$$\bar{\omega} - \bar{\Omega} = \omega - \Omega, \tag{A6}$$

where we recall that due the symmetries of the problem $\varphi = \varphi(r)$. From Eqs. (A4) and (A5) we get

$$e^{\bar{\lambda}} = \frac{\chi}{(1 + r\alpha\varphi')^2}e^{\lambda}. \tag{A7}$$

Introducing μ and $\bar{\mu}$ in the Einstein and Jordan frames by

$$e^{-\lambda} = 1 - \frac{2\mu}{r}, \qquad e^{-\bar{\lambda}} = 1 - \frac{2\bar{\mu}}{\bar{r}}, \tag{A8}$$

and using Eqs. (A5) and (A7) we find

$$\bar{\mu} = -\frac{rA(\varphi)}{2}\left[\left(1 - \frac{2\mu}{r}\right)\frac{(1 + r\alpha(\varphi)\varphi')^2}{\chi} - 1\right]. \tag{A9}$$

As it is reasonable to set $\varphi_0 = 0$ and $\varphi_0' = 0$ at asymptotic infinity, in the class of models considered in the text [Eq. (73)], $A(\varphi_0) = 1$, $\alpha(\varphi_0) = 0$ and $\chi(\varphi_0, \varphi_0') = 1$, we find that the ADM mass obtained from the leading-order values of μ and $\bar{\mu}$ at asymptotic infinity is disformally invariant

$$\bar{M} = M. \tag{A10}$$

The energy-momentum tensors of the matter fields in the Einstein and Jordan frames are defined by

$$T_{(\mathrm{m})\mu\nu} = \rho c^2 u_\mu u_\nu + p_r k_\mu k_\nu + p_t(g_{\mu\nu} + u_\mu u_\nu - k_\mu k_\nu),$$
$$\bar{T}_{(\mathrm{m})\mu\nu} = \bar{\rho} c^2 \bar{u}_\mu \bar{u}_\nu + \bar{p}_{\bar{r}} \bar{k}_\mu \bar{k}_\nu + \bar{p}_t(\bar{g}_{\mu\nu} + \bar{u}_\mu \bar{u}_\nu - \bar{k}_\mu \bar{k}_\nu), \tag{A11}$$

where u^μ ($\bar{u}^\mu$) and k^μ ($\bar{k}^\mu$) are the four-velocity and unit radial vectors in the Einstein (Jordan) frame, respectively [55]. Within the first order of Hartle-Thorne's slow-rotation approximation [106], in the Einstein frame

$$u^\mu = \left(\frac{1}{\sqrt{-g_{tt}}}, 0, 0, \frac{\Omega}{\sqrt{-g_{tt}}}\right),$$
$$k^\mu = \left(0, \frac{1}{\sqrt{g_{rr}}}, 0, 0\right), \tag{A12}$$

and in the Jordan frame $\bar{u}^\mu$ and $\bar{k}^\mu$ are defined in the same way as Eq. (A12) with an overbar. The nonvanishing components of the energy-momentum tensors in both frames are then given by

$$T_{(m)t}{}^{t} = -\rho c^2, \quad T_{(m)r}{}^{r} = p_r, \quad T_{(m)\theta}{}^{\theta} = T_{(m)\phi}{}^{\phi} = p_t, \tag{A13a}$$

$$\bar{T}_{(m)t}{}^{t} = -\bar{\rho} c^2, \quad \bar{T}_{(m)\bar{r}}{}^{\bar{r}} = \bar{p}_{\bar{r}}, \quad \bar{T}_{(m)\theta}{}^{\theta} = \bar{T}_{(m)\phi}{}^{\phi} = \bar{p}_t, \tag{A13b}$$

and

$$T_{(m)\phi}{}^{t} = \left(\rho + \frac{p_t}{c^2}\right)e^{-\nu}\omega r^2\sin^2\theta, \tag{A14a}$$

$$\bar{T}_{(m)\phi}{}^{t} = \left(\bar{\rho} + \frac{\bar{p}_t}{c^2}\right)e^{-\bar{\nu}}\bar{\omega}\bar{r}^2\sin^2\theta. \tag{A14b}$$

In the Jordan frame, we then make a coordinate transformation from $\bar{x}^\mu = (t, \bar{r}, \theta, \phi)$ to $x^\mu = (t, r, \theta, \phi)$, such that

$$\tilde{T}_{(m)\mu}{}^{\nu} \coloneqq \frac{\partial \bar{x}^\rho}{\partial x^\mu}\frac{\partial x^\nu}{\partial \bar{x}^\sigma}\bar{T}_{(m)\rho}{}^{\sigma}. \tag{A15}$$

Introducing the components of the energy-momentum tensor $\tilde{T}_{(m)\mu\nu}$ as Eqs. (A13a)–(A13b) with a tilde, we find

$$\bar{\rho} = \tilde{\rho}, \qquad \bar{p}_{\bar{r}} = \tilde{p}_r, \qquad \bar{p}_t = \tilde{p}_t, \tag{A16}$$

and consequently

$$\bar{T}_{(m)\phi}{}^{t} = \tilde{T}_{(m)\phi}{}^{t}. \quad \text{(A17)}$$

The components of the energy-momentum tensor in the Einstein and Jordan frames are related by Eq. (23) and

$$T_{(m)\phi}{}^{t} = A^4(\varphi)\sqrt{\chi}\tilde{T}_{(m)\phi}{}^{t}. \quad \text{(A18)}$$

Substituting Eqs. (23), (A3), (A5), (A14a)–(A14b), (A16) and (A17) into Eq. (A18) we find

$$\bar{\omega} = \omega. \quad \text{(A19)}$$

Thus from Eq. (A6),

$$\bar{\Omega} = \Omega. \quad \text{(A20)}$$

The angular momenta in the Einstein and Jordan frames are given by

$$J = \int dr d\theta d\phi r^2 \sin\theta e^{\frac{\nu+\lambda}{2}} T_{(m)\phi}{}^{t}, \quad \text{(A21)}$$

$$\bar{J} = \int d\bar{r} d\theta d\phi \bar{r}^2 \sin\theta e^{\frac{\bar{\nu}+\bar{\lambda}}{2}} \bar{T}_{(m)\phi}{}^{t}. \quad \text{(A22)}$$

Using again Eqs. (A3), (A4), (A5), (A17) and (A18), we find that the angular momentum is disformally invariant

$$\bar{J} = J. \quad \text{(A23)}$$

From Eqs. (A20) and (A23) we find that the moments of inertia in the Einstein and Jordan frames, $I = J/\Omega$ and $\bar{I} = \bar{J}/\bar{\Omega}$, are also disformally invariant

$$\bar{I} = I. \quad \text{(A24)}$$

Thus all quantities associated with rotation are disformally invariant. Our arguments in this appendix can be applied to a generic class of the Horndeski theory connected by the disformal transformation [60].

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