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Mechanisms of the isospin-breaking decay $f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$

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Estimated are the contributions of the following mechanisms responsible for the decay $f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$: (1) the contribution of the $a_0^0(980)-f_0(980)$ mixing, $f_1(1285)\to a_0(980)\pi^0\to(K^+K^-+K^0\bar K^0)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$, (2) the contribution of the transition $f_1(1285)\to(K^+K^-+K^0\bar K^0)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$, arising due to the pointlike decay $f_1(1285)\to K\bar K\pi^0$, (3) the contribution of the transition $f_1(1285)\to(K^*\bar K+\bar K^*K)\to(K^+K^-+K^0\bar K^0)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$, where $K^*=K^*(892)$, and (4) the contribution of the transition $f_1(1285)\to(K_0^*\bar K+\bar K_0^*K)\to(K^+K^-+K^0\bar K^0)\pi^0\to f_0(980)\to\pi^+\pi^-\pi^0$, where $K_0^*=K_0^*(800)$ (or κ) and $K_0^*(1430)$. These mechanisms break the conservation of the isospin due to the nonzero mass difference of the K^+ and K^0 mesons. They result in the appearance of the narrow resonance structure in the $\pi^+\pi^-$ mass spectrum in the region of the $K\bar K$ thresholds, with the width $\approx 2m_{K^0}-2m_{K^+}\approx 8$ MeV. The observation of such a structure in experiment is the direct indication on the $K\bar K$ loop mechanism of the breaking of the isotopic invariance. We point out that existing data should be more precise, and it is difficult to explain them using the single specific mechanism from those listed above. Taking the decay $f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ as the example, we discuss the general approach to the description of the $K\bar K$ loop mechanism of the breaking of isotopic invariance.

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I. INTRODUCTION

At the end of the 1970s, a threshold phenomenon known as the mixing of $a_0^0(980)$ and $f_0(980)$ resonances that breaks the isotopic invariance was theoretically discovered in Ref. [1]; see also Ref. [2]. Since that time many new proposals appeared, concerning both searching it and estimating the effects related with this phenomenon [3–26]. Recently, the results of the first experiments on its discovery in the reactions

(a) $\pi^-N\to\pi^-f_1(1285)N\to\pi^-f_0(980)\pi^0N\to\pi^-\pi^+\pi^-\pi^0N$ [27,28],
(b) $J/\psi\to\phi f_0(980)\to\phi a_0(980)\to\phi\eta\pi^0$ [29],
(c) $\chi_{c1}\to a_0(980)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ [29],
(d) $J/\psi\to\gamma\eta(1405)\to\gamma f_0(980)\pi^0\to\gamma 3\pi$ [30],
(e) $J/\psi\to\phi f_0(980)\pi^0\to\phi 3\pi$ [31],
(f) $J/\psi\to\phi f_1(1285)\to\phi f_0(980)\pi^0\to\phi 3\pi$ [31]

have been obtained with the help of detectors VES in Protvino [27,28] and BESIII in Beijing [29–31]. The theoretical considerations concerning the BESIII data [30] on the reaction (d), that is, on the decay $\eta(1405)\to f_0(980)\pi^0\to 3\pi$, were presented in [32–35].

Interest in the $a_0^0(980)-f_0(980)$ mixing [1–35] is primarily due to the fact that the amplitude of the isospin breaking transition $a_0^0(980)\to(K^+K^-+K^0\bar K^0)\to f_0(980)$, caused by the mass difference of the K^+K^- and $K^0\bar K^0$ intermediate states, in the region between $K\bar K$ thresholds turns out to be of the order of $\sqrt{(m_{K^0}-m_{K^+})/m_{K^0}}$ [1] [i.e. of the order of the modulus of difference of the phase space volumes of the K^+K^- and $K^0\bar K^0$ intermediate states: $|\rho_{K^+K^-}(s)-\rho_{K^0\bar K^0}(s)|$, where $\rho_{K^+K^-}(s)=\sqrt{1-4m^2_{K^+}/s}$, $\rho_{K^0\bar K^0}(s)=\sqrt{1-4m^2_{K^0}/s}$, s stands for the square the invariant mass of $K\bar K$ system], but not $(m_{K^0}-m_{K^+})/m_{K^0}$, i.e., by the order of magnitude greater than it could be expected from the naive considerations. It is natural to expect the relative magnitude of the isospin violation to be suppressed outside the $K\bar K$ threshold region, i.e., at the level of $(m_{K^0}-m_{K^+})/m_{K^0}$. To the first approximation, one can neglect these not really calculable contributions. Thus, in corresponding reactions the $a_0^0(980)-f_0(980)$ mixing has to manifest itself in the form of the narrow peaks (with the width of about 10 MeV) in the mass spectra of the final $\pi^+\pi^-$ or $\eta\pi^0$ mesons.

The narrow resonancelike structure breaking of the isotopic invariance have been observed in the $\pi^+\pi^-$ and $\eta\pi^0$ mass spectra in all the above reactions (a)–(f). At the same time, the very large isospin breaking effects discovered in the decays $f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ and $\eta(1405)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ in the reactions (a), (d), and (f) are indicative of the more general $K\bar K$ loop mechanism of the isospin breaking in these decays. Of course, the data need further confirmation.

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In the present work, we study the mechanisms which could be responsible for the isospin breaking decay $f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$. The paper is organized as follows. The experimental data on this decay are presented in Sec. II. In Sec. III, the estimates are given of the coupling constants squared of the $f_0(980)$ and $a_0(980)$ resonances, $g^2_{f_0\pi^+\pi^-}$, $g^2_{f_0K^+K^-}$, $g^2_{a_0\eta\pi}$, and $g^2_{a_0K^+K^-}$, obtained by us using the data on the intensities of the $a_0^0(980) - f_0(980)$ mixing, ξ_{fa} and ξ_{af}, measured in the reactions (b) and (c), respectively. These estimates are used in the subsequent analysis. The contribution of the transition $f_1(1285) \to a_0(980)\pi^0 \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ arising due to the $a_0^0(980) - f_0(980)$ mixing is discussed in Sec. IV. The contributions of the following transitions, $f_1(1285) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ arising due to the pointlike decay $f_1(1285) \to K\bar{K}\pi^0$, $f_1(1285) \to (K^*\bar{K} + \bar{K}^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$, where $K^* = K^*(892)$, and $f_1(1285) \to (K_0^*\bar{K} + \bar{K}_0^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980) \to \pi^+\pi^-\pi^0$, where $K_0^* = K_0^*(800)$ (or κ) and $K_0^*(1430)$, are scrutinized in Secs. V, VI, and VII, respectively. Note that here we consider the effect of the isospin violation in the decay $f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ as being due solely to the mass difference of the stable charged and neutral K mesons. In Sec. VIII, the general approach to the description of the $K\bar{K}$ loop mechanism of the violation of the isotopic invariance is discussed. Some general comments about our estimates are given in Sec. IX. The conclusions concerning the role of the considered mechanisms of the decay $f_1(1285) \to \pi^+\pi^-\pi^0$ and the discussion of the further studies are presented in Sec. X.

II. THE DATA ON $f_1(1285) \to \pi^+\pi^-\pi^0$

In the VES experiment [28] on reaction (a), the isotopic symmetry breaking decay $f_1(1285) \to \pi^+\pi^-\pi^0$ was observed, and the ratio

$$\frac{\mathrm{BR}(f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0)}{\mathrm{BR}(f_1(1285) \to \eta\pi^+\pi^-)} = (0.86 \pm 0.16 \pm 0.20)\% \tag{2.1}$$

was measured. From this ratio, taking into account the Particle Data Group (PDG) data [36] on $\mathrm{BR}(f_1(1285) \to \eta\pi^+\pi^-)$, it was found in Ref. [28] (see also [37]) that

$$\mathrm{BR}(f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0) = (0.30 \pm 0.09)\%. \tag{2.2}$$

Taking into account the PDG data [37] on $\mathrm{BR}(f_1(1285) \to a_0^0(980)\pi^0 \to \eta\pi^0\pi^0)$, this results in

$$\frac{\mathrm{BR}(f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0)}{\mathrm{BR}(f_1(1285) \to a_0^0(980)\pi^0 \to \eta\pi^0\pi^0)} = (2.5 \pm 0.9)\%. \tag{2.3}$$

The decay $f_1(1285) \to \pi^+\pi^-\pi^0$ was also observed in the BESIII experiment [31] on reaction (f), and the ratio of the branching fractions

$$\frac{\mathrm{BR}(f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0)}{\mathrm{BR}(f_1(1285) \to a_0^0(980)\pi^0 \to \eta\pi^0\pi^0)} = (3.6 \pm 1.4)\% \tag{2.4}$$

was obtained.

One more indication on the decay $f_1(1285)/\eta(1295) \to \pi^+\pi^-\pi^0$ was obtained in the BESIII experiment [30], together with the data on reaction (d). If one attributes it solely to the $f_1(1285)$ resonance then the following ratio of intensities will be obtained:

$$\frac{\mathrm{BR}(f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0)}{\mathrm{BR}(f_1(1285) \to a_0^0(980)\pi^0 \to \eta\pi^0\pi^0)} = (1.3 \pm 0.7)\%. \tag{2.5}$$

So, according to the data of the first experiments, the portion of the isospin-forbidden decay $f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ relative to the isospin-allowed decay $f_1(1285) \to a_0^0(980)\pi^0 \to \eta\pi^0\pi^0$ could amount to the quantity from one to four percent. This is large for the quantity which, at the first sight, could be naturally expected to have the magnitude at the level of 10^{-4}. The data indicate undoubtedly on the existence of the mechanisms that enhance the intensity of the decay $f_1(1285) \to \pi^+\pi^-\pi^0$. Also, the characteristic feature of this decay is the dominance of the narrow resonance structure in the $\pi^+\pi^-$ mass spectrum in the vicinity of the $K\bar{K}$ thresholds [28,31]. Notice that the enhancement of the decay $f_1(1285) \to \pi^+\pi^-\pi^0$ and the narrow structure in the $\pi^+\pi^-$ mass spectrum were expected as being due to the isospin breaking mechanism of the $a_0^0(980) - f_0(980)$ mixing [1,2].

To be specific, when comparing below the theoretical estimates with the experimental data, we will base our treatment on the VES data considering them as average.

III. THE COUPLINGS OF THE $f_0(980)$ AND $a_0(980)$ FROM THEIR MIXING

When calculating the $f_1(1285) \to \pi^+\pi^-\pi^0$ decay probability, we need the values of the coupling constants of the $f_0(980)$) and $a_0(980)$) resonances with the $\pi\pi$, $K\bar{K}$, and $\eta\pi$ channels. Here, we evaluate these coupling constants using the data on the $a_0^0(980) - f_0(980)$ mixing [29].

Such estimation is of interest because earlier it was not realized.

The BESIII collaboration [29] made the measurements of the intensity of the $a_0^0(980)-f_0(980)$ mixing in the decays $J/\psi\to\phi f_0(980)\to\phi a_0(980)\to\phi\eta\pi$ and $\psi'\to\gamma\chi_{c1}\to\gamma a_0(980)\pi^0\to\gamma f_0(980)\pi^0\to\gamma\pi^+\pi^-\pi^0$. As a result, the intensities ξ_{fa} and ξ_{af} of the transitions $f_0(980)\to a_0^0(980)$ and $a_0^0(980)\to f_0(980)$, respectively, were obtained:

$$\xi_{fa}=\frac{\mathrm{BR}(J/\psi\to\phi f_0(980)\to\phi a_0^0(980)\to\phi\eta\pi^0)}{\mathrm{BR}(J/\psi\to\phi f_0(980)\to\phi\pi\pi)}=(0.60\pm0.20(\mathrm{stat})\pm0.12(\mathrm{sys})\pm0.26(\mathrm{para}))\%, \tag{3.1}$$

$$\xi_{af}=\frac{\mathrm{BR}(\chi_{c1}\to a_0^0(980)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0)}{\mathrm{BR}(\chi_{c1}\to a_0^0(980)\pi^0\to\eta\pi^0\pi^0)}=(0.31\pm0.16(\mathrm{stat})\pm0.14(\mathrm{sys})\pm0.03(\mathrm{para}))\%. \tag{3.2}$$

The information concerning the denominators of Eqs. (3.1) and (3.2) was taken in Ref. [29] from the works [38] and [36], respectively. Since the $a_0^0(980)-f_0(980)$ mixing is determined mainly by the contribution of the $K\bar K$ loops [1,2,17], we take in what follows [39]

$$\xi_{fa}=\frac{\mathrm{BR}(f_0(980)\to K\bar K\to a_0^0(980)\to\eta\pi^0)}{\mathrm{BR}(f_0(980)\to\pi\pi)}, \tag{3.3}$$

$$\xi_{af}=\frac{\mathrm{BR}(a_0^0(980)\to K\bar K\to f_0(980)\to\pi^+\pi^-)}{\mathrm{BR}(a_0^0(980)\to\eta\pi^0)}, \tag{3.4}$$

where

$$\mathrm{BR}(f_0(980)\to K\bar K\to a_0^0(980)\to\eta\pi^0)=\int\left|\frac{\sqrt{s}M_{a_0^0f_0}(s)}{D_{a_0^0}(s)D_{f_0}(s)-sM^2_{a_0^0f_0}(s)}\right|^2\frac{2s\Gamma_{a_0^0\to\eta\pi^0}(s)}{\pi}d\sqrt{s}, \tag{3.5}$$

$$\mathrm{BR}(a_0^0(980)\to K\bar K\to f_0(980)\to\pi^+\pi^-)=\int\left|\frac{\sqrt{s}M_{a_0^0f_0}(s)}{D_{a_0^0}(s)D_{f_0}(s)-sM^2_{a_0^0f_0}(s)}\right|^2\frac{2s\Gamma_{f_0\to\pi^+\pi^-}(s)}{\pi}d\sqrt{s}, \tag{3.6}$$

$$\mathrm{BR}(f_0(980)\to\pi\pi)=\int\frac{2s\Gamma_{f_0\to\pi\pi}(s)}{\pi|D_{f_0}(s)|^2}d\sqrt{s}, \tag{3.7}$$

$$\mathrm{BR}(a_0^0(980)\to\eta\pi^0)=\int\frac{2s\Gamma_{a_0^0\to\eta\pi^0}(s)}{\pi|D_{a_0^0}(s)|^2}d\sqrt{s}. \tag{3.8}$$

In the above expressions, $D_r(s)$ is the inverse propagator of the unmixed resonance r $[r=a_0^0(980),f_0(980)]$ with the mass m_r,

$$D_r(s)=m_r^2-s+\sum_{ab}[\mathrm{Re}\Pi_r^{ab}(m_r^2)-\Pi_r^{ab}(s)], \tag{3.9}$$

$ab=(\eta\pi^0,K^+K^-,K^0\bar K^0,\eta'\pi^0)$ for $r=a_0^0(980)$ and $ab=(\pi^+\pi^-,\pi^0\pi^0,K^+K^-,K^0\bar K^0,\eta\eta)$ for $r=f_0(980)$; s is the square of the invariant mass of the system ab; $\Pi_r^{ab}(s)$ stands for the diagonal matrix element of the polarization operator of the resonance r corresponding to the contribution of the ab intermediate state [40]; at $s>(m_a+m_b)^2$,

$$\mathrm{Im}\Pi_r^{ab}(s)=\sqrt{s}\Gamma_{r\to ab}(s)=\frac{g^2_{rab}}{16\pi}\rho_{ab}(s), \tag{3.10}$$

where g_{rab} is the coupling constant of r with ab, $\rho_{ab}(s)=\sqrt{s-m_{ab}^{(+)2}}\sqrt{s-m_{ab}^{(-)2}}/s$ and $m_{ab}^{(\pm)}=m_a\pm m_b$. The expressions for $\Pi_r^{ab}(s)$ in different domains of s are given in Appendix A. The propagators of the scalar resonances $1/D_{a_0^0}(s)$ and $1/D_{f_0}(s)$ constructed with taking into account the finite width corrections [see Eqs. (3.9), (3.10), (A1)–(A4)] satisfy the Källén-Lehman representation and, due to this fact, provide the normalization of the total decay probability to unity: $\sum_{ab}BR(r\to ab)=1$ [41].

The amplitude of $a_0^0(980)-f_0(980)$ mixing, $\sqrt{s}M_{a_0^0f_0}(s)$, in Eqs. (3.5) and (3.6) is determined by the sum of the one-loop diagrams $a_0^0(980)\to K^+K^-\to f_0(980)$ and $a_0^0(980)\to K^0\bar K^0\to f_0(980)$ and, by taking into account the isotopic symmetry for the coupling constants, can be written in the form [1,2,17]

$$\sqrt{s}M_{a_0^0f_0}(s)=\frac{g_{a_0^0K^+K^-}g_{f_0K^+K^-}}{16\pi}\left[i[\rho_{K^+K^-}(s)-\rho_{K^0\bar K^0}(s)]-\frac{\rho_{K^+K^-}(s)}{\pi}\ln\frac{1+\rho_{K^+K^-}(s)}{1-\rho_{K^+K^-}(s)}+\frac{\rho_{K^0\bar K^0}(s)}{\pi}\ln\frac{1+\rho_{K^0\bar K^0}(s)}{1-\rho_{K^0\bar K^0}(s)}\right], \tag{3.11}$$

where $\rho_{K\bar{K}}(s) = \sqrt{1 - 4m_K^2/s}$ at $\sqrt{s} \geq 2m_K$; if $\sqrt{s} \leq 2m_K$ then $\rho_{K\bar{K}}(s)$ should be replaced by $i|\rho_{K\bar{K}}(s)|$. In the energy domain with the width of about 8 MeV between K^+K^- and $K^0\bar{K}^0$ thresholds one has

$$|\sqrt{s}M_{a_0^0 f_0}(s)| \approx \left|\frac{g_{a_0^0K^+K^-}g_{f_0K^+K^-}}{16\pi}\right|\sqrt{\frac{m_{K^0}^2 - m_{K^+}^2}{m_{K^0}^2}} \approx 0.127\left|\frac{g_{a_0^0K^+K^-}g_{f_0K^+K^-}}{16\pi}\right|. \quad (3.12)$$

When $\sqrt{s} > 2m_{K^0}$ and when $\sqrt{s} < 2m_{K^+}$ the quantity $|\sqrt{s}M_{a_0^0 f_0}(s)|$ drops sharply, so that the integrands in Eqs. (3.5) and (3.6) become the narrow resonance peaks located near the $K\bar{K}$ thresholds.

Upon inserting the central values of ξ_{fa} and ξ_{af} from Eqs. (3.1) and (3.2), respectively, to the left-hand side of the expressions (3.3) and (3.2) one obtains some equations for the coupling constants of the $a_0^0(980)$ and $f_0(980)$ resonances which can be solved numerically. When doing this in such a way, we have obtained the following estimates:

$$\frac{g_{f_0\pi\pi}^2}{16\pi} \equiv \frac{3}{2}\frac{g_{f_0\pi^+\pi^-}^2}{16\pi} = 0.098\ \text{GeV}^2, \quad (3.13)$$

$$\frac{g_{f_0K\bar{K}}^2}{16\pi} \equiv 2\frac{g_{f_0K^+K^-}^2}{16\pi} = 0.4\ \text{GeV}^2, \quad (3.14)$$

$$\frac{g_{a_0^0\eta\pi^0}^2}{16\pi} = 0.2\ \text{GeV}^2, \quad (3.15)$$

$$\frac{g_{a_0^0K\bar{K}}^2}{16\pi} \equiv 2\frac{g_{a_0^0K^+K^-}^2}{16\pi} = 0.5\ \text{GeV}^2. \quad (3.16)$$

When so doing, we fix the masses of the a_0^0 and f_0 resonances to be $m_{a_0^0} = 0.985$ GeV and $m_{f_0} = 0.985$ GeV, while the relations of the $q^2\bar{q}^2$ model, $g_{a_0^0\eta'\pi^0}^2 = g_{a_0^0\eta\pi^0}^2$ and $g_{f_0\eta\eta}^2 = g_{f_0K^+K^-}^2$, see, e.g., Refs. [2,42], are invoked for the estimates of their couplings with the $\eta'\pi^0$ and $\eta\eta$ channels. The integration in Eqs. (3.5) and (3.6) is made over the region from 0.9 to 1.05 GeV, while the corresponding integration interval in Eqs. (3.7) and (3.8) is from the $\pi\pi$ and $\eta\pi^0$ thresholds, respectively, to 1.3 GeV.

Using Eqs. (3.12), (3.14), and (3.16), one obtains that the "mass" of the $a_0(980) - f_0(980)$ transition $|M_{a_0^0 f_0}(4m_K^2)| \approx 28$ MeV. The mass spectra for the isospin-violating and isospin-conserving decays of the $f_0(980)$ and $a_0^0(980)$ resonances evaluated as the functions of $\sqrt{s}$ at the earlier found magnitudes of coupling constants are shown in Figs. 1 and 2. The curves in Fig. 1 correspond to the integrands in Eqs. (3.5) and (3.6). The curves in Fig. 2 correspond to the integrands in Eqs. (3.7) and (3.8),

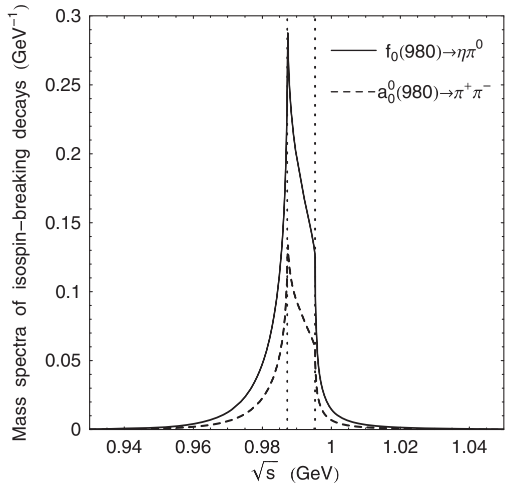

FIG. 1. Mass spectra in the isospin-violating decays $f_0(980) \to \eta\pi^0$ and $a_0(980) \to \pi^+\pi^-$, caused by the $a_0^0(980) - f_0(980)$ mixing. The solid and dashed lines are generally similar to each other. The dotted vertical lines show the locations of the K^+K^- and $K^0\bar{K}^0$ thresholds.

and to the analogous expressions for the mass spectra of the decays into $K\bar{K}$. The shown spectra look rather usual.

There are a sizable number of works devoted to the evaluation, estimation, and determination from the fits of the square of the coupling constants of the $f_0(980)$ and $a_0(980)$ resonances with the $\pi\pi$, $K\bar{K}$ and $\eta\pi$ channels, see, e.g., Refs. [1,2,19,20,40–48] (this list does not pretend on completeness). The spectrum of their possible values is rather wide, so that the magnitudes of couplings determined from different reactions by different methods agree within the factor of 2 or greater. The values (3.13)–(3.16) occupy some average position among those cited in the literature, so it seems to us to be natural to use them as the guide in the subsequent analysis.

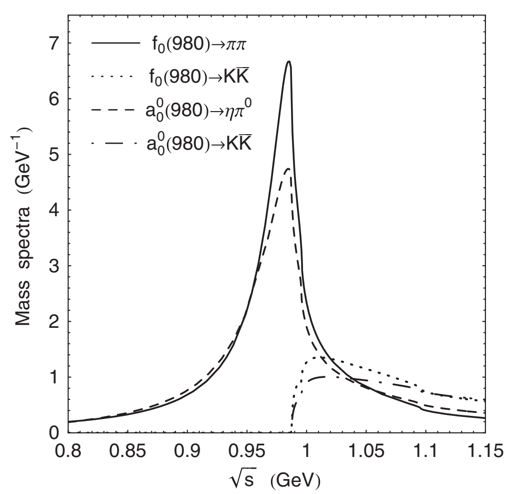

FIG. 2. Mass spectra in the isospin conserving decays of the $f_0(980)$ and $a_0^0(980)$ resonances.

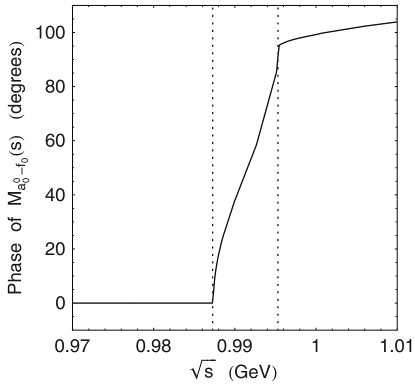

FIG. 3. The phase of the $a_0^0(980) - f_0(980)$ mixing amplitude $M_{a_0^0 f_0}(s)$ [See Eq. (3.11)].

Notice that the phase of the amplitude of the $a_0^0(980) - f_0(980)$ mixing, $M_{a_0^0 f_0}(s)$, in the region between K^+K^- and $K^0\bar{K}^0$ thresholds changes by about 90° [see Eq. (3.11) and Fig. 3]. This fact is crucial for the observation of the $a_0^0(980) - f_0(980)$ mixing effect in polarization experiments [17,18]. A similar sharp and large variation of the phase of the amplitude $f_1(1285) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0$ in the $f_0(980)$ channel takes place for all mechanisms of the decay $f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ considered below. This fact should be also taken into account in suitable polarization and interference experiments.

IV. $a_0^0(980) - f_0(980)$ MIXING IN THE $f_1(1285) \to \pi^+\pi^-\pi^0$ DECAY

Let us calculate the widths of the decays $f_1(1285) \to a_0^0(980)\pi^0 \to \eta\pi^0\pi^0$, $f_1(1285) \to a_0(980)\pi \to K\bar{K}\pi$ and the width of the decay $f_1(1285) \to a_0^0(980)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ caused by the $a_0^0(980) - f_0(980)$ mixing [49].

Let us write the $f_1(1285) \to a_0^0(980)\pi^0 \to \eta\pi^0\pi^0$ decay width in the form

$$\Gamma_{f_1\to a_0^0\pi^0\to\eta\pi^0\pi^0} = \frac{1}{3}\Gamma_{f_1\to a_0\pi\to\eta\pi\pi} = \frac{g^2_{f_1 a_0^0\pi^0} g^2_{a_0^0\eta\pi^0}}{192\pi^3 m^3_{f_1}} \int_{(m_\eta+m_\pi)^2}^{(m_{f_1}-m_\pi)^2} ds \int_{a_-(s)}^{a_+(s)} dt \mathcal{T}(s,t), \quad (4.1)$$

where

$$\mathcal{T}(s,t) = \frac{p^2(s)}{|D_{a_0^0}(s)|^2} + \mathrm{Re}\frac{p(s)p(t)\cos\theta}{D_{a_0^0}(s)D^*_{a_0^0}(t)}, \quad (4.2)$$

$$a_\pm(s) = \frac{1}{2}(m^2_{f_1} + m^2_\eta + 2m^2_\pi - s) + \frac{(m^2_{f_1} - m^2_\pi)(m^2_\eta - m^2_\pi)}{2s} \pm \frac{2m_{f_1}}{\sqrt{s}} p(s)q(s), \quad (4.3)$$

$$p(s) = \sqrt{m^4_{f_1} - 2m^2_{f_1}(s + m^2_\pi) + (s - m^2_\pi)^2}/(2m_{f_1}), \quad (4.4)$$

$$p(t) = \sqrt{m^4_{f_1} - 2m^2_{f_1}(t + m^2_\pi) + (t - m^2_\pi)^2}/(2m_{f_1}), \quad (4.5)$$

$$q(s) = \sqrt{s^2 - 2s(m^2_\eta + m^2_\pi) + (m^2_\eta - m^2_\pi)^2}/(2\sqrt{s}), \quad (4.6)$$

$$p(s)p(t)\cos\theta = \frac{1}{2}(s + t - m^2_{f_1} - m^2_\eta) + \frac{(m^2_{f_1} + m^2_\pi - s)(m^2_{f_1} + m^2_\pi - t)}{4m^2_{f_1}}, \quad (4.7)$$

s is the square of the invariant mass of the state $\eta\pi^0_1$ and t stands for the square of the invariant mass of the state $\eta\pi^0_2$ in the decay $f_1(1285) \to \eta\pi^0_1\pi^0_2$. The expression $V_{f_1 a_0^0\pi^0} = g_{f_1 a_0^0\pi^0}(\epsilon_{f_1}, p_{\pi^0} - p_{a_0^0})$ is used for the effective vertex of the $f_1(1285)a_0^0(980)\pi^0$ interaction, where ϵ_{f_1} is the four-vector of the $f_1(1285)$ polarization while p_{π^0} and $p_{a_0^0}$ are the four-momenta of π^0 and $a_0^0(980)$, respectively.

The width of the $f_1(1285) \to a_0(980)\pi^0 \to K\bar{K}\pi$ decay in the approximation of isotopic symmetry is written in the following form:

$$\Gamma_{f_1\to a_0\pi\to K\bar{K}\pi} = 6\Gamma_{f_1\to a_0^0\pi^0\to K^+K^-\pi^0} = \frac{g^2_{f_1 a_0^0\pi^0}}{\pi m^2_{f_1}} \int_{2m_{K^+}}^{m_{f_1}-m_{\pi^0}} p^3(s) \frac{2s\Gamma_{a_0^0\to K^+K^-}(s)}{\pi|D_{a_0^0}(s)|^2} d\sqrt{s}. \quad (4.8)$$

The width of the decay $f_1(1285) \to a_0^0(980)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ caused by the $a_0^0(980) - f_0(980)$ mixing is represented by the expression

$$\Gamma_{f_1\to a_0^0\pi^0\to f_0\pi^0\to\pi^+\pi^-\pi^0} = \frac{g^2_{f_1 a_0^0\pi^0}}{6\pi m^2_{f_1}} \int_{0.9\text{ GeV}}^{1.05\text{ GeV}} \left| \frac{\sqrt{s}M_{a_0^0 f_0}(s)}{D_{a_0^0}(s)D_{f_0}(s) - sM^2_{a_0^0 f_0}(s)} \right|^2 \times p^3(s)\frac{2s\Gamma_{f_0\to\pi^+\pi^-}(s)}{\pi} d\sqrt{s}. \quad (4.9)$$

The form of $\pi^+\pi^-$ mass spectrum given by the integrand in Eq. (4.9) is practically indistinguishable from the curves shown in Fig. 1.

As a result of numerical integration Eqs. (4.1), (4.8), and (4.9) we obtain

$$\frac{\Gamma_{f_1\to a_0^0\pi^0\to f_0\pi^0\to\pi^+\pi^-\pi^0}}{\Gamma_{f_1\to a_0^0\pi^0\to\eta\pi^0\pi^0}} = \frac{\mathrm{BR}(f_1\to a_0^0(980)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0)}{\mathrm{BR}(f_1\to a_0^0(980)\pi^0\to\eta\pi^0\pi^0)} \approx 0.29\%, \quad (4.10)$$

$$\frac{\Gamma_{f_1\to a_0\pi\to K\bar{K}\pi}}{\Gamma_{f_1\to a_0\pi\to\eta\pi\pi}} = \frac{\mathrm{BR}(f_1\to a_0(980)\pi\to K\bar{K}\pi)}{\mathrm{BR}(f_1\to a_0(980)\pi\to\eta\pi\pi)} \approx 0.11. \quad (4.11)$$

The magnitude of the ratio (4.10) is close to the central value of ξ_{af} from Eq. (3.2) but approximately by an order of magnitude lower than that resulting from the VES data, see Eq. (2.3); see also Eqs. (2.4) and (2.5). Hence, it is difficult to explain the VES data by the $a_0^0(980)-f_0(980)$ mixing mechanism only. In due turn, the PDG data [37] for the ratio

$$\frac{\mathrm{BR}(f_1\to K\bar{K}\pi)}{\mathrm{BR}(f_1\to a_0(980)\pi\to\eta\pi\pi)} \approx 0.25\pm 0.05 \quad (4.12)$$

do not contradict Eq. (4.11) but indicate that the mechanism $f_1(1285)\to a_0(980)\pi\to K\bar{K}\pi$ could be a nonunique source of the decay $f_1(1285)\to K\bar{K}\pi$.

It is interesting to reveal at what coupling constants of the $a_0^0(980)$ and $f_0(980)$ resonances the ratio

$$\frac{\mathrm{BR}(f_1\to a_0^0(980)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0)}{\mathrm{BR}(f_1\to a_0^0(980)\pi^0\to\eta\pi^0\pi^0)},$$

calculated for the $a_0^0(980)-f_0(980)$ mixing mechanism, can be compatible with the VES data shown in Eq. (2.3), i.e., ≈0.025. Using Eqs. (3.9), (3.10), (4.1)–(4.7), and (4.9) we find that the relation

$$\frac{\mathrm{BR}(f_1\to a_0^0(980)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0)}{\mathrm{BR}(f_1\to a_0^0(980)\pi^0\to\eta\pi^0\pi^0)} = 0.025 \quad (4.13)$$

is fulfilled if

$$\frac{g^2_{f_0\pi\pi}}{16\pi} \equiv \frac{3}{2}\frac{g^2_{f_0\pi^+\pi^-}}{16\pi} = 0.46\ \mathrm{GeV}^2, \quad (4.14)$$

$$\frac{g^2_{f_0K\bar{K}}}{16\pi} \equiv 2\frac{g^2_{f_0K^+K^-}}{16\pi} = 2.87\ \mathrm{GeV}^2, \quad (4.15)$$

$$\frac{g^2_{a_0^0\eta\pi^0}}{16\pi} = 0.48\ \mathrm{GeV}^2, \quad (4.16)$$

$$\frac{g^2_{a_0^0K\bar{K}}}{16\pi} \equiv 2\frac{g^2_{a_0^0K^+K^-}}{16\pi} = 4.97\ \mathrm{GeV}^2, \quad (4.17)$$

i.e., at rather exotic (large) values of the coupling constants. The values of $g^2_{f_0K\bar{K}}/(16\pi)$ and $g^2_{a_0^0K\bar{K}}/(16\pi)$ in Eqs. (4.15) and (4.17) are by the factors of 7 and 10 greater than in Eqs. (3.14) and (3.16), respectively. In this connection, the parameter ξ_{af}, evaluated according to Eq. (3.4), turns out to be 9 times larger than its central experimental value in Eq. (3.2). Due to the very strong coupling of $a_0^0(980)$ with the $K\bar{K}$ channel, the width of the $a_0^0(980)$ peak in the $\eta\pi^0$ mass spectrum turns out to be near 15 MeV in all, and $BR(a_0^0(980)\to\eta\pi^0)$ evaluated over the interval from the $\eta\pi^0$ threshold to 1.3 GeV, reduces to the magnitude of 6.5% only. The magnitude of $BR(f_0(980)\to\pi\pi)$ evaluated over the region from the $\pi\pi$ threshold to 1.3 GeV reduces to approximately 12%.

Since the experimental situation is far from being clear, these estimates, despite the obtained not-too-satisfactory resonance characteristics, allow one to guess the possible role of the $a_0^0(980)-f_0(980)$ mixing mechanism in the decay $f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$. In what follows, we will not base our considerations on the values (4.14)–(4.17).

V. POINTLIKE DECAY $f_1(1285)\to K\bar{K}\pi$

Let us consider the pointlike mechanism of the $f_1(1285)$ decay into the π meson and the S wave $K\bar{K}$ system. Let us write the corresponding effective vertex of the $f_1(1285)K^+K^-\pi^0$ interaction in the form $V_{f_1K^+K^-\pi^0} = g_{f_1K^+K^-\pi^0}(\epsilon_{f_1}, p_{\pi^0})$. One has, assuming the isotopic symmetry,

$$\Gamma_{f_1\to K\bar{K}\pi} = 6\Gamma_{f_1\to K^+K^-\pi^0} = \frac{g^2_{f_1K^+K^-\pi^0}}{4\pi}\int_{2m_{K^+}}^{m_{f_1}-m_{\pi^0}} \frac{p^3(s)\rho_{K^+K^-}(s)}{16\pi m^2_{f_1}}\frac{2\sqrt{s}}{\pi}d\sqrt{s} = \frac{g^2_{f_1K^+K^-\pi^0}}{4\pi}\times 1.46\times 10^{-6}\ \mathrm{GeV}^3. \quad (5.1)$$

For the width of the isospin-breaking transition $f_1(1285)\to(K^+K^-+K^0\bar{K}^0)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ caused by the pointlike decay $f_1(1285)\to K\bar{K}\pi^0$ one gets [see Eq. (3.11)] the expression

$$\Gamma_{f_1\to(K^+K^-+K^0\bar{K}^0)\pi^0\to f_0\pi^0\to\pi^+\pi^-\pi^0} = \frac{g^2_{f_1K^+K^-\pi^0}}{4\pi}\frac{1}{6}\int_{0.9\ \mathrm{GeV}}^{1.05\ \mathrm{GeV}}\left|\frac{\sqrt{s}M_{a_0^0f_0}(s)}{g_{a_0^0K^+K^-}}\right|^2 \times\frac{p^3(s)}{m^2_{f_1}}\frac{2s\Gamma_{f_0\to\pi^+\pi^-}(s)}{\pi|D_{f_0}(s)|^2}d\sqrt{s} = \frac{g^2_{f_1K^+K^-\pi^0}}{4\pi}\times 3.28\times 10^{-9}\ \mathrm{GeV}^3. \quad (5.2)$$

The comparison of Eq. (5.2) with (5.1) gives

$$\frac{\Gamma_{f_1\to(K^+K^-+K^0\bar{K}^0)\pi^0\to f_0\pi^0\to\pi^+\pi^-\pi^0}}{\Gamma_{f_1\to K\bar{K}\pi}} = 0.224\times10^{-2}. \quad (5.3)$$

This value is by approximately 15 times lower than the corresponding central experimental value

$$\frac{\mathrm{BR}(f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0)}{\mathrm{BR}(f_1(1285)\to K\bar{K}\pi)} = 0.033\pm0.010 \quad (5.4)$$

resulted from the VES [28], see Eq. (2.2), and PDG [37] data.

The $\pi^+\pi^-$ mass spectrum in the decay $f_1(1285)\to(K^+K^-+K^0\bar{K}^0)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ whose expression is given by the integrand in Eq. (5.2), looks similar to the curves in Fig. 1. However, it is clear that the pointlike mechanism of the decay $f_1(1285)\to K\bar{K}\pi$ cannot by itself provide the considerable probability of the $f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ transition.

VI. DECAY $f_1(1285)\to(K^*\bar{K}+\bar{K}^*K)\to(K^+K^-+K^0\bar{K}^0)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$

If the meson $f_1(1285)$ decays into $(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi$, then, due to the final state interaction of the K and $\bar{K}$ mesons, i.e., due to the transitions $K^+K^-\to f_0(980)\to\pi^+\pi^-$ and $K^0\bar{K}^0\to f_0(980)\to\pi^+\pi^-$, the isospin-breaking decay $f_1(1285)\to(K^*\bar{K}+\bar{K}^*K)\to(K^+K^-+K^0\bar{K}^0)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ is induced (see Fig. 4). This occurs because the contributions from the K^+K^- and $K^0\bar{K}^0$ pair production are not compensated entirely. Naturally, the compensation is less pronounced in the region $m_{\pi^+\pi^-}$ ($\sqrt{s}$) between the K^+K^- and $K^0\bar{K}^0$ thresholds. Below we shall obtain the estimate for the ratio of the branching fractions of the decays $f_1(1285)\to\pi^+\pi^-\pi^0$ and $f_1(1285)\to K\bar{K}\pi$, caused by the mechanisms graphically represented by Figs. 4 and 5, respectively.

The $f_1(1285)\to K^*\bar{K}$ decay amplitude is determined, in general, by the two independent effective coupling constants. However, at the present state of experimental data the general form of this amplitude is in fact unknown.

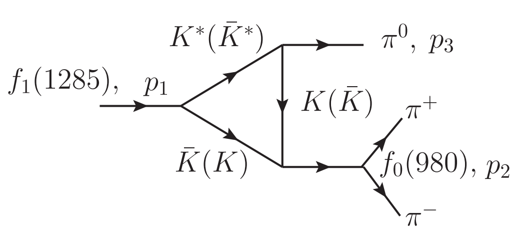

FIG. 4. The diagram of the decay $f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ via the $K^*\bar{K}+\bar{K}^*K$ intermediate states; p_1, p_2, p_3 stand for the four-momenta of particles participating in the reaction, $p_1^2=m_{f_1}^2$, $p_2^2=s=m_{\pi^+\pi^-}^2$ is the invariant mass squared of the $f_0(980)$ or of the final $\pi^+\pi^-$ system, $p_3^2=m_{\pi^0}^2$.

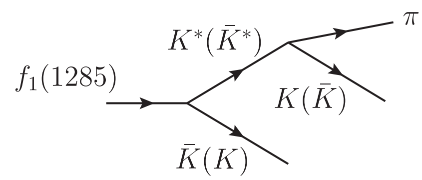

FIG. 5. The diagram of the decay $f_1(1285)\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi$. The four-momenta of $=f_1(1285)$, K, $\bar{K}$, and π are, respectively, p_1, p_K, $p_{\bar{K}}$, and p_π; the four-momenta of the intermediate K^* and $\bar{K}^*$ are k_1 and k_2, respectively.

Hence, for the sake of definiteness we restrict ourselves with the particular expression for it (in the spirit of the effective chiral Lagrangian approach [50–52]) of the form

$$V_{f_1K^*\bar{K}} = g_{f_1K^*\bar{K}}F_{\mu\nu}^{(f_1)}(F^{(K^*)\mu\nu})^*, \quad (6.1)$$

where $F_{\mu\nu}^{(f_1)}=p_{1\mu}\epsilon_{f_1\nu}-p_{1\nu}\epsilon_{f_1\mu}$, $F_{\mu\nu}^{(K^*)}=k_{1\mu}\epsilon_{K^*\nu}-k_{1\nu}\epsilon_{K^*\mu}$, ϵ_{K^*} stands for the polarization four-vector of the K^* meson. Notice that the K^* produced in the result of such transverse interaction carries the unit spin off the mass shell. The $K^*\to K\pi$ decay amplitude is written as

$$V_{K^*K\pi} = g_{K^*K\pi}(\epsilon_{K^*}, p_\pi-p_K), \quad (6.2)$$

where $g_{K^{*+}K^+\pi^0}=-g_{\bar{K}^{*0}\bar{K}^0\pi^0}$, $g_{K^{*+}K^0\pi^+}=\sqrt{2}g_{K^{*+}K^+\pi^0}$. The analogous expressions are valid for the $f_1(1285)\to\bar{K}^*K$ and $\bar{K}^*\to\bar{K}\pi$ decays. According to Eqs. (6.1) and (6.2), the product of the vertices in the amplitude of the diagram shown in Fig. 4 turns out to be of the third order in momenta. But two momenta out of three refer to the momenta of external particles, so that the diagram is convergent (see Appendix B).

The width of the decay $f_1(1285)\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi$ (see Fig. 5) to all charged modes under assumption of the isotopic invariance is written in the form

$$\Gamma_{f_1\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi} = \frac{g^2_{f_1K^{*+}K^-}g^2_{K^{*+}K^+\pi^0}}{4\pi^3m^3_{f_1}} \times\int_{(m_K+m_\pi)^2}^{(m_{f_1}-m_K)^2}dk_1^2\int_{\tilde{a}_-(k_1^2)}^{\tilde{a}_+(k_1^2)}dk_2^2\mathcal{F}(k_1^2,k_2^2), \quad (6.3)$$

where

$$\mathcal{F}(k_1^2,k_2^2) = \frac{-Q_1^2}{|D_{K^*}(k_1^2)|^2}+\mathrm{Re}\frac{-(Q_1,Q_2)}{D_{K^*}(k_1^2)D^*_{K^*}(k_2^2)}, \quad (6.4)$$

$$\tilde{a}_\pm(k_1^2) = \frac{1}{2}(m_{f_1}^2+m_\pi^2+2m_K^2-k_1^2) + \frac{(m_{f_1}^2-m_K^2)(m_K^2-m_\pi^2)}{2k_1^2}\pm\frac{2m_{f_1}}{\sqrt{k_1^2}}\tilde{p}(k_1^2)\tilde{q}(k_1^2), \quad (6.5)$$

$$\tilde{p}(k_1^2) = \sqrt{m_{f_1}^4 - 2m_{f_1}^2(k_1^2 + m_K^2) + (k_1^2 - m_K^2)^2}/(2m_{f_1}), \quad (6.6)$$

$$\tilde{q}(k_1^2) = \sqrt{k_1^4 - 2k_1^2(m_K^2 + m_\pi^2) + (m_K^2 - m_\pi^2)^2}/\left(2\sqrt{k_1^2}\right), \quad (6.7)$$

$1/D_{K^*}(k_{1(2)}^2)$ stands for the propagator of the $K^*(\bar{K}^*)$, $Q_{1\mu} = (p_1, p_K)p_{\pi\mu} - (p_1, p_\pi)p_{K\mu}$, $Q_{2\mu} = (p_1, p_{\bar{K}})p_{\pi\mu} - (p_1, p_\pi)p_{\bar{K}\mu}$; the functions Q_1^2 and (Q_1, Q_2) are given in Appendix B.

The invariant mass of the $K\pi$ pair, $\sqrt{k_1^2}$, in the decay $f_1(1285) \to K\bar{K}\pi$ variates in the interval from 629 to 788 MeV. Since $m_{K^*} \approx 895$ MeV and $\Gamma_{K^*} \approx 50$ [37], then, it is easy to convince, the influence of the width of the virtual intermediate K^* resonance in Eqs. (6.3) and (6.4) turns out to be negligible. So, we set in what follows that $1/D_{K^*}(k_{1(2)}^2) = 1/(m_{K^*}^2 - k_{1(2)}^2)$, i.e., we neglect the width of the $K^*(\bar{K}^*)$ in its propagator.

The numerical integration in Eq. (6.3) gives

$$\Gamma_{f_1\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi} = \frac{g_{f_1K^{*+}K^-}^2 g_{K^{*+}K^+\pi^0}^2}{4\pi^3} \times 0.976 \times 10^{-2}\ \mathrm{GeV}^3. \quad (6.8)$$

The width of the decay $f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ in the case of the mechanism shown in Fig. 4 is represented in the form

$$\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0} = \int_{0.9\ \mathrm{GeV}}^{1.05\ \mathrm{GeV}} \frac{|G_{f_1f_0\pi^0}(s)|^2 p^3(s)}{6\pi m_{f_1}^2} \frac{2s\Gamma_{f_0\to\pi^+\pi^-}(s)}{\pi|D_{f_0}(s)|^2} d\sqrt{s}, \quad (6.9)$$

where $G_{f_1f_0\pi^0}(s)$ is the invariant amplitude which determines the effective vertex of the $f_1f_0\pi^0$ interaction,

$$V_{f_1f_0\pi^0} = G_{f_1f_0\pi^0}(s)(\epsilon_{f_1}, p_3 - p_2). \quad (6.10)$$

The detailed calculation of $G_{f_1f_0\pi^0}(s)$ is give in Appendix B. The function in the integrand in Eq. (6.9) gives the mass spectrum of the $\pi^+\pi^-$ pair, $d\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0}(s)/d\sqrt{s}$. Its sharp enhancement in the region of the $K\bar{K}$ thresholds (see Fig. 6) is determined by the corresponding behavior of the amplitude $G_{f_1f_0\pi^0}(s)$. Let us turn attention to the fact that the shape of the $\pi^+\pi^-$ spectrum in Fig. 6 practically coincides with the corresponding spectrum shown in Fig. 1, caused by the $a_0^0(980) - f_0(980)$ mixing. The integration in Eq. (6.9) gives

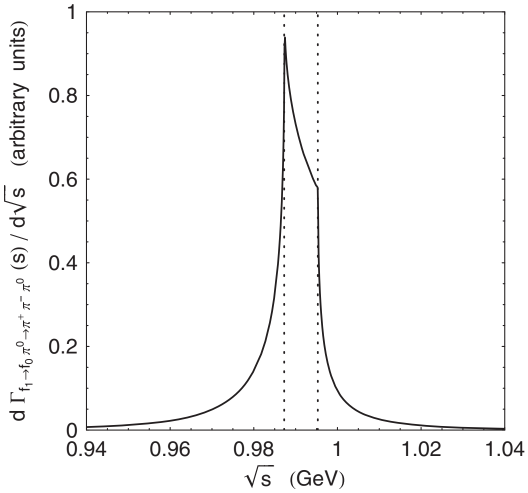

FIG. 6. The $\pi^+\pi^-$ mass spectrum in the decay $f_1(1285) \to (K^*\bar{K} + \bar{K}^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$.

$$\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0} = \frac{g_{f_1K^{*+}K^-}^2 g_{K^{*+}K^+\pi^0}^2}{4\pi^3} \times 0.277 \times 10^{-4}\ \mathrm{GeV}^3. \quad (6.11)$$

Comparing Eq. (6.11) with Eq. (6.8) we obtain

$$\frac{\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0}}{\Gamma_{f_1\to(K^*\bar{K}+\bar{K}^*K)\to K\bar{K}\pi}} = \frac{\mathrm{BR}(f_1 \to f_0\pi^0 \to \pi^+\pi^-\pi^0)}{\mathrm{BR}(f_1 \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi)} = 0.284 \times 10^{-2}. \quad (6.12)$$

Assuming that the whole decay branching $BR(f_1 \to K\bar{K}\pi) = (9.0 \pm 0.4)\%$ [37] results from the $f_1 \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi$ decay mode, the following estimate for $BR(f_1 \to f_0\pi^0 \to \pi^+\pi^-\pi^0)$ follows from Eq. (6.12):

$$\mathrm{BR}(f_1 \to f_0\pi^0 \to \pi^+\pi^-\pi^0) \approx 0.255 \times 10^{-3}. \quad (6.13)$$

This value is approximately 12 times lower than the central experimental value in Eq. (2.2) obtained by VES. If, in addition, one takes into account the relations (4.11) and (4.12), then the estimate Eq. (6.13) should be further divided by approximately 1.8. So, the $f_1(1285) \to (K^*\bar{K} + \bar{K}^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ transition mechanism alone is also insufficient to understand the experimental data.

VII. DECAY $f_1(1285) \to (K_0^*\bar{K} + \bar{K}_0^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$

Let us try to reveal the possible role of the decay mechanism $f_1(1285) \to (K_0^*\bar{K} + \bar{K}_0^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ with the participation of the scalar meson K_0^*. The variant with the $K_0^*(800)$ resonance (or κ) [37] should be rejected. The fact is that for the κ resonance with the mass m_κ which is approximately equal or less than 800 MeV and the width $\Gamma_\kappa \approx (400\text{–}550)$ MeV [37], the shapes of the $K\pi$ and $K\bar{K}$ spectra in the decay $f_1(1285) \to (\kappa\bar{K} + \bar{\kappa}K) \to K\bar{K}\pi$ are literally opposite to those observed in the experiment [53–55]. According to the data on the $f_1(1285) \to K\bar{K}\pi$ decay [37,53–55], there is considerable enhancement in the $K\bar{K}$ spectrum near the $K\bar{K}$ threshold, while in the $K\pi$ mass spectrum one observes the large enhancement near the upper border of the spectrum, i.e., near $m_{f_1} - m_K \approx 788$ MeV. Such a picture agrees well with the $f_1(1285) \to a_0(980)\pi \to K\bar{K}\pi$ decay mechanism and does not contradict to the mechanism $f_1(1285) \to (K^*\bar{K} + \bar{K}^*K) \to K\bar{K}\pi$. On the contrary, the $f_1(1285) \to (\kappa\bar{K} + \bar{\kappa}K) \to K\bar{K}\pi$ mechanism results in the sharp enhancement near the upper border of the $K\bar{K}$ spectrum, i.e., near $m_{f_1} - m_\pi \approx 1147$ MeV, and to the enhancement of the $K\pi$ spectrum close to its threshold. Clearly, such a mechanism cannot be responsible for a sizable portion of the decay $f_1(1285) \to K\bar{K}\pi$. Also, we cannot point to some special enhancement of the decay $f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ due to this mechanism.

Increasing the mass of the K_0^* resonance (pushing it from the $K\pi$ threshold, $m_K + m_\pi \approx 0.629$ GeV) makes the disagreement with the data on the $K\bar{K}$ and $K\pi$ mass spectra less pronounced. The resonance $K_0^*(1430)$ with the mass $m_{K_0^*} \approx 1425$ MeV and the width $\Gamma_{K_0^*} \approx 270$ MeV [37] could be considered as the candidate responsible for the decay $f_1(1285) \to (K_0^*\bar{K} + \bar{K}_0^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$.

Along the lines similar to Sec. VI, first let us calculate the width of the decay $f_1(1285) \to (K_0^*\bar{K} + \bar{K}_0^*K) \to K\bar{K}\pi$ to all charge modes [see Fig. 5, where the resonance $K_0^*(\bar{K}_0^*)$ should be inserted instead of $K^*(\bar{K}^*)$]. The amplitude of the $f_1(1285) \to K_0^*\bar{K}$ transition looks as $V_{f_1K_0^*\bar{K}} = g_{f_1K_0^*\bar{K}}(\epsilon_{f_1}, p_{\bar{K}} - p_{K_0^*})$ [analogously for the $f_1(1285) \to \bar{K}_0^*K$ one]. One has, assuming the isotopic symmetry,

$$\Gamma_{f_1\to(K_0^*\bar{K}+\bar{K}_0^*K)\to K\bar{K}\pi} = \frac{g^2_{f_1K_0^{*+}K^-}g^2_{K_0^{*+}K^+\pi^0}}{16\pi^3 m_{f_1}^3} \times \int_{(m_K+m_\pi)^2}^{(m_{f_1}-m_K)^2} dk_1^2 \int_{\tilde{a}_-(k_1^2)}^{\tilde{a}_+(k_1^2)} dk_2^2 \tilde{\mathcal{F}}(k_1^2, k_2^2), \quad (7.1)$$

where

$$\tilde{\mathcal{F}}(k_1^2, k_2^2) = \frac{|\tilde{p}(k_1^2)|^2}{|D_{K_0^*}(k_1^2)|^2} + \mathrm{Re}\frac{\tilde{p}(k_1^2)\tilde{p}(k_2^2)\cos\tilde{\theta}}{D_{K_0^*}(k_1^2)D^*_{K_0^*}(k_2^2)}, \quad (7.2)$$

$$\tilde{p}(k_1^2)\tilde{p}(k_2^2)\cos\tilde{\theta} = \frac{1}{2}(k_1^2 + k_2^2 - m_{f_1}^2 - m_\pi^2) + \frac{(m_{f_1}^2 + m_K^2 - k_1^2)(m_{f_1}^2 + m_K^2 - k_2^2)}{4m_{f_1}^2}, \quad (7.3)$$

and $1/D_{K_0^*}(k^2_{1(2)})$ is the propagator of the $K_0^*(\bar{K}_0^*)$. In what follows we set $1/D_{K_0^*}(k^2_{1(2)}) = 1/(m^2_{K_0^*} - k^2_{1(2)})$, i.e., we neglect the width of the $K_0^*(\bar{K}_0^*)$ resonance in its propagator. This is a good approximation for the $f_1(1285) \to (K_0^*\bar{K} + \bar{K}_0^*K) \to K\bar{K}\pi$ decay that considerably simplifies the calculations. The numerical integration in Eq. (7.1) gives

$$\Gamma_{f_1\to(K_0^*\bar{K}+\bar{K}_0^*K)\to K\bar{K}\pi} = \frac{g^2_{f_1K_0^{*+}K^-}g^2_{K_0^{*+}K^+\pi^0}}{16\pi^3} \times 0.971 \times 10^{-4}\ \mathrm{GeV}^{-1}. \quad (7.4)$$

Notice that the strong destructive interference occurs between the $K_0^*\bar{K}$ and $\bar{K}_0^*K$ intermediate state contributions in the decay $f_1(1285) \to (K_0^*\bar{K} + \bar{K}_0^*K) \to K\bar{K}\pi$. Namely, the second interfering term in Eq. (7.2) turns out to be large in magnitude and negative in practically the entire physical region of the variables k_1^2 and k_2^2. As a result, the interference reduces the result obtained without interference taking into account by approximately 74%.

The $f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ transition width for the mechanism $f_1(1285) \to (K_0^*\bar{K} + \bar{K}_0^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ [see Fig. 4 in which $K_0^*(\bar{K}_0^*)$ should be substituted instead of $K^*(\bar{K}^*)$] can be represented in the form

$$\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0} = \int_{0.9\ \mathrm{GeV}}^{1.05\ \mathrm{GeV}} \frac{|\tilde{G}_{f_1f_0\pi^0}(s)|^2 p^3(s)}{6\pi m_{f_1}^2} \frac{2s\Gamma_{f_0\to\pi^+\pi^-}(s)}{\pi|D_{f_0}(s)|^2} d\sqrt{s}, \quad (7.5)$$

where $\tilde{G}_{f_1f_0\pi^0}(s)$ is the invariant amplitude that determines the effective vertex of the $f_1f_0\pi^0$ interaction,

$$\tilde{V}_{f_1f_0\pi^0} = \tilde{G}_{f_1f_0\pi^0}(s)(\epsilon_{f_1}, p_3 - p_2). \quad (7.6)$$

The evaluations of $\tilde{G}_{f_1f_0\pi^0}(s)$ and $\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0}$ are analogous to those made in Sec. VI and Appendix B. We will not dwell on them here. We restrict ourselves only by pointing out that the $\pi^+\pi^-$ mass spectrum in the decay $f_1(1285) \to (K_0^*\bar{K} + \bar{K}_0^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ looks similar to the $\pi^+\pi^-$ mass spectra in Figs. 1 and 6 and cite the final result of the $\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0}$ evaluation:

$$\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0} = \frac{g^2_{f_1K_0^{*+}K^-}g^2_{K_0^{*+}K^+\pi^0}}{16\pi^3} \times 0.263\times 10^{-6}\ \mathrm{GeV}^{-1}. \quad (7.7)$$

The comparison of (7.7) with (7.4) gives

$$\frac{\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0}}{\Gamma_{f_1\to(K_0^*\bar K+\bar K_0^*K)\to K\bar K\pi}} = \frac{\mathrm{BR}(f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0)}{\mathrm{BR}(f_1\to(K_0^*\bar K+\bar K_0^*K)\to K\bar K\pi)} = 0.271\times 10^{-2}. \quad (7.8)$$

Since the estimate (7.8) practically coincides with Eq. (6.12), then the statements made about $BR(f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0)$ after Eq. (6.12) are valid in the present case, too. So, the mechanism of the $f_1(1285)\to(K_0^*\bar K+\bar K_0^*K)\to(K^+K^-+K^0\bar K^0)\pi^0\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ transition cannot by itself explain the experimental data.

VIII. GENERAL APPROACH TO DESCRIPTION OF THE $K\bar K$ LOOP BREAKING OF ISOTOPIC INVARIANCE

Let us write the $\pi^+\pi^-$ mass spectrum in the decay $f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ in the form

$$\frac{d\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0}(s)}{d\sqrt{s}} = \frac{1}{16\pi}|\mathcal{M}_{f_1\to f_0\pi^0}(s)|^2 p^3(s) \times\frac{2s\Gamma_{f_0\to\pi^+\pi^-}(s)}{\pi|D_{f_0}(s)|^2}. \quad (8.1)$$

The isospin breaking amplitude $\mathcal{M}_{f_1\to f_0\pi^0}(s)$ can be expanded near the $K\bar K$ threshold into the series in $\rho_{K\bar K}(s)=\sqrt{1-4m_K^2/s}$:

$$\mathcal{M}_{f_1\to f_0\pi^0}(s) = g_{f_0K^+K^-}\{A(s)\times i[\rho_{K^+K^-}(s)-\rho_{K^0\bar K^0}(s)] + B(s)[\rho^2_{K^+K^-}(s)-\rho^2_{K^0\bar K^0}(s)] + O[\rho^3_{K^+K^-}(s)-\rho^3_{K^0\bar K^0}(s)]+\cdots\}. \quad (8.2)$$

The character of the behavior of the functions $|\rho^n_{K^+K^-}(s)-\rho^n_{K^0\bar K^0}(s)|$ near the $K\bar K$ threshold is shown in Fig. 7.

Let us restrict ourselves in Eq. (8.2) by the dominant term proportional to $i[\rho_{K^+K^-}(s)-\rho_{K^0\bar K^0}(s)]$, i.e., let us set

$$\mathcal{M}_{f_1\to f_0\pi^0}(s) = g_{f_0K^+K^-}\times A(s)i[\rho_{K^+K^-}(s)-\rho_{K^0\bar K^0}(s)]. \quad (8.3)$$

The amplitude $A(s)$ contains the information about all possible mechanisms of production of the $K\bar K$ system with isospin $I=1$ in S wave in the process $f_1(1285)\to K\bar K\pi$.

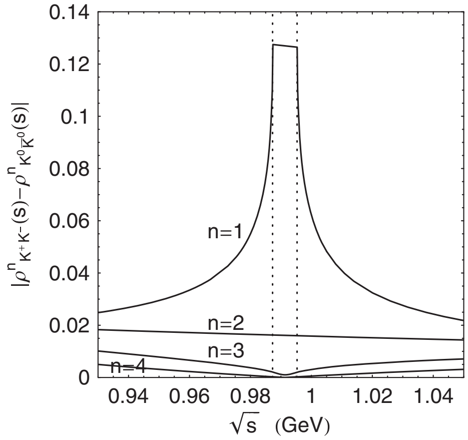

FIG. 7. The functions $|\rho^n_{K^+K^-}(s)-\rho^n_{K^0\bar K^0}(s)|$ for $n=1$, 2, 3, and 4 near the $K\bar K$ thresholds.

From the data on the decay $f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$ one can extract the information about $|A(s)|^2$ in the region above the $K^0\bar K^0$ threshold, between the K^+K^- and $K^0\bar K^0$ thresholds, and below the K^+K^- threshold. A simplest variant of the description of the data on $d\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0}(s)/d\sqrt{s}$ with the help of Eqs. (8.1) and (8.3) can be realized by setting $|A(s)|^2$ to be constant, for instance, upon setting $|A(s)|^2=|A(4m^2_{K^+})|^2$). Then resulting from this fit of the $\pi^+\pi^-$ mass spectrum will be determination of this constant.

The information about $|A(s)|^2$ at $\sqrt{s}>2m_K$ can be obtained from the data on the $K\bar K$ mass spectra measured in the decays $f_1(1285)\to K\bar K\pi$. Unfortunately, the data on these spectra are poor as yet [53–55]. However, possessing the high statistics and good resolution in the invariant mass of $K\bar K$ ($\sqrt{s}$), the simple scheme of obtaining the information about $|A(s)|^2$ at $\sqrt{s}$ above the K^+K^-, or $K^0\bar K^0$, or $K^\pm K^0_S$ thresholds could be consisted in the following.

The $K\bar K$ system, due to the essential restriction of the admissible phase space in the decay $f_1(1285)\to K\bar K\pi$ ($2m_K<\sqrt{s}<2m_K+150$ MeV), should be produced predominantly in S wave. Then, for instance, the K^+K^- spectrum in the decay $f_1(1285)\to K^+K^-\pi^0$ can be represented in the form

$$\frac{d\Gamma_{f_1\to K^+K^-\pi^0}}{d\sqrt{s}} = \frac{2\sqrt{s}}{\pi}\rho_{K^+K^-}(s)p^3(s)|A(s)|^2. \quad (8.4)$$

Fitting the data on $d\Gamma_{f_1\to K^+K^-\pi^0}/d\sqrt{s}$, one can construct the function $|A(s)|^2$. Using its value at the K^+K^- threshold, $|A(4m^2_{K^+})|^2$ (which, for granted, corresponds to the contribution of the S wave) and Eqs. (8.1) and (8.3), one can obtain the estimate for the quantity

$$\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0} = \int \frac{d\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0}(s)}{d\sqrt{s}} d\sqrt{s}$$
$$= \int |A(4m^2_{K^+})|^2 |\rho_{K^+K^-}(s) - \rho_{K^0\bar{K}^0}(s)|^2 \times p^3(s) \frac{g^2_{f_0K^+K^-}}{16\pi} \frac{2s\Gamma_{f_0\to\pi^+\pi^-}(s)}{\pi|D_{f_0}(s)|^2}\sqrt{s}. \quad (8.5)$$

Its comparison with the data on the decay $f_1(1285) \to \pi^+\pi^-\pi^0$ permits one to verify their consistence with the data on the decay $f_1(1285) \to K\bar{K}\pi$ and with the idea of the breaking of isotopic invariance caused by the mass difference of K^+ and K^0 mesons.

Upon using the coupling constants found by us, the integration in Eq. (8.5) over the interval from 0.9 to 1.05 GeV gives

$$\Gamma_{f_1\to f_0\pi^0\to\pi^+\pi^-\pi^0} = |A(4m^2_{K^+})|^2 2.59 \times 10^{-6}\ \text{GeV}^5. \quad (8.6)$$

The proposed approach is applicable to the estimates of other decays of similar sort.

If the isospin-violating amplitude contains in the physical region of kinematic variables (in the region of the $K\bar{K}$ thresholds) the logarithmic (triangle) singularities, as in the case of the $\eta(1405) \to (K^*\bar{K} + \bar{K}^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ decay, then its structure near the $K\bar{K}$ thresholds becomes more sophisticated and the consistency condition of the type of Eq. (8.5) cannot be obtained.

IX. SOME COMMENTS ABOUT ESTIMATES

The effect under consideration is caused by the K meson mass difference and manifests itself in the vicinity of the K^+K^- and $K^0\bar{K}^0$ thresholds, where kaons are near their mass shells. All resonance contributions taken into account by us in the intermediate states of the tree diagrams and in the imaginary parts of the triangle loop diagrams appear also near the mass shells, that is, at $s \approx m^2_{\text{res}}$, etc. It means that the vertex form factors, usually suppressing the hadronic amplitudes, do not play an essential role in the present case. As for the real parts of the triangle loop diagrams, the insertion of the form factor for obtaining their numerical estimate would have some meaning in the case of the divergent diagrams. In our case, the triangle diagrams with the charged and neutral intermediate kaon states are convergent separately, hence their estimates are possible without introduction of any phenomenological form factors. Moreover, the result of compensation of the charged and neutral intermediate states in the channels $f_1(1285) \to (K^*\bar{K} + \bar{K}^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ and $f_1(1285) \to (K^*_0\bar{K} + \bar{K}^*_0K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980) \to \pi^+\pi^-\pi^0$ (i.e., the shape of the basic contribution near the $K\bar{K}$ thresholds) turns out to be practically insensitive to the form factor behavior off the mass shell.

It should be also emphasized that the convergence or divergence of the triangle diagrams as well as of the $K\bar{K}$ loops in the case of the $a^0_0(980) \to (K^+K^- + K^0\bar{K}^0) \to f_0(980)$ transition is not related with the effect under discussion. The sum of the subtraction constants for the contributions of the charged and neutral intermediate states in the dispersion representation for the isospin breaking amplitude should have the natural order of smallness $\sim(m_{K^0} - m_{K^+})$, and it cannot be responsible for the enhancement of the symmetry violation in the vicinity of the K^+K^- and $K^0\bar{K}^0$ thresholds neither in the magnitude nor in the shape.

Let us call attention to the fact that the estimates obtained for the ratios (4.10), (5.3), (6.12), and (7.8), which characterize the isospin breaking for different mechanisms, do not depend on the magnitudes of the coupling constants $g_{f_1a^0_0\pi^0}$, $g_{f_1K^+K^-\pi^0}$, $g_{f_1K^{*+}K^-}$, and $g_{f_1K^{*+}_0K^-}$, respectively. By themselves, these constants are either ill defined or simply unknown. Hence, in order to combine meaningfully the different theoretical mechanisms of the decay $f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$, the considerably improved quality of the data on the main decay channels $f_1(1285) \to \eta\pi\pi$ and $f_1(1285) \to K\bar{K}\pi$ is necessary. The partial wave analysis of the three-particle events is required for the clarification of the relative role of the specific mechanisms in the above channels. In this route, we would persuade the experimenters to measure in the first place the decays $f_1(1285) \to \pi^+\pi^-\pi^0$ and $f_1(1285) \to K^+K^-\pi^0$ simultaneously (at the same facility and in the same experiment) and to obtain the $\pi^+\pi^-$- and K^+K^- mass spectra. As it is explained in Sec. VIII, this would give the possibility of checking the consistency of the data on the $\pi^+\pi^-$ and K^+K^- mass spectra before the detailed partial-wave analysis.

X. CONCLUSION AND OUTLOOK

The phenomenon of the $a^0_0(980) - f_0(980)$ mixing [1] gave the impetus to doing experiments on reactions (a)–(f) which were made by the collaborations VES [27,28] and BESIII [29–31]. In the present work, we show the principal possibility of the estimate of coupling constants of the $a_0(980)$ and $f_0(980)$ resonances using the BESIII data [29] on the $a_0(980) - f_0(980)$ mixing. Notice that the relations among the couplings found in Sec. III agree well with the predictions of the $q^2\bar{q}^2$ model. Interesting for physics and a promising problem is the task of making more precise the BESIII data [29] on reactions (b) and (c) [see Eqs. (3.1) and (3.2)].

We have analyzed in detail four possible mechanisms for the isospin-breaking decay $f_1(1285) \to \pi^+\pi^-\pi^0$:

(1) $f_1(1285) \to a_0(980)\pi^0 \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$,

(2) $f_1(1285) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$,
(3) $f_1(1285) \to (K^*\bar{K} + \bar{K}^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$,
(4) $f_1(1285) \to (K_0^*\bar{K} + \bar{K}_0^*K) \to (K^+K^- + K^0\bar{K}^0)\pi^0 \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$.

Our conclusions from the estimates are the following. The experimental data are difficult to explain by the single specific mechanism from those listed above. On the other hand, these mechanisms are united by the fact that for each of them the $\pi^+\pi^-$ mass spectrum in the decay $f_1(1285) \to \pi^+\pi^-\pi^0$ turns out to be located between the K^+K^- and $K^0\bar{K}^0$ thresholds in view of the $K\bar{K}$ loop mechanism of the isospin breaking. It is clear that the considered mechanisms of the decay $f_1(1285) \to \pi^+\pi^-\pi^0$ underlie the observable isospin breaking phenomenon. It is apparent also that considerable experimental efforts are yet required to eliminate the uncertainties in the available data [for example, it is desirable to measure the various decay modes of $f_1(1285)$ simultaneously at the same experimental setup].

Taking the decay $f_1(1285) \to f_0(980)\pi^0 \to \pi^+\pi^-\pi^0$ as an example we have discussed also the general approach to the description of the $K\bar{K}$ loop mechanism of the breaking of isotopic invariance in the absence of logarithmic singularities.

Since the existing data on the $f_1(1285) \to \pi^+\pi^-\pi^0$ decay probability have a rather large spread [see Eqs. (2.3)–(2.5)], the task of making them to be precise is an extremely interesting and important problem.

Among the numerous production reactions of the $f_1(1285)$ resonance we want to call attention to the reaction of the $f_1(1285)$ production in the central region via the two-pomeron exchange, and to the possibility of the study in this reaction the decay $f_1(1285) \to \pi^+\pi^-\pi^0$,

$$pp \to p_f f_1(1285) p_s \to p_f(\pi^+\pi^-\pi^0)p_s.$$

It is interesting also to study the related reaction $pp \to p_f f_1(1420) p_s \to p_f(\pi^+\pi^-\pi^0)p_s$.

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APPENDIX A: POLARIZATION OPERATORS

The polarization operator $\Pi_r^{ab}(s)$ [see Eqs. (3.9) and (3.10)] can be written as

$$\Pi_r^{ab}(s) = g_{rab}^2 \tilde{B}_0(s; m_a, m_b). \tag{A1}$$

The function $\tilde{B}_0(s; m_a, m_b)$ at $s > m_{ab}^{(+)2}$ looks as

$$\tilde{B}_0(s; m_a, m_b) = \frac{1}{16\pi}\left[\frac{m_{ab}^{(+)} m_{ab}^{(-)}}{\pi s}\ln\frac{m_b}{m_a} + \rho_{ab}(s) \times \left(i - \frac{1}{\pi}\ln\frac{\sqrt{s - m_{ab}^{(-)2}} + \sqrt{s - m_{ab}^{(+)2}}}{\sqrt{s - m_{ab}^{(-)2}} - \sqrt{s - m_{ab}^{(+)2}}}\right)\right], \tag{A2}$$

where $\rho_{ab}(s) = \sqrt{s - m_{ab}^{(+)2}}\sqrt{s - m_{ab}^{(-)2}}/s$, $m_{ab}^{(\pm)} = m_a \pm m_b$ and $m_a \geq m_b$. At $m_{ab}^{(-)2} < s < m_{ab}^{(+)2}$

$$\tilde{B}_0(s; m_a, m_b) = \frac{1}{16\pi}\left[\frac{m_{ab}^{(+)} m_{ab}^{(-)}}{\pi s}\ln\frac{m_b}{m_a} - \rho_{ab}(s)\left(1 - \frac{2}{\pi}\arctan\frac{\sqrt{m_{ab}^{(+)2} - s}}{\sqrt{s - m_{ab}^{(-)2}}}\right)\right], \tag{A3}$$

where $\rho_{ab}(s) = \sqrt{m_{ab}^{(+)2} - s}\sqrt{s - m_{ab}^{(-)2}}/s$. At $s \leq m_{ab}^{(-)2}$

$$\tilde{B}_0(s; m_a, m_b) = \frac{1}{16\pi}\left[\frac{m_{ab}^{(+)} m_{ab}^{(-)}}{\pi s}\ln\frac{m_b}{m_a} + \rho_{ab}(s)\frac{1}{\pi}\ln\frac{\sqrt{m_{ab}^{(+)2} - s} + \sqrt{m_{ab}^{(-)2} - s}}{\sqrt{m_{ab}^{(+)2} - s} - \sqrt{m_{ab}^{(-)2} - s}}\right], \tag{A4}$$

where $\rho_{ab}(s) = \sqrt{m_{ab}^{(+)2} - s}\sqrt{m_{ab}^{(-)2} - s}/s$.

APPENDIX B: TRIANGLE DIAGRAM

The functions Q_1^2 and (Q_1, Q_2) in Eq. (6.4) look as follows:

$$Q_1^2 = \frac{1}{4}[m_\pi^2(m_{f_1}^2 + m_K^2 - k_2^2)^2 + m_K^2(k_1^2 + k_2^2 - 2m_K^2)^2 - (m_{f_1}^2 + m_K^2 - k_2^2)(k_1^2 + k_2^2 - 2m_K^2) \times (k_1^2 - m_\pi^2 - m_K^2)], \tag{B1}$$

$$(Q_1,Q_2)=\frac{1}{8}[2m_\pi^2(m_{f_1}^2+m_K^2-k_2^2)(m_{f_1}^2+m_K^2-k_1^2)+(k_1^2+k_2^2-2m_K^2)^2(m_{f_1}^2+m_\pi^2-k_1^2-k_2^2)-(m_{f_1}^2+m_K^2-k_2^2)(k_1^2+k_2^2-2m_K^2)\times(k_2^2-m_\pi^2-m_K^2)-(m_{f_1}^2+m_K^2-k_1^2)\times(k_1^2+k_2^2-2m_K^2)(k_1^2-m_\pi^2-m_K^2)]. \quad \text{(B2)}$$

The invariant amplitude $G_{f_1f_0\pi^0}(s)$ introduced in Eqs. (6.9) and (6.10) is represented as the sum of the amplitudes corresponding to the charged (c) and neutral (n) intermediate states in the kaon triangle loop in Fig. 4,

$$G_{f_1f_0\pi^0}(s)=G^{(c)}_{f_1f_0\pi^0}(s)+G^{(n)}_{f_1f_0\pi^0}(s). \quad \text{(B3)}$$

In the approximation of the isotopic invariance for coupling constants of the $f_1(1285)$, K^*, and $f_0(980)$ resonances and at $m_{K^{*+}}=m_{K^{*0}}$, the amplitude $G^{(c)}_{f_1f_0\pi^0}(s)$ and $G^{(n)}_{f_1f_0\pi^0}(s)$ differ by only the overall sign and by the masses of the $K^+(K^-)$ and $K^0(\bar{K}^0)$ mesons.

Using Eqs. (6.1) and (6.2), let us write the amplitude of the triangle diagram in Fig. 4 for the charged intermediate kaon states in the following way:

$$V^{(c)}_{f_1f_0\pi^0}=\bar{g}_{f_1f_0\pi^0}C^{(c)}_\mu[(\epsilon_{f_1},p_3)p_{1\mu}-(p_1,p_3)\epsilon_{f_1\mu}], \quad \text{(B4)}$$

where $\bar{g}_{f_1f_0\pi^0}=2(2g_{f_1K^{*+}K^-}2g_{K^{*+}K^+\pi^0}g_{f_0K^+K^-})$ and

$$C^{(c)}_\mu=\frac{i}{(2\pi)^4}\int\frac{k_\mu d^4k}{(k^2-m_{K^*}^2)((p_1-k)^2-m_{K^+}^2)}\times\frac{1}{((k-p_3)^2-m_{K^-}^2)}. \quad \text{(B5)}$$

Expanding the amplitude $C^{(c)}_\mu$ in the momenta of external particles, $C^{(c)}_\mu=p_{1\mu}C^{(c)}_{11}+p_{3\mu}C^{(c)}_{12}$ [56], we can rewrite Eq. (B4) in the form

$$V^{(c)}_{f_1f_0\pi^0}=G^{(c)}_{f_1f_0\pi^0}(s)(\epsilon_{f_1},p_3-p_2) \quad \text{(B6)}$$

and determine $G^{(c)}_{f_1f_0\pi^0}(s)$ in Eq. (B3) as

$$G^{(c)}_{f_1f_0\pi^0}(s)=m_{f_1}^2\bar{g}_{f_1f_0\pi^0}C^{(c)}_{11}/2, \quad \text{(B7)}$$

where the amplitude

$$C^{(c)}_{11}=\frac{1}{4m_{f_1}^2p^2(s)}\{C_0(s;m_{K^*},m_{K^+},m_{K^-})\times[m_\pi^4-m_\pi^2(m_{f_1}^2+s+m_{K^*}^2-m_{K^+}^2)+(m_{K^*}^2-m_{K^+}^2)(m_{f_1}^2-s)]-2m_\pi^2[B_0(m_\pi^2;m_{K^*},m_{K^+})-B_0(s;m_{K^+},m_{K^-})]+[B_0(m_{f_1}^2;m_{K^*},m_{K^+})-B_0(s;m_{K^+},m_{K^-})]\times(m_{f_1}^2+m_\pi^2-s)\}. \quad \text{(B8)}$$

Here, $C_0(s;m_{K^*},m_{K^+},m_{K^-})$ is the amplitude of the triangle loop (see Fig. 4) in the case of the scalar particles, in which as the arguments shown are the square of the virtual invariant mass of the produced $f_0(980)$ resonance (s) and the masses of the particles inside the loop, $B_0(p_i^2;m_a,m_b)-B_0(p_j^2;m_c,m_d)$ are the differences of the amplitudes of the two-point diagrams also for the case of the scalar particles; these differences are related with the functions $\tilde{B}_0(p_i^2;m_a,m_b)$, given in Appendix A, by the relations $B_0(p_i^2;m_a,m_b)-B_0(p_j^2;m_c,m_d)=\tilde{B}_0(p_i^2;m_a,m_b)-\tilde{B}_0(p_j^2;m_c,m_d)-[\ln(m_am_b/m_cm_d)]/(16\pi^2)$; $4m_{f_1}^2p^2(s)=m_{f_1}^4-2m_{f_1}^2(s+m_\pi^2)+(s-m_\pi^2)^2$. At points where $1/p^2(s)$ goes to infinity, the function $C^{(c)}_{11}$ is finite. The numerical evaluation of the amplitude $C_0(s;m_{K^*},m_{K^+},m_{K^-})$ was fulfilled along the lines suggested in Ref. [57]. Notice that the amplitude for the diagram in Fig. 4 does not contain the logarithmic (triangle) singularity in the physical region of the decay $f_1(1285)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$, contrary to the case of the decay $\eta(1405)\to f_0(980)\pi^0\to\pi^+\pi^-\pi^0$, where it is important to take into account the finite width of the $K^*(892)$ meson when calculating [35].

Analogously, for the neutral intermediate states we have $G^{(n)}_{f_1f_0\pi^0}(s)=-m_{f_1}^2\bar{g}_{f_1f_0\pi^0}C^{(n)}_{11}/2$, where $C^{(n)}_{11}$ is obtained from $C^{(c)}_{11}$ upon the substitution $m_{K^+(K^-)}$ by $m_{K^0(\bar{K}^0)}$.

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