General analytic solutions of scalar field cosmology with arbitrary potential

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We present the solution space for the case of a minimally coupled scalar field with arbitrary potential in a Friedmann-Lemaître-Robertson-Walker metric. This is made possible due to the existence of a nonlocal integral of motion corresponding to the conformal Killing field of the two-dimensional minisuperspace metric. Both the spatially flat and nonflat cases are studied first in the presence of only the scalar field and subsequently with the addition of noninteracting perfect fluids. It is verified that this addition does not change the general form of the solution, but only the particular expressions of the scalar field and the potential. The results are applied in the case of parametric dark energy models where we derive the scalar field equivalence solution for some proposed models in the literature.

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I. INTRODUCTION

The search for particular solutions to Einstein's general relativity has been both intensive and fruitful over the past 100 years. As the years passed, less and less simple-form solutions were found. This occurrence is very well depicted in the case of cosmology: for many decades, the only known solutions were simplified spatially homogeneous models; the only solutions that were subsequently recognized as the most general were the Bianchi type I (Kasner, 1921 [1]), type II (Taub, 1951 [2]) and type V solutions (Joseph, 1966 [3]). Subsequently, the automorphisms of the various Bianchi-type groups were used in order to count the number of expected essential constants (Ellis and MacCallum, 1969 [4]), while later on automorphisms were seen to be induced by particular spacetime coordinate transformations (Samuel and Ashtekar, 1991 [5], Christodoulakis *et al.* [6]). Finally, the automorphisms were used as Lie point symmetries of the corresponding Einstein equations with the result of uncovering the entire solution space for Bianchi types I–VII [7–10]. The situation as described above naturally led many people working in the field to turn to extended or alternative theories of gravity. It is fair to say that such novelties are supported by recent cosmological data (for instance, Refs. [11–13]).

The cosmological constant Λ, leading to the ΛCDM cosmology, is one of the simplest extensions of the Einstein-Hilbert action since it keeps the gravitational action linear, the degrees of freedom, and the order of the gravitational theory. However, while the ΛCDM cosmological model fits some of the cosmological data, it suffers from two major drawbacks, i.e., fine-tuning and the coincidence problem (for details, see Refs. [14,15]).

In order to surpass these problems, other theoretical models which include new matter sources or higher-order curvature invariants in the gravitational action have been proposed (see, for instance, Refs. [16–20] and references therein). More specifically, in scalar field cosmology, the new terms added in the gravitational Lagrangian increase the number of degrees of freedom by one, when considering a single scalar field, and the terms in the gravitational field equations are the components of an energy-momentum tensor. The scalar field models are categorized into two classes, according to whether the basic fields are defined in the coordinate (Einstein) frame or in its conformally equivalent (Jordan frame); in the latter, there exists a coupling term in the action of the scalar field with the curvature (for instance, the Brans-Dicke theory). Moreover, other modified theories can be seen with the use of a Lagrange multiplier as scalar-field theories, such as $f(R)$ gravity in the metric formalism, $f(\mathcal{R})$ hybrid gravity (where $\mathcal{R}$ is the scalar curvature in the Palatini formalism and coincides with the Ricci scalar only for a linear function f), or the higher-order $f(R,\Box R,\ldots)$ gravity [21–23]. However, under conformal transformations the two different frames (Einstein and Jordan) are related, which means that a cosmological solution can pass from one frame to the other [24,25].

In this work we follow the old path, focusing our attention on an isotropic and spatially homogeneous model with matter and directing our analysis towards the investigation of the entire solution space. Specifically, we consider a Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime and a minimally coupled scalar field with arbitrary potential.

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(For relevant results, see Ref. [26], in which the local integrals of motion corresponding to the automorphisms are used to completely integrate the relevant equations for the Bianchi type I and V diagonal metrics; thus, the $k = 0, -1$ cases of the present work should somehow be included in this reference, if only one could find the relevant transformation.) We are interested in the analytic solutions of that cosmological model. Exact closed-form solutions without a matter term or with a dust fluid can be found in Refs. [27–37] and in the case of a nonminimal coupling in Refs. [38–40], while for non-spatially-flat FLRW spacetimes some exact solutions are contained in Refs. [41–43]. Further analytic solutions for a scalar field model with the presence of a perfect fluid have been presented in Refs. [44–48].

The major ingredient which enables the exhibition of the entire solution space of our model is the use of nonlocal conservation laws, which are generated by the elements of the minisuperspace conformal algebra. The general analytic solution is presented for every arbitrary scalar field potential $V(\phi)$, for a quintessence, or for a phantom field in both spatially flat and/or nonflat FLRW metrics. Moreover, we show that the same result also holds when additional noninteracting perfect fluids are introduced in the field equations.

The plan of the paper is as follows. The basic definitions for our model and the mathematical properties that we use are discussed in Sec. II. In Secs. III and IV we derive the analytic solution for a minimally coupled scalar field in a FLRW spacetime for arbitrary potential for the cases in which the spatial curvature is $k = 0$ and $k \neq 0$, respectively. In Sec. V we consider that our cosmological model admits perfect fluids which are minimally coupled with the scalar field and we derive the analytic solution of the model for an arbitrary potential. Furthermore, in Sec. VI we demonstrate the usefulness and analytic power of our results by deriving some particular solutions of the field equations for specific equation of state parameters of the total cosmological fluid. In Sec. VII we discuss our results and draw our conclusions.

II. BASIC DEFINITIONS

It is a known fact that for many cosmological systems there is a procedure of deriving valid minisuperspace Lagrangians, whose dynamical content is the same as that of the original system. These Lagrangians are by construction singular in nature, since their equations of motion are not all independent of each other (as is also true for the general theory). This property is reflected in the time reparametrization invariance of the system $t \mapsto f(t)$, which is a remnant of the four-dimensional diffeomorphism invariance of the full theory.

It was proven in Ref. [49] that for singular systems described by Lagrangians of the form

$$\bar{L} = \frac{1}{2N(t)} \bar{G}_{\mu\nu}(q)\dot{q}^\mu(t)\dot{q}^\nu(t) - N(t)U(q) \tag{2.1}$$

all conformal Killing fields of $\bar{G}_{\mu\nu}$ can be used to write down integrals of motion for the system. To make it explicit, let us for simplicity perform a reparametrization of the form $N \to n = NU$. Then, it can be seen that for the equivalent system

$$L = \frac{1}{2n} G_{\mu\nu}(q)\dot{q}^\mu\dot{q}^\nu - n, \tag{2.2}$$

where $G_{\mu\nu} = U\bar{G}_{\mu\nu}$, if there exist vector fields $\xi^\alpha(q)$ defined on the configuration space for which the relation

$$\pounds_\xi G_{\mu\nu} = \omega(q) G_{\mu\nu} \tag{2.3}$$

holds, then the quantity

$$Q = \xi^\alpha p_\alpha + \int n(t)\omega(q(t))dt, \tag{2.4}$$

with $p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}$, defines integrals of motion on the phase space due to the existence of the Hamiltonian constraint

$$\mathcal{H} = \frac{1}{2} G^{\mu\nu} p_\mu p_\nu + 1 \approx 0. \tag{2.5}$$

It can be easily verified that, by virtue of Eq. (2.3), Q is a constant of motion,

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \{Q, \mathcal{H}\} = \omega\mathcal{H} \approx 0.$$

In the special case were $\omega = 0$, i.e., the ξ's are Killing vectors of the scaled minisupermetric $G_{\mu\nu}$, Eq. (2.4) leads to autonomous integrals of motion that do not exhibit any explicit time dependence. On the other hand, when $\omega \neq 0$, nonlocal integrals of motion emerge that are actually rheonomic, due to the explicit time dependence brought by the integral in Eq. (2.4).

The autonomous conserved quantities are commonly used in the literature as a kind of selection rule to constrain the potential U, or any arbitrary functions of the configuration variables appearing in Eq. (2.1), so that the system is forced to become integrable [33,36,48,50–54]. In our case, we shall refrain from doing that. In the context of Einstein's relativity and a spatially flat/nonflat FLRW spacetime minimally coupled with a scalar field ϕ, we use the most general rheonomic integrals of motion we can write down, so that the equations of motion can be solved for any arbitrary potential $V(\phi)$. The importance of this fact is twofold: not only does it provide a mapping between metrics and scalar field potentials for which the former are solutions to Einstein's equations, but it also proves that, for this particular configuration (FLRW plus minimally coupled scalar field), the induced system is completely integrable for any (smooth enough) function $V(\phi)$.

The action of the system under consideration is

$$S = \int d^4x\sqrt{-g}(R + \epsilon\phi_{,\mu}\phi^{,\mu} + 2V(\phi)), \quad (2.6)$$

where R is the Ricci scalar and g is the determinant of the spacetime metric $g_{\mu\nu}$. We also allow for the existence of a phantom field by introducing a pure sign constant $\epsilon = \pm 1$. As is well known, the variation of the above action with respect to $g^{\mu\nu}$ yields the equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \quad (2.7)$$

with

$$T_{\mu\nu} = \epsilon\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}(\epsilon\phi^{,\kappa}\phi_{,\kappa} - 2V(\phi))g_{\mu\nu} \quad (2.8)$$

being the energy-momentum tensor for the matter part. The variation of Eq. (2.6) with respect to the scalar field leads to the Klein-Gordon equation (with arbitrary potential)

$$\epsilon\Box\phi - V'(\phi) = 0, \quad (2.9)$$

where the prime denotes differentiation with respect to the field ϕ.

It can be easily shown that assuming that the metric has the form

$$ds^2 = -N(t)^2dt^2 + a(t)^2\left(\frac{1}{1-kr^2}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\right) \quad (2.10)$$

leads to the necessity that the scalar field is spatially homogeneous, i.e., $\phi = \phi(t)$. The set of equations (2.7)–(2.9), when reduced by the above demands, is equivalent to the Euler-Lagrange equations derived by the Lagrangian

$$L = \frac{2a^2}{n}(a^2V(\phi) - 3k)(-6\dot{a}^2 + \epsilon a^2\dot{\phi}^2) - n, \quad (2.11)$$

with the dots indicating the time derivatives. Note that we have written the Lagrangian in the form (2.2), and the scaled lapse function n is related to the original N appearing in Eq. (2.10) by

$$N = \frac{n}{2a(a^2V(\phi) - 3k)}. \quad (2.12)$$

We choose to start working in the lapse parametrization which leads to a Lagrangian with a constant potential $U(q) = 1$, due to the fact that it simplifies the relations for the corresponding integrals of motion. The scaled minisupermetric in this case is

$$G_{\mu\nu} = 4a^2(a^2V(\phi) - 3k)\begin{pmatrix} -6 & 0 \\ 0 & \epsilon a^2 \end{pmatrix} \quad (2.13)$$

and any of its conformal Killing fields can be used to define integrals of motion for the corresponding system.

III. SPATIALLY FLAT FLRW SPACETIME

As we shall see in the next section, although the cases $k = 0$ and $k \neq 0$ can be treated simultaneously, we choose to express the solution for the spatially flat case separately, since its simplicity helps us better understand the implemented methodology.

In the $k = 0$ case, the minisupermetric (2.13) exhibits a homothetic vector

$$\xi = \frac{a}{6}\frac{\partial}{\partial a} \quad (3.1)$$

which is independent of the scalar field potential $V(\phi)$ and satisfies $\pounds_\xi G_{\mu\nu} = G_{\mu\nu}$. This results in the existence of a conserved quantity in phase space that is written as

$$Q = \frac{a}{6}p_a + \int n(t)dt = \frac{a}{6}\frac{\partial L}{\partial \dot{a}} + \int n(t)dt = -\frac{4a^5\dot{a}V(\phi)}{n} + \int n(t)dt. \quad (3.2)$$

Thus, we are led to consider the relation

$$Q = \kappa \quad (3.3)$$

(where κ is a constant) as a first integral for the relevant system of equations of motion.

Before proceeding let us mention a few facts from the theory of constrained systems. The number of true degrees of freedom is found by the relation $\frac{1}{2}(M - 2F - S)$, where M is the dimension of the full phase space, while F and S are the numbers of first- and second-class constraints, respectively. In our case the full phase space is spanned by n, a, ϕ together with the corresponding momenta. At the same time there exist two first-class constraints $p_n \approx 0$ and $\mathcal{H} \approx 0$, both representing the invariance of the action under arbitrary time reparametrizations $t = f(\tilde{t})$, which means that there exists only one true degree of freedom. This can be seen by algebraically solving the constraint equation $\frac{\partial L}{\partial n} = 0$ with respect to n and substituting the result into the two remaining Euler-Lagrange equations. Then, it is found that only one independent equation remains. Thus, the general solution can be obtained by solving any convenient combination of $t = f(a, \phi)$ as the time parameter. This interesting property of constrained systems can be exploited for adopting different and more convenient gauge choices, in order for the system of equations to be integrated. In what follows we appoint ϕ as the time variable t, while n and a are to be derived with the help of the integrals of motion and one of the Euler-Lagrange equations. Of course, the choice $\phi = t$ has as a prerequisite the assumption that ϕ can be considered (at least locally) as an invertible function of time. The latter is guaranteed by the inverse function theorem as long as $\phi(t)$ is differentiable with $\dot{\phi} \neq 0$. As a result, the case of a constant scalar field—which corresponds to the known pure cosmological constant solution—cannot be included in the following analysis. Of course, it can be considered as a separate case, for $\phi = c$ and by selecting the scale factor a as the time parameter. Let us now proceed by fixing the gauge through the choice $\phi(t) = t$. At the same time we express the scaled lapse function $n(t)$ with the help of a new nonconstant function $h(t)$, and we reparametrize the potential—which now can be considered as a function of time $V(t)$—with respect to a new (again nonconstant) function $A(t)$:

$$n(t) = \dot{h}(t), \tag{3.4a}$$

$$V(t) = \frac{(h(t) - \kappa)\dot{h}(t)}{4\dot{A}(t)}. \tag{3.4b}$$

As a result, Eq. (3.3) reduces to a local expression,

$$a(t)^5 \dot{a}(t) - \dot{A}(t) = 0,$$

that can be easily integrated to give

$$a(t) = \pm 6^{1/6}(A(t) + c_1)^{1/6}. \tag{3.5}$$

Substituting the above solution into the Euler-Lagrange equation for $a(t)$ leads to

$$2(A(t) + c_1)\dot{A}(t)\dot{h}(t) + (\kappa - h(t))(\dot{A}(t)^2 - 6\epsilon(A(t) + c_1)^2) = 0, \tag{3.6}$$

which implies that

$$A(t) = \frac{\mu^4}{6}\exp\left(-\int \frac{\dot{h} \pm (\dot{h}^2 + \epsilon(\kappa - h)^2)^{1/2}}{\kappa - h} dt\right) - c_1, \tag{3.7}$$

where μ is a nonzero constant. It can be easily checked that Eqs. (3.4a)–(3.4b) together with Eqs. (3.5) and (3.7) completely solve the system of the Euler-Lagrange equations of the Lagrangian (2.11) for $k = 0$ in the gauge $\phi = t$ [and hence Einstein's equations (2.7)]. The function $h(t)$ remains free, reflecting the arbitrariness of the potential $V(t)$ through Eq. (3.4a).

It can be seen from Eqs. (3.5) and (3.7) that the constant c_1 is not important for the solution, so we might as well consider it to be zero. The same is true for κ, since one only needs to define a new parametrization as $h(t) = \kappa + \exp(\frac{\omega}{2} - 3\int \frac{\epsilon}{\dot{\omega}} dt)$. It can be verified that with this form for $h(t)$, the lapse function $N(t)$ [as given by Eq. (2.12)] together with the scale factor $a(t)$ and the potential $V(t)$ assume the values

$$N(t) = \frac{1}{3}\mu^2 \dot{\omega} e^{3\int \frac{\epsilon}{\dot{\omega}} dt}, \qquad a(t) = \mu^{2/3} e^{\omega/6},$$
$$V(t) = \frac{3(\dot{\omega}^2 - 6\epsilon)e^{-6\int \frac{\epsilon}{\dot{\omega}} dt}}{4\mu^4 \dot{\omega}^2} \tag{3.8}$$

when $\dot{\omega} > 0$ and

$$N(t) = -\frac{2\epsilon\mu^2 e^{-\frac{\omega}{2}}}{\dot{\omega}}, \qquad a(t) = \mu^{2/3} e^{-\int \frac{\epsilon}{\dot{\omega}} dt},$$
$$V(t) = -\frac{e^{\omega}(\dot{\omega}^2 - 6\epsilon)}{8\epsilon\mu^4} \tag{3.9}$$

when $\dot{\omega} < 0$. [We have chosen to adopt the plus solution in Eq. (3.7); the same applies for the minus case with an interchange of the previous relations with respect to the sign of $\dot{\omega}$.] However, it is easy to check that in the second case of Eq. (3.9), if we choose to express ω with respect to a new function $g(t)$, through a relation $\omega = -6\int \frac{\epsilon}{\dot{g}} dt$, then we acquire the first set of relations (3.8) with $g(t)$ in place of $\omega(t)$. Henceforth, given the fact that ω in the solution is arbitrary, without loss of generality we can consider only Eq. (3.8), since the solution can always be brought into this form. The resulting line element with the additional help of a scaling $r \mapsto r\mu^{-2/3}$ and a reparametrization $\mu = \sqrt{3}m$ can be written as

$$ds^2 = -m^4 \dot{\omega}^2 e^{6\int(\epsilon/\dot{\omega})dt} dt^2 + e^{\omega/3}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2), \tag{3.10}$$

and the respective scalar field potential for each $\omega(t)$ is

$$V(t) = \frac{(\dot{\omega}^2 - 6\epsilon)e^{-6\int(\epsilon/\dot{\omega})dt}}{12 m^4 \dot{\omega}^2}. \tag{3.11}$$

So, given any nonconstant function ω, there is a line element (3.10) satisfying the equations of motion with the corresponding potential (3.11).

The solution (3.10) can be further simplified by performing a change in the time variable from t to ω. Since ω is an arbitrary function of t, we choose to invert the relation $\omega(t)$ by the use of an arbitrary function $F(\omega)$ defined as follows:

$$t = \int \sqrt{\epsilon \frac{F'(\omega)}{6}} d\omega, \tag{3.12}$$

with the prime denoting differentiation with respect to the argument ω. The line element (3.10)—with a slight redefinition of the $F(\omega)$ function $[F(\omega) \mapsto F(\omega) - \log m^4]$ that does not alter Eq. (3.12)—can be written as

$$ds^2 = -e^{F(\omega)}d\omega^2 + e^{\omega/3}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2), \tag{3.13}$$

while the potential (3.11) transforms to

$$V(\omega) = \frac{1}{12}e^{-F(\omega)}(1 - F'(\omega)), \tag{3.14}$$

and of course the scalar field that completes the solution is

$$\phi(t) = t \mapsto \phi(\omega) = \int \sqrt{\epsilon \frac{F'(\omega)}{6}} d\omega. \tag{3.15}$$

The set (3.13)–(3.15) satisfies the Einstein plus Klein-Gordon equations in the new time variable ω, with $F(\omega)$ remaining of course an arbitrary function due to the fact that we have not adopted a particular form for the potential. It is to be noted that exactly the same procedure applies if, instead of Eq. (3.12), we consider the time change $t = -\int \sqrt{\epsilon \frac{F'(\omega)}{6}} d\omega$. The only thing that changes is the sign in front of the integral in Eq. (3.15). Thus, for the line element (3.13) and potential (3.14), both $+/-$ solutions for ϕ are valid.

Given the energy-momentum tensor (2.8), it is a well-known fact that the behavior of matter due to the scalar field can be effectively simulated by a perfect fluid, whose energy density and pressure are

$$\rho_\phi(t) = T^{\mu\nu}u_\mu u_\nu, \tag{3.16a}$$

$$P_\phi(t) = \frac{1}{3}T^{\mu\nu}h_{\mu\nu}, \tag{3.16b}$$

where $u_\mu = \frac{\phi_{,\mu}}{\sqrt{-g^{\kappa\lambda}\phi_{,\kappa}\phi_{,\lambda}}}$ is the comoving four-velocity and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the metric of the three-surfaces normal to the direction of u_μ.

In our case, the solution (3.13)–(3.15) leads to the simple expressions

$$\rho_\phi(\omega) = \frac{1}{12}e^{-F(\omega)}, \tag{3.17a}$$

$$P_\phi(\omega) = \frac{1}{12}e^{-F(\omega)}(2F'(\omega) - 1), \tag{3.17b}$$

from which we can deduce the equation of state

$$P_\phi = (2F'(\omega) - 1)\rho_\phi, \tag{3.18}$$

and we can see that the parameter $\gamma_\phi = \frac{P_\phi}{\rho_\phi}$ now contains the arbitrary function $F(\omega)$. This expression allows for the study of general cases for the scalar field potential.

IV. FLRW WITH SPATIAL CURVATURE

Due to the fact that, when $k \neq 0$, the configuration-space vector (3.1) no longer generates a homothecy of $G_{\mu\nu}$, the situation becomes more complicated. However, it can be seen that there exists a conformal vector $\xi = \frac{\partial}{\partial\phi}$ with a corresponding factor $\frac{a^2V'(\phi)}{a^2V(\phi)-3k}$, i.e.,

$$\pounds_\xi G_{\mu\nu} = \frac{a^2V'(\phi)}{a^2V(\phi) - 3k}G_{\mu\nu}. \tag{4.1}$$

Subsequently, the following nonlocal integral of motion can be defined:

$$\begin{aligned} Q &= p_\phi + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k} dt \\ &= \frac{\partial L}{\partial\dot\phi} + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k} dt \\ &= \frac{4\epsilon a^4 \dot\phi(a^2V(\phi) - 3k)}{n} + \int \frac{a(t)^2 n(t) V'(\phi(t))}{a(t)^2 V(\phi(t)) - 3k} dt. \end{aligned} \tag{4.2}$$

It can be straightforwardly checked that the equation $Q = \kappa$ is a first integral of the Klein-Gordon equation (2.9) (or equivalently, of the Euler-Lagrange equation with respect to ϕ).

Again, we choose the gauge $\phi(t) = t$ and parametrize the dependent variables as

$$n(t) = \frac{2\dot h(a^2V - 3k)}{a^2\dot V}, \tag{4.3a}$$

$$V(t) = \int \frac{\dot w}{a^6} dt, \tag{4.3b}$$

with $w(t)$ and $h(t)$ being nonconstant functions of time. Now, the corresponding equation (3.3) reduces to

$$\frac{2\epsilon\dot w}{\dot h} + 2h - \kappa = 0$$

and can be immediately integrated to yield

$$h(t) = \frac{1}{2}\left(\kappa \pm \sqrt{4c_1 + \kappa^2 - 8\epsilon w}\right). \tag{4.4}$$

By choosing to express $w(t)$ as

$$w(t) = \frac{a\dot v}{\dot a} + \frac{1}{8\epsilon}(4c_1 + \kappa^2) - 6v, \tag{4.5}$$

where $v(t)$ is a new function of time, we can substitute Eq. (4.3) together with Eq. (4.4) into the quadratic constraint $\frac{\partial L}{\partial n} = 0$ to get

$$-\frac{6\dot{a}\,\dot{v}}{\epsilon a}+\frac{36v\dot{a}^2}{\epsilon a^2}+3ka^4-6v=0, \tag{4.6}$$

with the solution

$$v(t)=\exp\left(\int\left(\frac{6\dot{a}}{a}-\frac{\epsilon a}{\dot{a}}\right)dt\right)\left(\int\frac{ka^5\exp\left(-\int\left(\frac{6\dot{a}}{a}-\frac{\epsilon a}{\dot{a}}\right)dt\right)}{2\dot{a}}dt+c_2\right), \tag{4.7}$$

where c_2 is an integration constant. Thus, the process is complete and the resulting line element [where for simplicity we set $a(t)=e^{\omega/6}$] can be written as

$$ds^2=\frac{-e^{\omega}\dot{\omega}^2}{36\left(2e^{\omega-6\int(\epsilon/\dot{\omega})dt}\left(c_2+3k\int\frac{\exp\left(6\int(\epsilon/\dot{\omega})dt-\frac{\omega}{3}\right)}{\dot{\omega}}dt\right)-ke^{\frac{2\omega}{3}}\right)}dt^2+e^{\omega/3}\left(\frac{1}{1-kr^2}dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2\right), \tag{4.8}$$

which for $k=0$ can, by an appropriate reparametrization of the integration constant $c_2=\frac{1}{72m^4}$, be brought exactly into the form (3.10). The corresponding scalar field potential for each nonconstant function ω is

$$V(t)=\frac{6e^{-\omega}\left((\dot{\omega}^2-6\epsilon)e^{\omega-6\int(\epsilon/\dot{\omega})dt}\left(c_2+3k\int\frac{\exp\left(6\int(\epsilon/\dot{\omega})-\frac{\omega}{3}dt\right)}{\dot{\omega}}dt\right)+3ke^{\frac{2\omega}{3}}\right)}{\dot{\omega}^2}, \tag{4.9}$$

which again, for $k=0$ and $c_2=\frac{1}{72m^4}$, becomes Eq. (3.11).

By adopting a suitable time change, as in the previous section, the result can be significantly simplified. We perform the transformation [where again we utilize a nonconstant function $S(\omega)$]

$$t=\pm\int\left[\frac{1}{6\epsilon}\left(\frac{S''(\omega)}{S'(\omega)}+\frac{1}{3}\right)\right]^{1/2}d\omega \tag{4.10}$$

(for any of the two signs in the above equation the treatment is exactly the same). Then, the line element (4.8)—with the help of an allowable ($k\neq 0$) redefinition

$$S(\omega)=\exp\left(12k\int e^{F(\omega)-\omega/3}d\omega\right)-\frac{6c_2}{k} \tag{4.11}$$

which leads to the absorption of the nonessential constant c_2—simplifies to

$$ds^2=-e^{F(\omega)}d\omega^2+e^{\omega/3}\left(\frac{1}{1-kr^2}dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2\right), \tag{4.12}$$

while the corresponding potential becomes

$$V(\omega)=\frac{1}{12}e^{-F(\omega)}(1-F'(\omega))+2ke^{-\omega/3}, \tag{4.13}$$

with the scalar field $\phi(\omega)$ [since in the previous gauge $\phi(t)=t$] given now by Eq. (4.10) after the substitution of Eq. (4.11),

$$\phi(\omega)=\pm\int\left[\frac{1}{6\epsilon}\left(F'(\omega)+12ke^{F(\omega)-\omega/3}\right)\right]^{1/2}d\omega, \tag{4.14}$$

where the $+$ or $-$ sign corresponds to the relevant choice of time transformation in Eq. (4.10).

As a result, the set of relations (4.12)–(4.14) satisfies Einstein's equation plus a scalar field for the case $k\neq 0$, with $F(\omega)$ remaining again arbitrary. It can be seen that if one considers $k=0$, then the solution (4.12)–(4.14) becomes exactly (3.13)–(3.15). It is noteworthy that this occurs despite the fact that in the process of deriving the relations (4.12)–(4.14) the assumption $k\neq 0$ has been taken into account [see Eq. (4.11)].

As in the previous case, starting from the relations (3.16a)–(3.16b), we can compute the energy density and the pressure of the matter content in terms of the function $F(\omega)$,

$$\rho_\phi(\omega)=\frac{1}{12}e^{-F(\omega)}+3ke^{-\omega/3}, \tag{4.15a}$$

$$P_\phi(\omega)=\frac{1}{12}e^{-F(\omega)}(2F'(\omega)-1)-ke^{-\omega/3}, \tag{4.15b}$$

leading to the equation of state

$$P_\phi=\left(\frac{2e^{\omega/3}(3F'(\omega)-1)}{3(36ke^{F(\omega)}+e^{\omega/3})}-\frac{1}{3}\right)\rho_\phi. \tag{4.16}$$

Again, we can notice the difference with respect to Eqs. (3.17a)–(3.17b) due to the contribution of k.

V. INCLUSION OF AN ADDITIONAL PERFECT FLUID

All the previous considerations can be slightly modified to consider an additional perfect-fluid matter source together with the scalar field. The extra contribution to the energy-momentum tensor is given by

$$\mathcal{T}_{\mu\nu} = (\rho + P)\tilde{u}_\mu \tilde{u}_\nu + P g_{\mu\nu}, \tag{5.1}$$

where $\tilde{u}^\mu = (1/N(t), 0, 0, 0)$ is the four-velocity of the comoving observer and ρ, P are the energy density and the pressure of the fluid, respectively. In what follows we shall assume a barotropic equation of state of the form $P = \gamma\rho$, with γ being a constant. It is known that the energy-momentum tensor of a perfect fluid (5.1) can be recovered by varying the matter Lagrangian density $\mathcal{L}_m \propto \rho$ with respect to $g_{\mu\nu}$ when a continuity equation $\mathcal{T}^{\mu\nu}{}_{;\nu} = 0$ is assumed to be *a priori* valid [55,56].

The same procedure can also be applied in the minisuperspace approach. By considering a FLRW spacetime, the continuity equation for $\mathcal{T}_{\mu\nu}$ becomes a differential equation that involves ρ, P, and the scale factor a. Substitution of the equation of state that we mentioned leads to the well-known solution

$$\rho = m a^{-3(1+\gamma)}, \tag{5.2}$$

where m is a constant of integration. The addition of an extra term $L_m = -2\sqrt{-g}\rho = -2Nma^{-3\gamma}$ to the minisuperspace Lagrangian can be seen, which correctly reproduces the set of the reduced Einstein's equations. In the parametrization in which the potential is constant, i.e., when we set

$$N = \frac{n}{2(ma^{-3\gamma} - 3ka + a^3 V(\phi))}, \tag{5.3}$$

with n being the new "lapse" function, the aforementioned Lagrangian is written as

$$L = \frac{2a}{n}(ma^{-3\gamma} + a^3 V(\phi) - 3ak)(-6\dot{a}^2 + \epsilon a^2\dot{\phi}^2) - n. \tag{5.4}$$

It is an easy task to verify that the Euler-Lagrange equations of Eq. (5.4) are equivalent to Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} + \mathcal{T}_{\mu\nu} \tag{5.5}$$

when reduced by the ansatz of an FLRW spacetime.

Thus, the corresponding minisuperspace metric we are interested in is

$$G_{\mu\nu} = 4a(ma^{-3\gamma} + a^3 V(\phi) - 3ak)\begin{pmatrix} -6 & 0 \\ 0 & \epsilon a^2 \end{pmatrix}. \tag{5.6}$$

We aim to consider both cases of a spatially flat and a nonflat universe, but since the treatment is quite similar to what has already been done we shall state the results in a more condensed manner.

A. The $k = 0$ case

The vector $\xi = \frac{\partial}{\partial\phi}$ remains a conformal Killing vector satisfying the relation

$$\pounds_\xi G_{\mu\nu} = \frac{a^{3(\gamma+1)}V'(\phi)}{a^{3(\gamma+1)}V(\phi) + m} G_{\mu\nu}, \tag{5.7}$$

and the corresponding nonlocal integral of motion is

$$\begin{aligned} Q &= p_\phi + \int n \frac{a^{3(\gamma+1)}V'(\phi)}{a^{3(\gamma+1)}V(\phi) + m} dt \\ &= \frac{4\epsilon a^{3(1-\gamma)}\dot{\phi}(a^{3(\gamma+1)}V(\phi) + m)}{n} \\ &\quad + \int n \frac{a^{3(\gamma+1)}V'(\phi)}{a^{3(\gamma+1)}V(\phi) + m} dt. \end{aligned} \tag{5.8}$$

It can be seen that $Q = \text{const}$ together with the quadratic constraint equation $\frac{\partial L}{\partial n} = 0$ satisfy the complete set of Euler-Lagrange equations for the case $k = 0$ and thus Einstein's equations.

We proceed in the same manner as before: at first we fix the gauge by setting $\phi(t) = t$, which allows us to easily reparametrize the potential as a time function. Next, a suitable parametrization of the "lapse" n can be chosen so that the equation $Q = \text{const}$ can be turned into a local expression through writing down the integrand as a total derivative. The procedure is almost similar to the case $k \neq 0$ without a fluid. So, for the sake of brevity, we skip the full calculations and present the complete solution of the system in the gauge $\phi = t$:

$$a(t) = e^{\omega/6}, \tag{5.9}$$

$$\begin{aligned} n(t) = {} & \left(\frac{6}{\dot{\omega}} - \frac{\dot{\omega}}{\epsilon}\right)\left(-\frac{1}{2}c_1 e^I - 2\epsilon m e^I \right. \\ & \left. \times \int \frac{e^{-(\frac{1}{2}(\gamma-1)\omega + I)}}{\dot{\omega}} dt + \frac{1}{3} m e^{-\frac{1}{2}(\gamma-1)\omega}\right)^{1/2}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} V(t) = {} & \frac{3e^{-\frac{1}{2}(\gamma+2)\omega}}{2\dot{\omega}^2}\left[\left(c_1 + 4\epsilon m \int \frac{e^{(-\frac{1}{2}(\gamma-1)\omega - I)}}{\dot{\omega}} dt\right)\right. \\ & \left. \times (6\epsilon - \dot{\omega}^2) e^{(\frac{1}{2}\gamma\omega + I)} - 4\epsilon m e^{\frac{\omega}{2}}\right], \end{aligned} \tag{5.11}$$

with c_1 being a constant of integration, $\omega(t)$ an unspecified nonconstant function (reflecting the arbitrariness of the scalar field potential), and I being given by

$$I = \omega - 6\int \frac{\epsilon}{\dot{\omega}} dt. \tag{5.12}$$

Note here that Eq. (5.10) is the scaled lapse and not the one that enters the metric. The latter is given by Eq. (5.3) (with $k = 0$).

As was also done in the previous sections, the above expressions can be significantly simplified when performing a suitable time parametrization. In this case if one adopts the transformation

$$\phi(t) = t = \pm\int \sqrt{\frac{1}{6\epsilon}\left(\frac{\gamma+1}{2} + \frac{S''(\omega)}{S'(\omega)}\right)} d\omega \tag{5.13}$$

with $S(\omega)$ being parametrized as

$$S(\omega) = \exp\left(-6(\gamma+1)m\int e^{F(\omega)-\frac{1}{2}(\gamma+1)\omega} d\omega\right) - \frac{3c_1}{(\gamma+1)m}, \tag{5.14}$$

the emerging line element takes the exact general form of Eq. (3.13); of course, in this case the corresponding potential becomes

$$V(\omega) = \frac{1}{12}e^{-F(\omega)}(1 - F'(\omega)) + \frac{1}{2}(\gamma - 1)me^{-\frac{1}{2}(\gamma+1)\omega} \tag{5.15}$$

while the scalar field $\phi(\omega)$ is given, through Eqs. (5.13) and (5.14), as

$$\phi(\omega) = \pm\int \left[\frac{1}{6\epsilon}\left(F'(\omega) - 6(\gamma+1)me^{F(\omega)-\frac{1}{2}(\gamma+1)\omega}\right)\right]^{1/2} d\omega. \tag{5.16}$$

It can be easily checked that Eqs. (3.13), (5.15), and (5.16) solve Einstein's equation (5.5) with a scalar field plus a perfect fluid for an arbitrary function $F(\omega)$. As a result, even in this case, the system has been fully integrated without having to choose a specific form for the potential.

Once more, with the help of Eqs. (3.16a)–(3.16b), one can derive in this gauge the relations for the energy density ρ_ϕ and the pressure P_ϕ of the scalar field,

$$\rho_\phi = \frac{1}{12}e^{-F(\omega)} - me^{-\frac{1}{2}(\gamma+1)\omega}, \tag{5.17a}$$

$$P_\phi = \frac{1}{12}e^{-F(\omega)}(2F'(\omega) - 1) - \gamma me^{-\frac{1}{2}(\gamma+1)\omega}. \tag{5.17b}$$

On the other hand, one can easily verify that for the perfect fluid

$$\rho = me^{-\frac{1}{2}(\gamma+1)\omega}, \tag{5.18}$$

with the pressure being given of course by $P = \gamma\rho$. The comoving velocities for the perfect fluid $\tilde{u}_\mu$ and the one constructed by the scalar field $u_\mu = \frac{\phi_{,\mu}}{\sqrt{-g^{\kappa\lambda}\phi_{,\kappa}\phi_{,\lambda}}} = e^{F(\omega)/2}$ are by definition the same for the given line element (a possible difference in sign is of no significance since only quadratic expressions appear in the energy-momentum tensor). Thus, one can immediately add the energy densities and pressures to get the net quantities

$$\rho_{\mathrm{tot}} = \rho_\phi + \rho = \frac{1}{12}e^{-F(\omega)}, \tag{5.19a}$$

$$P_{\mathrm{tot}} = P_\phi + P = \frac{1}{12}e^{-F(\omega)}(2F'(\omega) - 1), \tag{5.19b}$$

which are identical to Eqs. (3.17a)–(3.17b). The same result can also be obtained formally by adding the two energy-momentum tensors. The sum of $T_{\mu\nu} + \mathcal{T}_{\mu\nu}$ leads to the same energy-momentum tensor that is obtained in the $k = 0$ case without an additional fluid.

As a result, we can conclude that, as expected in the context of a FLRW geometry, the perfect fluid can always be "absorbed" by the scalar field in the following sense: a system with a perfect fluid (with a linear barotropic equation) and a (minimally coupled) scalar field (5.16) with potential (5.15), exhibits the same dynamical behavior in comparison to another cosmological system possessing a single (minimally coupled) scalar field (3.15) with potential (3.14).

B. The $k \neq 0$ case

In this section we conclude our analysis by considering the open/closed universe cases. The vector $\xi = \frac{\partial}{\partial\phi}$ is once more a conformal Killing vector, for which

$$\pounds_\xi G_{\mu\nu} = \frac{a^{3\gamma+3}V'(\phi)}{-3ka^{3\gamma+1} + a^{3\gamma+3}V(\phi) + m} G_{\mu\nu} \tag{5.20}$$

holds. The nonlocal integral of motion that corresponds to ξ is

$$Q = p_\phi + \int n(t) \frac{a^{3(\gamma+1)}V'(\phi)}{-3ka^{3\gamma+1} + a^{3(\gamma+1)}V(\phi) + m} dt$$
$$= \frac{4\epsilon a^{3(1-\gamma)}\dot{\phi}(-3ka^{3\gamma+1} + a^{3(\gamma+1)}V(\phi) + m)}{n} + \int \frac{na^{3(\gamma+1)}V'(\phi)}{-3ka^{3\gamma+1} + a^{3(\gamma+1)}V(\phi) + m} dt. \tag{5.21}$$

As in all previous cases, the relations $Q = \text{const}$ and $\frac{\partial L}{\partial n} = 0$ are sufficient to completely integrate the system of the Euler-Lagrange equations of motion. In the gauge $\phi = t$ the solution becomes

$$a(t) = e^{\omega/6}, \tag{5.22}$$

$$n(t) = \frac{(\frac{6}{\dot{\omega}} - \frac{\dot{\omega}}{\epsilon})}{\sqrt{6}} \left(2me^{-\frac{1}{2}(\gamma-1)\omega} - 3c_1 e^I - 12\epsilon e^I \int \frac{e^{-I-\frac{1}{2}(\gamma-1)\omega}(m - 3ke^{\frac{1}{6}(3\gamma+1)\omega})}{\dot{\omega}} dt - 6ke^{\frac{2\omega}{3}}\right)^{1/2}, \tag{5.23}$$

$$V(t) = -\frac{3e^{-\frac{1}{2}(\gamma+2)\omega}}{2\dot{\omega}^2}\left[\left(c_1 + 4\epsilon \int \frac{e^{-I-\frac{1}{2}(\gamma-1)\omega}(m - 3ke^{\frac{1}{6}(3\gamma+1)\omega})}{\dot{\omega}} dt\right)(\dot{\omega}^2 - 6\epsilon)e^{I+\frac{1}{2}\gamma\omega} - 12ke^{\frac{1}{6}(3\gamma+4)\omega} + 4me^{\frac{\omega}{2}}\right], \tag{5.24}$$

where c_1 is a constant of integration, I is given again by Eq. (5.12), and $\omega(t)$ remains an arbitrary nonconstant function.

By performing a time transformation and introducing a new function $S(\omega)$

$$\phi(\omega) = t(\omega) = \pm \int \left(\frac{-18S'''(\omega) + 3(3\gamma - 1)S''(\omega) + (3\gamma + 1)S'(\omega)}{18\epsilon((3\gamma + 1)S'(\omega) - 6S''(\omega))}\right)^{1/2} d\omega, \tag{5.25}$$

which we can associate to an another function $F(\omega)$ through the relation

$$S''(\omega) = \frac{1}{6}e^{-(\gamma+\frac{7}{6})\omega}(18c_1 e^{(\gamma+\frac{5}{6})\omega+F(\omega)} + S'(\omega)((3\gamma + 1)e^{(\gamma+\frac{7}{6})\omega} + 72ke^{(\gamma+\frac{5}{6})\omega+F(\omega)} - 36(\gamma + 1)me^{\frac{1}{6}(3\gamma+4)\omega+F(\omega)})$$
$$-12(3\gamma + 1)kS(\omega)e^{(\gamma+\frac{5}{6})\omega+F(\omega)}), \tag{5.26}$$

we are led to line element (4.12). The potential is parametrized with respect to another function $F(\omega)$ as

$$V(\omega) = \frac{1}{12}e^{-F(\omega)}(1 - F'(\omega)) + 2ke^{-\omega/3} + \frac{1}{2}(\gamma - 1)me^{-\frac{1}{2}(\gamma+1)\omega}, \tag{5.27}$$

which, together with the aforementioned metric (4.12) and

$$\phi(\omega) = \pm \int \left[\frac{1}{6\epsilon}(F'(\omega) + 12ke^{-\omega/3+F(\omega)} - 6(\gamma + 1)me^{F(\omega)-\frac{1}{2}(\gamma+1)\omega})\right]^{1/2} d\omega, \tag{5.28}$$

solves the Einstein's equations for a minimally coupled scalar field in the presence of the perfect fluid that we considered. As in all previous cases the function $F(\omega)$ remains free, since we have not chosen a particular scalar field potential.

Of course, the relations regarding the energy density and the pressure of the scalar field can also be derived as functions of ω, namely,

$$\rho_\phi = \frac{1}{12}e^{-F(\omega)} + 3ke^{-\frac{\omega}{3}} - me^{-\frac{1}{2}(\gamma+1)\omega}, \tag{5.29a}$$

$$P_\phi = \frac{1}{12}e^{-F(\omega)}(2F'(\omega) - 1) - ke^{-\omega/3} - m\gamma e^{-\frac{1}{2}(\gamma+1)\omega}, \tag{5.29b}$$

respectively.

Again, the energy density and pressure of the fluid are given by Eq. (5.18) and $P = \gamma\rho$. As in the previous case (and for the same reason), the total quantities ρ_{tot} and P_{tot} can be retrieved by a simple addition,

$$\rho_{\text{tot}} = \rho_\phi + \rho = \frac{1}{12}e^{-F(\omega)} + 3ke^{-\frac{\omega}{3}}, \tag{5.30a}$$

$$P_{\rm tot} = P_\phi + P = \frac{1}{12}e^{-F(\omega)}(2F'(\omega)-1) - ke^{-\omega/3}, \tag{5.30b}$$

and they are identical to Eqs. (4.15a)–(4.15b) obtained in the case $k \neq 0$ of a single scalar field. The same is also true for the total energy-momentum tensor $T_{\mu\nu} + \mathcal{T}_{\mu\nu}$. Henceforth, the statement that we made in the case $k = 0$ also applies here. A system that possesses both a scalar field and a perfect fluid can be simulated by another with just a single appropriate scalar field.

In addition, we can consider an arbitrary number (say, $\nu \in \mathbb{N}$) of perfect fluids, each one satisfying an equation of state of the form

$$P_i = \gamma_i \rho_i, \qquad i = 1, \ldots, \nu. \tag{5.31}$$

The Einstein plus Klein-Gordon system of equations in this case,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} + \sum_{i=1}^{\nu}\mathcal{T}^{(i)}_{\mu\nu}, \tag{5.32a}$$

$$\epsilon\Box\phi(\omega) + \frac{1}{\phi'(\omega)}V'(\omega) = 0, \tag{5.32b}$$

where $\mathcal{T}^{(i)}_{\mu\nu}$ is the energy-momentum tensor of the ith labeled fluid, have a solution given by the line element (4.12) with the scalar field being

$$\phi(\omega) = \pm\int\left[\frac{1}{6\epsilon}(F'(\omega) + 12ke^{-\omega/3+F(\omega)} - 6\sum_{i=1}^{\nu}(\gamma_i+1)m_i e^{F(\omega)-\frac{1}{2}(\gamma_i+1)\omega})\right]^{1/2} d\omega, \tag{5.33}$$

while the corresponding potential is

$$V(\omega) = \frac{1}{12}e^{-F(\omega)}(1-F'(\omega)) + 2ke^{-\omega/3} + \frac{1}{2}\sum_{i=1}^{\nu}(\gamma_i-1)m_i e^{-\frac{1}{2}(\gamma_i+1)\omega}. \tag{5.34}$$

The energy density of each fluid is of course given by

$$\rho_i = m_i e^{-\frac{1}{2}(\gamma_I+1)\omega}, \tag{5.35}$$

with the m_i's being constants of integration. Finally, the expressions for the energy density and pressure of the effective fluid that simulates the behavior of the scalar field are

$$\rho_\phi = \frac{1}{12}e^{-F(\omega)} + 3ke^{-\frac{\omega}{3}} - \sum_{i=1}^{\nu}m_i e^{-\frac{1}{2}(\gamma_i+1)\omega}, \tag{5.36a}$$

$$P_\phi = \frac{1}{12}e^{-F(\omega)}(2F'(\omega)-1) - ke^{-\omega/3} - \sum_{i=1}^{\nu}m_i\gamma_i e^{-\frac{1}{2}(\gamma_i+1)\omega}. \tag{5.36b}$$

It is straightforward to check that the previous relations identically satisfy the system of equations (5.32a)–(5.32b). Thus, we have completed the solution of a general scalar field in the presence of an arbitrary number of perfect fluids.

VI. PARTICULAR SOLUTIONS

In this section, in order to demonstrate the power and applicability of our results, we show how specific models can be studied by using the equations of state (3.18) and (4.16), depending on whether or not we are in the spatially flat case. We do this for three special forms for the equation of state parameter of the total fluid in which our cosmological model is that of a quintessence scalar field without matter source. The cases that we study are (a) a constant equation of state parameter for vanishing and nonvanishing spatial curvature, (b) an exponentially dependent parametric dark energy model, and (c) the logarithmic dark energy model.

A. A constant equation of state parameter

Let us first consider the $k = 0$ case: thus, due to Eqs. (3.17a)–(3.17b), we demand that

$$\frac{P}{\rho} = 2F'(\omega) - 1 = \gamma, \tag{6.1}$$

with γ being a constant different from -1 (since in our definition of F we needed it to be a nonconstant function). Hence, we are led to the solution

$$F(\omega) = \frac{1}{2}(\gamma+1)\omega, \tag{6.2}$$

where we have omitted the integration constant, because—as can be seen from the induced line element (3.13)—it is not essential and can be absorbed with an appropriate coordinate transformation. The corresponding scalar field and potential are given by [in what follows we choose to work just with the plus solution of every $\phi(\omega)$]

$$\phi(\omega) = \frac{1}{2}\sqrt{\frac{1+\gamma}{3\epsilon}}\omega, \quad V(\phi(\omega)) = -\frac{1}{24}(\gamma-1)e^{-\frac{1}{2}(\gamma+1)\omega}, \tag{6.3}$$

which imply the exponential relation $V(\phi) = -\frac{1}{24}(\gamma-1)e^{\epsilon\sqrt{3\epsilon(\gamma+1)}\phi}$. We can go over to the gauge where the lapse function of the metric is one, i.e., we shall adopt the cosmological time variable. The transformation we need is

$$\int e^{F(\omega)/2} d\omega = \tau \Rightarrow \omega = \frac{4}{\gamma+1} \ln\left(\frac{1}{4}(\gamma+1)\tau\right) \quad (6.4)$$

and in this gauge the corresponding solution becomes the well-known power law for the scale factor and the logarithm relation for the scalar field [31],

$$a(\omega) = e^{\omega/6} \Rightarrow a(\tau) \propto \tau^{\frac{2}{3(\gamma+1)}}, \quad \phi(\tau) = \frac{2\ln\left(\frac{1}{4}(\gamma+1)\tau\right)}{\sqrt{3\epsilon(\gamma+1)}}. \quad (6.5)$$

The situation is slightly more complicated in the $k \neq 0$ case. The differential equation at hand is

$$\frac{P}{\rho} = \gamma \Rightarrow (\gamma+1)e^{\omega/3} - 2e^{\omega/3}F'(\omega) + 12(3\gamma+1)ke^{F(\omega)} = 0, \quad (6.6)$$

where now of course ρ and P are given by Eqs. (4.15a)–(4.15b).

First of all, let us check the specific case when $\gamma = -\frac{1}{3}$. The solution of Eq. (6.6) is

$$F(\omega) = \frac{\omega}{3}, \quad (6.7)$$

where once more the constant of integration has been assumed to be zero, since it is not essential for the geometry, as one can see from the line element (4.12). This solution for $\gamma = -\frac{1}{3}$ is distinguished from all other values of γ, for which the corresponding models depend on two parameters, as we shall immediately see.

When $\gamma \neq -\frac{1}{3}$, the general solution of Eq. (6.6) is

$$F(\omega) = \frac{1}{3}\left[\omega - 3\log\left(e^{\frac{1}{6}(3\gamma+1)(\mu-\omega)} - 36k\right)\right], \quad (6.8)$$

where μ is the integration constant which, unlike the previous case, is essential for the line element (4.12). The scalar field and potential become (for simplicity, in what follows we choose to express the results for the standard scalar field, i.e., we set $\epsilon = +1$)

$$\phi(\omega) = -\frac{2\sqrt{3}\sqrt{\gamma+1}\ln\left(e^{\frac{1}{12}(3\gamma+1)(\mu-\omega)} + \sqrt{e^{\frac{1}{6}(3\gamma+1)(\mu-\omega)} - 36k}\right)}{3\gamma+1}, \quad (6.9)$$

$$V(\omega) = -\frac{1}{24}(\gamma-1)e^{\frac{1}{6}(\mu(3\gamma+1)-3(\gamma+1)\omega)}, \quad (6.10)$$

while the line element can be written as

$$ds^2 = -\frac{e^{\omega/3}}{e^{\frac{1}{6}(3\gamma+1)(\mu-\omega)} - 36k} d\omega^2 + \frac{e^{\omega/3}}{1-kr^2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (6.11)$$

It is clear by the form of the above metric that, had we tried to express the result in the cosmological time gauge, it would not be possible to obtain the solution in terms of elementary functions, since

$$\int \left(\frac{e^{\omega/3}}{e^{\frac{1}{6}(3\gamma+1)(\mu-\omega)} - 36k}\right)^{1/2} d\omega = \tau \Rightarrow \tau = \frac{e^{\omega/6}}{\sqrt{-k}}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{-3\gamma-1}; \frac{3\gamma}{3\gamma+1}; \frac{e^{\frac{1}{6}(3\gamma+1)(\mu-\omega)}}{36k}\right). \quad (6.12)$$

where ${}_2F_1(a,b;c;x)$ is the Gauss hypergeometric function. Note that here the $-k$ in the square root does not restrict k to being -1 for this transformation to be valid, since there exist values of the parameters μ, γ, and ω for which ${}_2F_1$ can be purely imaginary.

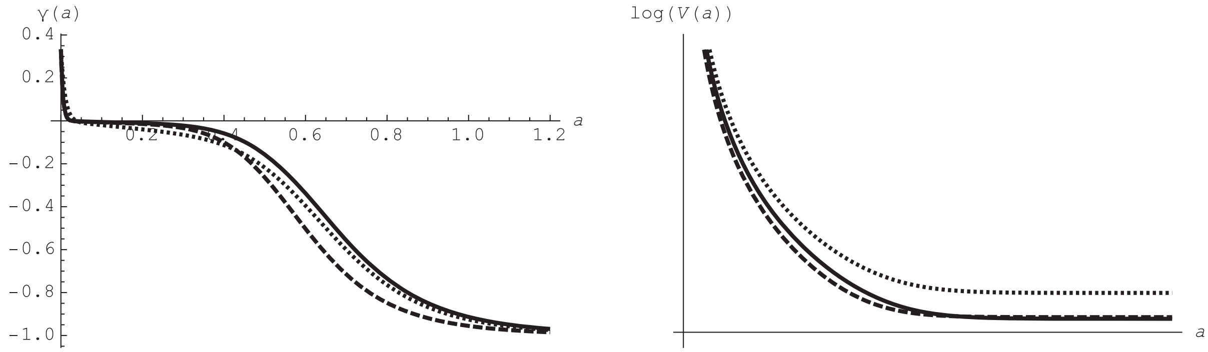
However, useful results can be extracted in the gauge where time is ω, if the various cosmological parameters are expressed in parametric form. It is true that the Hubble H, deceleration q, and jerk j parameters can be given in an arbitrary gauge $N(t)$ by

$$H(t) = \frac{1}{aN}\frac{da}{dt}, \quad (6.13a)$$

$$q(t) = -a\left(\frac{1}{N}\frac{da}{dt}\right)^{-2} \frac{1}{N}\frac{d}{dt}\left(\frac{1}{N}\frac{da}{dt}\right), \quad (6.13b)$$

$$j(t) = \frac{1}{aH(t)^3 N}\frac{d}{dt}\left(\frac{1}{N}\frac{d}{dt}\left(\frac{1}{N}\frac{da}{dt}\right)\right). \quad (6.13c)$$

With the time variable being ω we can easily derive all the previous parameters as functions of the scale factor



(a) EoS parameter $\gamma(a)$

(b) Potential $V(a)$

FIG. 1. (a) The qualitative behavior of the equation-of-state (EoS) parameter $\gamma(a)$ and (b) the respective behavior of the potential $V(a)$. The solid lines are for the constants $(\lambda, \mu, \nu) = (1/10, 1/20, 200)$, the dashed lines are for $(\lambda, \mu, \nu) = (1/20, 1/15, 200)$, and the dotted lines are for the constants $(\lambda, \mu, \nu) = (1/10, 1/5, 100)$.

$a = e^{\omega/6}$, by simply setting $\omega = 6 \ln a$ in the expressions. Of course, the gauge function is given by $N(\omega) = e^{F(\omega)/2}$, where $\omega = 6 \ln a$, and $F(\omega)$ is taken to be Eq. (6.8).

The corresponding relations are

$$H(a) = \sqrt{\frac{e^{\frac{1}{6}(3\gamma+1)\mu}}{36 a^{3(\gamma+1)}} - \frac{k}{a^2}}, \tag{6.14a}$$

$$q(a) = \frac{(3\gamma+1) e^{\frac{1}{6}(3\gamma+1)\mu}}{2(e^{\frac{1}{6}(3\gamma+1)\mu} - 36 k a^{3\gamma+1})}, \tag{6.14b}$$

$$j(a) = \frac{(9\gamma^2 + 9\gamma + 2) e^{\frac{1}{6}(3\gamma+1)\mu}}{2(e^{\frac{1}{6}(3\gamma+1)\mu} - 36 k a^{3\gamma+1})}, \tag{6.14c}$$

and their behavior with respect to the scale factor a and according to various values of the essential constants γ and μ can be easily derived.

In the present epoch, in which by convention we take $a = 1$, we deduce from Eq. (6.14a) that the integration constant μ is related to the Hubble constant; specifically, we find that

$$\mu = \frac{12}{3\gamma+1} \ln[6(H_0 + k)], \qquad \gamma \neq -\frac{1}{3}. \tag{6.15}$$

For $\gamma = -\frac{1}{3}$, the function $H(a)$ can be calculated from Eqs. (6.49a)–(6.49c) with the use of the solution (6.7), again with the substitution $\omega = 6 \ln a$. In this particular case one can see that $H(a) = \frac{1}{6a}$, while $q(a) = j(a) = 0$.

B. An exponential equation-of-state parameter

As a second example we consider that the equation-of-state parameter for the total fluid has the form

$$\gamma(a) = -1 + \frac{\lambda}{\lambda + a^\sigma} e^{-\mu a} + \frac{1}{3} e^{-\nu a}, \tag{6.16}$$

the reason being that, as the current experimental data suggest, in the early universe we must have $\gamma(a \to 0) \simeq \frac{1}{3}$, while in the late universe $\gamma(a \to 1) \simeq -1$ for positive values of σ, μ, ν. However, there will be an epoch in which the equation-of-state parameter will have a linear behavior around $\gamma = 0$. For simplicity, in the following we consider that $\sigma = 6$, where the evolution of Eq. (6.16) is given in Fig. 1(a).

Furthermore, from Eq. (3.18), when we assume that the scalar field has the behavior (6.16) we find that

$$F(\omega) = \frac{1}{2} \lambda \int \frac{\exp(-\mu e^{\omega/6})}{\lambda + e^\omega} d\omega + \mathrm{Ei}(-\nu e^{\omega/6}), \tag{6.17}$$

with $\mathrm{Ei}(x) = -\int_{-x}^{+\infty} \frac{e^{-s}}{s} ds$ being the exponent integral function. The integration constant in $F(\omega)$ has been set to zero.

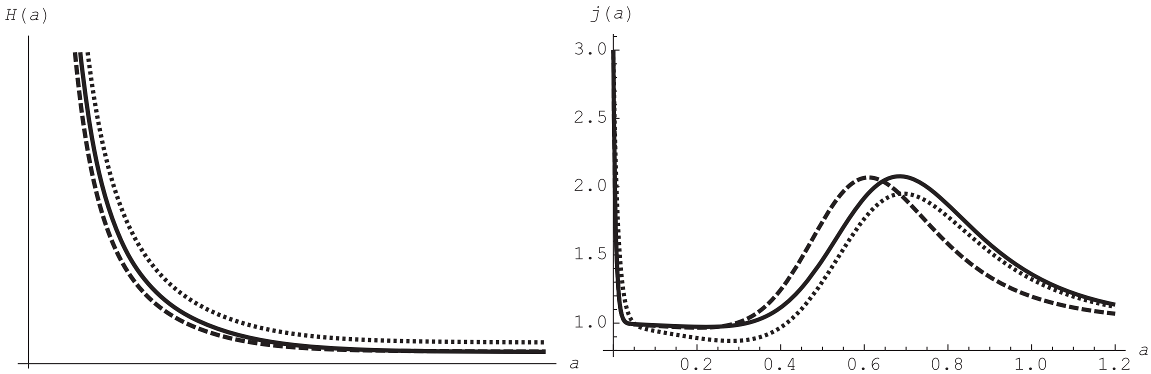
In the case of a quintessence scalar field, from Eq. (3.15) we have that

$$\phi(a) = \pm \int \frac{1}{a} \left(\frac{3\lambda e^{-\mu a}}{a^6 + \lambda} + e^{-\nu a} \right)^{1/2} da, \tag{6.18a}$$

while for the potential we have that

$$V(a) = \frac{1}{72} \left(-\frac{3\lambda e^{-a\mu}}{a^6 + \lambda} - e^{-a\nu} + 6 \right) \exp\left[-\mathrm{Ei}(-\nu a) - 3\lambda \int \frac{e^{-\mu a}}{a(a^6 + \lambda)} da \right], \tag{6.19}$$

where in Fig. 1(b) the evolution of $V(a)$ is given. Here we would like to remark that if we had considered extra fluid terms, then the solutions for the scalar field, i.e., $\phi(a)$, $V(a)$, would be different. But in any case the solution (6.17) would be the same if we assume that Eq. (6.16) describes the equation-of-state parameter for the total fluid.



(a) Hubble parameter $H(a)$ (b) Jerk parameter $j(a)$

FIG. 2. (a) The qualitative behavior of the cosmological parameter $H(a)$ and (b) the respective for $j(a)$. The solid lines are for the constants $(\lambda,\mu,\nu)=(1/10,1/20,200)$, the dashed lines are for $(\lambda,\mu,\nu)=(1/20,1/15,200)$, and the dotted lines are for the constants $(\lambda,\mu,\nu)=(1/10,1/5,100)$.

Finally, the Hubble function and the jerk parameters are given as follows, while the qualitative evolution is given in Fig. 2:

$$H(a)=\frac{1}{6}\left[\exp\left(\mathrm{Ei}(-\nu a)+3\lambda\int\frac{e^{-a\mu}}{a(a^6+\lambda)}da\right)\right]^{-1/2}, \tag{6.20}$$

$$\begin{aligned}j(a)=\frac{e^{-2(\mu+\nu)a}}{2(a^6+\lambda)^2}&[\nu a^{13}e^{(2\mu+\nu)a}+a^{12}e^{2\mu a}(2e^{2\nu a}-3e^{\nu a}+1)+\lambda a^7(3\mu e^{(\mu+2\nu)a}+2\nu e^{(2\mu+\nu)a})\\&+\lambda a^6(4e^{2(\mu+\nu)a}-6e^{(2\mu+\nu)a}+9e^{(\mu+2\nu)a}+2e^{2\mu a})+6\lambda(a^6+\lambda)e^{(\mu+\nu)a}+\lambda^2a(3\mu e^{(\mu+2\nu)a}+\nu e^{(2\mu+\nu)a})\\&+\lambda^2(2e^{2(\mu+\nu)a}-3e^{(2\mu+\nu)a}-9e^{(\mu+2\nu)a}+e^{2\mu a}+9e^{2\nu a})].\end{aligned} \tag{6.21}$$

C. Logarithmic parametric dark energy model

We consider the logarithmic parametric model [57]

$$\gamma(a)=\gamma_0-\gamma_1\ln(a), \tag{6.22}$$

which gives that the unknown function in the line element should be

$$F(\omega)=\frac{1}{24}\omega(12\gamma_0-\gamma_1\omega+12), \tag{6.23}$$

from which we have that the scalar field and cosmological parameters of the model are as follows:

$$\phi(a)=\mp\frac{2(-\gamma_1\frac{\omega}{6}+\gamma_0+1)^{3/2}}{\sqrt{3}\gamma_1}+c_1, \tag{6.24a}$$

$$V(a)=\frac{1}{24}a^{\frac{3}{2}\gamma_1\ln(a)-3(\gamma_0+1)}(\gamma_1\ln(a)-\gamma_0+1), \tag{6.24b}$$

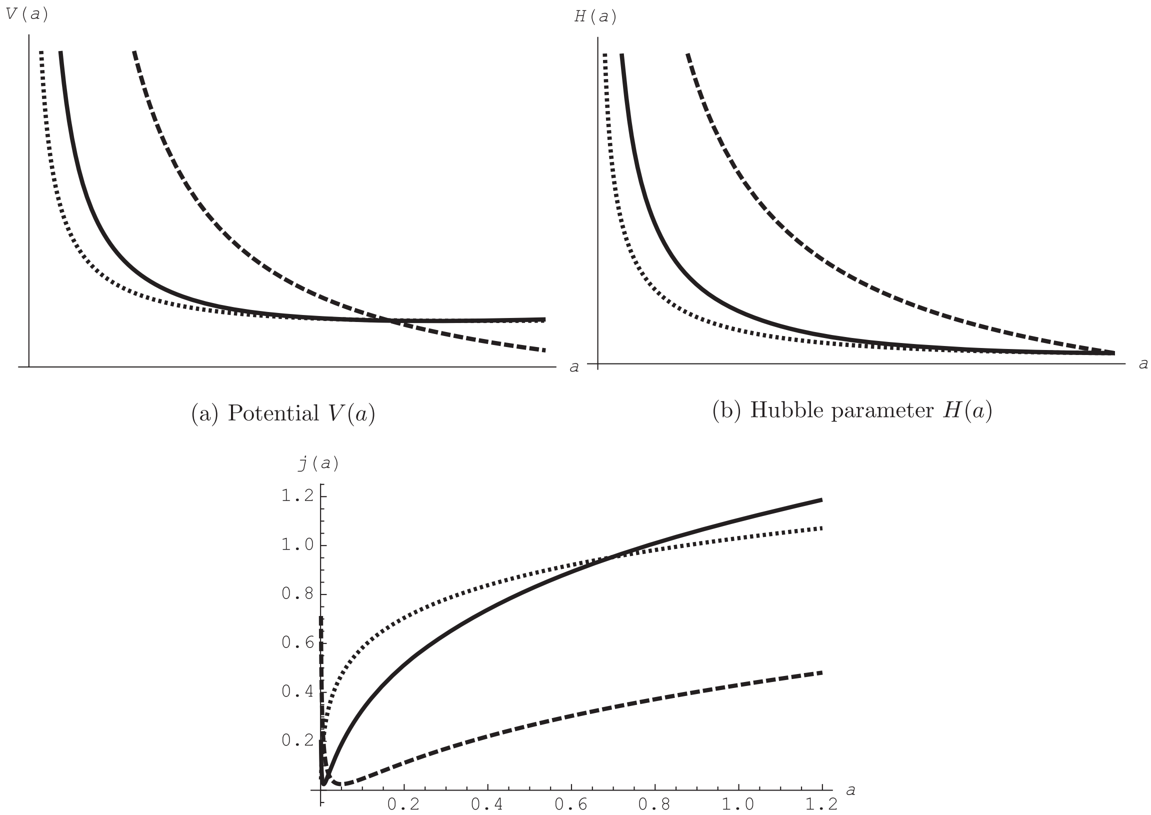
$$H(a)=\frac{1}{6}(a^{-\frac{3}{2}\gamma_1\ln(a)+3(\gamma_0+1)})^{-1/2}, \tag{6.25a}$$

$$\begin{aligned}j(a)=\frac{1}{2}&[9\gamma_1\ln(a)(\gamma_1\ln(a)-2\gamma_0-1)\\&+9\gamma_0(\gamma_0+1)+3\gamma_1+2].\end{aligned} \tag{6.25b}$$

Finally, for $c_1=0$ we have that the functional form of the potential is

$$\begin{aligned}V(\phi)=&\frac{4-6^{1/3}\gamma_1^{2/3}\phi^{2/3}}{48}\\&\times\exp\left[\frac{3(6^{2/3}\gamma_1^{4/3}\phi^{4/3}-4\gamma_0(\gamma_0+2)-4)}{8\gamma_1}\right].\end{aligned} \tag{6.26}$$

The evolutions of the equation-of-state parameter, the scalar field potential $V(\phi)$, and the cosmological parameters $H(a)$, $j(a)$ are given in Figs. 3(a)–3(c).



(a) Potential $V(a)$

(b) Hubble parameter $H(a)$

(c) Jerk parameter $j(a)$

FIG. 3. The qualitative evolution of (a) the potential function $V(a)$, (b) the potential and (c) the jerk parameter for the logarithmic dark energy model. The solid lines are for the parameters $(\gamma_0, \gamma_1) = (-0.99, 0.1)$, the dashed lines are for $(\gamma_0, \gamma_1) = (-0.9, 0.1)$, and the dotted lines are for the constants $(\gamma_0, \gamma_1) = (-0.99, 0.05)$.

VII. CONCLUSIONS

In this paper, in the context of a FLRW geometry, we were able to derive the general solution for a scalar field, with an arbitrary potential, minimally coupled to Einstein's gravity with or without including perfect fluids. This was made possible mainly by exploiting the reparametrization invariance inherent in constrained systems characterized by Eq. (2.1).

The prescription we used was the following. A minisuperspace Lagrangian was constructed in each case we considered. The existence of the Hamiltonian constraint allowed us to define nonlocal integrals of motion corresponding to the conformal symmetries of the minisuperspace. One of these infinite (for a two-dimensional configuration space) conserved quantities, together with the quadratic constraint provided us with enough equations to solve the system. In order to turn the nonlocal expression into a first-order differential equation, we adopted an appropriate gauge and performed specific reparametrizations so as to succeed in integrating both the first integral and the quadratic constraint. As a result, we can now state that the solution we presented encompasses all possible cosmological configurations regarding a minimally coupled scalar field (apart from the specific case $\phi = \text{const}$ that we had to exclude from the analysis, although of course it can be treated in a similar manner).

It could be argued that a general solution of this form, obtained in an arbitrary gauge, might not be of major physical importance; this is mainly due to the fact that the inversion of the function $F(\omega)$, which is essential for expressing the potential as a function of the scalar field, may be transcendental and thus not of particular use. However, as we demonstrated in the examples, it is a relatively easy task to derive the general expressions regarding the effective perfect fluid related to the scalar field. From this point, the association of a physical behavior for a given equation of state, written in parametric form, is just a matter of solving a first-order ordinary differential equation (or simply an integration with respect to ω in the spatially flat case without fluids). Additionally, from a mathematical perspective, it is an interesting fact that this two-dimensional minisuperspace under consideration is integrable for every well-behaved function $V(\phi)$ and an analytic solution can be derived without the need to impose any restrictions on the potential.

Moreover, the solutions we obtained in this work can also prove to be a useful tool for other gravitational configurations. There are several types of theories whose action, under specific transformations, can be mapped to the minimally coupled scalar field of general relativity. For instance, we can mention $f(R)$ or several scalar-tensor theories of gravitation (like Brans-Dicke cosmology). The transformations that one uses to go from the Jordan to the Einstein frame are well known in the literature and we refrain from presenting them here. This link between these theories can be used at any point to transform results from one frame to the other. Nevertheless, we have to note that in the presence of fluids, one should be careful when making this transition. While we have considered that the fluid terms are not interacting with the scalar field, this property is lost under a conformal transformation and interactions among the former and the scalar field shall arise.

For the completeness of the method that we presented here, we plan (in a forthcoming work) to extend it via applications to cosmological models in scalar-tensor theory with perfect fluids which are not interacting with the scalar field. The possible derivation of analytic solutions, among the two different theories/frames, is important in order to better understand the differences between them.

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