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Analyzing modified unimodular gravity via Lagrange multipliers

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The so-called unimodular version of general relativity is revisited. Unimodular gravity is constructed by fixing the determinant of the metric, which leads to the trace-free part of the equations instead of the usual Einstein field equations. Then a cosmological constant naturally arises as an integration constant. While unimodular gravity turns out to be equivalent to general relativity (GR) at the classical level, it provides important differences at the quantum level. Here we extend the unimodular constraint to some extensions of general relativity that have drawn a lot of attention over the last years—$f(R)$ gravity (or its scalar-tensor picture) and Gauss-Bonnet gravity. The corresponding unimodular version of such theories is constructed as well as the conformal transformation that relates the Einstein and Jordan frames for these nonminimally coupled theories. From the classical point of view, the unimodular versions of such extensions are completely equivalent to their originals, but an effective cosmological constant arises naturally, which may provide a richer description of the evolution of the Universe. Here we analyze the case of Starobisnky inflation and compare it with the original one.

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I. INTRODUCTION

Over the last decades, the so-called "cosmological constant problem" has been one of the major challenges in theoretical physics. The issue refers to the absence of gravitational effects, particularly at the cosmological level, of the vacuum energy density predicted by quantum field theories or, better said, the impossibility of fine-tuning properly its counterterms, which is known as radiative instability (for a review, see Refs. [1] and [2]). On the other hand, after the discovery of some deviations of the luminosity distances of Supernovae Ia in 1998, which was then interpreted as a consequence of the acceleration of the Universe's expansion (and later confirmed by other proofs), the best and most accepted model that can explain such behaviour lies on the presence of a cosmological constant in the gravitational field equations, which in principle should be connected somehow to the vacuum energy density. However, the required cosmological constant for the acceleration of the expansion (of the order the Hubble parameter today) is around 120 orders of magnitude smaller than the one predicted by quantum field theories. Hence, here the problem arises, how to drop down the huge value of vacuum fluctuations. In this sense, there have been plenty of proposals, which include a possible symmetry that protects the cosmological constant in the same sense that chiral symmetry does with the electron mass as well as supersymmetry attempts. In addition, an alternative way, which may include the dark energy models and modified gravities, tries to suppress such a large value by additional fields or modifications of general relativity (GR). In this sense, there have been plenty of dark energy models proposed, which may play that role; see [3] and [4]. However, rather than solving the problem, the former always requires a precise fine-tuning as well (Weinberg's no-go theorem).

An alternative widely studied in the literature is the so-called unimodular gravity (see Refs. [1,2,5–13]). The theory fixes the determinant of the metric, such that the field equations are given by the trace-free part of GR's field equations. From the classical point of view, fixing the determinant of the metric provides a cosmological constant that naturally arises as an integration constant after applying the corresponding geometrical identities, which at the cosmological level may be a way of understanding the problem of dark energy [5]. The unimodular constraint can be implemented in several ways, all of them leading to the same classical theory, as shown in the literature [5–13]. However, in spite of the theory being equivalent to GR at the classical level, the equivalence is not clear at the quantum one, where great effort has been made to get a better understanding of the features of the theory [6,7]. When the theory is analyzed in quantum mechanics, radiative instability is absent for this effective cosmological constant, which is one of the most interesting features of the theory since it can suppress the large contribution of the vacuum energy density [8]. The absence of radiative instability has been shown in the literature by using different approaches, for instance, by the existence of a shift symmetry in the classical field equations that remove the contributions from the quantum vacuum and also by the evaluation of the renormalization group equation for the cosmological constant. In addition, unimodular gravity may be distinguished from GR by some observables, as it may lead to a different concept of mass [9]. Hence, unimodular gravity may provide a way of better understanding not only the cosmological constant problem but also the dark energy issue.

In this sense, an extension of unimodular gravity has been recently proposed, where more general actions rather than the Hilbert-Einstein action are considered [14,15]. Note that modified gravities such as $f(R)$ gravity have drawn a lot of attention in recent years, as alternatives to dark energy and inflation, since they can realize the cosmological history. In addition, some of them, such as Starobinsky inflation [16] and the so-called Hu-Sawicki model for late-time acceleration [17], are able to satisfy the last observational constraints with great accuracy. Hence, the analysis of such extensions within the unimodularlike framework may provide interesting features.

With this aim, this paper is devoted to the analysis of some generalizations of unimodular gravity at the classical level. First, we carefully reconstruct such extensions departing from variational principles by using a Lagrange multiplier which imposes the unimodular constraint, leading to the trace-free part of the field equations. We show that, as in the case of unimodular gravity, any extension leads to the same result; i.e., a cosmological constant arises naturally in the field equations, recovering full diffeomorphisms. We also analyze how conformal transformations affect the gauge choice imposed initially and the effects of unimodular gravity in the Einstein frame. Finally, some cosmological solutions are obtained for $f(R)$ gravity and Gauss-Bonnet gravities, while Starobinsky inflation is also analyzed, where we find a constraint on the merging constant in order to keep Starobinsky predictions.

The paper is organized as follows: Section II gives a brief review of unimodular gravity. In Sec. III, we introduce $f(R)$ and Gauss-Bonnet unimodular gravity. In Sec. IV, the conformal transformation is analyzed and the corresponding unimodular version is obtained in the Einstein frame. Then, Sec. V is devoted to the analysis of cosmological solutions. Finally, Sec. VI gathers the conclusions.

II. UNIMODULAR GRAVITY

Unimodular gravity is constructed in such a way that the determinant of the spacetime metric is not dynamical but is restricted to be

$$\sqrt{-g} = s_0, \tag{1}$$

which fixes the determinant of the metric to be a constant s_0. As stated in [1], "just because we use a generally covariant formalism does not mean that we are committed to treating all components of the metric as dynamical fields." Then, restricted variations of the action with respect to the metric have to be null only for those which keep the determinant fixed,

$$g_{\mu\nu}\delta g^{\mu\nu} = 0. \tag{2}$$

The variation of the metric can be written then in terms of the unconstrained variation as

$$\delta g^{\mu\nu} = \delta_u g^{\mu\nu} - \frac{1}{4} g^{\mu\nu} g_{\lambda\gamma} \delta_u g^{\lambda\gamma}. \tag{3}$$

The gravitational field equations are obtained by varying the gravitational action S_G, which can also be expressed in terms of the unconstrained variation, leading to

$$\frac{\delta S_G}{\delta g^{\mu\nu}} = \frac{\delta S_G}{\delta_u g^{\mu\nu}} - \frac{1}{4} g_{\mu\nu} g^{\lambda\gamma} \frac{\delta S_G}{\delta_u g^{\lambda\gamma}}. \tag{4}$$

These are precisely the traceless part of the gravitational field equations, which for the Hilbert-Einstein action leads to the traceless part of the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = \kappa^2 \left(T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T \right), \tag{5}$$

where $R_{\mu\nu}$ is the usual Ricci tensor and R its trace, while $T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial S_m}{\partial g_{\mu\nu}}$ is the matter energy-momentum tensor, and $\kappa^2 = 8\pi G$. Contrary to general relativity, the field equations (5) are not divergence free:

$$\nabla_\mu \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R - \kappa^2 T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T \right) = 0. \tag{6}$$

Then, by using the Bianchi identities, which hold

$$\nabla_\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0, \tag{7}$$

and the energy conservation,

$$\nabla_\mu T_{\mu\nu} = 0, \tag{8}$$

The divergence of the field equations (6) yields

$$\nabla_\mu (R + \kappa^2 T) = 0, \tag{9}$$

the so-called integrability condition, which after integrating leads to

$$R + \kappa^2 T = 4\lambda_0 = \text{const}, \tag{10}$$

where λ_0 is an integration constant. Hence, by inserting (10) in the field equations (5), we get

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\lambda_0 = \kappa^2 T_{\mu\nu}, \tag{11}$$

where the usual Einstein field equations are recovered, with λ_0 being a cosmological constant. This is the great success of unimodular gravity, since a cosmological constant emerges naturally as an integration constant by departing from the trace-free part of the Einstein equations. Such a constant may compensate for the large value of the vacuum energy density. In addition, since the integrability condition (9) recovers the usual general relativity equations, any prediction from the former turns out to be a prediction of unimodular gravity, which avoids any possible discrepancy with well-tested experiments.

Alternatively, unimodular gravity can be implemented through a variational principle with unrestricted variations of the metric by assuming transverse diffeomorphisms (TDiff) instead of the full diffeomorphisms [6,8,10], which gives rise to the appearance of a scalar field that represents the determinant of the metric. Such extra degrees of freedom can be removed by an additional Weyl symmetry (WTDiff) [6,10,11]. Moreover, unimodular gravity can also be obtained by using a Lagrange multiplier in the action as follows [12],

$$S = \frac{1}{2\kappa^2}\int dx^4[\sqrt{-g}R - 2\lambda(\sqrt{-g} - s_0)] + S_m, \qquad (12)$$

where s_0 is a constant and λ is the Lagrange multiplier, which in principle is dynamical. Note that the last term in (12) breaks the full diffeomorphism invariance, since it fixes the determinant of the metric, restricting the group of symmetries. Then, by varying the action with respect to the metric, the field equations yield

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\lambda = \kappa^2 T_{\mu\nu}, \qquad (13)$$

while the variation with respect to λ leads to the unimodular restriction (1). Taking the trace of the field equations (13),

$$R + \kappa^2 T = 4\lambda(x). \qquad (14)$$

This looks like (10) except that, in principle, $\lambda = \lambda(x)$ is not a constant. Nevertheless, taking the divergence of equations (13) together with the energy conservation $\nabla_\mu T_{\mu\nu} = 0$, yields

$$\nabla_\mu \lambda = 0 \rightarrow \lambda = \lambda_0. \qquad (15)$$

Then, the trace-free part of the equations follows, and the previous result (11) is obtained, in this case by means of the action (12). Moreover, as in (12), one can depart from the Henneaux-Teitelboim action, leading to the same result [13]. Note that while all these implementations of unimodular gravity are classically equivalent, they are not at the quantum level (see [6]). However, as we focus just on classical aspects throughout this paper, we are assuming, for convenience, the action (12) as the departing point, as shown below.

III. GENERALIZATIONS OF UNIMODULAR GRAVITY

In recent years, some modifications of the Hilbert-Einstein action have been considered, particularly as infrared corrections to GR, in order to provide a natural explanation to the late-time acceleration of the expansion [4]. Moreover, such modifications have been widely applied to inflation since the data seem to favor such models. Within modified gravities, the so-called $f(R)$ gravity has drawn a lot of attention. Its principle states on a gravitational action given precisely by,

$$S = \frac{1}{2\kappa^2}\int dx^4\sqrt{-g}f(R) + S_m, \qquad (16)$$

whose field equations are obtained by varying the action with respect to the metric, leading to

$$R_{\mu\nu}f_R - \frac{1}{2}g_{\mu\nu}f + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R = \kappa^2 T_{\mu\nu}, \qquad (17)$$

where $f_R = \frac{\partial f}{\partial R}$. Generalization of unimodular gravity turns out now to be clear. As noted in [15], the action (16) has a unimodular $f(R)$ version which is constructed by fixing the determinant to be a constant,

$$S = \frac{1}{2\kappa^2}\int dx^4[\sqrt{-g}f(R) - 2\lambda(\sqrt{-g} - s_0)] + S_m. \qquad (18)$$

The field equations are then given by

$$R_{\mu\nu}f_R - \frac{1}{2}g_{\mu\nu}f + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R + g_{\mu\nu}\lambda = \kappa^2 T_{\mu\nu}, \qquad (19)$$

While the variation of the action with respect to λ leads to $\sqrt{-g} = s_0$. As in the previous section, taking the divergence of the field equations (19) yields

$$\nabla_\mu \lambda = 0 \rightarrow \lambda = \lambda_0, \qquad (20)$$

where we have used the identities $\nabla_\mu(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}) = 0$ and $(\nabla_\nu\Box - \Box\nabla_\nu)f_R = R_{\mu\nu}\nabla^\mu f_R$. Then, by using the trace of the field equations (19), the following condition is provided,

$$-Rf_R + 2f - 3\Box f_R + \kappa^2 T = 4\lambda_0, \qquad (21)$$

which is the generalization of the integrability condition (10). Hence, the usual $f(R)$ equations are recovered with an additional cosmological constant:

$$R_{\mu\nu}f_R - \frac{1}{2}g_{\mu\nu}f + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R + g_{\mu\nu}\lambda_0 = \kappa^2 T_{\mu\nu}, \qquad (22)$$

Equivalently, one may proceed to obtain the same result by starting from the trace-free part of (17) as the field equations:

$$R_{\mu\nu}f_R - \frac{1}{2}g_{\mu\nu}f + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R - \frac{1}{4}(Rf_R - 2f + 3\Box f_R)g_{\mu\nu} = \kappa^2\left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right). \quad (23)$$

By using $\nabla_\mu T^{\mu\nu} = 0$ and the Bianchi identities, the divergence of (23) yields

$$\nabla_\mu(Rf_R - 2f + 3\Box f_R - \kappa^2 T) = 0, \quad (24)$$

which is equivalent to (21) after integrating, and the field equations (22) are recovered.

Hence, it is straightforward to construct other generalizations of unimodular gravity by following the procedure described above. For instance, we may consider the so-called modified Gauss-Bonnet gravity,

$$S = \frac{1}{2\kappa^2}\int dx^4[\sqrt{-g}(R + f(G) - 2\lambda(\sqrt{-g} - s_0)] + S_m, \quad (25)$$

where $G = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss-Bonnet topological invariant. The field equations are obtained by varying the action (25) with respect to the metric [18],

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}f(G) + 2f_G RR_{\mu\nu} - 4f_G R_{\mu\rho}R_\nu{}^\rho + 2f_G R^{\mu\rho\sigma\tau}R_{\nu\rho\sigma\tau} + 4f_G R_{\mu\rho\sigma\nu}R^{\rho\sigma} - 2R\nabla_\mu\nabla_\nu f_G + 2g_{\mu\nu}R\Box f_G + 4R_{\nu\rho}\nabla^\rho\nabla_\mu f_G + 4R_{\mu\rho}\nabla^\rho\nabla_\nu f_G - 4R_{\mu\nu}\Box f_G - 4g_{\mu\nu}R^{\rho\sigma}\nabla_\rho\nabla_\sigma f_G + 4R_{\mu\rho\nu\sigma}\nabla^\rho\nabla^\sigma f_G + \lambda g_{\mu\nu} = \kappa^2 T^{\mu\nu}. \quad (26)$$

As above, by taking the divergence of the field equations, the condition $\nabla_\mu\lambda = 0$ arises again, which leads to the integrability condition for this case:

$$R + 2f - 2f_G R^2 + 4f_G R_{\mu\nu}R^{\mu\nu} - 2f_G R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma} - 4f_G R^{\mu\nu\lambda}{}_\mu R_{\nu\lambda} - 2R\Box f_G + 8R^{\mu\nu}\nabla_\mu\nabla_\nu f_G + 4R^{\mu\nu\lambda}{}_\mu\nabla_\nu\nabla_\lambda f_G + T = 4\lambda_0. \quad (27)$$

Then, we get the usual modified Gauss-Bonnet gravity with an additional cosmological constant. Note that the same result is obtained when departing from the trace-free part of the field equations for Gauss-Bonnet gravity, as was shown above for the case of $f(R)$ gravity. Hence, following any of the above procedures, unimodular gravity can be easily extended to other more complex actions. The result basically adds a cosmological constant to the field equations, as in the case of Hilbert-Einstein unimodular gravity.

Alternatively to the Lagrange multiplier, one may depart from restricting variations over the gravitational action (4), leading to the traceless part of the corresponding $f(R)$ or $f(R, G)$ action, as above. Other implementations of unimodular gravity can also be applied for these cases equivalently at the classical level. However, by using a Lagrange multiplier instead of other implementations of the unimodular condition, calculations are simplified when dealing with theories with higher-order derivatives. In the next section, we analyze unimodular scalar-tensor theories (equivalent to $f(R)$ gravities) and their transformation to the so-called Einstein frame when applying a conformal transformation, which becomes also simpler when forcing the unimodular constraint by a Lagrange multiplier than other alternative -classically- equivalent implementations.

IV. CONFORMAL FRAMES

As well known, $f(R)$ gravities can be expressed in terms of an scalar field with a null kinetic term through the action:

$$S = \frac{1}{2\kappa^2}\frac{1}{2\kappa^2}\int dx^4\sqrt{-g}(\phi R - V(\phi)) + S_m, \quad (28)$$

Varying the action with respect to the scalar field, the corresponding equivalence is found:

$$V'(\phi) = R \to \phi = \phi(R), \qquad f(R) = \phi(R)R - V(\phi(R)),$$

which yields the relations:

$$\phi = f_R, \qquad V = Rf_R - f, \quad (29)$$

As in the previous section, the reconstruction of the unimodular theory for the action (28) is given by fixing the determinant of the metric,

$$S = \frac{1}{2\kappa^2}\int dx^4\sqrt{-g}(\phi R - V(\phi)) - 2\lambda(\sqrt{-g} - s_0) + S_m, \quad (30)$$

The field equations are given by:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(\phi R - V(\phi)) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\phi + g_{\mu\nu}\lambda = \kappa^2 T^{(m)}_{\mu\nu}, \quad (31)$$

Taking the divergence of the field equations, the condition $\nabla_\mu \lambda = 0$ results and the integrability condition (21) is obtained, which is now given by

$$\phi R - 2V - 3\Box\phi + \kappa^2 T^{(m)} = 4\lambda_0. \tag{32}$$

Consequently, the field equations (31) become the usual equations for the scalar-tensor theory (28) with an additional cosmological constant. The question now arises, does the action (30) have a counterpart in the Einstein frame? To do so, let us transform the action (30) into the Einstein frame, which basically means recovering the usual Hilbert-Einstein action by applying the following conformal transformation:

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \text{where } \Omega^2 = \phi. \tag{33}$$

Here the tilde refers to the Einstein frame. Then, the Ricci scalar is transformed as follows:

$$\tilde{R} = \frac{2}{\Omega^2}\left(R - \frac{6\Box\Omega}{\Omega}\right). \tag{34}$$

And the action (30) becomes

$$\tilde{S} = \int dx^4 \left[\sqrt{-\tilde{g}}\left(\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tilde{V}(\varphi)\right) - 2\tilde{\lambda}(\sqrt{-\tilde{g}}e^{-2\sqrt{2/3}\kappa\varphi} - s_0),\right] \tag{35}$$

where we have redefined the scalar field,

$$\phi = e^{\sqrt{2/3}\kappa\varphi} \tilde{V}(\varphi) = \frac{e^{-2\sqrt{2/3}\kappa\varphi}}{2\kappa^2} V(\varphi), \qquad \tilde{\lambda} = \frac{\lambda}{2\kappa^2}. \tag{36}$$

The field equations are obtained by varying the action with respect to the metric,

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = \kappa^2 (T^{(\varphi)}_{\mu\nu} + T^{(m)}_{\mu\nu}), \tag{37}$$

where we have defined the energy-momentum tensor of the scalar field as,

$$T^{(\varphi)}_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial_\sigma\varphi\partial^\sigma\varphi + \tilde{V}\right) - 2\tilde{\lambda} g_{\mu\nu} e^{-2\sqrt{2/3}\kappa\varphi} \tag{38}$$

The scalar field equation is obtained by varying the action (35) with respect to the scalar field,

$$\Box\varphi - V'(\varphi) + 4\tilde{\lambda}\sqrt{\frac{2}{3}}\kappa e^{-2\sqrt{2/3}\kappa\varphi} = 0. \tag{39}$$

While the variation of the action with respect to the Lagrange multiplier leads to the constraint,

$$\sqrt{-\tilde{g}} = s_0 \times e^{2\sqrt{2/3}\kappa\varphi}. \tag{40}$$

Hence, contrary to the case of the Jordan frame, the determinant of the metric $\tilde{g}_{\mu\nu}$ is not constant. Taking the divergence of the field equations (39), and applying the identity $\nabla_\mu(\tilde{R}^{\mu\nu} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{R}) = 0$ and the matter-energy conservation $\nabla_\mu T^{\mu\nu(m)} = 0$, yields,

$$\nabla_\mu T^{\mu\nu(\varphi)} = \left(\Box\varphi - V' + 4\tilde{\lambda}\sqrt{\frac{2}{3}}\kappa e^{-2\sqrt{2/3}\kappa\varphi}\right)\partial^\nu\varphi - 2e^{-2\sqrt{2/3}\kappa\varphi}\partial^\nu\tilde{\lambda} = 0. \tag{41}$$

The first term in (41) is the scalar field equation (39) which becomes null, leading to

$$\partial_\nu\tilde{\lambda} = 0, \qquad \to \tilde{\lambda} = \tilde{\lambda}_0. \tag{42}$$

Then the energy-momentum tensor for the scalar field (38) is obtained:

$$T^{(\varphi)}_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial_\sigma\varphi\partial^\sigma\varphi + \tilde{V}\right) - 2\tilde{\lambda}_0 g_{\mu\nu} e^{-2\sqrt{2/3}\kappa\varphi}. \tag{43}$$

Hence, the field equations (37) are basically the equations of the action,

$$\tilde{S} = \frac{1}{2\kappa^2}\int dx^4 \left[\sqrt{-\tilde{g}}\left(\frac{\tilde{R}}{2\kappa^2} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \tilde{V}_{\text{eff}}(\varphi)\right)\right], \tag{44}$$

where the effective potential is defined as,

$$\tilde{V}_{\text{eff}}(\varphi) = \tilde{V}(\varphi) + 2\tilde{\lambda}_0 e^{-2\sqrt{2/3}\kappa\varphi}. \tag{45}$$

In comparison with the case in the Jordan frame, where a cosmological constant naturally arises, here the scalar potential is modified, which may introduce corrections to some solutions.

In the next section, we explore some cosmological solutions within the context of $f(R)$ and modified Gauss-Bonnet gravities, but also solutions in the Einstein frame are analyzed, particularly we study Starobinsky inflation within the context of unimodular gravity by applying the results obtained above.

V. COSMOLOGICAL SOLUTIONS

Let us now explore some cosmological solutions in the generalizations of unimodular gravity studied above. Here we intend to analyze dark energy solutions as well as some inflationary models.

A. Late-time acceleration

Since we are interested in late-time cosmological solutions, we assume a flat Friedmann-Lemaître-Robertson. Walker metric,

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3} dx^{i2} \tag{46}$$

Let us start by studying solutions in $f(R)$ unimodular gravity, whose field equations (22) for the metric (46) turn out to be

$$\begin{aligned} H^2 &= \frac{1}{3f_R}\left[\kappa^2\rho_m + \frac{Rf_R - f}{2} - 3H\dot{R}f_{RR} + \lambda_0\right], \\ -3H^2 - 2\dot{H} &= \frac{1}{f_R}\left[\kappa^2 p_m + \dot{R}^2 f_{RRR} + 2H\dot{R}f_{RR} + \ddot{R}f_{RR}\right. \\ &\quad \left. + \frac{1}{2}(f - Rf_R - 2\lambda_0)\right]. \end{aligned} \tag{47}$$

Note that every solution of a particular $f(R)$ gravity is also a solution of unimodular $f(R)$ gravity just by shifting the action $f \to f + 2\lambda_0$. Nevertheless, the additional constant in the FLRW equations (47) may provide a wider set of solutions. In order to show this, let us analyze some particular and illustrative cosmological solutions. Since the Universe goes through several accelerating stages, the de Sitter solution plays an important role, where the Hubble parameter is given by

$$H(t) = H_0. \tag{48}$$

Moreover, $H = H_0$ is a critical point in every $f(R)$ gravity [19], such that the possible critical points of a particular gravitational action can be identified with the dark energy epoch and also inflation. Then, for the de Sitter solution (48), the first FLRW (in vacuum) is given by,

$$3f_{R_0}H_0^2 - \frac{1}{2}(R_0 f_{R_0} - f_0 - \lambda_0) = 0. \tag{49}$$

Hence, every root of this equation is a critical point and becomes a possible de Sitter stage along which the Universe evolves. The presence of λ_0 introduces a correction that some particular $f(R)'s$, which lead to an effective cosmological constant (as in the Hu-Sawicky model [17]), may require.

Let us now explore power-law solutions in cosmology, which also have great importance in the history of the Universe:

$$a(t) = a_0 t^m, \qquad H(t) = \frac{m}{t}. \tag{50}$$

Note that for pressureless matter $m = 2/3$, for radiation $m = 1/2$ and for an accelerating universe $m > 1$. The above solution has been analyzed in standard $f(R)$ gravity, where the following action holds [20],

$$f(R) = A_{\pm}R^{\frac{1}{4}(3-m\pm\sqrt{1+10m+m^2})}. \tag{51}$$

Whether we assume the above $f(R)$ gravity in unimodular gravity with $m < 1$, the effective cosmological constant λ_0 may become important at late-times, when the dark energy epoch starts, while the terms in (51) may contribute during the matter/radiation epochs when they dominate over λ_0. Moreover, whether $m > 1$ the unimodular $f(R)$ gravity (51) contributes to the acceleration of the expansion, leading to corrections over a de Sitter expansion which would depend on the weight of $A_\pm$ in comparison with λ_0.

Let us now consider the unimodular version of Gauss-Bonnet gravity (25), whose FLRW equation becomes:

$$3H^2 = \kappa^2\rho_m + \frac{1}{2}(Gf_G - f) - 12f_{GG}\dot{G}H^3 + \lambda_0, \tag{52}$$

where $G = 24(\dot{H}H^2 + H^4)$. As in the previous case, we can analyze de Sitter solutions (48) by introducing (48) into Eq. (52), which turns out to be an algebraic equation,

$$3H_0^2 + \frac{1}{2}(f_0 - G_0 f_{G_0}) - \lambda_0 = 0. \tag{53}$$

Hence, the merging cosmological constant λ_0 would determine the de Sitter points and, consequently, the accelerating stages of the Universe. In the case of power-law solutions (50), the exact action within pure Gauss-Bonnet gravity (with no Ricci scalar in the action) that reproduces such solutions in vacuum are [21]:

$$f(G) = AG^{\frac{1-m}{4}}, \tag{54}$$

which may play the same role as in the case of $f(R)$ gravity, as shown above. Nevertheless, the most important feature of the action $f(R,G) = R + f(G)$ is that reproduces exact ΛCDM model,

$$H^2 = \frac{\Lambda}{3} + \frac{\kappa^2}{3}\rho_0 a^{-3}, \tag{55}$$

by means of the gravitational action given by [18],

$$\begin{aligned} f(R,G) = R &+ a_1\left(\Lambda \pm \sqrt{9\Lambda^2 - 3G}\right)^2 \\ &+ a_2\left(\Lambda \pm \sqrt{9\Lambda^2 - 3G}\right) + a_3, \end{aligned} \tag{56}$$

where a_1 is an integration constant, $a_2 = \frac{6-30a_1\Lambda}{15}$ and $a_3 = 3(1 - 6a_1\Lambda)$ are constants. Then, by identifying the last term of (56) with the cosmological constant λ_0,

$$\lambda_0 = -\frac{3}{2}(1 - 6a_1\Lambda) \tag{57}$$

The unimodular version of Gauss-Bonnet gravity described by the action (56) arises naturally as the gravitational action which leads to the ΛCDM model (55).

Hence, extensions of unimodular gravity provide reliable descriptions of the late-time acceleration in a natural way.

B. Inflation

Let us now study how these extensions of unimodular gravity may affect the inflationary paradigm. In particular, here we analyze Starobinsky inflation [16] when considering the unimodular $f(R)$ theory (18), which for the case of Starobinsky inflation is given by

$$S = \frac{1}{2\kappa^2}\int dx^4\left[\sqrt{-g}\left(R + \frac{R^2}{6m^2}\right) - 2\lambda(\sqrt{-g} - s_0)\right], \tag{58}$$

where m^2 is a constant. In order to simplify the calculations, we work in the scalar-tensor equivalence (30), whose correspondence to the action (58) is provided by

$$\phi = 1 + \frac{R}{3m^2}, \qquad V(\phi) = 3m^2(\phi - 1)^2. \tag{59}$$

Applying the conformal transformation (33) and the definitions (36), the action (44) is constructed following the steps described in Section IV, where the effective potential for the case (58) is given by,

$$\tilde{V}_{\mathrm{eff}}(\varphi) = \frac{1}{2\kappa^2}\left[\frac{3m^2}{2}\left(1 - e^{-2\sqrt{2/3}\kappa\varphi}\right)^2 - 2\lambda_0 e^{-2\sqrt{2/3}\kappa\varphi}\right]. \tag{60}$$

Then, the FLRW equations are:

$$\frac{3}{\kappa^2}H^2 = \frac{1}{2}\dot{\varphi}^2 + \tilde{V}_{\mathrm{eff}}(\varphi),$$
$$-\frac{1}{\kappa^2}(3H^2 + 2\dot{H}) = \frac{1}{2}\dot{\varphi}^2 - \tilde{V}_{\mathrm{eff}}(\varphi), \tag{61}$$

While the scalar field satisfies

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial\tilde{V}_{\mathrm{eff}}(\varphi)}{\partial\varphi} = 0 \tag{62}$$

Slow-roll inflation occurs in the regime $\kappa\varphi \gg 1$, where the friction term in (62) dominates, and the expansion grows exponentially approximately, being the Hubble parameter $H \sim H_0$. Then, the following relations hold

$$H\dot{\phi} \gg \ddot{\varphi}, \qquad \tilde{V} \gg \dot{\varphi}^2. \tag{63}$$

Equivalently, we can define the slow-roll parameters,

$$\epsilon = \frac{1}{2\kappa^2}\left(\frac{\tilde{V}'_{\mathrm{eff}}(\varphi)}{\tilde{V}_{\mathrm{eff}}(\varphi)}\right)^2, \qquad \eta = \frac{1}{\kappa^2}\frac{\tilde{V}''_{\mathrm{eff}}(\varphi)}{\tilde{V}_{\mathrm{eff}}(\varphi)}, \tag{64}$$

Hence, during inflation $\epsilon \ll 1$ and $\eta < 1$, while after an enough number of e-foldings, usually around $N = 50 - 65$, $\epsilon \geq 1$, when the scalar field φ rolls down the potential slope and the kinetic term becomes important and eventually dominates. Then the field oscillates around the minimum of the potential, emitting particles and reheating the Universe. Hence, by using these approximations and combining the FLRW equations (61) and the scalar field equation (62), the equations during inflation are given approximately by

$$H^2 \simeq \frac{\kappa^2}{3}\tilde{V}_{\mathrm{eff}}(\varphi),$$
$$3H\dot{\varphi} \simeq -\tilde{V}'_{\mathrm{eff}}(\varphi). \tag{65}$$

The slow-roll parameters (64) for the potential (60) are given by

$$\epsilon = \frac{4}{3}\frac{\left[3m^2\left(-1 + e^{\sqrt{2/3}\kappa\varphi}\right) - 4\lambda_0\right]^2}{\left[3m^2\left(-1 + e^{\sqrt{2/3}\kappa\varphi}\right)^2 + 4\lambda_0\right]^2},$$
$$\eta = \frac{4}{3}\frac{-3m^2\left(-2 + e^{\sqrt{2/3}\kappa\varphi}\right) + 8\lambda_0}{3m^2\left(-1 + e^{\sqrt{2/3}\kappa\varphi}\right)^2 + 4\lambda_0}, \tag{66}$$

Starobinsky inflation is recovered by setting $\lambda_0 = 0$. Nevertheless, since $m^2/\lambda_0 > \mathrm{e}^{-2\sqrt{2/3}\kappa\varphi_{\mathrm{start}}}$ in order to ensure a large enough number of e-foldings before the field rolls down, together with $\kappa\varphi \gg 1$, it gives the following the slow-roll parameters,

$$\epsilon = \frac{4}{3}e^{-2\sqrt{2/3}\kappa\varphi},$$
$$\eta = -\frac{4}{3}e^{-\sqrt{2/3}\kappa\varphi}. \tag{67}$$

In addition, the spectral index and the scalar-to-tensor ratio are given in terms of the slow-roll parameters by

$$n_s - 1 = -3\epsilon + 2\eta r = 16\epsilon. \tag{68}$$

It is straightforward to calculate the number of e-foldings during inflation, which is given by

$$N \equiv \int_{t_{\mathrm{start}}}^{t_{\mathrm{end}}}\tilde{H}dt = -\kappa^2\int_{\varphi_{\mathrm{start}}}^{\varphi_{\mathrm{end}}}\frac{\tilde{V}_{\mathrm{eff}}(\varphi)}{\tilde{V}'_{\mathrm{eff}}(\varphi)} \simeq \frac{3}{4}e^{\sqrt{2/3}\kappa\tilde{\varphi}_{\mathrm{start}}}. \tag{69}$$

Note that the number of e-foldings is related to the slow-roll parameters as

$$\epsilon \simeq \frac{3}{4}\frac{1}{N^2}, \qquad \eta \simeq -\frac{1}{N}. \tag{70}$$

Then, assuming a number of e-foldings $N \sim 65$, the following values of the inflationary observables are obtained:

$$n_s = 0.968, \qquad r = 0.00284. \tag{71}$$

This is exactly the same result as in Starobisnky inflation, which satisfies quite well the constraints provided by the previous data [22]. Hence, as far as $m^2/\lambda_0 > e^{-2\sqrt{2/3}\kappa\varphi_{\rm start}}$, the unimodular version of Starobinsky inflation is also successful, but it includes, in addition, the corresponding cosmological constant that may dominate at late times, leading to a complete description of the evolution of the Universe.

VI. CONCLUSIONS

To summarize, in this paper we have extended the so-called unimodular gravity to other more general actions other than the Hilbert-Einstein action. As in the original case, extensions of unimodular gravity can be constructed by departing from the trace-free part of the field equations or, alternatively, by the gauge choice that fixes the determinant of the metric to be a constant. As has been widely noted in the literature, any implementation of the unimodular constraint leads to the same results at the classical level, but may provide differences when quantum mechanics are considered. Nevertheless, since the paper is devoted to the classical aspects, we have forced the unimodular constraint in the action through a Lagrange multiplier, so that the calculations become simpler when dealing with gravitational Lagrangians that have more general functions of curvature invariants. Hence, following this procedure, and in spite of the apparent lack of symmetries, extensions of unimodular gravity lead to the covariant field equations of the originals with the presence of a cosmological constant.

The issue is more subtle when dealing with conformal transformations. As is shown, by transforming the gravitational action from the Jordan to the Einstein frame, the determinant is no longer fixed to be a constant. However, the Lagrange multiplier used to fix the determinant of the metric turns out to be a constant as well, so that the corresponding counterpart in the Einstein frame becomes the usual quintessencelike model, but in this case with a correction in the scalar potential. Such an additional term may have consequences when studying some particular solutions.

Finally, some cosmological solutions have been studied within the unimodular version of Gauss-Bonnet gravity and $f(R)$ gravity (together with its scalar-tensor equivalence). As shown, the unimodular version of these theories provides a richer set of solutions and is able to give a complete picture of the evolution of the Universe in a natural way. In addition, predictions from Starobinsky inflation are fully recovered as long as the correction in the scalar potential is well set. Moreover, the unimodular version of Starobinsky inflation may provide an explanation for the late-time acceleration through the effective cosmological constant that naturally arises. Hence, such results point to $R + R^2$ as a reliable cosmological model for describing the whole history of the Universe.

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