

Degenerate higher order scalar-tensor theories beyond Horndeski and disformal transformations

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We consider all degenerate scalar-tensor theories that depend quadratically on second-order derivatives of a scalar field, which we have identified in a previous work. These theories, whose degeneracy, in general, ensures the absence of Ostrogradsky’s instability, include the quartic Horndeski Lagrangian and its quartic extension beyond Horndeski, as well as other Lagrangians. We study how all these theories transform under general disformal transformations and find that they can be separated into three main classes that are stable under these transformations. This leads to a complete classification modulo disformal transformations. Finally, we show that these higher order theories include mimetic gravity and some particular khronometric theories. They also contain theories that do not correspond, to our knowledge, to already studied theories, even up to disformal transformations.

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I. INTRODUCTION

Scalar-tensor theories play a prominent role in theories of modified gravity. As ever more sophisticated models have been considered, special attention was lately devoted to scalar-tensor Lagrangians that contain second order derivatives of a scalar field. A crucial requirement for such theories is the absence of the so-called Ostrogradsky ghost, in order to avoid disastrous instabilities [1]. It has been thought for a long time that the absence of an Ostrogradsky ghost demands the Euler-Lagrange equations to be at most second order, which explains why the literature has been mostly limited to the study of Horndeski’s theories [2,3] until recently. But the discovery of viable theories “beyond Horndeski” [4–6], i.e. possessing Euler-Lagrange equations for the metric and scalar field whose order is higher than two, has challenged this preconception.

A higher order scalar-tensor theory generically contains four degrees of freedom, including the Ostrogradsky ghost. As we proposed in [7], a systematic way to identify scalar-tensor theories that contain only three degrees of freedom is to consider Lagrangians that are degenerate, in a generalized sense involving the coupling between the metric and the scalar field. From the Hamiltonian point of view, this degeneracy implies the existence of phase space constraints, in addition to the usual Hamiltonian and momentum constraints due to diffeomorphism invariance, and explains why one degree of freedom is eliminated, even if the equations of

motion are higher order. A detailed Hamiltonian analysis has confirmed the direct link between this degeneracy and the elimination of the Ostrogradsky ghost [8].

The degeneracy criterium, which provides a powerful and simple method to identify viable theories, was used in [7] to find all scalar-tensor theories based on a Lagrangian quadratic in second-order derivatives of a scalar field, together with a term proportional to the scalar curvature. Within these degenerate higher order scalar-tensor (DHOST) theories, we recovered, as particular cases, the (quadratic) Horndeski Lagrangian L_4^H as well as its extension L_4^{bH} introduced in [5,6] (see section II for an explicit definition of these Lagrangians). We also considered the quintic extension beyond Horndeski of [5], L_5^{bH} , which is degenerate by itself or combined with L_4^{bH} but not with an arbitrary L_4^H . By using the same degeneracy argument, the combinations involving L_5^H too were studied in [9]. In particular, the results of [7,9] show that only specific combinations of Horndeski Lagrangians with their quartic and quintic extensions beyond Horndeski are viable and they coincide with the combinations obtained in [6] via disformal transformation of Horndeski.

The goal of the present work is to examine in more detail all the quadratic DHOST theories of [7] and investigate whether they can be related, or not, to already known theories via generalized disformal transformations [10], i.e. redefinitions of the metric of the form

$$\tilde{g}_{\mu\nu} = A(X, \phi)g_{\mu\nu} + B(X, \phi)\nabla_\mu\phi\nabla_\nu\phi, \quad (1.1)$$

where $X \equiv g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. Several results concerning the disformal transformations of Horndeski theories have

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already been established in previous works. It was shown, in [11], that Horndeski theories transform into themselves under special disformal transformations where A and B depend on ϕ only, not on X . The general disformal transformation of the Einstein-Hilbert Lagrangian was computed in [4], providing the first example of theory “beyond Horndeski,” i.e. a ghost-free theory with higher order Euler-Lagrange equations of motion. In [6], it was shown that disformal transformations of Horndeski theories with $A = 1$ lead to the extensions beyond Horndeski proposed in [5]. The X -dependent disformal transformations have also been studied recently in several papers (see e.g. [12–16], and [17] for scalar-tensor theories that explicitly break spacetime covariance [18].)

In this work, we present the general disformal transformation of all quadratic DHOST theories identified in [7]. This is useful to show that the three main classes of theories, as well as the few subclasses within each, are stable under disformal transformations. Part of our results coincides with the conclusions of [19], which also studies the theories of [7] and especially the class of theories related to (quadratic) Horndeski via disformal transformations. Here, we derive the transformation laws induced on the arbitrary functions in the general action, which enables us to study the disformal transformations in the other classes as well.

Interestingly, the quadratic DHOST theories contain a few theories which have been well studied in the literature. Indeed, the Lagrangians that remain invariant under a field redefinition of the scalar field correspond to khronometric theories [20], which are a subset of Einstein-aether theories [21]. Note that khronometric theories are not in general degenerate and only a subset of them appear among DHOST theories. Finally, we also discuss mimetic gravity [22] and related theories [23], which are obtained from the Einstein-Hilbert action by a disformal transformation that is not invertible (see also [4]).

Our paper is organized as follows. In the next section, we introduce the general form of the Lagrangians we will study. In Sec. III, we summarize the main results obtained in [7] and present the classification of quadratic DHOST theories. In Sec. IV, we derive the general disformal transformation of any quadratic DHOST Lagrangian. This enables us to show that all classes, and subclasses within them, are stable under these transformations. In Sec. V, we consider the theories related to Horndeski. Other classes are analysed in the subsequent section. In Sec. VII, we discuss the degenerate khronometric theories as well as mimetic theories. We conclude in the final section.

II. SCALAR-TENSOR THEORIES

A. The action

In this work, we consider scalar-tensor theories whose dynamics is governed by an action of the general form

$$S = S_g + S_\phi, \quad (2.1)$$

where the first contribution involves the Ricci scalar R of the metric $g_{\mu\nu}$,

$$S_g \equiv \int d^4x \sqrt{-g} f(\phi, X) R, \quad (2.2)$$

and the second contribution depends quadratically on the second derivatives of the scalar field ϕ ,

$$S_\phi \equiv \int d^4x \sqrt{-g} C^{\mu\nu,\rho\sigma} \nabla_\mu \nabla_\nu \phi \nabla_\rho \nabla_\sigma \phi, \quad (2.3)$$

with $C^{\mu\nu,\rho\sigma}$ being an arbitrary tensor that depends only on ϕ and $\nabla_\mu \phi$. We recall that $X \equiv g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$. Note that S_g reduces to the familiar Einstein-Hilbert action when the function f is constant.

We stress that our analysis is also valid if we add to the above action extra contributions that depend at most linearly on $\phi_{,\mu\nu}$, i.e. of the form

$$S_{\text{other}} = \int d^4x \sqrt{-g} \{ P(\phi, X) + Q_1(\phi, X) \square \phi + Q_2(\phi, X) \phi^\mu \phi_{,\mu\nu} \phi^\nu \}, \quad (2.4)$$

where we have used the compact notation $\phi_{,\mu} \equiv \nabla_\mu \phi$, $\phi_{,\mu\nu} \equiv \nabla_\mu \nabla_\nu \phi$ and $\square \phi = g^{\mu\nu} \phi_{,\mu\nu}$. These additional contributions do not modify the degeneracy conditions derived in [7], which will be summarized in the next section. For simplicity, we will not include these terms explicitly in our study but one should keep in mind that they can be present.

Without loss of generality, we require the tensor $C^{\mu\nu,\rho\sigma}$ in (2.3) to satisfy the index symmetries

$$C^{\mu\nu,\rho\sigma} = C^{\nu\mu,\rho\sigma} = C^{\rho\sigma,\mu\nu}, \quad (2.5)$$

which implies that the most general form of this tensor is

$$\begin{aligned} C^{\mu\nu,\rho\sigma} = & \frac{1}{2} \alpha_1 (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) + \alpha_2 g^{\mu\nu} g^{\rho\sigma} \\ & + \frac{1}{2} \alpha_3 (\phi^\mu \phi^\nu g^{\rho\sigma} + \phi^\rho \phi^\sigma g^{\mu\nu}) \\ & + \frac{1}{4} \alpha_4 (\phi^\mu \phi^\rho g^{\nu\sigma} + \phi^\nu \phi^\rho g^{\mu\sigma} + \phi^\mu \phi^\sigma g^{\nu\rho} + \phi^\nu \phi^\sigma g^{\mu\rho}) \\ & + \alpha_5 \phi^\mu \phi^\nu \phi^\rho \phi^\sigma, \end{aligned} \quad (2.6)$$

where the α_i are five arbitrary functions of ϕ and X . Defining the five elementary Lagrangians quadratic in second derivatives,

$$\begin{aligned} L_1^\phi & \equiv \phi^{\mu\nu} \phi_{,\mu\nu}, & L_2^\phi & \equiv (\square \phi)^2, & L_3^\phi & \equiv \square \phi (\phi^\rho \phi_{,\rho\sigma} \phi^\sigma), \\ L_4^\phi & \equiv \phi^\mu \phi_{,\mu\nu} \phi^{\nu\rho} \phi_{,\rho}, & L_5^\phi & \equiv (\phi^\rho \phi_{,\rho\sigma} \phi^\sigma)^2, \end{aligned} \quad (2.7)$$

the action S_ϕ in (2.3) now reads

$$\begin{aligned} S_\phi &= \int d^4x \sqrt{-g} (\alpha_1 L_1^\phi + \alpha_2 L_2^\phi + \alpha_3 L_3^\phi + \alpha_4 L_4^\phi + \alpha_5 L_5^\phi) \\ &\equiv \int d^4x \sqrt{-g} \alpha_I L_I^\phi, \end{aligned} \quad (2.8)$$

where the summation over the index I ($I = 1, \dots, 5$) is implicit in the last expression.

B. Other curvature terms

It is not difficult to see that the general action (2.1) also includes terms of the form

$$S_{\text{Ricci}} \equiv \int d^4x \sqrt{-g} \theta(\phi, X) R_{\mu\nu} \phi^\mu \phi^\nu, \quad (2.9)$$

where θ is an arbitrary function. Indeed, using the definition of the Ricci tensor and the properties of the Riemann tensor, one can write

$$\begin{aligned} \phi^\mu R_{\mu\nu} \phi^\nu &= -\phi^\mu g^{\rho\sigma} R_{\rho\mu\nu\sigma} \phi^\nu = -\phi^\mu g^{\rho\sigma} (\nabla_\rho \nabla_\nu - \nabla_\nu \nabla_\rho) \phi_\sigma \\ &= -\phi^\mu \nabla_\mu \nabla_\nu \phi^\nu + \phi^\mu \nabla_\nu \nabla_\mu \phi^\nu. \end{aligned} \quad (2.10)$$

Substituting this into the action (2.9), one gets, after integration by parts,

$$\begin{aligned} S_{\text{Ricci}} &\equiv \int d^4x \sqrt{-g} \{ -\theta(L_1^\phi - L_2^\phi) + 2\theta_X(L_3^\phi - L_4^\phi) \\ &\quad + \theta_\phi(X \square \phi - \phi^\mu \phi_{\mu\nu} \phi^\nu) \}, \end{aligned} \quad (2.11)$$

where the contribution proportional to θ_ϕ is of the form (2.4).

C. Particular cases

The theories (2.1) include as a particular case the quartic Horndeski term,

$$L_4^{\text{H}} = G_4(\phi, X)R - 2G_{4,X}(\phi, X)(\square\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu}), \quad (2.12)$$

which corresponds to (2.2) and (2.8) with

$$f = G_4, \quad \alpha_1 = -\alpha_2 = 2G_{4,X}, \quad \alpha_3 = \alpha_4 = \alpha_5 = 0. \quad (2.13)$$

The action (2.1) also includes the extension beyond Horndeski introduced in [5], which can be written as

$$L_4^{\text{bH}} = F_4(\phi, X) \epsilon^{\mu\nu\rho\sigma} \epsilon^{\mu'\nu'\rho'\sigma} \phi_\mu \phi_{\mu'} \phi_{\nu\nu'} \phi_{\rho\rho'}. \quad (2.14)$$

This is of the form (2.8) with

$$\alpha_1 = -\alpha_2 = XF_4, \quad \alpha_3 = -\alpha_4 = 2F_4, \quad \alpha_5 = 0. \quad (2.15)$$

Of course, any combination of L_4^{H} and L_4^{bH} is also among the theories (2.1).

III. CLASSIFICATION OF DEGENERATE THEORIES

In this section, we summarize the main results obtained in [7], as well as some additional elements derived in [8], and present all the quadratic DHOST theories, i.e. all the theories of the form (2.1) which are degenerate.

A. Degeneracy conditions

In order to study the degeneracy of (2.1), it is useful to introduce the notation $A_\mu \equiv \nabla_\mu \phi$. For an arbitrary foliation of spacetime by spacelike hypersurfaces $\Sigma(t)$, endowed with spatial metric h_{ij} , the metric in ADM form reads

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (3.1)$$

where N is the lapse and N^i the shift vector. The (3 + 1) decomposition of the action (2.1) leads to a kinetic term of the form [7]

$$S_{\text{kin}} = \int dt d^3x N \sqrt{h} \left[\frac{1}{N^2} \mathcal{A} \dot{A}_*^2 + \frac{2}{N} \mathcal{B}^{ij} \dot{A}_* K_{ij} + \mathcal{K}^{ijkl} K_{ij} K_{kl} \right], \quad (3.2)$$

where we have introduced the quantity

$$A_* \equiv \frac{1}{N} (A_0 - N^i A_i), \quad (3.3)$$

and the extrinsic curvature tensor

$$K_{ij} \equiv \frac{1}{2N} (\dot{h}_{ij} - D_i N_j - D_j N_i). \quad (3.4)$$

The coefficients that appear in (3.2) depend on the six arbitrary functions f and α_I of (2.1). They are explicitly given by [7,8]

$$\mathcal{A} = \alpha_1 + \alpha_2 - (\alpha_3 + \alpha_4) A_*^2 + \alpha_5 A_*^4, \quad \mathcal{B}^{ij} = \beta_1 h^{ij} + \beta_2 \hat{A}^i \hat{A}^j, \quad (3.5)$$

$$\begin{aligned} \mathcal{K}^{ij,kl} &= \kappa_1 h^{i(k} h^{l)j} + \kappa_2 h^{ij} h^{kl} + \frac{1}{2} \kappa_3 (\hat{A}^i \hat{A}^j h^{kl} + \hat{A}^k \hat{A}^l h^{ij}) \\ &\quad + \frac{1}{2} \kappa_4 (\hat{A}^i \hat{A}^{(k} h^{l)j} + \hat{A}^j \hat{A}^{(k} h^{l)i}) + \kappa_5 \hat{A}^i \hat{A}^j \hat{A}^k \hat{A}^l, \end{aligned} \quad (3.6)$$

with

$$\beta_1 = \frac{A_*}{2} (2\alpha_2 - \alpha_3 A_*^2 + 4f_{,X}), \quad \beta_2 = \frac{A_*}{2} (2\alpha_5 A_*^2 - \alpha_3 - 2\alpha_4), \quad (3.7)$$

$$\begin{aligned}\kappa_1 &= \alpha_1 A_*^2 + f, & \kappa_2 &= \alpha_2 A_*^2 - f, & \kappa_3 &= -\alpha_3 A_*^2 + 4f_X, \\ \kappa_4 &= -2\alpha_1, & \kappa_5 &= \alpha_5 A_*^2 - \alpha_4.\end{aligned}\quad (3.8)$$

The three-dimensional vector \hat{A}^i is defined by $\hat{A}_i \equiv A_i$ and $\hat{A}^i \equiv h^{ij}\hat{A}_j$.

By choosing an appropriate basis of the six-dimensional vector space of symmetric 3×3 matrices, where the K_{ij} take their values, the kinetic matrix associated with (3.2) can be written as a 7×7 block diagonal symmetric matrix of the form [8]

$$\begin{pmatrix} \mathcal{M} & \mathbf{0} \\ \mathbf{0} & \mathcal{D} \end{pmatrix}, \quad (3.9)$$

with the 3×3 matrix

$$\mathcal{M} \equiv \left(\begin{array}{c|cc} A & \frac{1}{2}(\beta_1 + \hat{A}^2\beta_2) & \frac{1}{\sqrt{2}}\beta_1 \\ \hline \frac{1}{2}(\beta_1 + \hat{A}^2\beta_2) & & \\ \frac{1}{\sqrt{2}}\beta_1 & & \mathcal{M}_K \end{array} \right) \quad (3.10)$$

where

$$\mathcal{M}_K \equiv \left(\begin{array}{c|c} \kappa_1 + \kappa_2 + \hat{A}^2(\kappa_3 + \kappa_4) + (\hat{A}^2)^2\kappa_5 & \sqrt{2}(\kappa_2 + \frac{1}{2}\hat{A}^2\kappa_3) \\ \hline \sqrt{2}(\kappa_2 + \frac{1}{2}\hat{A}^2\kappa_3) & \kappa_1 + 2\kappa_2 \end{array} \right), \quad (3.11)$$

and the diagonal matrix

$$\mathcal{D} = \text{Diag} \left[\kappa_1, \kappa_1, \kappa_1 + \frac{1}{2}\hat{A}^2\kappa_4, \kappa_1 + \frac{1}{2}\hat{A}^2\kappa_4 \right]. \quad (3.12)$$

The elements of the first row (or first column) of \mathcal{M} are the coefficients of the kinetic terms involving the scalar field related variable A_* , including its mixing with the metric sector. As for the metric sector alone, it is described by the right lower 2×2 submatrix of \mathcal{M} , which we call \mathcal{M}_K , together with \mathcal{D} . As our goal is to eliminate the extra degree of freedom due to the higher derivatives of the scalar field, we are looking for a degeneracy of the kinetic matrix that arises from the scalar sector. Therefore, we will be mainly interested in theories for which \mathcal{M} is degenerate while \mathcal{M}_K and \mathcal{D} remain nondegenerate, in order to preserve the usual tensor structure of gravity.

Requiring the determinant of the matrix \mathcal{M} to vanish¹ yields an expression of the form

$$D_0(X) + D_1(X)A_*^2 + D_2(X)A_*^4 = 0, \quad (3.13)$$

¹Note that we have not used the same matrix in [7] but another, related, matrix constructed by solving for null eigenvectors of the kinetic matrix. The two methods are equivalent.

where we have substituted the expressions (3.5)–(3.8) into (3.10) and replaced all \hat{A}^2 by $X + A_*^2$. The functions D_0 , D_1 and D_2 can be written in terms of the six arbitrary functions \tilde{f} and α_i of the initial Lagrangian:

$$D_0(X) \equiv -4(\alpha_1 + \alpha_2)[Xf(2\alpha_1 + X\alpha_4 + 4f_X) - 2f^2 - 8X^2f_X^2], \quad (3.14)$$

$$\begin{aligned}D_1(X) &\equiv 4[X^2\alpha_1(\alpha_1 + 3\alpha_2) - 2f^2 - 4Xf\alpha_2]\alpha_4 \\ &\quad + 4X^2f(\alpha_1 + \alpha_2)\alpha_5 + 8X\alpha_1^3 \\ &\quad - 4(f + 4Xf_X - 6X\alpha_2)\alpha_1^2 - 16(f + 5Xf_X)\alpha_1\alpha_2 \\ &\quad + 4X(3f - 4Xf_X)\alpha_1\alpha_3 - X^2f\alpha_3^2 \\ &\quad + 32f_X(f + 2Xf_X)\alpha_2 - 16ff_X\alpha_1 \\ &\quad - 8f(f - Xf_X)\alpha_3 + 48ff_X^2,\end{aligned}\quad (3.15)$$

$$\begin{aligned}D_2(X) &\equiv 4[2f^2 + 4Xf\alpha_2 - X^2\alpha_1(\alpha_1 + 3\alpha_2)]\alpha_5 + 4\alpha_1^3 \\ &\quad + 4(2\alpha_2 - X\alpha_3 - 4f_X)\alpha_1^2 + 3X^2\alpha_1\alpha_3^2 \\ &\quad - 4Xf\alpha_3^2 + 8(f + Xf_X)\alpha_1\alpha_3 - 32f_X\alpha_1\alpha_2 \\ &\quad + 16f_X^2\alpha_1 + 32f_X^2\alpha_2 - 16ff_X\alpha_3.\end{aligned}\quad (3.16)$$

Since the determinant must vanish for any value of A_* , we deduce that degenerate theories are characterized by the three conditions

$$D_0(X) = 0, \quad D_1(X) = 0, \quad D_2(X) = 0. \quad (3.17)$$

By solving these three conditions, one can determine and classify all DHOST theories, as discussed in [7].

B. Degenerate theories

The condition $D_0(X) = 0$ is the simplest of all three and allows us to distinguish several classes of theories. Indeed, D_0 can vanish either if $\alpha_1 + \alpha_2 = 0$, which defines our first class of solutions, or if the term between brackets in (3.14) vanishes, which defines our second class, as well as our third class corresponding to the special case where $f = 0$.

1. Class I ($\alpha_1 + \alpha_2 = 0$)

Class I is defined by

$$\alpha_1 + \alpha_2 = 0. \quad (3.18)$$

One can then use the conditions $D_1(X) = 0$ and $D_2(X) = 0$ to express, respectively, α_4 and α_5 in terms of α_2 and α_3 , provided $f + X\alpha_2 = f - X\alpha_1 \neq 0$. This defines the subclass Ia, characterized by

$$\alpha_4 = \frac{1}{8(f + X\alpha_2)^2} [16X\alpha_2^3 + 4(3f + 16Xf_X)\alpha_2^2 + (16X^2f_X - 12Xf)\alpha_3\alpha_2 - X^2f\alpha_3^2 + 16f_X(3f + 4Xf_X)\alpha_2 + 8f(Xf_X - f)\alpha_3 + 48ff_X^2] \quad (\text{class Ia}) \quad (3.19)$$

and

$$\alpha_5 = \frac{(4f_X + 2\alpha_2 + X\alpha_3)(-2\alpha_2^2 + 3X\alpha_2\alpha_3 - 4f_X\alpha_2 + 4f\alpha_3)}{8(f + X\alpha_2)^2}. \quad (\text{class Ia}) \quad (3.20)$$

Degenerate theories in class Ia thus depend on three arbitrary functions α_2 , α_3 and f .

In the special case $f = X\alpha_1$, we find another subclass of solutions characterized by

$$\alpha_1 = -\alpha_2 = \frac{f}{X}, \quad \alpha_3 = \frac{2}{X^2}(f - 2Xf_X), \quad (\text{class Ib}), \quad (3.21)$$

where f , α_4 and α_5 are arbitrary functions. In the following, we will not explore this class much further because the metric sector is degenerate. Indeed, the last two eigenvalues of \mathcal{D} in (3.12) are equal to $f - \alpha_1 X$ and thus vanish in this case.

2. Class II ($f \neq 0$, $\alpha_1 + \alpha_2 \neq 0$)

If $\alpha_1 + \alpha_2 \neq 0$, the condition $D_0(X) = 0$ is equivalent to

$$Xf(2\alpha_1 + X\alpha_4 + 4f_X) - 2f^2 - 8X^2f_X^2 = 0. \quad (3.22)$$

We can then proceed as previously by solving $D_1(X) = 0$ and $D_2(X) = 0$ to express α_4 and α_5 in terms of the three other functions. Substituting the obtained expression for α_4 into the condition (3.22), one finally gets

$$(X\alpha_1 - f)[(4f^2 + Xf(8\alpha_2 + 2\alpha_1 + X\alpha_3 - 4f_X) - 4X^2f_X(\alpha_1 + 3\alpha_2))] = 0. \quad (3.23)$$

Assuming that $f - X\alpha_1 \neq 0$, this leads to the expressions

$$\alpha_3 = \frac{1}{X^2f} [-4f(f - Xf_X) - 2X(f - 2Xf_X)\alpha_1 + 4X(-2f + 3Xf_X)\alpha_2], \quad (3.24)$$

$$\alpha_4 = \frac{2}{X^2f} [f^2 - 2fXf_X + 4X^2f_X^2 - Xf\alpha_1], \quad (3.25)$$

$$\alpha_5 = \frac{2}{f^2X^3} [4f(f^2 - 3fXf_X + 2X^2f_X^2) + (3Xf^2 - 8X^2ff_X + 6X^3f_X^2)\alpha_1 + 2X(2f - 3Xf_X)^2\alpha_2], \quad (3.26)$$

while f , α_1 and α_2 are arbitrary. This describes our class IIa, characterized by three arbitrary functions.

The case $f = X\alpha_1$ defines another class, similar to class Ib, which we will call class IIb, described by

$$\alpha_1 = \frac{f}{X}, \quad \alpha_4 = 4f_X \left(2\frac{f_X}{f} - \frac{1}{X} \right), \quad (3.27)$$

$$\alpha_5 = \frac{1}{4X^3f(f + X\alpha_2)} [8X(4Xf_Xf - f^2 - 4X^2f_X^2)\alpha_2 + Xf(8X^2f_X + X^3\alpha_3 - 4Xf)\alpha_3 + 4(Xf_Xf^2 - 2X^3f_X^3 + 2X^2f_X^2f - f^3)], \quad (3.28)$$

where α_2 and α_3 are arbitrary functions. Like class Ib, the metric sector is degenerate for these theories, and we will not consider them further in the following.

3. Class III ($f = 0$)

Finally, we devote a special class to the case $f = 0$, which also leads automatically to $D_0 = 0$. Using $D_1 = 0$ and $D_2 = 0$ to determine α_4 and α_5 , one gets

$$\alpha_4 = -\frac{2}{X}\alpha_1, \quad \alpha_5 = \frac{4\alpha_1^2 + 8\alpha_1\alpha_2 - 4\alpha_1\alpha_3X + 3\alpha_3^2X^2}{4X^2(\alpha_1 + 3\alpha_2)} \quad (\text{class IIIa}), \quad (3.29)$$

while α_1 , α_2 and α_3 are arbitrary, provided $\alpha_1 + 3\alpha_2 \neq 0$. This defines our class IIIa. Note that the intersection of IIIa with the class Ia is described by

$$\alpha_2 = -\alpha_1, \quad \alpha_4 = -\frac{2}{X}\alpha_1, \quad \alpha_5 = \frac{(2\alpha_1 - X\alpha_3)(2\alpha_1 + 3X\alpha_3)}{8X^2\alpha_1}, \quad (\text{IIIa} \cap \text{Ia}), \quad (3.30)$$

which depends on two arbitrary functions, α_1 and α_3 , and includes the Lagrangian L_4^{bH} (for which $\alpha_1/X = \alpha_3/2 = F_4$).

The case $\alpha_1 + 3\alpha_2 = 0$ yields another subclass,

$$f = 0, \quad \alpha_1 = \frac{3}{2}X\alpha_3, \quad \alpha_2 = -\frac{X}{2}\alpha_3 \quad (\text{class IIIb}), \quad (3.31)$$

which leads to a degenerate metric sector. Another special case corresponds to the class

$$f = 0, \quad \alpha_1 = 0, \quad (\text{class IIIc}) \quad (3.32)$$

which depends on four arbitrary functions. Since $f - \alpha_1 X = 0$, this class is also degenerate in the metric sector.

4. Degeneracy of the scalar sector alone

Among all the degenerate theories that we have listed above, it is not difficult to identify the theories that remain degenerate even when the metric becomes nondynamical, as noted in [7]. In this limit, only the kinetic term for A_* is relevant and the degeneracy of the scalar sector alone thus requires $\mathcal{A} = 0$, which imposes simultaneously the three constraints

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_3 + \alpha_4 = 0, \quad \alpha_5 = 0. \quad (3.33)$$

The first condition implies that the theories satisfying these conditions belong to class I. Ignoring class Ib, whose metric sector is degenerate, we turn to class Ia. For theories satisfying (3.33), the functions α_2 and α_3 are no longer independent, but related by

$$4f_X + 2\alpha_2 + X\alpha_3 = 0. \quad (3.34)$$

This means that the condition $\mathcal{A} = 0$ restricts the degenerate theories to a subclass that depends on two arbitrary functions only. It is easy to see that this family of theories in fact coincides with the sum of L_4^H and L_4^{bH} , upon using the identification

$$f = G_4, \quad \alpha_1 = -\alpha_2 = 2G_{4X} + XF_4, \quad \alpha_3 = -\alpha_4 = 2F_4. \quad (3.35)$$

This implies that the quartic Lagrangian $L_4 = L_4^H + L_4^{bH}$ represents the most general theories that are degenerate when the metric is nondynamical (with a nondegenerate metric sector).

IV. DISFORMAL TRANSFORMATIONS

We now study the effect of generalized disformal transformations (which include conformal transformations), introduced in [10], in which the ‘‘disformed’’ metric $\tilde{g}_{\mu\nu}$ is expressed in terms of $g_{\mu\nu}$ and ϕ as

$$\tilde{g}_{\mu\nu} = A(X, \phi)g_{\mu\nu} + B(X, \phi)\phi_\mu\phi_\nu. \quad (4.1)$$

Via this transformation, any action \tilde{S} given as a functional of $\tilde{g}_{\mu\nu}$ and ϕ induces a new action S for $g_{\mu\nu}$ and ϕ , when one substitutes the above expression for $\tilde{g}_{\mu\nu}$ in \tilde{S} :

$$S[\phi, g_{\mu\nu}] \equiv \tilde{S}[\phi, \tilde{g}_{\mu\nu} = Ag_{\mu\nu} + B\phi_\mu\phi_\nu]. \quad (4.2)$$

We will say that the actions S and \tilde{S} are related by the disformal transformation (4.1).

Starting from an action \tilde{S} of the form (2.1),

$$\tilde{S} = \tilde{S}_g + \tilde{S}_\phi = \int d^4x \sqrt{-\tilde{g}} [\tilde{f} \tilde{R} + \tilde{\alpha}_I \tilde{L}_I^\phi], \quad (4.3)$$

we show below that the action S , related to \tilde{S} via a disformal transformation, is also of the form (2.1), up to terms of the form (2.4), and we compute explicitly the relations between the functions that appear in the two Lagrangians. Interestingly, if the disformal transformation is invertible, in the sense that the metric $g_{\mu\nu}$ can be fully expressed in terms of $\tilde{g}_{\mu\nu}$, then the number of degrees of freedom associated with S and \tilde{S} must be the same. One thus expects that the disformal transformations of all the degenerate theories described in the previous section are also degenerate. We will also discuss the special case where the transformation is noninvertible in subsection VII B.

A. Relations between the two metrics and their covariant derivatives

In order to write explicitly the above action in terms of $g_{\mu\nu}$ and ϕ , we will need the expression of the inverse metric

$$\tilde{g}^{\mu\nu} = A^{-1} \left(g^{\mu\nu} - \frac{B}{A + BX} \nabla^\mu \phi \nabla^\nu \phi \right). \quad (4.4)$$

Contracting this relation with $\phi_\mu \phi_\nu$ gives \tilde{X} as a function of X :

$$\tilde{X} = \frac{X}{A + BX}. \quad (4.5)$$

It is also useful to introduce the ratio:

$$\mathcal{J}_g \equiv \frac{\sqrt{-\tilde{g}}}{\sqrt{-g}} = A^{3/2} \sqrt{A + BX}. \quad (4.6)$$

The difference between the two covariant derivatives $\tilde{\nabla}$ and ∇ , associated respectively to the two metrics $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$, is fully characterized by the difference of their respective Christoffel symbols,

$$C_{\mu\nu}^\lambda \equiv \tilde{\Gamma}_{\mu\nu}^\lambda - \Gamma_{\mu\nu}^\lambda, \quad (4.7)$$

which defines a tensor. In particular, the relation between the respective second-order covariant derivatives of ϕ reads

$$\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi = \nabla_\mu \nabla_\nu \phi - C_{\mu\nu}^\lambda \phi_\lambda. \quad (4.8)$$

The explicit expression for $C_{\mu\nu}^\lambda$ is given by

$$\begin{aligned}
C_{\mu\nu}^\lambda = & \frac{A_X}{A} \left[2\delta_{(\mu}^\lambda \phi_{\nu)\sigma} \phi^\sigma - \phi^{\lambda\sigma} \phi_\sigma g_{\mu\nu} + \frac{B}{A+BX} (-2\phi^\lambda \phi_{(\mu} \phi_{\nu)\sigma} \phi^\sigma + \phi^\lambda \phi^\rho \phi_{\rho\sigma} \phi^\sigma g_{\mu\nu}) \right] \\
& + B_X \left[-\frac{1}{A} \phi_\mu \phi_\nu \phi^{\lambda\sigma} \phi_\sigma + \frac{1}{A+BX} \left(2\phi^\lambda \phi_{(\mu} \phi_{\nu)\sigma} \phi^\sigma + \frac{B}{A} \phi^\rho \phi_{\rho\sigma} \phi^\sigma \phi^\lambda \phi_\mu \phi_\nu \right) \right] + \frac{B}{A+BX} \phi^\lambda \phi_{\mu\nu} \\
& + \frac{A_\phi}{2A} \left[\delta_\mu^\lambda \phi_\nu + \delta_\nu^\lambda \phi_\mu - \frac{1}{A+BX} (A\phi^\lambda g_{\mu\nu} + 2B\phi^\lambda \phi_\mu \phi_\nu) \right] + \frac{B_\phi}{2(A+BX)} \phi^\lambda \phi_\mu \phi_\nu. \tag{4.9}
\end{aligned}$$

The last line does not depend on second order gradients of ϕ . As we will see, this implies that the terms in A_ϕ and B_ϕ appear only in the transformed action as terms of the form (2.4), which we will not compute explicitly.

B. Curvature term

Let us first concentrate on the term depending on the Ricci scalar of $\tilde{g}_{\mu\nu}$ in \tilde{S} . Following the derivation presented in [4], the Ricci scalar \tilde{R} can be written in terms of the tensor $C_{\mu\nu}^\lambda$ and of the metric $g_{\mu\nu}$, according to the expression

$$\begin{aligned}
\tilde{R} \equiv \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu} = & A^{-1} \left(g^{\mu\nu} - \frac{B}{A+BX} \phi^\mu \phi^\nu \right) \\
& \times (R_{\mu\nu} + C_{\mu\rho}^\sigma C_{\nu\sigma}^\rho - C_{\mu\nu}^\rho C_{\rho\sigma}^\sigma) + \tilde{\nabla}_\rho \xi^\rho, \tag{4.10}
\end{aligned}$$

with

$$\gamma_4 = \frac{(6A^2 + 8ABX + 2B^2X^2)A_X^2 + 4AX(A+BX)A_XB_X + A^2B(B+XB_X)}{A^3(A+BX)^2}, \tag{4.14}$$

$$\gamma_5 = -\frac{2A_X(BA_X + 2AB_X)}{A^3(A+BX)}. \tag{4.15}$$

The dots in (4.12) indicate terms that are at most linear in $\phi_{\mu\nu}$, i.e. of the form (2.4), which we will not write down explicitly.

The total derivative $\tilde{\nabla}_\rho \xi^\rho$ can be ignored if the function \tilde{f} multiplying the scalar curvature is a constant. Otherwise, one also needs to reexpress this term as a function of $g_{\mu\nu}$ and ϕ . This can be done after an integration by parts so that one gets

$$\begin{aligned}
\int d^4x \sqrt{-\tilde{g}} \tilde{f} \tilde{\nabla}_\mu \xi^\mu &= - \int d^4x \sqrt{-\tilde{g}} \xi^\mu \nabla_\mu \tilde{f} \\
&= -2 \int d^4x \sqrt{-\tilde{g}} \tilde{f}_{;\tilde{X}} \tilde{X}_X \xi^\mu \phi_{\mu\nu} \phi^\nu + (\dots), \tag{4.16}
\end{aligned}$$

with

$$\tilde{X}_X \equiv \frac{\partial \tilde{X}}{\partial X} = \frac{A}{(A+BX)^2}. \tag{4.17}$$

$$\xi^\rho \equiv \tilde{g}^{\mu\nu} C_{\mu\nu}^\rho - \tilde{g}^{\rho\mu} C_{\mu\nu}^\nu. \tag{4.11}$$

The terms quadratic in $C_{\mu\nu}^\lambda$ can be rewritten in terms of the elementary Lagrangians L_I^ϕ . One finds

$$\begin{aligned}
A^{-1} \left(g^{\mu\nu} - \frac{B}{A+BX} \phi^\mu \phi^\nu \right) (C_{\mu\rho}^\sigma C_{\nu\sigma}^\rho - C_{\mu\nu}^\rho C_{\rho\sigma}^\sigma) \\
= \gamma_I L_I^\phi + (\dots), \tag{4.12}
\end{aligned}$$

with

$$\gamma_1 = \gamma_2 = 0, \quad \gamma_3 = -\frac{B(BXA_X + A(2A_X + XB_X + B))}{A^2(A+BX)^2}, \tag{4.13}$$

Since ξ^μ contains second derivatives of ϕ , the scalar quantity $\xi^\mu \phi_{\mu\nu} \phi^\nu$ can be decomposed as a combination of the elementary terms L_I^ϕ . One finds

$$\xi^\mu \phi_{\mu\nu} \phi^\nu = \lambda_I L_I^\phi + (\dots), \tag{4.18}$$

with

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \frac{B}{A^2 + ABX}, \tag{4.19}$$

$$\lambda_4 = -\frac{4BXA_X + A(6A_X + 2XB_X + B)}{A^2(A+BX)}, \tag{4.20}$$

$$\lambda_5 = \frac{2(2BA_X + AB_X)}{A^2(A+BX)}, \tag{4.20}$$

and the dots stand as usual for the terms at most linear in $\phi_{\mu\nu}$.

Gathering up everything, one finds that the scalar curvature term yields

$$\int d^4x \sqrt{-\tilde{g}} \tilde{f} \tilde{R} = \int d^4x \sqrt{-g} \mathcal{J}_g \left\{ \frac{\tilde{f}}{A} \left[R - \frac{B}{A+BX} R_{\mu\nu} \phi^\mu \phi^\nu \right] + (\gamma_I - 2\tilde{X}_X \tilde{f}_{\tilde{X}} \lambda_I) L_I^\phi \right\} + (\dots), \quad (4.21)$$

where the term in $R_{\mu\nu} \phi^\mu \phi^\nu$ is of the form (2.9) with the function

$$\theta = -\mathcal{J}_g \frac{B}{A(A+BX)} \tilde{f}, \quad (4.22)$$

and the dots correspond to terms of the form (2.4).

C. Scalar field terms

Let us now consider the terms quadratic in second derivatives of the scalar field. Each of the five terms in \tilde{S}_ϕ can be decomposed, after substitution of (4.1), into the five terms that appear in the final action S_ϕ .

Let us illustrate this with the first term $\tilde{L}_1^\phi \equiv \tilde{\phi}_{\mu\nu} \tilde{\phi}^{\mu\nu}$, which can be decomposed as follows,

$$\begin{aligned} \mathcal{T}_{11} &= \frac{1}{(A+BX)^2}, & \mathcal{T}_{13} &= \frac{2A_X}{A(A+BX)^2}, & \mathcal{T}_{14} &= \frac{2(X(A_X+XB_X)^2 - A(2(A_X+XB_X)+B))}{A(A+BX)^3}, \\ \mathcal{T}_{15} &= \frac{1}{A^2(A+BX)^4} [2A^3B_X + A^2(-2XA_XB_X + 2A_X^2 + B^2 - X^2B_X^2 + 4BXB_X) + 3B^2X^2A_X^2 \\ &\quad - 2ABX(2XA_XB_X + A_X(B-2A_X) + X^2B_X^2)], \\ \mathcal{T}_{22} &= \frac{1}{(A+BX)^2}, & \mathcal{T}_{23} &= -\frac{2(A(-2A_X+XB_X+B) - 3BXA_X)}{A(A+BX)^3}, \\ \mathcal{T}_{25} &= \frac{(A(-2A_X+XB_X+B) - 3BXA_X)^2}{A^2(A+BX)^4}, \\ \mathcal{T}_{33} &= \frac{A - X(A_X+XB_X)}{(A+BX)^4}, & \mathcal{T}_{35} &= -\frac{(A(-2A_X+XB_X+B) - 3BXA_X)(A - X(A_X+XB_X))}{A(A+BX)^5}, \\ \mathcal{T}_{44} &= \frac{(A - X(A_X+XB_X))^2}{A(A+BX)^4}, & \mathcal{T}_{45} &= -\frac{B(A - X(A_X+XB_X))^2}{A(A+BX)^5}, & \mathcal{T}_{55} &= \frac{(A - X(A_X+XB_X))^2}{(A+BX)^6}. \end{aligned} \quad (4.25)$$

It can be noticed that these coefficients form a triangular matrix.

D. Transformation of the total action

Collecting all the results obtained above, one can now write the functions that appear in the action S in terms of the functions \tilde{f} and $\tilde{\alpha}_I$ of \tilde{S} . We find

$$f = \mathcal{J}_g A^{-1} \tilde{f}, \quad (4.26)$$

$$\alpha_1 = -\theta + \mathcal{J}_g \mathcal{T}_{11} \tilde{\alpha}_1, \quad (4.27)$$

$$\alpha_2 = \theta + \mathcal{J}_g \mathcal{T}_{22} \tilde{\alpha}_2, \quad (4.28)$$

$$\begin{aligned} \tilde{L}_1^\phi &= \tilde{g}^{\mu\rho} \tilde{g}^{\nu\sigma} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \tilde{\nabla}_\rho \tilde{\nabla}_\sigma \phi \\ &= A^{-2} \left(g^{\mu\rho} - \frac{B}{A+BX} \phi^\mu \phi^\rho \right) \left(g^{\nu\sigma} - \frac{B}{A+BX} \phi^\nu \phi^\sigma \right) \\ &\quad \times (\phi_{\mu\nu} - C_{\mu\nu}^\lambda \phi_\lambda) (\phi_{\rho\sigma} - C_{\rho\sigma}^\tau \phi_\tau) \\ &= \mathcal{T}_{11} L_1^\phi + \mathcal{T}_{13} L_3^\phi + \mathcal{T}_{14} L_4^\phi + \mathcal{T}_{15} L_5^\phi + (\dots), \end{aligned} \quad (4.23)$$

where the coefficients are determined explicitly by substituting the expression (4.9) for $C_{\mu\nu}^\lambda$. Note that the term L_2^ϕ does not appear in the decomposition.

Proceeding similarly with all the other terms, one finally gets five similar decompositions, which can be summarized by the expression

$$\tilde{L}_I^\phi = \mathcal{T}_{IJ} L_J^\phi + (\dots), \quad (4.24)$$

where the summation with respect to the index J is implicit. The nonvanishing coefficients \mathcal{T}_{IJ} are given by

$$\alpha_3 = 2\theta_X + \mathcal{J}_g [\tilde{f} \gamma_3 - 2\tilde{X}_X \tilde{f}_{\tilde{X}} \lambda_3 + \mathcal{T}_{13} \tilde{\alpha}_1 + \mathcal{T}_{23} \tilde{\alpha}_2 + \mathcal{T}_{33} \tilde{\alpha}_3], \quad (4.29)$$

$$\alpha_4 = -2\theta_X + \mathcal{J}_g [\tilde{f} \gamma_4 - 2\tilde{X}_X \tilde{f}_{\tilde{X}} \lambda_4 + \mathcal{T}_{14} \tilde{\alpha}_1 + \mathcal{T}_{44} \tilde{\alpha}_4], \quad (4.30)$$

$$\alpha_5 = \mathcal{J}_g [\tilde{f} \gamma_5 - 2\tilde{X}_X \tilde{f}_{\tilde{X}} \lambda_5 + \mathcal{T}_{15} \tilde{\alpha}_1 + \mathcal{T}_{25} \tilde{\alpha}_2 + \mathcal{T}_{35} \tilde{\alpha}_3 + \mathcal{T}_{45} \tilde{\alpha}_4 + \mathcal{T}_{55} \tilde{\alpha}_5]. \quad (4.31)$$

By substituting all the formulas given in the previous subsections, one obtains the explicit expressions of f and

α_I in terms of \tilde{f} , $\tilde{\alpha}_I$, A and B . One can verify that the f and α_I satisfy the degeneracy conditions (3.14)–(3.16). In fact, it turns out that this is a very efficient way to check the expressions for f and α_I . A first conclusion is thus that all quadratic DHOST theories transform into quadratic DHOST theories, as expected.

For a more detailed analysis, the relations (4.26)–(4.31) in their abridged form are useful to see how the main families of DHOST theories transform. First of all, let us note that if $\tilde{f} = 0$ then necessarily $f = 0$. Therefore, the transformed version of theories in class III remains in class III. As a consequence of $\mathcal{T}_{11} = \mathcal{T}_{22}$, we also find the relation

$$\alpha_1 + \alpha_2 = \mathcal{J}_g \mathcal{T}_{11}(\tilde{\alpha}_1 + \tilde{\alpha}_2), \quad (4.32)$$

which shows that the property $\alpha_1 + \alpha_2 = 0$ (or $\alpha_1 + \alpha_2 \neq 0$) is invariant under disformal transformations. This implies that class I, characterized by $\alpha_1 + \alpha_2 = 0$, is stable under disformal transformations. Therefore all the three main classes are stable. We study more precisely the impact of disformal transformations in the next two sections.

V. DISFORMAL TRANSFORMATIONS IN CLASS IA

Disformal transformations for theories in class Ia have been partially studied in several previous works. In particular, it has been shown that Horndeski theories are stable under X -independent disformal transformations [11]. The first example of theory beyond Horndeski, i.e. with higher order equations of motion, was exhibited in [4] by considering the general disformal transformation of the Einstein-Hilbert action, which is also in class Ia. It was also shown in [6] that the extended quadratic (quintic) Lagrangian proposed in [5] can be generated from the quadratic (quintic) Horndeski Lagrangian via a purely disformal transformation with $A = 1$. All these examples are particular cases of disformal transformations within class Ia.

If we now consider class Ia theories such that $\tilde{f} \neq 0$, which depend on three arbitrary functions, it is natural to expect that generic theories can be “generated” from the subset of (quadratic) Horndeski theories, characterized by a single arbitrary function \tilde{f} , via general disformal transformations, which depend on two arbitrary functions. We can check that this is indeed the case,² by starting from the quartic Horndeski Lagrangian expressed in terms of the metric $\tilde{g}_{\mu\nu}$ and of the scalar field ϕ ,

$$\begin{aligned} \tilde{S}[\phi, \tilde{g}_{\mu\nu}] = & \int d^4x \sqrt{-\tilde{g}} \{ \tilde{f}(\tilde{X}, \phi) \tilde{R} - 2\tilde{f}_{,\tilde{X}}(\tilde{X}, \phi) \\ & \times [(\tilde{\nabla}^\mu \tilde{\nabla}_\mu \phi)^2 - \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi \tilde{\nabla}^\mu \tilde{\nabla}^\nu \phi] \}. \end{aligned} \quad (5.1)$$

Substituting (4.1), we obtain an action S for $g_{\mu\nu}$ and ϕ , which is characterized by the functions

$$f = A^{1/2} \sqrt{A + BX} \tilde{f}, \quad (5.2)$$

and

$$\alpha_1 = -\alpha_2 = -\frac{2A^{3/2}}{(A + BX)^{3/2}} [(B + XB^2)\tilde{f} + 2\tilde{f}_{,\tilde{X}}], \quad (5.3)$$

$$\alpha_3 = -\frac{2(BA_X + AB_X)}{A^{1/2}(A + BX)^{1/2}} \tilde{f} + \frac{4(XB_X - A_X)A^{1/2}}{(A + BX)^{3/2}}, \quad (5.4)$$

$$\begin{aligned} \alpha_4 = & \frac{2(A^2B_X + AA_X(2XB_X + B) + A_X^2(3A + BX))}{A^{3/2}(A + BX)^{1/2}} \tilde{f} \\ & - \frac{4(-A_X(A - 2X^2B_X) + AXB_X + 2XA_X^2)}{A^{1/2}(A + BX)^{3/2}} \tilde{f}_{,\tilde{X}}, \end{aligned} \quad (5.5)$$

$$\alpha_5 = -\frac{2A_X(BA_X + 2AB_X)}{A^{3/2}(A + BX)^{1/2}} \tilde{f} + \frac{4A_X(2XB_X - A_X)}{A^{1/2}(A + BX)^{3/2}} \tilde{f}_{,\tilde{X}}. \quad (5.6)$$

If one starts from a generic theory in class Ia, defined by the functions f and α_I , it is possible to determine two functions A and B such that this theory is disformally related to Horndeski, as we now show. According to (5.2), the Horndeski function \tilde{f} is related to f , A and B by

$$\tilde{f} = A^{-1/2}(A + BX)^{-1/2} f. \quad (5.7)$$

Substituting this expression for \tilde{f} into (5.3) yields

$$\alpha_1 = -\alpha_2 = \frac{2Af_X - f(2A_X + XB_X)}{A - XA_X - X^2B_X}, \quad (5.8)$$

which one can solve to find B_X in terms of α_2 , f and A :

$$B_X = \frac{(2f_X + \alpha_2)A - (2f + X\alpha_2)A_X}{X(f + X\alpha_2)}. \quad (5.9)$$

Substituting (5.7) and (5.9) in α_3 gives

$$\frac{A_X}{A} = \frac{4f_X + 2\alpha_2 + X\alpha_3}{4(f + X\alpha_2)}. \quad (5.10)$$

Finally, by substituting successively (5.7), (5.9) and (5.10), one can rewrite α_4 and α_5 in terms of f , α_2 and α_3 and check that one recovers exactly the expressions (3.19) and (3.20). This proves that generic theories in class Ia are “generated”

²The same calculation has been performed independently in the recent paper [19].

from the Horndeski quadratic Lagrangians via disformal transformations (4.1).

In analogy with the choice between the ‘‘Jordan frame’’ and ‘‘Einstein frame’’ for traditional scalar-tensor theories, the above construction shows that theories belonging to class Ia with $f \neq 0$ can be defined either in the ‘‘Jordan frame’’, where the metric is minimally coupled to matter,

$$S_{\text{total}} = \int d^4x \sqrt{-g} [fR + \alpha_I L_I^\phi] + S_m[g_{\mu\nu}, \Psi_m], \quad (5.11)$$

or in the ‘‘Horndeski frame,’’ where the gravitational part of the action is described by Horndeski,

$$\begin{aligned} \tilde{S}_{\text{total}} = & \int d^4x \sqrt{-\tilde{g}} \{ \tilde{f} \tilde{R} - 2\tilde{f}_{,\tilde{\chi}} [(\tilde{\square}\phi)^2 - \tilde{\phi}_{,\mu\nu} \tilde{\phi}^{\mu\nu}] \} \\ & + (\dots) + S_m[g_{\mu\nu}, \Psi_m]. \end{aligned} \quad (5.12)$$

In the ‘‘Horndeski frame,’’ the matter action is nonminimally coupled, but can be expressed explicitly in terms of the ‘‘Horndeski metric’’ by inverting the transformation (4.1).

Note that the Einstein-Hilbert Lagrangian, with \tilde{f} constant and $\tilde{\alpha}_I = 0$, is a particular case of Horndeski. It generates, via disformal transformations, the family characterized by the expressions (5.2)–(5.6) with $\tilde{f}_{,\tilde{\chi}} = 0$. If the disformal transformation is invertible, one thus gets a family of scalar-tensor theories which are in fact general relativity in disguise and, as such, are doubly degenerate and contain only two tensor modes. Of course, one can always add another term of the form (2.4) in the action, which does not modify the quadratic part of the action (2.1), in order to break the second degeneracy. One then obtains a degenerate scalar-tensor theory with one scalar mode and two tensor modes. This is precisely how a theory ‘‘beyond Horndeski’’ was constructed in [4].

VI. DISFORMAL TRANSFORMATIONS IN OTHER CLASSES

We now turn to the other classes of DHOST theories.

A. Stability of all classes

We have already pointed out the stability, under disformal transformations, of the signs (including zero) of f and of $\alpha_1 + \alpha_2$, which guarantees the stability of the classes I, II and III separately. We now consider the criteria that distinguish the subclasses within these classes.

One can first notice the relation

$$\begin{aligned} f - \alpha_1 X &= \mathcal{J}_g \left[\frac{1}{A + BX} \tilde{f} - \frac{X}{(A + BX)^2} \tilde{\alpha}_1 \right] \\ &= \frac{\mathcal{J}_g}{A + BX} (\tilde{f} - \tilde{X} \tilde{\alpha}_1), \end{aligned} \quad (6.1)$$

where we have substituted the expression (4.22) for θ and the coefficient \mathcal{T}_{11} in (4.27). If we start from a theory in class Ib or in class IIb, characterized by $\tilde{\alpha}_1 = \tilde{f}/\tilde{X}$, the above relation implies that the disformally transformed theory verifies $\alpha_1 = f/X$ and thus belongs to the same subclass, either Ib or IIb, as the original theory. Therefore, the classes Ia, Ib, IIa and IIb are separately stable.

We find the same properties for the subclasses in class III. Indeed, when $f = 0$, we have $\alpha_1 = \mathcal{J}_g \mathcal{T}_{11} \tilde{\alpha}_1$ and $\alpha_2 = \mathcal{J}_g \mathcal{T}_{11} \tilde{\alpha}_2$. Therefore the signs of α_1 and $\alpha_1 + 3\alpha_2$ which distinguish the subclasses IIIa, IIIb and IIIc are conserved under a disformal transformation.

In summary, all the classes and subclasses that we have distinguished are separately stable under disformal transformations. In particular, the intersections of two classes or subclasses are also stable. This applies for instance to the intersection of Ia and IIIa, which contains L_4^{bH} .

B. Disformal transformations in class IIa

It is straightforward to specialize the general disformal transformations to class IIa. One just needs to impose that the tilted functions \tilde{f} and the $\tilde{\alpha}_I$ satisfy the properties (3.26). Since Lagrangians in class IIa depend on three arbitrary functions, one can try to proceed as in class Ia by choosing a particular family that depends on a single arbitrary function and then produce generic theories by applying a disformal transformation. There is no natural candidate for this one-function family, in contrast with Horndeski. One could choose, for example, the family

$$\begin{aligned} \tilde{f} &= 1, & \tilde{\alpha}_1 &= 0, & \tilde{\alpha}_3 &= -\frac{4}{X^2} (1 + 2X\alpha_2), \\ \tilde{\alpha}_4 &= \frac{2}{X^2}, & \tilde{\alpha}_5 &= \frac{8}{X^3} (1 + 2X\alpha_2), \end{aligned} \quad (6.2)$$

which depends only on the arbitrary function $\tilde{\alpha}_2$.

One finds that the disformal transformation of this family leads, in particular, to

$$f = A^{1/2} (A + BX)^{1/2} \quad (6.3)$$

and

$$\alpha_2 = -\frac{A^{1/2} (A(B - \tilde{\alpha}_2) + B^2 X)}{(A + BX)^{3/2}}. \quad (6.4)$$

We can solve the first equation to determine B in terms of A and f :

$$B = \frac{f^2 - A^2}{AX}. \quad (6.5)$$

Substituting in (6.4) and solving for $\tilde{\alpha}_2$, we get

$$\tilde{\alpha}_2 = \frac{f^2}{A^3 X} (f^2 - A^2 + Xf\alpha_2). \quad (6.6)$$

One can then substitute these relations into the other coefficients obtained by disformal transformations. The coefficient α_4 , for instance, is particularly simple:

$$\alpha_4 = \frac{2}{fX^2} (A^2 + 4X^2 f_X^2 - 2Xff_X). \quad (6.7)$$

In this way, one can determine the function A in terms of α_4 and f , and then B . Note however that this procedure works only if the above equation can be solved for A^2 . This means that there will be restrictions on the theories generated by the family (6.2).

C. Disformal transformation in class IIIa

Although class IIIa contains L_4^{bH} , which depends on a single function, this Lagrangian cannot generate all theories in IIIa because it belongs to the intersection of IIIa and Ia and any of its transformed Lagrangians will also belong to this intersection. This means in particular that L_4^{bH} cannot be connected to Horndeski, as already pointed out in [9].

In class IIIa, $f = 0$ and therefore the ratio between α_1 and α_2 is conserved under disformal transformations. This implies that one cannot choose a “seed” family with $\tilde{\alpha}_1 = 0$ or $\tilde{\alpha}_2 = 0$ in order to obtain generic theories of IIIa. Instead, one can take, for instance, the family

$$\begin{aligned} \tilde{f} &= 0, & \tilde{\alpha}_1 &= 1, & \tilde{\alpha}_3 &= 0, & \tilde{\alpha}_4 &= -\frac{2}{X}, \\ \tilde{\alpha}_5 &= \frac{1 + 2\tilde{\alpha}_2}{X^2(1 + 3\tilde{\alpha}_2)}, \end{aligned} \quad (6.8)$$

which depends on the arbitrary function $\tilde{\alpha}_2$. Applying a disformal transformation on this family, one gets

$$\alpha_1 = A^{3/2}(A + BX)^{-3/2}, \quad \alpha_2 = A^{3/2}(A + BX)^{-3/2}\tilde{\alpha}_2, \quad (6.9)$$

which can be solved to give

$$B = A \frac{1 - \alpha_1^{2/3}}{X\alpha_1^{2/3}}, \quad \tilde{\alpha}_2 = A^{-3/2}(A + BX)^{3/2}\alpha_2. \quad (6.10)$$

Substituting into the expression for α_3 , one finds

$$\frac{A_X}{A} = \frac{2\alpha_1\alpha_3 - 4\alpha_3\alpha_{1X}}{6\alpha_1(\alpha_1 + 2\alpha_2)}. \quad (6.11)$$

This enables us to determine A and then B .

VII. LINK WITH KHROMETRIC AND MIMETIC THEORIES

In this section we show that, in addition to Horndeski and its extension, quadratic DHOST theories also contain, as particular cases, other theories that have already been studied elsewhere. In the first part, we identify some “khronometric” theories in both class I and class II. In the second part, we discuss “mimetic” theories, associated with a noninvertible disformal transformation.

A. Khronometric theories

Khronometric theories [20] represent a subset of Einstein-aether theories [21] where the unit timelike vector u^μ , which defines a special frame, is expressed as the normalized gradient of a scalar field,

$$u_\mu = \frac{\nabla_\mu \phi}{\sqrt{-X}}. \quad (7.1)$$

Their dynamics is described by the action

$$S = \int d^4x \sqrt{-g} (fR + K^{\mu\nu}{}_{\rho\sigma} \nabla_\mu u^\rho \nabla_\nu u^\sigma), \quad (7.2)$$

with

$$K^{\mu\nu}{}_{\rho\sigma} \equiv c_1 g^{\mu\nu} g_{\rho\sigma} + c_2 \delta_\rho^\mu \delta_\sigma^\nu + c_3 \delta_\sigma^\mu \delta_\rho^\nu + c_4 u^\mu u^\nu g_{\rho\sigma}, \quad (7.3)$$

where the c_a are constant. This is the most general Lagrangian which depends only quadratically on the vector field u^μ and it is clearly invariant under arbitrary scalar field redefinitions $\phi \mapsto \psi(\phi)$. Since f is constant, we can set $f = 1$ without loss of generality.

Substituting (7.1) into (7.2)–(7.3), we find that the action (7.2) is of the form (2.1), with the functions

$$\begin{aligned} f &= 1, & \alpha_1 &= -\frac{1}{X}(c_1 + c_3), & \alpha_2 &= -\frac{1}{X}c_2, & \alpha_3 &= \frac{2}{X^2}c_2, \\ \alpha_4 &= \frac{1}{X^2}(c_1 + 2c_3 + c_4), & \alpha_5 &= -\frac{1}{X^3}(c_2 + c_3 + c_4). \end{aligned} \quad (7.4)$$

One immediately sees that the parameter c_1 can be absorbed in c_3 and c_4 by redefining $c_3 + c_1 \rightarrow c_3$ and $c_4 - c_1 \rightarrow c_4$. In the following, we will thus assume $c_1 = 0$ without loss of generality (one can always return to the original form by using the inverse redefinitions).

Among khronometric theories, which represent a subset of higher order theories, one can look for degenerate theories by examining the degeneracy conditions (3.14)–(3.16). Substituting (7.4), with $c_1 = 0$, into the first degeneracy condition yields

$$D_0 = \frac{4}{X}(c_2 + c_3)(c_4 - 2) = 0, \quad (7.5)$$

whose solutions give several families of degenerate khronons.

1. Khronons in class I

Let us first consider the case

$$c_2 + c_3 = 0, \quad (7.6)$$

which corresponds to class I, since this implies $\alpha_1 + \alpha_2 = 0$. Substituting this condition into D_1 and D_2 , one finds

$$D_1 = XD_2 = -\frac{8}{X^2}(c_2 - 1)^2 c_4, \quad (7.7)$$

which leads to two possibilities.

The first family, characterized by

$$\alpha_1 = -\alpha_2 = \frac{1}{X}, \quad \alpha_3 = \frac{2}{X^2}, \quad \alpha_4 = \frac{1}{X^2}(c_4 - 2), \quad \alpha_5 = -\frac{c_4}{X^3}, \quad (7.8)$$

with c_4 arbitrary, belongs to the class Ib.

The second family, described by

$$\alpha_1 = -\alpha_2 = \frac{c_2}{X}, \quad \alpha_3 = -\alpha_4 = \frac{2c_2}{X^2}, \quad \alpha_5 = 0, \quad (7.9)$$

with c_2 arbitrary, belongs to class Ia (except the case $c_2 = 1$ which is also in the previous family) and corresponds to L_4^{BH} with $F_4 = c_2/X^2$.

2. Khronometric class II

The second possibility to satisfy $D_0 = 0$ is given by

$$c_4 = 2, \quad (7.10)$$

which leads to

$$D_1 = XD_2 = \frac{8}{X^2}(1 + c_3)(3c_2 + c_3 - 2). \quad (7.11)$$

Once again, we get two families, but now belonging to class II.

The first family, corresponding to $c_3 = -1$, is described by

$$\alpha_1 = \frac{1}{X}, \quad \alpha_2 = -\frac{c_2}{X}, \quad \alpha_3 = \frac{2c_2}{X^2}, \quad \alpha_4 = 0, \quad \alpha_5 = -\frac{c_2 + 1}{X^3}, \quad (7.12)$$

and depends on the arbitrary parameter c_2 . These theories are in IIb, except for the case $c_2 = 1$, which is in Ib.

The second family, corresponding to $c_3 = 2 - 3c_2$, also depends on the single parameter c_2 :

$$\alpha_1 = \frac{3c_2 - 2}{X}, \quad \alpha_2 = -\frac{c_2}{X}, \quad \alpha_3 = \frac{2c_2}{X^2}, \quad \alpha_4 = \frac{6(1 - c_2)}{X^2},$$

$$\alpha_5 = \frac{2(c_2 - 2)}{X^3}. \quad (7.13)$$

These Lagrangians belong to class IIa, except if $c_2 = 1$.

3. Disformal transformations

The set of khronometric theories is stable under the action of disformal transformations of the form

$$\tilde{g}_{\mu\nu} = ag_{\mu\nu} + bu_{\mu}u_{\nu}, \quad (7.14)$$

i.e. for $A = a$ and $B = -b/X$, where a and b are constant.

Without loss of generality, one can restrict our analysis to one-parameter transformations such that $b = a - a^{-1}$, which preserve $f = 1$ in the khronometric action (7.2). It is straightforward to study the action of disformal transformations of this type on the four degenerate khronometric families identified above. One finds that each theory of the first family (7.8) remains invariant. Each of the three other families is stable, the transformed theory being obtained by the following modification of the parameter c_2 :

$$c_2 \rightarrow (c_2 - 1)a^2 + 1. \quad (7.15)$$

In particular, one notes that, in the family (7.9), any theory with $c_2 < 1$ can be transformed into general relativity by choosing $a^2 = 1/(1 - c_2)$.

B. Mimetic theories

In previous sections, we have assumed that the disformal transformation (4.1) is invertible, in the sense that one can also express the metric $g_{\mu\nu}$ in terms of $\tilde{g}_{\mu\nu}$. When one relaxes this condition, one obtains so-called ‘‘mimetic’’ theories, in analogy with the first model of this kind, investigated in [22], defined from a noninvertible disformal transformation of the Einstein-Hilbert action (see e.g. [23–26] for subsequent works on mimetic theories).

1. Noninvertible disformal transformation and symmetries

By differentiating the expression

$$\tilde{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\phi_{\mu}\phi_{\nu}, \quad (7.16)$$

one obtains

$$\delta\tilde{g}_{\mu\nu} = F_{\mu\nu}\delta\phi + H_{\mu\nu}^{\alpha}\nabla_{\alpha}\delta\phi + J_{\mu\nu}^{\alpha\beta}\delta g_{\alpha\beta} \quad (7.17)$$

with

$$F_{\mu\nu} = A_{\phi}g_{\mu\nu} + B_{\phi}\phi_{\mu}\phi_{\nu}, \quad (7.18)$$

$$H_{\mu\nu}^{\alpha} = 2(A_X g_{\mu\nu} + B_X \phi_{\mu} \phi_{\nu}) \phi^{\alpha} + B(\phi_{\nu} \delta_{\mu}^{\alpha} + \phi_{\mu} g_{\nu}^{\alpha}), \quad (7.19)$$

$$J_{\mu\nu}^{\alpha\beta} = A \delta_{(\mu}^{\alpha} \delta_{\nu)}^{\beta} - \phi^{\alpha} \phi^{\beta} (A_X g_{\mu\nu} + B_X \phi_{\mu} \phi_{\nu}). \quad (7.20)$$

As discussed in [4], the disformal transformation is noninvertible; (i.e., $g_{\mu\nu}$ cannot be determined from $\tilde{g}_{\mu\nu}$) if the determinant of the Jacobian matrix $J_{\mu\nu}^{\alpha\beta} \equiv \frac{\partial \tilde{g}_{\mu\nu}}{\partial g_{\alpha\beta}}$ vanishes.

This happens when $J_{\mu\nu}^{\alpha\beta}$ admits a null vector $v_{\alpha\beta}$ such that

$$J_{\mu\nu}^{\alpha\beta} v_{\alpha\beta} = 0. \quad (7.21)$$

It is straightforward to check that the combination

$$v_{\alpha\beta} = A_X g_{\alpha\beta} + B_X \phi_{\alpha} \phi_{\beta} \quad (7.22)$$

is a null vector of the Jacobian matrix, provided the functions A and B verify

$$B_X = \frac{A - X A_X}{X^2}. \quad (7.23)$$

After integration, this yields

$$B = -\frac{A}{X} + \mu(\phi), \quad (7.24)$$

corresponding to the disformal transformation

$$\tilde{g}_{\mu\nu} = A(\phi, X) \left(g_{\mu\nu} - \frac{1}{X} \phi_{\mu} \phi_{\nu} \right) + \mu(\phi) \phi_{\mu} \phi_{\nu}. \quad (7.25)$$

Note that if we insert (7.24) into (4.5), one gets $\tilde{X} = 1/\mu(\phi)$, which shows that \tilde{X} does not depend on X .

2. Mimetic action

If we start from an action of the form

$$\tilde{S}[\phi, \tilde{g}_{\mu\nu}] = \int d^4x \sqrt{-\tilde{g}} (\tilde{f}(\phi) \tilde{R} + \alpha_I(\phi) \tilde{L}_I^{\phi}), \quad (7.26)$$

and substitute (7.25), we obtain a new action S , given as a functional of $g_{\mu\nu}$ and ϕ . This leads to a subclass of our DHOST theories with particular properties. This procedure has been used in [22] for the Einstein-Hilbert action, i.e. $\tilde{f} = 1$ and $\tilde{\alpha}_I = 0$, with the disformal transformation characterized by $A = X$ and $B = 0$, to introduce the model of mimetic dark matter. It has been extended in [23] to a general noninvertible transformation, with (7.24). In contrast with the generic case where the disformal transformation is invertible, the number of degrees of freedom is not necessarily the same for \tilde{S} and S . In particular, if \tilde{S} is the Einstein-Hilbert action, with only two degrees of freedom, one ends up with three degrees of freedom for S , as discussed in [22] and [23].

Interestingly, the mimetic action S is invariant under the local symmetry

$$\delta\phi = 0 \quad \text{and} \quad \delta g_{\mu\nu} = \varepsilon v_{\mu\nu} = \varepsilon (A_X g_{\mu\nu} + B_X \phi_{\mu} \phi_{\nu}), \quad (7.27)$$

where ε is an infinitesimal space-time function. This symmetry follows immediately from (7.17), together with the property that $v_{\mu\nu}$ is a null eigenvector of the Jacobian matrix.

In the Hamiltonian framework, such a symmetry implies the existence of an extra first class constraint in addition to the usual Hamiltonian and momentum constraints associated with diffeomorphism invariance. This is in contrast with the standard quadratic DHOST theories, for which the extra constraints are generically second class, as shown in [8]. The secondary (second-class) constraint is necessary to eliminate the Ostrogradsky ghost and we thus expect that, even though mimetic theories contain three degrees of freedom, the Ostrogradsky ghost is still present.

This is indeed the case for the simplest model of mimetic gravity, obtained from Einstein-Hilbert with $A = X$ and $B = 0$. In that case, the symmetry (7.27) reduces to an invariance under conformal transformations of $g_{\mu\nu}$. In the Hamiltonian description, this symmetry is necessarily associated to a first class constraint. Following the analysis of [8], and introducing the conjugate momenta π^{ij} and p_* of h_{ij} and A_* , respectively, one can show that the primary constraint reduces to

$$\Psi \equiv \gamma_{ij} \pi^{ij} - \frac{1}{2} p_*, \quad (7.28)$$

which is indeed the generator of infinitesimal conformal transformations. As a consequence, the primary constraint is first class and it Poisson commutes with the Hamiltonian and momentum constraints. Hence, there is no secondary constraint that eliminates the Ostrogradsky ghost. This has already been noticed in [27] and we expect this to remain true for any mimeticlike theory.

VIII. CONCLUSIONS

The degeneracy, in the generalized sense introduced in [7], of scalar-tensor Lagrangians is a powerful tool to classify viable alternative theories of gravity. In the present work we have revisited all the quadratic DHOST theories identified in [7] and studied how they transform under generalized disformal transformations. In order to do so, we have obtained the general transformation laws of the six arbitrary functions that appear in the Lagrangian of these theories. This shows that, as expected, any quadratic DHOST theory is transformed, via a disformal transformation, into another quadratic DHOST theory, up to terms that are at most linear in $\phi_{\mu\nu}$ (which do not affect the degeneracy of the theory). Moreover, we have found that the three main classes of quadratic DHOST theories, as well as the two or three subclasses within each, are all stable under disformal transformations. One of these subclasses (class Ia) contains the theories “generated” from the (quadratic) Horndeski Lagrangian via disformal transformations.

Two disformally related theories describe distinct physics if matter is assumed to be minimally coupled for both theories. Conversely, a given scalar-tensor theory can be described by different disformally related Lagrangians, provided the coupling to matter is modified accordingly. In this sense, the situation is very similar to traditional scalar-tensor theories for which one can use the Jordan frame, in which matter is minimally coupled to the metric, or the Einstein frame, where the gravitational dynamics is described by the usual Einstein-Hilbert Lagrangian but at the price of a nonminimal coupling of matter to the metric. Similarly, for the quadratic DHOST theories in class Ia, one can work either in the Jordan frame, where matter is minimally coupled but the equations of motion are in general higher order, or in the “Horndeski” frame, where the equations of motion are second order but with a nonminimal coupling of matter to the metric.

Apart from Horndeski and its extension, we have also recognized other known theories among the quadratic DHOST theories. Khronometric theories, which are a subclass of Einstein-aether theories where the unit vector

is proportional to the gradient of a scalar field, lead to higher order scalar-tensor theories when covariantized. These covariantized theories are, in general, not degenerate but we have found that a subset of them are indeed degenerate. Our theories also encompass mimetic gravity, which is obtained from the Einstein-Hilbert Lagrangian via a specific, noninvertible, disformal transformation.

Quadratic DHOST theories also contain theories that cannot be related, up to disformal deformations, to known theories, at least to our knowledge, and thus seem to represent genuinely new scalar-tensor theories, independently of their specific coupling to matter. It would be interesting to study the cosmology of these new theories by using, for instance, the general effective formalism presented in e.g. [28].

Note added.—Reference [19], which appeared as an e-print during the preparation of this manuscript, also studies the theories introduced in [7] and has some partial overlap with the present work, in particular concerning the disformal transformations in class Ia.

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