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Type II seesaw supersymmetric neutrino model for $\theta_{13} \neq 0$

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Using the type II seesaw approach and properties of discrete flavor symmetry group representations, we build a supersymmetric $A_4 \times A_3$ neutrino model with $\theta_{13} \neq 0$. After describing the basis of this model—which is beyond the minimal supersymmetric Standard Model—with a superfield spectrum containing flavons in $A_4 \times A_3$ representations, we first generate the tribimaximal neutrino mixing which is known to be in agreement with the mixing angles θ_{12} and θ_{23}. Then, we give the scalar potential of the theory where the A_3 discrete subsymmetry is used to avoid the so-called sequestering problem. We next study the deviation from the tribimaximal mixing matrix which is produced by perturbing the neutrino mass matrix with a nontrivial A_4 singlet. Normal and inverted mass hierarchies are discussed numerically. We also study the breaking of A_4 down to Z_3 in the charged lepton sector, and use the branching ratio of the decay $\tau \to \mu\mu e$—which is allowed by the residual symmetry Z_3—to get estimations on the mass of one of the flavons and the cutoff scale Λ of the model. Key words: Neutrino family symmetry, supersymmetry, deviation from TBM

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I. INTRODUCTION

In the Standard Model (SM) of electroweak interactions, neutrinos $(\nu_i)_{i=1,2,3}$ are left-handed and massless; this is because in the SM there are no right-handed neutrino singlets ν_{iR} that allow gauge-invariant Yukawa couplings to the Higgs doublet $y(H.L_i)\nu_{iR}$. However, recent experimental data on neutrino oscillations have shown that they have very tiny masses m_i and that the different flavors ν_1, ν_2, ν_3 are mixed with some mixing angles θ_{ij}, as shown in Table I below. This important discovery led to awarding the Nobel Prize in Physics for 2015 to Takaaki Kajita (SUPER-KAMIOKANDE Collaboration) and Arthur B. McDonald (SNO Collaboration). Although we cannot determine the exact masses m_i of the neutrinos, many experiments performed in the last few years measured the squared-mass differences $\Delta m_{ij}^2 = m_i^2 - m_j^2$ and mixing angles θ_{ij}, as reported by several global fits of neutrino data [1–3], the most recent of which can be found in Ref. [4].

To deal with the small masses and mixing of neutrinos we need to go beyond the SM framework; for this purpose many neutrino models have been proposed in recent years, and it is common that the observed mixing angles θ_{12} and θ_{23} are close to the tribimaximal mixing matrix (TBM), which predicts them to be in the 2σ and 3σ ranges, as in Table I [5]. The remaining θ_{13} is however not compatible with TBM, as announced by recent experiments, [6–9] although TBM still remains a good approach to the present data. We recall that one way to reproduce TBM at leading order (LO) is to go beyond the usual spectrum of the Standard Model via discrete non-Abelian groups like the alternating A_4 symmetry, which is admitted as the most natural discrete group that captures the family symmetry, as motivated in the literature. Following Altarelli and Feruglio [10], A_4 models have a particularly economical and attractive structure, e.g., in terms of group representations and field content [11–14]. For neutrino models based on other discrete groups see, for instance, Ref. [15], and for an introduction to non-Abelian discrete symmetries and representations see Ref. [16] and references therein. Recall also that there are several ways to generate masses for neutrinos beyond the standard model, such as the implementation of dimension-five nonrenormalizable operators [17]; or by using the three types of the seesaw mechanism: type I with extra SU(2) singlet fermions, type II with an

TABLE I. The global fit values for the mass squared differences Δm_{ij}^2 and mixing angles θ_{ij} as reported by Ref. [2]. NH and IH stand for normal and inverted hierarchies, respectively.

Parameters	Best fit$^{(+1\sigma,+2\sigma,+3\sigma)}_{(-1\sigma,-2\sigma,-3\sigma)}$
$\Delta m_{21}^2[10^{-5}\ \mathrm{eV}^2]$	$7.60^{(+0.19,+0.39,+0.58)}_{(-0.18,-0.34,-0.49)}$
$\Delta m_{31}^2[10^{-3}\ \mathrm{eV}^2]$(NH)	$2.48^{(+0.05,+0.11,+0.17)}_{(-0.07,-0.13,-0.18)}$
$\Delta m_{31}^2[10^{-3}\ \mathrm{eV}^2]$(IH)	$-2.38^{(+0.05,+0.10,+0.16)}_{(-0.06,-0.12,-0.18)}$
$\sin^2\theta_{12}$	$0.323^{(+0.016,+0.034,+0.052)}_{(-0.016,-0.031,-0.045)}$
$\sin^2\theta_{23}$(NH)	$0.567^{(+0.032,+0.056,+0.076)}_{(-0.128,-0.154,-0.175)}$
$\sin^2\theta_{23}$(IH)	$0.573^{(+0.025,+0.048,+0.067)}_{(-0.043,-0.141,-0.170)}$
$\sin^2\theta_{13}$(NH)	$0.0234^{(+0.0020,+0.004,+0.006)}_{(-0.0020,-0.0039,-0.0057)}$
$\sin^2\theta_{13}$(IH)	$0.0240^{(+0.0019,+0.0038,+0.0057)}_{(-0.0019,-0.0038,-0.0057)}$

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extra SU(2) triplet scalar, and type III with an extra SU(2) triplet fermion [18–22].

In this paper, we propose a supersymmetric neutrino model with discrete flavor symmetry $A_4 \times A_3$ that extends the minimal supersymmetric SM (MSSM), and whose theoretical predictions for Δm^2_{ij} and $\sin^2\theta_{ij}$ are compatible with experiments [6–9]. This field theory prototype is a supersymmetric type II seesaw neutrino theory based on a particular extension of the MSSM. In addition to the usual MSSM superfield spectrum and the chiral superfield triplets of the type II seesaw model, our model involves the extra flavon chiral superfields $\{\vec{\chi}, \vec{\chi}', \Phi, \Phi'\}$ carrying quantum numbers under $A_4 \times A_3$ discrete symmetry. $\vec{\chi}$ is needed by the A_4 symmetry in charged sector, while the three others concern the chargeless sector: $\vec{\chi}'$ to realize the tribimaximal texture, Φ to reproduce the correct mass squared difference $\Delta m^2_{31} \neq 0$, and Φ' to generate $\theta_{13} \neq 0$. By giving vacuum expectation values (VEVs) to these flavons, one generates Majorana mass terms and induces neutrino mixing compatible with the observations listed in Table I. Notice that supersymmetry plays a crucial rule in our construction; it is needed to have the right vacuum alignment and to overcome the sequestering problem, as was first noticed in Refs. [23,24]. Without supersymmetry there is no way to forbid terms of the form $\lambda_{\chi\chi'}|\chi|^2|\chi'|^2$ in the scalar potential which destroys the desired VEV structure in four-dimensional renormalizable theories. With supersymmetry, the scalar potential is derived from complex F terms in the chiral superpotential $W = W(\chi, \chi'; \ldots)$ sector, and Hermitian D terms of the Kahler $K(\chi, \chi^\dagger, \chi', \chi'^\dagger; \ldots\ldots)$ involving gauge interactions; terms like the undesirable $|\chi|^2|\chi'|^2$ come from a complex W and may be eliminated by an extra discrete symmetry having complex representations. Notice also that aspects of the type II seesaw mechanism for neutrinos with an A_4 flavor symmetry were considered before in Ref. [25] but without supersymmetry. In our supersymmetric extension, the two A_4 flavon superfield triplets $\vec{\chi}$ and $\vec{\chi}'$ act, respectively, in the charged lepton sector and neutrino sector; they carry different charges under the extra A_3 discrete subsymmetry which is needed to exclude unwanted terms in the superpotential W and to avoid the communication between charged and chargeless sectors. To engineer appropriate squared mass differences Δm^2_{ij} and mixing angles $\sin^2\theta_{ij}$ in the chargeless sector, we find that we also need to implement two A_4 scalar flavon chiral superfields Φ and Φ'. By giving them VEVs, we obtain TBM consistent with the experimental data on Δm^2_{ij} and $\sin^2\theta_{13}$. In this regard, we recall that several models use different approaches to generate a θ_{13} deviation from the TBM pattern; for instance, in Ref. [26], the deviation of TBM is obtained by adding a nonleading contribution coming from charged lepton mass diagonalization. In Ref. [25], the TBM was generated at LO with the type I seesaw mechanism and the deviation was made by perturbing the neutrino mass matrix with the type II seesaw mechanism. In our approach, we borrow techniques from the method used in Ref. [27] before $\theta_{13} = 0$ was ruled out. This method relies on perturbing the neutrino mass matrix by adding nontrivial A_4 singlets and has been used recently in Ref. [28] where neutrino masses were generated by dimension-five operators. After a numerical study, we show that normal and inverted hierarchies are both permitted. The VEV of the triplet $\vec{\chi}$ breaks A_4 down to Z_3 in the charged lepton sector; because of this residual symmetry, only the lepton-flavor-violating decays $\tau \to ee\mu$ and $\tau \to \mu\mu e$ are allowed in our model. We find that these decays are mediated by the flavon triplet χ_i, and by using the experimental upper bound of the branching ratio of the decay $\tau \to \mu\mu e$ we obtain an estimation on the mass of the flavon as well as the cutoff scale Λ of our model.

The presentation is as follows. In Sec. II we present the superfield content of the extended MSSM we are interested in here, and give their A_4 representations. Useful tools on A_4 tensor calculus, superpotential building, and the lepton charged sector are also given. In Sec. III, we first introduce our supersymmetric $A_4 \times A_3$ model and make some comments. Then, we focus on the chargeless sector; we first study the neutrino mass matrix and its diagonalization with the TBM matrix, then we analyze the scalar potential of flavons and describe the motivation beyond the need for the extra A_3 discrete symmetry. In Sec. IV we study the deviation of the TBM matrix with the help of the A_4 flavon singlets and give numerical results for both normal hierarchy (NH) and inverted hierarchy (IH). In Sec. V we study the lepton flavor violation (LFV) in the charged lepton sector to constrain the mass of the flavons χ_i and the cutoff scale Λ. In Sec. VI we give our conclusion and comments. In the three appendices, we report some relevant details and extra tools. In Appendix A, we recall useful properties of the A_4 group and irreducible representations. In Appendix B, we derive the vacuum alignments of $\vec{\chi}$ and $\vec{\chi}'$ used in this paper, and show that they are obtained without having to add extra superfields. In this regard, recall that in many models in the literature the problem of vacuum alignment is resolved by adding the so-called driving fields [29,30]. In Appendix C, we give explicit details on the tensor product of A_4-invariant terms used in the derivation of the flavon scalar potential (3.26) obtained in Sec. III. We also give details on solving the minimum condition of the scalar potential of the theory with respect to the two A_4 triplets $\vec{\chi}$ and $\vec{\chi}'$.

II. FLAVOR SYMMETRY IN SUPERSYMMETRIC MODELS

We begin by noticing that it is quite commonly admitted that the family symmetry relating flavors belonging to different generations of the SM might be behind the neutrino mass hierarchy and their mixing. This hypothetical flavor symmetry Γ is a discrete invariance that has been the subject of several studies, and particular interest has been focused on those Γ's given by non-Abelian discrete symmetries [16,31]. In this study, we consider the interesting case where flavor symmetry is given by $A_4 \times A_3$, and describe how this discrete symmetry can be implemented in models around the supersymmetric scale M^2_{SUSY} where the discrete Γ's are expected to follow from more basic symmetries, such as the breaking of E_8 gauge invariance of heterotic string or F-theory GUTs on Calabi-Yau manifolds [32–34].

A. Extending the MSSM

We start with the usual chiral superfield spectrum of the MSSM; then, we describe a particular extension of this minimal supersymmetric model by implementing flavon superfields carrying quantum numbers under a flavor symmetry $A_4 \times A_3$. This extension is one of the results of this paper; it will be further developed in forthcoming sections.

1. MSSM contents

In addition to the usual gauge superfield sector that we will omit for simplicity, the chiral superfield spectrum of the MSSM and their quantum numbers under $SU(3)_C \times SU(2)_L \times U(1)_Y$ invariance are as shown in Table II, with $i = 1, 2, 3$ referring to the number of matter generations. In superspace, these chiral superfields (and similar ones to be introduced later; see Tables III and V) may be generically denoted by Φ_m with the usual θ expansion [35]

$$\Phi_m = \phi_m + \sqrt{2}\theta.\psi_m + \theta^2 F_m. \tag{2.1}$$

Recall that properties and theoretical predictions of the MSSM are well established; the interacting dynamics of the MSSM spectrum is very well known, including both spontaneous and soft supersymmetry breaking. Recall also that this particular field theory dynamics is nicely described in superspace; we refer to the rich literature for details [36,37]. Moreover, notice that in this study we will focus on those relevant contributions to neutrino physics coming from couplings involving some ϕ_m's, auxiliary F_m's, and the usual auxiliary D's; that is, those contributions to the scalar potential of the model that lead to the computation of neutrino masses and mixing angles (for details, see Sec. III).

TABLE II. MSSM chiral superfield content.

sector	chiral superfields	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
leptons	$L_i = (\nu_i, e^-)_L$	1	2	-1
	$R^c_i = e^c_i$	1	1	$+2$
quarks	$Q_i = (u_i, d_i)_L$	3	2	$+\frac{1}{3}$
	$U^c_i = u^c_i$	$\bar{3}$	1	$-\frac{4}{3}$
	$D^c_i = d^c_i$	$\bar{3}$	1	$+\frac{2}{3}$
Higgs	$H_u = (H^+_u, H^0_u)$	1	2	$+1$
	$H_d = (H^0_d, H^-_d)$	1	2	-1

TABLE III. Chiral superfields added to the MSSM.

chiral superfields	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
$\Delta_u = (\Delta^0_u, \Delta^-_u, \Delta^{--}_u)$	1	3	-2
$\Delta_d = (\Delta^{++}_d, \Delta^+_d, \Delta^0_d)$	1	3	2

2. Extending the MSSM

There are several extensions of the MSSM that have been considered in literature. The extension of the MSSM we are interested in here concerns the enlargement of the *Higgs sector*; it is obtained by adding extra chiral superfields which carry quantum numbers under gauge symmetry and also under the discrete symmetry $A_4 \times A_3$. So the Higgs sector in our proposal may be thought of as consisting of three subsectors.

(i) The H subsector, involving the usual H_u, H_d of the MSSM.
(ii) The Δ subsector of the extended MSSM (type II seesaw); see Table III.
(iii) The χ subsector. This is our subsector; see Table V for its content.

Before giving the full superfield spectrum of our model, let us first focus on the Δ subsector; this is a particular extension of the Higgs sector of the MSSM given by adding two chiral superfield triplets $\vec{\Delta}_u$ and $\vec{\Delta}_d$ with gauge quantum numbers as in Table III. The $y = \pm 2$ hypercharge values are required by gauge invariance of the superfield couplings $H_{u,d}$ and $\Delta_{u,d}$ in the chiral superpotential $W = W(H, \Delta)$ of the extended supersymmetric model; this chiral superfield coupling has the form

$$W = \lambda_u \mathrm{Tr}(H_u \otimes \Delta_u \otimes H_u) + \lambda_d \mathrm{Tr}(H_d \otimes \Delta_d \otimes H_d), \tag{2.2}$$

where $\lambda_{u,d}$ are Yukawa coupling constants.

To describe the χ subsector, it is interesting to first collect some useful tools on discrete groups, in particular, on the group $A_4 \times A_3$ and its representations.

B. $A_4 \times A_3$ symmetry

First, notice that $A_3 \simeq Z_3$ is an Abelian group and so its irreducible representations $\mathbf{1}_{q^r}$ are one dimensional with charge $r = 0, \pm 1$ and $q = e^{\frac{2ir\pi}{3}}$. This group should not be confused with the A'_3 subgroup contained in A_4. In what follows, we will focus on describing pertinent properties of the discrete symmetry, in particular those concerning the non-Abelian A_4 factor and its representations. These realizations will be used later to refine the quantum numbers of the chiral superfield spectrum (see Tables II and III) as well as the content of the χ subsector given in Table V.

1. A_4 and its representations

The finite A_4 symmetry is a non-Abelian discrete group with order 12; it is a particular subgroup of the symmetric S_4 and is generated by two noncommuting elements S and T that satisfy the following cyclic relations:

$$S^2 = T^3 = (ST)^3 = 1. \tag{2.3}$$

Because of their noncommutativity, S and T cannot be diagonalized simultaneously; later, we use the basis where S is diagonal.

Representations and tensor products.—By using the group character relation $12 = \sum_i d_i^2$ relating the order 12 of the group A_4 to the dimensions d_k of the irreducible representations $\mathbf{R}_i$ of A_4, we have

$$12 = 1^2 + 1^2 + 1^2 + 3^2. \tag{2.4}$$

From this relation we learn a set of useful features, in particular

(i) the group A_4 has four $\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{R}_4$ with respective dimensions d_i as in Eq. (2.4),

(ii) it has four conjugacy classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ given by Eq. (A5) of Appendix A, and

(iii) it has one irreducible triplet 3, but three kinds of singlets 1, 1′, 1″.

Though interesting, the appearance of three singlets in the A_4 representation theory makes their use somehow subtle; this difficulty is apparent and can be overcome by using the characters $\chi_{R_i}(\mathcal{C}_j) = \chi_{ij}$ of the irreducible representations. The basic table of these characters, thought of as a matrix $\chi_{ij} \equiv \chi_{R_i}(\mathcal{C}_i)$, is given by Eq. (A6) in Appendix A. By restricting to the characters of the S and T generators of A_4, the above four irreducible representations $\mathbf{R}_i$ can be characterized as follows:

$$\begin{aligned} &\mathbf{1}:\mathbf{1}_{(1,1)}, \qquad \mathbf{1}':\mathbf{1}_{(1,\omega)}, \\ &\mathbf{3}:\mathbf{3}_{(-1,0)}, \qquad \mathbf{1}'':\mathbf{1}_{(1,\omega^2)}, \end{aligned} \tag{2.5}$$

where $\omega = e^{\frac{2i\pi}{3}}$ with the usual feature $1 + \omega + \bar{\omega} = 0$ and $\bar{\omega} = \omega^2$. These irreducible representations obey the following tensor product algebra [16,31]:

$$\begin{aligned} &\mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)} = \mathbf{1}_{(1,1)} \oplus \mathbf{1}_{(1,\omega)} \oplus \mathbf{1}_{(1,\omega^2)} \oplus \mathbf{3}_{(-1,0)} \oplus \mathbf{3}_{(-1,0)}, \\ &\mathbf{3}_{(-1,0)} \otimes \mathbf{1}_{(1,\omega^r)} = \mathbf{3}_{(-1,0)}, \\ &\mathbf{1}_{(1,\omega^r)} \otimes \mathbf{1}_{(1,\omega^s)} = \mathbf{1}_{(1,\omega^{r+s})}, \end{aligned} \tag{2.6}$$

where the integers r and s take the values 0, 1, 2 mod 3. Observe that these relations preserve total dimension and the total character. Observe also that the tensor product $\mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)}$ has a singlet $\mathbf{1}_{(1,1)}$; the same feature holds for higher product powers, in particular, for the cubic and quartic powers to be encountered later in our construction:

$$\begin{aligned} &\mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)} = \mathbf{1}_{(1,1)} \oplus \ldots, \\ &\mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)} = \mathbf{1}_{(1,1)} \oplus \ldots. \end{aligned} \tag{2.7}$$

Superpotential.—The superpotential of chiral superfields Φ_i in the extended MSSM is given by a superfunction $W(\Phi_i)$ that obeys two kinds of symmetries:

(i) invariance under the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ gauge group;

(ii) invariance under the flavor group $A_4 \times A_3$.

Since $W(\Phi_i)$ has a polynomial form in the chiral superfields Φ_i, the invariance of the superpotential under $A_4 \times A_3$ is obtained by performing tensor products of irreducible representations. Seeing that the tensor product of the $\mathbf{1}_{q^r}$ representation of A_3 is governed by the fusion relation $\mathbf{1}_{q^r} \otimes \mathbf{1}_{q^s} = \mathbf{1}_{q^{r+s}}$, the main difficulty comes from the non-Abelian A_4 when computing higher-order monomials of the type

$$\prod_i \Phi_i^{n_i} \tag{2.8}$$

with the fusion algebra (2.6). These computations are necessary since the A_4-invariant trace $\mathrm{Tr}_{A_4} W(\Phi_i)$ is given by the following restriction:

$$\mathrm{Tr}_{A_4} W(\Phi_i) = W(\Phi_i)|_{1_{(1,1)}}. \tag{2.9}$$

To illustrate how the method works let us focus on the A_4 subsymmetry and later extend the construction to the full discrete symmetry.

2. A_4-invariant superpotential

As a first step to implementing flavor symmetry in neutrino supersymmetric model building, we consider the superfield spectrum given in Tables II and III, to which we add flavon chiral superfields

$$\chi_k = (\chi_1, \chi_2, \chi_3), \tag{2.10}$$

which transform as a triplet under the discrete group A_4. Then, we attribute the following A_4 quantum numbers to the chiral superfield spectrum:

$$\begin{array}{c|c|c|c|c} \text{chiral superfields} & L_i & R_i^c Q_i U_i^c D_i^c & H_{u,d}\Delta_{u,d} & \chi_k \\ \hline A_4\,\text{symmetry} & \mathbf{1}_{(1,\bar{\omega}^{i-1})} & \mathbf{3}_{(-1,0)} & \mathbf{1}_{(1,1)} & \mathbf{3}_{(-1,0)} \end{array} \tag{2.11}$$

where the L_i's refer to the left doublets $(\nu_i, e^-)_L$, the R_i^c's to the right-handed e_i^c, and the others are as in Tables II and III. Notice the following remarkable features:

(i) The three lepton doublets (L_1, L_2, L_3) sit in different A_4 singlets, while the right leptons (R_1^c, R_2^c, R_3^c) sit together in an A_4 triplet [38].

(ii) The implementation of the A_4 discrete symmetry is not a soft operation; by attributing A_4 quantum numbers to leptons L_i and R_i^c, the usual superfield couplings for building the lepton mass matrix, such as

$$\mathrm{y}^{ij} R_i^c L_j H_d,$$

are forbidden by invariance under discrete A_4. Indeed, by focusing on the charged lepton sector, the chiral superpotential W_{lep^+} describing the usual gauge-invariant Yukawa couplings,

$$W_{\mathrm{lep}^+} = \mathrm{y}^{ij} R_i^c L_j H_d, \tag{2.12}$$

is no longer invariant under A_4 transformations, since from the view of the A_4 representation group theory this chiral superfield coupling has the following tensor product form:

$$\mathbf{3}_{(-1,0)} \otimes \mathbf{1}_{(1,\bar{\omega}^{i-1})} \otimes \mathbf{1}_{(1,1)} \sim \mathbf{3}_{(-1,0)}, \tag{2.13}$$

which does not contain the desired A_4 singlet $\mathbf{1}_{(1,1)}$ in the trace (2.9). We will see later that a similar feature to Eq. (2.12) also happens for the chiral superpotential W_{lep^0} describing couplings involving neutrinos.

To make the gauge-invariant W_{lep^+} symmetric as well under the discrete A_4, we have to modify the chiral superfield interaction (2.12) like $\tilde{\mathcal{W}}_{\mathrm{lep}^+} = \mathrm{Tr}_{A_4}(\tilde{W}_{\mathrm{lep}^+})$, with

$$\tilde{W}_{\mathrm{lep}^+} = \frac{1}{\Lambda} y^{ijk} (\chi_i R_j^c L_k H_d), \tag{2.14}$$

where y^{ijk} are Yukawa couplings, Λ denotes a cutoff scaling as mass (to be related in Sec. IV with a flavon VEV), and χ_i is an A_4 flavon triplet. The fourth-order superfield coupling $\chi_i R_j^c L_k H_d$ transforms under discrete symmetry as

$$\mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)} \otimes \mathbf{1}_{(1,\bar{\omega}^{i-1})} \otimes \mathbf{1}_{(1,1)}, \tag{2.15}$$

with the reduction containing the desired A_4 singlet type $\mathbf{1}_{(1,1)}$. Indeed, by using the fusion algebra (2.6)—in particular, the reduction $\mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)} = \mathbf{1}_{(1,\omega^{1-p})} \oplus \ldots$ with $p = 1, 2, 3$—it follows that the above chiral superfield product usually contains a term of the form $\mathbf{1}_{(1,\omega^{1-i})} \otimes \mathbf{1}_{(1,\omega^{i-1})}$, leading precisely to the desired singlet $\mathbf{1}_{(1,1)}$. To write down an explicit expression in terms of the superfields, it is interesting to work in the basis of A_4 where the generator S is diagonal. In this basis, the tensor product $R^c \otimes \chi$ between the two A_4 triplet superfields $R^c = (e_1^c, e_2^c, e_3^c)$ and $\chi = (\chi_1, \chi_2, \chi_3)$ reads as

$$R^c \otimes \chi = \begin{pmatrix} e_1^c\chi_1 & e_1^c\chi_2 & e_1^c\chi_3 \\ e_2^c\chi_1 & e_2^c\chi_2 & e_2^c\chi_3 \\ e_3^c\chi_1 & e_3^c\chi_2 & e_3^c\chi_3 \end{pmatrix}. \tag{2.16}$$

It is formally given by $\mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)}$ with nine components transforming in the $\mathbf{9}_{(1,0)}$ representation of A_4, which is reducible as in Eq. (2.6). The restrictions of this tensor product to the three A_4 singlet components $\mathbf{1}_{(1,\omega^r)}$ are given by

$$\begin{aligned} R^c \otimes \chi|_{1_{(1,1)}} &= e_1^c\chi_1 + e_2^c\chi_2 + e_3^c\chi_3, \\ R^c \otimes \chi|_{1_{(1,\omega)}} &= e_1^c\chi_1 + \omega e_2^c\chi_2 + \omega^2 e_3^c\chi_3, \\ R^c \otimes \chi|_{1_{(1,\omega^2)}} &= e_1^c\chi_1 + \omega^2 e_2^c\chi_2 + \omega e_3^c\chi_3, \end{aligned} \tag{2.17}$$

satisfying the properties

$$\begin{aligned} e_1^c\chi_1 &= \frac{1}{3} R^c \otimes \chi| + \frac{1}{3} R^c \otimes \chi|_\omega + \frac{1}{3} R^c \otimes \chi|_{\omega^2}, \\ e_2^c\chi_2 &= \frac{1}{3} R^c \otimes \chi| + \frac{\omega^2}{3} R^c \otimes \chi|_\omega + \frac{\omega}{3} R^c \otimes \chi|_{\omega^2}, \\ e_3^c\chi_3 &= \frac{1}{3} R^c \otimes \chi| + \frac{\omega}{3} R^c \otimes \chi|_\omega + \frac{\omega^2}{3} R^c \otimes \chi|_{\omega^2}, \end{aligned} \tag{2.18}$$

where we have used the notations

$$\begin{aligned} R^c \otimes \chi| &\equiv R^c \otimes \chi|_{1_{(1,1)}}, \\ R^c \otimes \chi|_\omega &\equiv R^c \otimes \chi|_{1_{(1,\omega)}}, \\ R^c \otimes \chi|_{\omega^2} &\equiv R^c \otimes \chi|_{1_{(1,\omega^2)}}. \end{aligned} \tag{2.19}$$

If we choose the VEVs of the A_4 triplet χ_i as in the Altarelli-Feruglio model (AF) [39] and the VEV of the Higgs H_d as usual

$$\langle \chi_i \rangle = \upsilon_\chi (1,1,1), \qquad \langle H_d \rangle = \upsilon_d, \tag{2.20}$$

then by substituting these expressions back into the superpotential (2.14) we obtain the charged lepton mass matrix M_{lep^+} as

$$M_{\mathrm{lep}^+} = \frac{\upsilon_\chi \upsilon_d}{\Lambda} \begin{pmatrix} y_e & y_e & y_e \\ y_\mu & \omega y_\mu & \omega^2 y_\mu \\ y_\tau & \omega^2 y_\tau & \omega y_\tau \end{pmatrix}, \tag{2.21}$$

where the Yukawa couplings $y_{e,\mu,\tau}$ are related to the ones in Eq. (2.14) as follows:

$$y_e = y^{ij1}, \qquad y_\mu = y^{ij2}, \qquad y_\tau = y^{ij3}, \tag{2.22}$$

where $i = j = 1, 2, 3$. Following Ref. [40], this matrix can be diagonalized by using asymmetric left and right transformations like $M^{\mathrm{diag}}_{\mathrm{lep}^+} = U_R M_{\mathrm{lep}^+} U_L^\dagger$ with eigenvalues $m_i (i = e, \mu, \tau)$ given by

$$M^{\mathrm{diag}}_{\mathrm{lep}^+} = \frac{\sqrt{3}\upsilon_\chi \upsilon_d}{\Lambda} \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}, \tag{2.23}$$

and where

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \qquad U_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.24}$$

In order to obtain the hierarchy among the three families of charged leptons, one may use the Froggatt-Nielsen (FN) mechanism which consists of adding a new $\mathrm{U}(1)_{\mathrm{FN}}$ symmetry with a new charge to be assigned to the right-handed charged leptons [41]; for more details we refer to Refs. [16,39]. Following the AF model [39], by taking $y_\tau \upsilon_d < 250$ GeV and by using the experimental value of the tau lepton mass, we get a constraint on the lower bound of the ratio of the triplet VEV υ_χ over the Λ cutoff scale as follows:

$$\frac{\upsilon_\chi}{\Lambda} > 0.004. \tag{2.25}$$

III. SUPERSYMMETRIC $A_4 \times A_3$ NEUTRINO MODEL

In this section, we use the tools introduced in the previous section to develop our supersymmetric $A_4 \times A_3$ neutrino model describing neutrino mixing and their masses. First, we give the superfield spectrum of the proposal; then, we study the contributions of the χ sector to the chargeless leptons of the model, in particular the aspects regarding neutrino masses and their mixing.

A. Superfield content

The superfield spectrum of the $A_4 \times A_3$ neutrino model involves—in addition to the usual superfields of the type II seesaw picture—extra flavon superfields with nontrivial quantum numbers under $A_4 \times A_3$.

1. Chiral superfields in type II seesaw

In our model, the Higgs sector has three subsectors: (a) the H subsector involving the H_u, H_d superfields of the MSSM, (b) the Δ subsector given in Table III, and (c) an extra χ subsector involving flavons. The quantum numbers of the chiral superfields of the H and Δ sectors are shown in Table IV (with explicit content like in Tables II and III).

The $A_4 \times A_3$-invariant superpotentials relevant for the neutrino physics will be studied explicitly once we introduce the superfield content of the χ subsector.

2. Flavon sector

Flavon superfields are chiral superfields which transform as singlets under gauge symmetry, but in general they carry nontrivial charges under the $A_4 \times A_3$ flavor symmetry; for our concern, we show the relevant flavons in Table V.

These flavons couple to the lepton superfields of the model; for instance, the chiral superfield triplet χ_i, which was introduced previously in Eq. (2.14), is needed to build the mass matrix for the charged leptons. The other chiral superfield triplet χ'_i is needed to engineer the Majorana mass matrix of the neutrinos; its coupling to leptons will be described in detail in the next subsection.

Moreover, the trivial singlet Φ is needed to reproduce the correct mass-squared difference $\Delta m^2_{31} \neq 0$, while the nontrivial singlet Φ' has been added in order to generate a nonzero mixing angle θ_{13}. Notice also that the discrete symmetry A_3 is required to satisfy the following:

(i) Exclude unwanted terms that appear in A_4-invariant superpotentials for charged and chargeless leptons.

TABLE IV. $A_4 \times A_3$ quantum numbers of the matter and Higgs superfields.

sector	superfields	$\mathrm{SU}(3)_C$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$	A_4	A_3
leptons	L_i	1	2	-1	$1_{(1,\bar{\omega}^{i-1})}$	1_0
	R_i^c	1	1	$+2$	$3_{(-1,0)}$	1_-
quarks	Q_i	3	2	$+\frac{1}{3}$	$3_{(-1,0)}$	1_0
	U_i^c	$\bar{3}$	1	$-\frac{4}{3}$	$3_{(-1,0)}$	1_0
	D_i^c	$\bar{3}$	1	$+\frac{2}{3}$	$3_{(-1,0)}$	1_0
Higgs	H_u	1	2	$+1$	$1_{(1,1)}$	1_0
	H_d	1	2	-1	$1_{(1,1)}$	1_0
	Δ_u	1	3	-2	$1_{(1,1)}$	1_0
	Δ_d	1	3	$+2$	$1_{(1,1)}$	1_0

TABLE V. The flavon superfields.

superfields	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	A_4	A_3
χ_i	1	1	0	$3_{(-1,0)}$	1_+
χ'_i	1	1	0	$3_{(-1,0)}$	1_0
Φ	1	1	0	$1_{(1,1)}$	1_0
Φ'	1	1	0	$1_{(1,\omega)}$	1_0

Without the extra A_3, generic A_4-invariant superpotentials $W(\chi,\chi')$ would be invariant under the exchange of the two flavon triplets, that is, by performing the permutation

$$\chi_i \leftrightarrow \chi'_i. \tag{3.1}$$

(ii) Prevent $\chi\chi'$ interactions in the superpotential through other intermediate superfields, and therefore between the charged and chargeless lepton subsectors of the supersymmetric $A_4 \times A_3$ model. It happens that this constraint coincides precisely with the so-called sequestering problem [23,24,42]. The A_3 subsymmetry is therefore a requirement of the sequestering problem.

B. Chargeless lepton sector

Before implementing $A_4 \times A_3$ invariance, it is interesting to notice that without flavons, the part W_{lep^0} of the chiral superpotential of the model that leads to the Majorana mass may be expressed as

$$\begin{aligned} W_{\mathrm{lep}^0} = {} & \lambda_\nu^{ee} L_e \Delta_d L_e + \lambda_\nu^{e\mu} L_e \Delta_d L_\mu + \lambda_\nu^{e\tau} L_e \Delta_d L_\tau \\ & + \lambda_\nu^{\mu e} L_\mu \Delta_d L_e + \lambda_\nu^{\mu\mu} L_\mu \Delta_d L_\mu + \lambda_\nu^{\mu\tau} L_\mu \Delta_d L_\tau \\ & + \lambda_\nu^{\tau e} L_\tau \Delta_d L_e + \lambda_\nu^{\tau\mu} L_\tau \Delta_d L_\mu + \lambda_\nu^{\tau\tau} L_\tau \Delta_d L_\tau, \end{aligned} \tag{3.2}$$

where $\lambda_\nu^{ij} = \lambda_\nu^{ji}$ are Yukawa coupling constants. By using the A_4 quantum charges given in Tables IV and V, it follows that the three terms $L_e\Delta_d L_e$, $L_\mu \Delta_d L_\tau$, and $L_\tau \Delta_d L_\mu$ are invariant under A_4 transformations, but not the other terms of Eq. (3.2) due to the fusion relation $\mathbf{1}_{(1,\omega^r)} \otimes \mathbf{1}_{(1,\omega^s)} = \mathbf{1}_{(1,\omega^{r+s})}$ which in general is not a trivial singlet. For example, by using Table IV, the superfield coupling $L_\mu \Delta_d L_\mu$ transforms under the A_4 representation like

$$\mathbf{1}_{(1,\omega^2)} \otimes \mathbf{1}_{(1,\omega^2)} \otimes \mathbf{1}_{(1,1)}, \tag{3.3}$$

which behaves as a nontrivial singlet representation since it is given by $\mathbf{1}_{(1,\omega)}$. To overcome this difficulty, we introduce an extra flavon superfield that transforms as $\mathbf{1}_{(1,\omega^2)}$; by using the fusion algebra (2.6), this nontrivial singlet of A_4 can be thought of in terms of a composite of the χ' triplet as

$$(\chi'\chi')|_{\omega^2}, \tag{3.4}$$

where the notation (2.19) has been used. The two other singlet composites appearing in the reduction of the tensor product $\chi' \otimes \chi'$, which are denoted as

$$(\chi'\chi')|_{\omega} \quad \text{and} \quad (\chi'\chi')|_{\omega^3}, \tag{3.5}$$

are needed to recover A_4 invariance of the other couplings, as shown below. Notice that if we use only the three A_4-invariant terms described above, the neutrino mass matrix will not agree with the TBM matrix and thus with the mixing angles θ_{12} and θ_{23}; with the three invariant terms $L_e\Delta_d L_e$, $L_\mu\Delta_d L_\tau$, and $L_\tau \Delta_d L_\mu$ the shape of neutrino mass matrix is given by

$$\begin{pmatrix} x & 0 & 0 \\ 0 & 0 & y \\ 0 & y & 0 \end{pmatrix}, \tag{3.6}$$

where the mixing matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{3.7}$$

which is clearly in conflict with the TBM matrix.

1. Implementing the flavon triplet χ'_i

To restore A_4 invariance in the chargeless lepton subsector, we add[1] the A_4 triplet $\chi'_i = (\chi'_1, \chi'_2, \chi'_3)$ and modify the superpotential W_{lep^0} of Eq. (3.2) as

$$\mathcal{W}_{\mathrm{lep}^0} = \mathrm{Tr}_{A_4}[W'_{\mathrm{lep}^0}] \equiv W'_{\mathrm{lep}^0}|_{1_{(1,1)}}, \tag{3.8}$$

with

$$\begin{aligned} W'_{\mathrm{lep}^0} = {} & \lambda_\nu^{ee} L_e \Delta_d L_e + \frac{\lambda_\nu^{\mu e}}{\Lambda^2} L_e \Delta_d L_\mu (\chi'\chi')|_\omega \\ & + \frac{\lambda_\nu^{e\tau}}{\Lambda^2} L_e \Delta_d L_\tau (\chi'\chi')|_{\omega^2} + \frac{\lambda_\nu^{e\mu}}{\Lambda^2} L_\mu \Delta_d L_e (\chi'\chi')|_\omega \\ & + \frac{\lambda_\nu^{\mu\mu}}{\Lambda^2} L_\mu \Delta_d L_\mu (\chi'\chi')|_{\omega^2} + \lambda_\nu^{\mu\tau} L_\mu \Delta_d L_\tau \\ & + \frac{\lambda_\nu^{\tau e}}{\Lambda^2} L_\tau \Delta_d L_e (\chi'\chi')|_{\omega^2} + \lambda_\nu^{\tau\mu} L_\tau \Delta_d L_\mu \\ & + \frac{\lambda_\nu^{\tau\tau}}{\Lambda^2} L_\tau \Delta_d L_\tau (\chi'\chi')|_{\omega}. \end{aligned} \tag{3.9}$$

In this relation, the term $(\chi'\chi')$ stands for $\chi' \otimes \chi'$ transforming in the $\mathbf{3}_{(-1,0)} \otimes \mathbf{3}_{(-1,0)}$ representation of the A_4

[1]The first triplet has been used in the charged lepton sector; see Eq. (2.14).

discrete symmetry whose reduction (2.6) contains (amongst others) three possible A_4 singlets. The notation $(\chi'\chi')|_\xi$ is as defined in Eq. (2.19), which for convenience we recall below:

$$\begin{aligned}(\chi'\chi')|_{1_{(1,1)}} &\equiv (\chi'\chi')|_1 = \chi_1'^2 + \chi_2'^2 + \chi_3'^2,\\ (\chi'\chi')|_{1_{(1,\omega)}} &\equiv (\chi'\chi')|_\omega = \chi_1'^2 + \omega\chi_2'^2 + \omega^2\chi_3'^2,\\ (\chi'\chi')|_{1_{(1,\omega^2)}} &\equiv (\chi'\chi')|_{\omega^2} = \chi_1'^2 + \omega^2\chi_2'^2 + \omega\chi_3'^2. \end{aligned} \tag{3.10}$$

2. Tribimaximal mixing matrix

For the sake of the TBM matrix, the neutrino mass matrix must respect the $\mu-\tau$ symmetry and the two following conditions [5,43]:

$$\begin{aligned}(M_\upsilon)_{11} + (M_\upsilon)_{12} &= (M_\upsilon)_{22} + (M_\upsilon)_{23},\\ (M_\upsilon)_{12} &= (M_\upsilon)_{13}.\end{aligned} \tag{3.11}$$

The implementation of the form of the TBM matrix for generating neutrino masses requires vacuum alignment of the A_4 triplet χ' and for Δ_d as follows[2]:

$$\langle\chi'\rangle = \upsilon_{\chi'}(1,0,0), \qquad \langle\Delta_d\rangle = \upsilon_{\Delta_d}. \tag{3.12}$$

Hence the neutrino mass matrix is

$$M_\upsilon = \upsilon_{\Delta_d}\begin{pmatrix} \lambda_\upsilon^{ee} & \lambda_\upsilon^{e\mu}b & \lambda_\upsilon^{e\tau}b \\ \lambda_\upsilon^{e\mu}b & \lambda_\upsilon^{\mu\mu}b & \lambda_\upsilon^{\mu\tau} \\ \lambda_\upsilon^{e\tau}b & \lambda_\upsilon^{\mu\tau} & \lambda_\upsilon^{\tau\tau}b \end{pmatrix}, \tag{3.13}$$

where we have set

$$\frac{\upsilon_{\chi'}^2}{\Lambda^2} \equiv \beta^2 = b. \tag{3.14}$$

Since the higher-dimensional operators involving $(\chi'\chi')$ contribute to the tiny mass of the neutrinos, the VEV of the flavon χ' should be small and close to the cutoff scale $\upsilon_{\chi'} \lesssim \Lambda$ which means that $b \lesssim 1$. Assuming for simplicity that the Yukawa couplings λ_υ^{ij} are of the order of unity,[3] and using the usual tribimaximal mixing matrix U, it results that the above mass matrix M_υ is diagonalized as $\mathcal{M}_\upsilon = U^T M_\upsilon U$ with

$$\mathcal{M}_\upsilon = \upsilon_{\Delta_d}\begin{pmatrix} 1-b & 0 & 0 \\ 0 & 1+2b & 0 \\ 0 & 0 & -1+b \end{pmatrix}. \tag{3.15}$$

Recall that the TBM mixing matrix has the form

$$U = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3.16}$$

It predicts the mixing angles as follows:

$$\sin^2\theta_{12} = \frac{1}{3}, \qquad \sin^2\theta_{23} = \frac{1}{2}, \qquad \sin^2\theta_{13} = 0. \tag{3.17}$$

However, a careful inspection of the eigenvalues of $\mathcal{M}_\upsilon$ reveals that we have $\Delta m_{31}^2 = 0$, which is in conflict with the data in Table I. For this reason, we need to correct the mass matrix (3.13), a correction that we realize by further enlarging the flavon spectrum of the model as described below.

3. An extra flavon singlet Φ

To generate appropriate masses for the neutrinos, we deform the superpotential (3.9) by adding $\delta W_{\mathrm{lep}^0}$ contributions inducing off-diagonal elements in the matrix $\mathcal{M}_\upsilon$ as a perturbation so that we can preserve the form of the matrix (3.13), which respects the $\mu-\tau$ symmetry and the conditions in Eq. (3.11) where the A_4 trivial singlet Φ is sufficient to solve the problem. Since the superpotential (3.9) is A_4 invariant, if we add one nontrivial singlet (such as $\Phi' \sim 1_{(1,\omega)}$ or $\Phi'' \sim 1_{(1,\omega^2)}$) we do not obtain invariant terms; this is why in the case of one singlet, the trivial $1_{(1,1)} \sim \Phi = \zeta + \theta\psi_\zeta + \theta^2 F_\zeta$ is the only representation that reproduces the TBM matrix. Hence, the desired deformed chiral superpotential reads as

$$\mathcal{W}''_{\mathrm{lep}^0} = \mathcal{W}'_{\mathrm{lep}^0} + \delta\mathcal{W}_{\mathrm{lep}^0}, \tag{3.18}$$

with an additional $\delta\mathcal{W}_{\mathrm{lep}^0} = \mathrm{Tr}_{A_4}[\delta W_{\mathrm{lep}^0}]$ term given by

$$\begin{aligned}\delta W_{\mathrm{lep}^0} = \frac{\lambda_\upsilon^{e\mu}}{\Lambda^3}&[L_e\Delta_d L_\mu + L_\mu\Delta_d L_e](\Phi(\chi'\chi')|_\omega)\\ &+\frac{\lambda_\upsilon^{e\tau}}{\Lambda^3}[L_e\Delta_d L_\tau + L_\tau\Delta_d L_e](\Phi(\chi'\chi')|_{\omega^2})\\ &+\frac{\lambda_\upsilon^{\tau\mu}}{\Lambda^3}[L_\mu\Delta_d L_\tau + L_\tau\Delta_d L_\mu](\Phi(\chi'\chi')|_{\omega^3}),\end{aligned} \tag{3.19}$$

where the scale Λ is the cutoff introduced before. Since the flavon Φ is introduced only to resolve the problem of the zero squared-mass difference $\Delta m_{31}^2 = 0$ its presence does not change the mixing angles, and also because it

[2]To avoid heavy notations, we denote the leading scalar components with the same letter as the superfields; see also the comment after Eq. ([27]).

[3]We can get the TBM matrix without assuming the Yukawa coupling of $\mathcal{O}(1)$, but to do so we have to impose some conditions on them in order to satisfy the relations (3.11); hence, for the matrix (3.13) we impose the following: $\lambda_\upsilon^{e\mu} = \lambda_\upsilon^{e\tau}$, $\lambda_\upsilon^{\mu\mu} = \lambda_\upsilon^{\tau\tau}$ and $\lambda_\upsilon^{ee} + \lambda_\upsilon^{e\mu}b = \lambda_\upsilon^{\mu\mu}b + \lambda_\upsilon^{\mu\tau}$.

transforms trivially under A_4 its VEV does not break A_4. Accordingly, we have two possible routes: (i) either we assume that $\langle\Phi\rangle = \upsilon_\Phi$ is much smaller than the cutoff scale $\upsilon_\Phi \ll \Lambda$ where invariant terms like the series $\sum_n L_e \Delta_d L_e (\frac{\Phi}{\Lambda})^n$ may be suppressed by the factor of $\frac{\upsilon_\Phi}{\Lambda} \ll 1$, or (ii) the VEV υ_Φ is of the order of the cutoff scale ($\upsilon_\Phi \sim \Lambda$) where the terms $\lambda_\upsilon^{ee} L_e \Delta_d L_e (\frac{\Phi}{\Lambda})^n$ are comparable to $\lambda_\upsilon^{ee} L_e \Delta_d L_e$. In this way, we assume that the additional factor coming from the combination of these operators is absorbed into the coupling constants λ_υ^{ee}. The previous neutrino mass matrix M_υ [Eq. (3.13)] gets corrected like $M'_\upsilon = M_\upsilon + \delta M_\upsilon$, whose expression can be put into the form

$$M'_\upsilon = \upsilon_{\Delta_d} \begin{pmatrix} 1 & b+c & b+c \\ b+c & b & 1+c \\ b+c & 1+c & b \end{pmatrix}, \tag{3.20}$$

where b is as in Eq. (3.14) and where we have set

$$c = \frac{\upsilon_{\chi'}^2}{\Lambda^2}\frac{\upsilon_\Phi}{\Lambda} = b\frac{\upsilon_\Phi}{\Lambda}. \tag{3.21}$$

Therefore, the convergence of the geometric series $L_e \Delta_d L_e \sum_n (\frac{\Phi}{\Lambda})^n$ turns into the condition $|c| < |b|$. The new mass matrix M'_υ is diagonalized by the TBM mixing matrix U as $\mathcal{M}'_\upsilon = \mathrm{diag}(m_1, m_2, m_2)$, with the neutrino mass eigenvalues (in units of υ_{Δ_d}) given as

$$\begin{aligned} m_1 &= 1 - c - b, \\ m_2 &= 2b + 2c + 1, \\ m_3 &= b - c - 1. \end{aligned} \tag{3.22}$$

From these new eigenvalues we learn that $\Delta m_{31}^2 = -4c(b-1)$ is no longer vanishing provided that we have $b \neq 1$ and $c \neq 0$. Notice that the same constraint on the parameter b ($b \lesssim 1$) holds for the parameter c for the same reasons we mentioned in the previous subsection; thus, $c \lesssim 1$, which means that $\upsilon_{\chi'}^2 \upsilon_\zeta \lesssim \Lambda^3$.

C. $A_4 \times A_3$-invariant scalar potential

Here we study the $A_4 \times A_3$-invariant scalar potential; the A_3 symmetry is needed for the reasons mentioned in Sec. III A.

1. Higgs and flavon sector

By using the notation of Ref. for monomials of flavons (in particular, the quadratic $\chi'^2 \equiv \chi' \otimes \chi'$ and the cubic $\chi'^3 \equiv \chi' \otimes \chi'^2$), the $A_4 \times A_3$-invariant superpotential restricted to the Higgs isodoublet $H_{u,d}$, isotriplet $\Delta_{u,d}$, and flavon superfields χ, χ', Φ is given by

$$\begin{aligned} W_{H-F} = {} & \mu H_u H_d + \mu_\Delta \mathrm{Tr}(\Delta_u \Delta_d) + \lambda_u H_u \Delta_u H_u \\ & + \lambda_d H_d \Delta_d H_d + \mu_\chi \chi'^2 + \lambda_{\zeta\chi} \Phi \chi'^2 + \mu_\zeta \Phi^2 + \lambda \chi^3 \\ & + \lambda' \chi'^3 + \lambda_\zeta \Phi^3 + k_\zeta \Phi + h_\zeta H_u \Phi H_d \\ & + \delta_\zeta \Phi \mathrm{Tr}(\Delta_u \Delta_d), \end{aligned} \tag{3.23}$$

where μ, μ_Δ, μ_ζ, μ_χ are mass parameters and λ_x, h_ζ, δ_ζ are coupling constants. To justify the choice of the A_3 symmetry instead of just Z_2 to discriminate the two flavon triplets, we need to analyze the scalar potential.

2. Scalar potential

Gathering all the contributions from F, D, and soft terms, the scalar potential $\mathcal{V}_{\mathrm{tot}}$ of the model is given by

$$\mathcal{V}_{\mathrm{tot}} = V_{\mathrm{SUSY}} + V_{\mathrm{soft}}, \tag{3.24}$$

with

$$\begin{aligned} V_{\mathrm{SUSY}} = {} & |F_u|^2 + |F_d|^2 + |F_{\Delta_d}|^2 + |F_{\Delta_u}|^2 \\ & + |F_\chi|^2 + |F_{\chi'}|^2 + |F_\Phi|^2 + \vec{D}^2 + D^2, \end{aligned} \tag{3.25}$$

where the explicit forms of V_{SUSY} and V_{soft} are given in Appendix B. So the $A_4 \times A_3$-invariant scalar potential is as follows:

$$\begin{aligned} \mathcal{V} = {} & 9\lambda^2|\chi|^4 + 4|\mu_\chi|^2|\chi'|^2 + 4\lambda_{\zeta\chi}^2|\chi'|^2|\Phi|^2 + 9\lambda'^2|\chi'|^4 \\ & + 8\mu_\chi\lambda_{\zeta\chi}|\chi'|^2\Phi + 12\mu_\chi\lambda'|\chi'|^3 + 12\lambda_{\zeta\chi}\lambda'|\chi'|^3\Phi \\ & + \lambda_{\zeta\chi}^2|\chi'|^4 + 2k_\zeta\lambda_{\zeta\chi}|\chi'|^2 + 6\lambda_{\zeta\chi}\lambda_\zeta|\chi'|^2|\Phi|^2 \\ & + 2h_\zeta\lambda_{\zeta\chi}H_uH_d|\chi'|^2 + 2\delta_\zeta\lambda_{\zeta\chi}\mathrm{Tr}(\Delta_u\Delta_d)|\chi'|^2 \\ & + 4\mu_\zeta\lambda_{\zeta\chi}\Phi|\chi'|^3 + m_\chi^2|\chi|^2 + m_{\chi'}^2|\chi'|^2 + 2b_{\chi'}|\chi'|^2 \\ & + 2A_{\zeta\chi'}\Phi|\chi'|^2 + 2A_\chi|\chi|^3 + 2A_{\chi'}|\chi'|^3 + \mathcal{V}_{\mathrm{ind}}, \end{aligned} \tag{3.26}$$

where $\mathcal{V}_{\mathrm{ind}}$ consists of terms that are irrelevant with two A_4 triplets. The tensor products for all possible A_4-invariant terms are reported in Appendix C.

As stated before, in order to avoid the communication between the charged and chargeless sectors (and thus the interaction between the two A_4 triplets χ_i and χ'_i), we impose invariance under the additional A_3 symmetry given in Table V. It is easy to check that without the charges of this symmetry, we can add to $W_{H\text{-}F}$ other A_4-invariant terms like

$$\lambda_{\zeta\chi}\Phi\chi^2. \tag{3.27}$$

But because of Eq. (3.1), the $W_{H\text{-}F}$ will also have $\lambda_{\zeta\chi}\Phi\chi'^2$, and thus an induced interaction between χ and χ' through Φ. This feature can be checked by first computing the F_Φ term of the singlet superfield Φ singlet and then $|F_\Phi|^2$. The resulting term

$$\lambda_{\chi\chi'}|\chi|^2|\chi'|^2 \tag{3.28}$$

spoils the vacuum alignment of the triplets (2.20) and (3.12). To prevent the existence of the term (3.28) in the scalar potential, one of the triplet-singlet interactions should be excluded; this has been achieved by the A_3 charges given in Table V [excluding the term (3.27)]. It is possible to choose χ' to carry a nonzero charge under A_3 instead of χ; this eliminates the term $\lambda_{\zeta\chi}\Phi\chi'^2$ from $W_{H\text{-}F}$ instead of $\lambda_{\zeta\chi}\Phi\chi^2$, but this choice would take apart the invariance of the superpotential (3.19) needed to obtain the TBM matrix consistent with the data. Therefore, the absence of the term (3.27) in $W_{H\text{-}F}$ implies the absence of the term (3.28) in $\mathcal{V}$, thus allowing us to get the desired vacuum alignment in Eqs. (2.20) and (3.12) after breaking the A_4 symmetry; see Appendix B for the details.

In addition, if we consider the interchange between χ_i and χ'_i for instance in Eq. (2.14), one generates the new gauge-invariant term

$$\tilde{W}'_{\mathrm{lep}^+} = \frac{y^{ijk}}{\Lambda}\chi'_i R^c_j L_k H_d, \tag{3.29}$$

which is also invariant under A_4. This extra term could be excluded with a Z_2 symmetry acting differently on the two A_4 triplets like

$$\begin{matrix} \chi_i \to +\chi_i, \\ \chi'_i \to -\chi'_i, \end{matrix} \quad \text{or} \quad \begin{matrix} \chi_i \to -\chi_i, \\ \chi'_i \to +\chi'_i. \end{matrix} \tag{3.30}$$

One may also assign Z_2 charges $(+1,-1)$ for the rest of the superfields so that the superpotential (2.14) and (3.18) is invariant under Z_2 symmetry while preventing Eq. (3.29). However, within this picture the term $\lambda_{\zeta\Upsilon}\Phi\chi^2$ cannot be banned with the two possible assignments in Eq. (3.30), thus allowing for the existence of Eq. (3.28) in the scalar potential which would spoil the vacuum alignment of the A_4 triplets, as mentioned before. This is why we choose the A_3 symmetry to exclude the unwanted terms (3.27)–(3.29) while keeping the required ones (2.14), (3.18), and (3.26) with respect to A_3 charges assigned to the various superfields listed in Tables IV and V.

As stated in Sec. III B 2, another chiral superfield is needed to study the deviation from TBM, so one may ask how this new flavon Φ' will affect the scalar potential (3.26). Since our aim is to study the vacuum alignment of the A_4 triplets (2.20) and (3.12) and (as we presented above) only one triplet is allowed to interact with the singlet Φ in order to avoid the sequestering problem thanks to the A_3 symmetry we have imposed, as the A_3 charge assignment for Φ' is the same as Φ only one triplet is able to interact with Φ', allowing for the vacuum alignment to be satisfied also with the presence of this extra flavon.

IV. DEVIATION FROM TBM MATRIX

In this section we study the angle deviation from TBM in order to reconcile the reactor angle θ_{13} with the recent data collected in Table I. First, we present the perturbation of the neutrino mass matrix (3.20); this perturbation is captured by the VEV of the extra chiral superfield singlet Φ' of the spectrum in Table V transforming as $\mathbf{1}_{(1,\omega)}$ under A_4. Then we study the effect of this deviation on the mixing angles θ_{13} and θ_{23}.

A. Deviation by A_4 singlet $\mathbf{1}_{1,\omega}$

Using the chiral superfield Φ' of Table V and the cutoff Λ, we see that we can perform a symmetric perturbation of the superpotential (3.2) that induces a deviation of the mass matrix M'_υ of Eq. (3.20). At leading order, the linear deviation in Φ' that respects the symmetries of the model is as follows:

$$\delta W'_\nu = \frac{\Phi'}{\Lambda}(L_e\Delta_d L_\mu + L_\mu\Delta_d L_e + L_\tau\Delta_d L_\tau), \tag{4.1}$$

where the deviation parameter $\varepsilon = \frac{\langle\Phi'\rangle}{\Lambda} \ll 1$. While local gauge and discrete A_3 symmetries are manifest, invariance may be explicitly exhibited by using the A_4 representation language,

$$\begin{aligned} L_e\Delta_d L_\mu\frac{\Phi'}{\Lambda} &\sim \mathbf{1}_{(1,1)} \otimes \mathbf{1}_{(1,1)} \otimes \mathbf{1}_{(1,\omega^2)} \otimes \mathbf{1}_{(1,\omega)}, \\ L_\tau\Delta_d L_\tau\frac{\Phi'}{\Lambda} &\sim \mathbf{1}_{(1,\omega)} \otimes \mathbf{1}_{(1,1)} \otimes \mathbf{1}_{(1,\omega)} \otimes \mathbf{1}_{(1,\omega)}. \end{aligned} \tag{4.2}$$

With this correction, the previous neutrino mass matrix M'_υ gets deformed as

$$M''_\upsilon = \upsilon_{\Delta_d}\begin{pmatrix} 1 & b+c+\varepsilon & b+c \\ b+c+\varepsilon & b & 1+c \\ b+c & 1+c & b+\varepsilon \end{pmatrix}. \tag{4.3}$$

This is a symmetric matrix that can be diagonalized by a similarity transformation like $M_{\mathrm{diag}} = \tilde{U}^T M''_\upsilon \tilde{U}$. The system of eigenvalues m_i and eigenvectors $\vec{\upsilon}_i$ can be computed perturbatively; we find, up to $o(\varepsilon^2)$, the eigenvalues (in units of υ_{Δ_d})

$$\begin{aligned} m_1 &= 1-c-b-\frac{\varepsilon}{2}+o(\varepsilon^2), \\ m_2 &= 2b+2c+1+\varepsilon, \\ m_3 &= b-c-1+\frac{\varepsilon}{2}+o(\varepsilon^2), \end{aligned} \tag{4.4}$$

and eigenvectors

$$v_1 = \begin{pmatrix} -\sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} + \frac{\sqrt{3}\varepsilon}{4\sqrt{2}(b-1)} \\ \frac{1}{\sqrt{6}} - \frac{\sqrt{3}\varepsilon}{4\sqrt{2}(b-1)} \end{pmatrix}, \qquad v_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} -\frac{\varepsilon}{2\sqrt{2}(b-1)} \\ -\frac{1}{\sqrt{2}} + \frac{\varepsilon}{4\sqrt{2}(b-1)} \\ \frac{1}{\sqrt{2}} + \frac{\varepsilon}{4\sqrt{2}(b-1)} \end{pmatrix},$$

with the condition $b \neq 1$ imposed previously. From these eigenvectors, we get the unitary matrix $\tilde{U}$ diagonalizing M''_υ; it reads, up to order $O(\varepsilon^2)$,

$$\tilde{U} = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & -\frac{\varepsilon}{2\sqrt{2}(b-1)} \\ \frac{1}{\sqrt{6}} + \frac{\sqrt{3}\varepsilon}{4\sqrt{2}(b-1)} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} + \frac{\varepsilon}{4\sqrt{2}(b-1)} \\ \frac{1}{\sqrt{6}} - \frac{\sqrt{3}\varepsilon}{4\sqrt{2}(b-1)} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} + \frac{\varepsilon}{4\sqrt{2}(b-1)} \end{pmatrix} + O(\varepsilon^2) \tag{4.5}$$

and coincides with TBM in the limit $\varepsilon \to 0$. The unitary property of the above matrix holds up to second order in the deformation parameter, i.e., $\tilde{U}^\dagger \tilde{U} \simeq I + O(\varepsilon^2)$. Notice, by the way, that Eq. (4.5) depends on two free parameters ε, b, in particular on $\frac{\varepsilon}{b-1}$ (which will be used later on). Notice also from Eq. (4.5) that the parameter of deviation ε does not affect the mixing angle θ_{12}, where we have the same value as in the case of TBM, $\sin\theta_{12} = \frac{1}{\sqrt{3}}$. Moreover, by using the usual relationships $\sin\theta_{13} = |U_{e3}|$ and $\cos\theta_{13}\sin\theta_{23} = |U_{\mu 3}|$, we get the link between the θ_{13} reactor and the θ_{23} atmospheric angles and b, ε as given below (see also Figs. 1–3):

$$\sin\theta_{13} = \left|\frac{\varepsilon}{2\sqrt{2}(b-1)}\right|,$$
$$\sin\theta_{23} = \left|\frac{\varepsilon}{4\sqrt{2}(b-1)} - \frac{1}{\sqrt{2}}\right|. \tag{4.6}$$

The deviation of the atmospheric angle θ_{23} from its TBM value can be seen as

$$\sin^2\theta_{23} = \frac{1}{2} - \frac{\varepsilon}{4(b-1)} + O(\varepsilon^2), \tag{4.7}$$

where, by looking at Table I, we understand that

$$-0.143 \leq \frac{\varepsilon}{4(b-1)} \leq 0.108 \quad \text{for NH},$$
$$-0.14 \leq \frac{\varepsilon}{4(b-1)} \leq 0.097 \quad \text{for IH}. \tag{4.8}$$

Using Eq. (4.4), the parameter c may be related to the neutrino mass-squared differences,

$$\Delta m_{31}^2 = 4 v_{\Delta_d}^2 \left(1 - b - \frac{\varepsilon}{2}\right) c,$$
$$\Delta m_{21}^2 = 3 v_{\Delta_d}^2 [(b+c)(b+c+2+\varepsilon) + \varepsilon]. \tag{4.9}$$

In the next subsection, we use the experimental values of $\sin\theta_{ij}$ and Δm_{ij}^2 to make predictions concerning numerical estimations of the parameters ε, b, and c capturing data on the VEVs of flavons.

B. Normal hierarchy

Focusing on relations in Eq. (4.6), we plot in Fig. 1 (left panel) $\sin\theta_{23}$ as a function of $\sin\theta_{13}$ in terms of the ratio

$$\frac{\varepsilon}{b-1} = \alpha \tag{4.10}$$

induced by the VEV of the singlet Φ' (provided the condition $b \neq 1$ holds) and from Eq. (3.14) the relations

$$\frac{\upsilon_{\chi'}^2}{\Lambda^2} \neq 1, \qquad \frac{\upsilon_{\chi'}}{\Lambda} \neq \pm 1. \tag{4.11}$$

Notice that although the matrix (4.5) involves two free parameters, the true dependence is only through their ratio α which generates the deviation of TBM we are interested in. Notice also that to draw this variation, we have assumed that ε and b are real parameters, and by using Eq. (4.6) we find the linear deviations

$$\sin\theta_{13} = \pm\frac{1}{2\sqrt{2}}\alpha. \tag{4.12}$$

The values of the parameter α that are compatible with both $\sin\theta_{13}$ and $\sin\theta_{23}$ are shown in the left panel of Fig. 1 within their 3σ allowed range for the normal hierarchy ($\Delta m_{31}^2 > 0$) case; see Table I. We observe that the best fit for θ_{13},

$$\sin\theta_{13} = 0.1529, \tag{4.13}$$

corresponds to

$$\alpha \simeq 0.43, \tag{4.14}$$

while for θ_{23}, we have

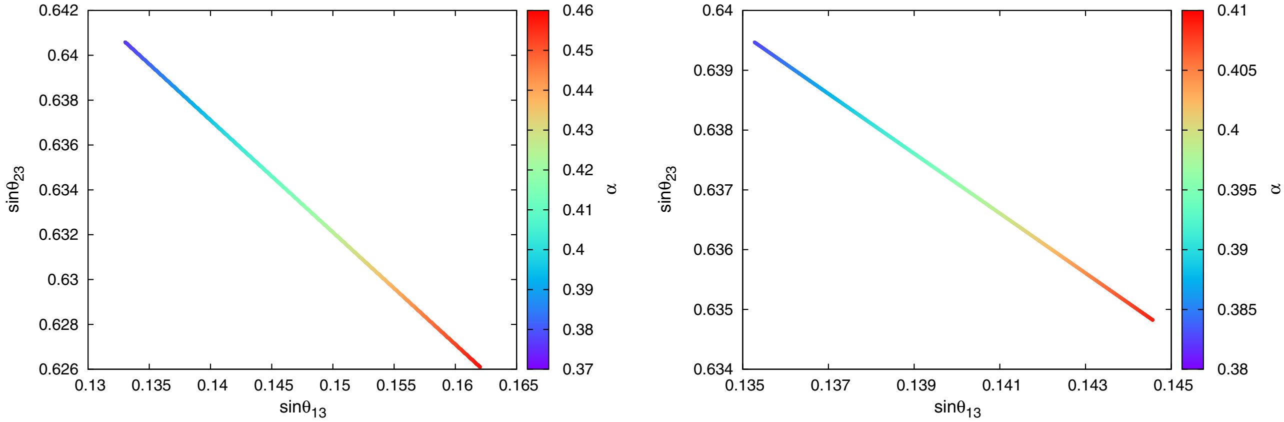

FIG. 1. Left: $\sin\theta_{23}$ as a function of $\sin\theta_{13}$ with the relative parameter $\alpha = \frac{\varepsilon}{b-1}$ shown in the palette on the right for normal hierarchy. Right: The same variation as in the left panel but for inverted hierarchy.

$$0.626 \leq \sin\theta_{23} \lesssim 0.641, \tag{4.15}$$

which is in the $[-2\sigma, -3\sigma]$ range (as can be read from Table I), and the interval of $\sin\theta_{23}$ corresponds to

$$0.37 \leq \alpha \lesssim 0.452. \tag{4.16}$$

1. Allowed interval for b

Since the parameter of deviation ε should be small we fix its value in the range of $O(\frac{1}{10})$, and from the equations in Eq. (4.6) we plot in the left panel in Fig. 2 $\sin\theta_{13}$ as a function of ε with the parameter b presented in the palette on the right. We plot the same variation in the right panel but for $\sin\theta_{23}$ instead of $\sin\theta_{13}$. We observe with the color palettes on the right of both panels in Fig. 2 that b is large for different values of ε. Moreover, as we discussed previously in Sec. III B 2, in order to have a tiny masses for neutrinos the parameter b should be less than approximately 1 ($b \lesssim 1$). Hence, with the order $O(\frac{1}{10})$ used for the range of ε, we read from Fig. 2 that b is positive and closely framed as

$$0.005 \lesssim b = \frac{\upsilon_{\chi'}^2}{\Lambda^2} < 1, \tag{4.17}$$

and by using Eq. (3.14) we conclude that the value of the cutoff Λ is around the value $\upsilon_{\chi'}$, the VEV of the flavon triplet χ'.

2. Allowed intervals for c

To get the allowed interval of the parameter c, we shall think of $(\upsilon_{\Delta_d}^2, b, \varepsilon)$ as spectral parameters and consider the first equation in Eq. (4.9) with the 3σ to express Δm_{31}^2 as a function of c. For $\varepsilon \sim \mathcal{O}(\frac{1}{10})$ the parameter b is as in Eq. (4.17), while in models with an extra Higgs triplet Δ_d the υ_{Δ_d} is fixed by using the relation $\upsilon_{\Delta_d} \sim \frac{m_\nu}{\lambda_\nu^{ij}}$ (λ_ν^{ij} are the Yukawa couplings). By using this relation and the recent cosmological upper bound on the sum of the neutrino

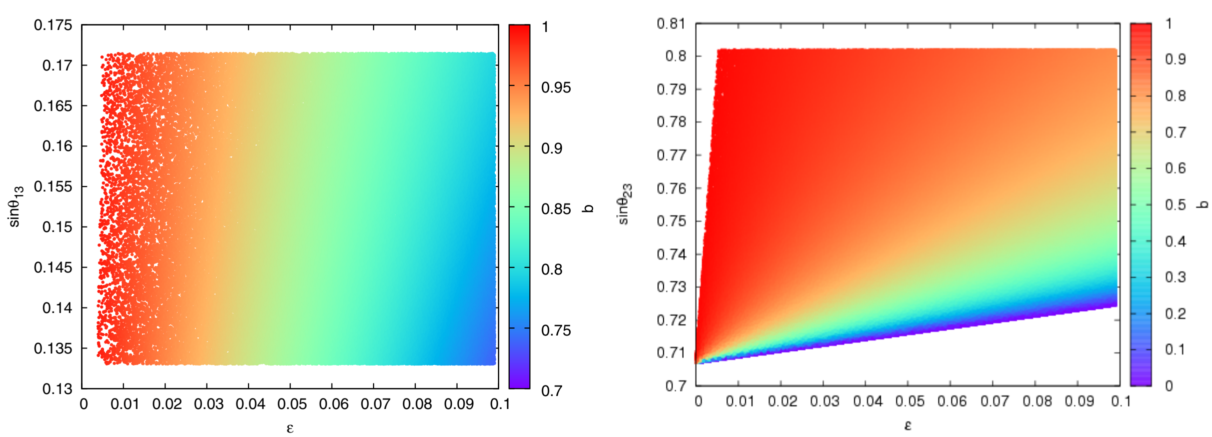

FIG. 2. Left: $\sin\theta_{13}$ as a function of ε with b shown in the palette on the right. Right: $\sin\theta_{23}$ as a function of ε with b shown in the palette on the right.

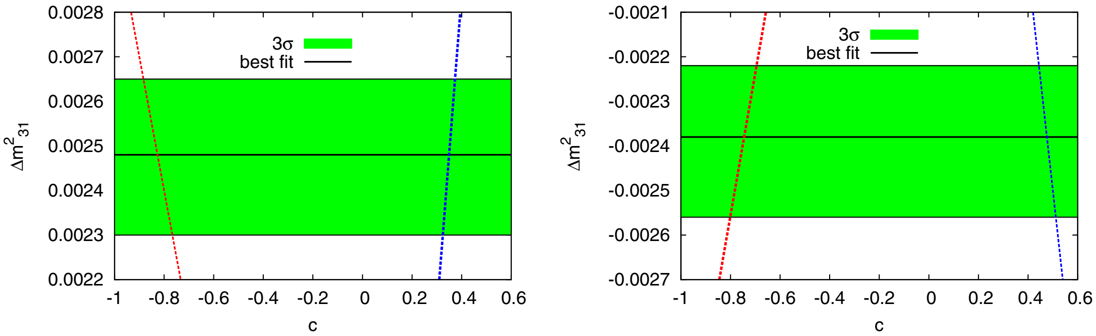

FIG. 3. Left (Right): Variation of Δm^2_{31} as a function of the parameter c for different inputs $(v^2_{\Delta_d}, b, \varepsilon)$ for NH (IH).

masses (which is constrained to $\sum m_\nu < 0.23$ eV [44]), the forthcoming inputs for $v^2_{\Delta_d}$ are reasonable.

In the left panel of Fig. 3 we plot the variation of Δm^2_{31} as a function of c in the case of normal hierarchy ($\Delta m^2_{31} > 0$) for two inputs:

$$v^2_{\Delta_d} \simeq 0.01 \text{ eV}^2, \qquad b \simeq 0.8, \qquad \varepsilon \simeq 0.09 \tag{4.18}$$

for the blue dashed line, and

$$v^2_{\Delta_d} \simeq 0.3 \text{ eV}^2, \qquad b \simeq 0.98, \qquad \varepsilon \simeq 0.045 \tag{4.19}$$

for the red dashed line. It is clear from the equation for Δm^2_{31} in Eq. (4.9) that the sign of c depends only on the value of b, which we found to be positive from Fig. 2, because Δm^2_{31} and $v^2_{\Delta_d}$ are positive-definite parameters. We observe in the left panel that c varies in the range

$$0.32 \lesssim c \lesssim 0.38 \tag{4.20}$$

for the blue dashed line, and

$$-0.83 \lesssim c \lesssim -0.78 \tag{4.21}$$

for the red dashed line. Notice that the NH depends strongly on the parameter b; for example, for values $0.96 \leq b < 1$ we remark that the factor $(1 - b - \frac{\varepsilon}{2})$ in the first equation of Eq. (4.9) is negative, so c has to be negative as well in order to respect $\Delta m^2_{31} > 0$ (red line in left panel of Fig. 3). On the other hand, for $0.005 \lesssim b \leq 0.95$, the factor $(1 - b - \frac{\varepsilon}{2})$ is positive for any allowed value of ε; this requires c to be positive in order to respect $\Delta m^2_{31} > 0$ (blue line in left panel of Fig. 3).

C. Inverted hierarchy

We represent in the right panel of Fig. 1 the same parameters $\sin\theta_{13}$, $\sin\theta_{23}$, and $\frac{\varepsilon}{b-1} = \alpha$ as in the left panel of the same figure, but this time for the inverted hierarchy with ($\Delta m^2_{31} < 0$). The allowed region for α is constrained by the values of the mixing angles $\sin\theta_{13}$ and $\sin\theta_{23}$ at 3σ; we observe that for the mixing angles θ_{23} and θ_{13} we have

$$0.6348 \lesssim \sin\theta_{23} \lesssim 0.6394, \tag{4.22}$$

which is in the range $[-2\sigma, -3\sigma]$ (as can be read from Table I) and

$$0.1348 \lesssim \sin\theta_{13} \lesssim 0.1354 \tag{4.23}$$

where this intervals corresponds to

$$0.385 \leq \alpha \lesssim 0.408. \tag{4.24}$$

We show in the right panel of Fig. 3 the variation of Δm^2_{31} as a function of the parameter c, where the latter is constrained by the 3σ allowed region of Δm^2_{31}. The input parameters b, ε, and $v^2_{\Delta_d}$ are as follows:

$$v^2_{\Delta_d} \simeq 0.5 \text{ eV}^2, \qquad b \simeq 0.98, \qquad \varepsilon \simeq 0.045 \tag{4.25}$$

for the blue dashed line, and

$$v^2_{\Delta_d} \simeq 0.0045 \text{ eV}^2, \qquad b \simeq 0.8, \qquad \varepsilon \simeq 0.08 \tag{4.26}$$

for the red dashed line. Thus, we observe that c varies in the range

$$0.42 \lesssim c \lesssim 0.5 \tag{4.27}$$

for the blue dashed line and

$$-0.8 \lesssim c \lesssim -0.7 \tag{4.28}$$

for the red dashed line.

V. LFV TO CONSTRAIN MASSES

In this section, we study LFV in the charged lepton sector in order to provide estimations on the mass of the flavon χ_i

and the cutoff scale Λ used in Eqs. (2.14) and (3.9). First, we break the A_4 symmetry down to Z_3 in order to induce LFV in the charged lepton sector; then, we calculate the analytic flavon masses. Next, we use the branching ratio of the allowed lepton-flavor-violating decays to give numerical lower bound estimations on the flavon masses and an upper bound on the cutoff scale Λ.

A. Breaking A_4 to Z_3

The discovery of neutrino oscillations provides clear evidence of lepton flavor violation in the chargeless lepton sector; however, in the charged sector LFV has not been observed yet. In this subsection, we study the breaking of the A_4 group to its subgroup Z_3 in order to get the allowed lepton-flavor-violating decays mediated by the flavon χ_i in the charged lepton sector. To start, we recall that in Sec. II B 2 the VEV of the flavon triplet was taken as $\langle\chi\rangle = \upsilon_\chi(1,1,1)$ [Eq. (2.20)], and because we are working in a basis of A_4 where the matrix generator S_{ij} is diagonal[4] this structure of the triplet VEV breaks A_4 down to its subgroup Z_3, with the matrix T_{ij} as a generator,

$$T_{ij}\langle\chi_j\rangle = 0, \qquad S_{ij}\langle\chi_j\rangle \neq 0. \tag{5.1}$$

By looking at the characters of the S and T generators of A_4 for the lepton superfields (2.11), it is not difficult to check that leptons l_i transform in different manners under the three possible representations $\mathbf{1}_{\omega^r}$ of the residual symmetry Z_3 characterized by the phases $\omega^r = e^{\frac{2i\pi r}{3}}$, with $r = 0, 1, 2$ and sum $1 + \omega + \omega^2 = 0$. Indeed, because A_4 singlets are also singlets of its subgroup Z_3, the left-handed charged leptons L_x live in the representations

$$L_e \sim \mathbf{1}_1, \qquad L_\mu \sim \mathbf{1}_{\omega^2}, \qquad L_\tau \sim \mathbf{1}_\omega, \tag{5.2}$$

and because of the decomposition of the A_4 triplet $\mathbf{3}$ in terms of irreducible Z_3 representations (namely, $\mathbf{3}_0 = \mathbf{1}_1 \oplus \mathbf{1}_\omega \oplus \mathbf{1}_{\omega^2}$), the right-handed A_4 triplets $(e_i^c) \sim \mathbf{3}$ are now combined into three Z_3 singlets with different characters as follows:

$$\begin{aligned} e^c &= \frac{1}{\sqrt{3}}(e_1^c + e_2^c + e_3^c) \sim \mathbf{1}_1, \\ \mu^c &= \frac{1}{\sqrt{3}}(e_1^c + \omega e_2^c + \omega^2 e_3^c) \sim \mathbf{1}_\omega, \\ \tau^c &= \frac{1}{\sqrt{3}}(e_1^c + \omega^2 e_2^c + \omega e_3^c) \sim \mathbf{1}_{\omega^2}. \end{aligned} \tag{5.3}$$

Consequently, the radiative decays $l_i \to l_j\gamma$ $(i \neq j)$ are all excluded in our model by the residual symmetry Z_3; this is because l_i and l_j live in different representations $\mathbf{1}_{\omega^i}$ and $\mathbf{1}_{\omega^j}$, and the photon γ is a singlet of Z_3. On the other hand, by using Eqs. (5.2) and (5.3), the LFV three-body decays

$$\begin{aligned} &\tau^+ \to e^+e^+\mu^-, \\ &\tau^+ \to \mu^+\mu^+e^- \end{aligned} \tag{5.4}$$

and their charged conjugates are allowed due to the representation character property $\mathbf{1}_{\omega^n} \otimes \mathbf{1}_{\omega^m} = \mathbf{1}_{\omega^{n+m}}$. As these decay modes are mediated by the flavon triplet χ_i, we start by calculating its mass.

B. Mass matrix of flavons

In order to calculate the mass matrix of field modes ξ_i describing the χ_i fluctuations near the vacuum expectation value $(\upsilon_\chi, \upsilon_\chi, \upsilon_\chi)$ of the flavon triplet χ_i, we proceed as follows. First, we consider the pure χ contribution $\mathcal{V}_\chi$ to the full scalar potential (3.26) of the model; it is given by $\mathcal{V}_\chi = \mathrm{Tr}_{A_4} V_\chi$ with

$$V_\chi = (|3\lambda\chi^2|^2 + m_\chi^2|\chi|^2 + 2A_\chi\chi^3) \tag{5.5}$$

[where χ^2 stands for $\chi \otimes \chi \equiv (\chi_i\chi_j)$], and a similar relation for the other χ^3 and χ^4 terms. Second, we use A_4 representation properties to decompose these tensor products into sums over irreducible representations of A_4 and take the trace afterwards; the explicit expression for $\mathrm{Tr}_{A_4} V_\chi$ can be read by substituting Eqs. (C4)–(C12) from Appendix C. Then, we expand the flavon field triplet (χ_1, χ_2, χ_3) around the vacuum expectation value as follows:

$$\begin{aligned} \chi_1 &= \upsilon_\chi + \xi_1, \\ \chi_2 &= \upsilon_\chi + \xi_2, \\ \chi_3 &= \upsilon_\chi + \xi_3, \end{aligned} \tag{5.6}$$

where the ξ_i's are field fluctuations; they will be thought of as real fields. This step, which breaks A_4 to its subgroup Z_3, leads to a quartic scalar potential $\mathcal{V}_\chi = \mathcal{V}(\xi_1, \xi_2, \xi_3)$ from which we can determine the mass matrix

$$(m_\xi^2)_{ij} = \frac{1}{2}\frac{\partial^2\mathcal{V}_\chi}{\partial\xi_i\partial\xi_j}\bigg|_{\xi=0}. \tag{5.7}$$

It reads explicitly as follows:

[4]The alternating group A_4 has two noncommuting generators S and T with the property $S^2 = T^3 = I$; because of the noncommutativity $ST \neq TS$, only one of them can be chosen diagonal. In Eqs. (A2) and (A3), the diagonal S and nondiagonal T are, respectively given by the matrices a_2 and b_1.

$$(m_{\xi}^2)_{ij} = \frac{1}{2}\begin{pmatrix} m_\chi^2 + 234\lambda^2 v_\chi^2 & 144\lambda^2 v_\chi^2 + 12A_\chi v_\chi & 144\lambda^2 v_\chi^2 + 12A_\chi v_\chi \\ 144\lambda^2 v_\chi^2 + 12A_\chi v_\chi & m_\chi^2 + 234\lambda^2 v_\chi^2 & 144\lambda^2 v_\chi^2 + 12A_\chi v_\chi \\ 144\lambda^2 v_\chi^2 + 12A_\chi v_\chi & 144\lambda^2 v_\chi^2 + 12A_\chi v_\chi & m_\chi^2 + 234\lambda^2 v_\chi^2 \end{pmatrix}. \quad (5.8)$$

The next step is to diagonalize the above mass matrix; we find

$$\begin{aligned} m_{\xi_1}^2 &= \frac{1}{2}m_\chi^2 + 45\lambda^2 v_\chi^2 - 6A_\chi v_\chi, \\ m_{\xi_2}^2 &= m_{\xi_1}^2, \\ m_{\xi_3}^2 &= \frac{1}{2}m_\chi^2 + 261\lambda^2 v_\chi^2 + 12A_\chi v_\chi, \end{aligned} \quad (5.9)$$

with two degenerate values.

C. Mass scale Λ

To get the order of magnitude of the cutoff scale, we need extra information in addition to the above flavon masses (5.9), in particular the structure of the flavon Yukawa couplings $L_{\mathrm{Yuk}}|_\xi$ in the charged lepton sector. To be able to use the experimental results on branching ratios (5.4), the explicit expression for $L_{\mathrm{Yuk}}|_\xi$ is also needed to extract information about which of the fields ξ_i is exchanged in lepton-flavor-violating decays. The fields ξ_i transform under Z_3 symmetry like

$$\xi_1 \sim \mathbf{1}_1, \qquad \xi_2 \sim \mathbf{1}_\omega, \qquad \xi_3 \sim \mathbf{1}_{\omega^2}. \quad (5.10)$$

Hence, we obtain the desired expression for $L_{\mathrm{Yuk}}|_\xi$ which, by using Eqs. (5.2), (5.3), and (5.10), reads as follows:

$$\begin{aligned} L_{\mathrm{Yuk}}|_\xi = &\frac{y_e v_d}{\Lambda}(e^c\xi_1 + \mu^c\xi_3 + \tau^c\xi_2)L_e \\ &+ \frac{y_\mu v_d}{\Lambda}(e^c\xi_2 + \mu^c\xi_1 + \tau^c\xi_3)L_\mu \\ &+ \frac{y_\tau v_d}{\Lambda}(e^c\xi_3 + \mu^c\xi_2 + \tau^c\xi_1)L_\tau. \end{aligned} \quad (5.11)$$

Moreover, by substituting the expression for the lepton masses we obtained in Sec. II B 2 [Eq. (2.23)], the flavon Yukawa interactions of the charged leptons in terms of the flavons ξ_i are given by

$$\begin{aligned} L_{\mathrm{Yuk}}|_\xi = &\left(\frac{m_e}{\sqrt{3}v_\chi}e^c L_e + \frac{m_\mu}{\sqrt{3}v_\chi}\mu^c L_\mu + \frac{m_\tau}{\sqrt{3}v_\chi}\tau^c L_\tau\right)\xi_1 \\ &+ \left(\frac{m_e}{\sqrt{3}v_\chi}\tau^c L_e + \frac{m_\mu}{\sqrt{3}v_\chi}e^c L_\mu + \frac{m_\tau}{\sqrt{3}v_\chi}\mu^c L_\tau\right)\xi_2 \\ &+ \left(\frac{m_e}{\sqrt{3}v_\chi}\mu^c L_e + \frac{m_\mu}{\sqrt{3}v_\chi}\tau^c L_\mu + \frac{m_\tau}{\sqrt{3}v_\chi}e^c L_\tau\right)\xi_3. \end{aligned} \quad (5.12)$$

Accordingly, we find that the flavon exchange ξ_1 does not lead to flavor violation, while the flavons ξ_2 and ξ_3 contribute to the lepton flavor violation processes (5.4). Following Ref. [45] and assuming that the contribution of supersymmetric particles in the decay modes (5.4) is negligible, the branching ratios of the these decays are as follows:

$$\begin{aligned} \mathrm{Br}(\tau^+ \to e^+e^+\mu^-) &= t_\tau \frac{m_\tau^5}{3072\pi^3}\left(\left|\frac{m_\tau m_e}{3v_\chi^2 m_{\xi_3}^2}\right|^2 + \left|\frac{m_e m_\mu}{3v_\chi^2 m_{\xi_2}^2}\right|^2\right), \\ \mathrm{Br}(\tau^+ \to \mu^+\mu^+e^-) &= t_\tau \frac{m_\tau^5}{3072\pi^3}\left(\left|\frac{m_\tau m_\mu}{3v_\chi^2 m_{\xi_2}^2}\right|^2 + \left|\frac{m_\mu m_e}{3v_\chi^2 m_{\xi_3}^2}\right|^2\right), \end{aligned} \quad (5.13)$$

where t_τ is the mean life of the tau lepton. To get an estimate on $m_{\xi_2}^2$, we consider the second equation in Eq. (5.13) and we assume that all terms proportional to $m_e^2 m_\mu^2$ and $m_\tau^2 m_e^2$ are negligible because $m_e \ll m_\mu \ll m_\tau$; we obtain the branching ratio

$$\mathrm{Br}(\tau^+ \to \mu^+\mu^+e^-) \simeq t_\tau \frac{m_\tau^7 m_\mu^2}{27648\pi^3 v_\chi^4}\frac{1}{m_{\xi_2}^4} \quad (5.14)$$

which, after substituting t_τ as well as the numerical values of the leptons masses from the Particle Data Group [46], we obtain

$$\mathrm{Br}(\tau^+ \to \mu^+\mu^+e^-) \simeq \frac{3.21}{v_\chi^4 m_{\xi_2}^4} \times 10^5\ \mathrm{GeV}^8. \quad (5.15)$$

Using the current upper bound of the branching ratio (5.15), which is $\mathrm{Br}(\tau^+ \to \mu^+\mu^+e^-) < 1.7 \times 10^{-8}$ at 90% C.L. [46], we get the following lower bound on the mass:

$$m_{\xi_2}^2 \gtrsim \frac{10^2}{v_\chi^2}\sqrt{t_\tau \frac{m_\tau^7 m_\mu^2}{4.7\pi^3}}. \quad (5.16)$$

If we assume that the mass of the flavon ξ_2 is of same order of magnitude as v_χ—say, $m_{\xi_2} \simeq v_\chi$—we get a lower bound on its mass $m_{\xi_2} \gtrsim 45.6$ GeV, which is surprisingly very light. With this limit, such kind of flavons could be generated through several decays; for instance, if the flavon mass m_{ξ_2} could be lighter than the Z^0 boson, the decay $Z^0 \to f\bar{f}\xi_2$ could occur at tree level. Moreover, using

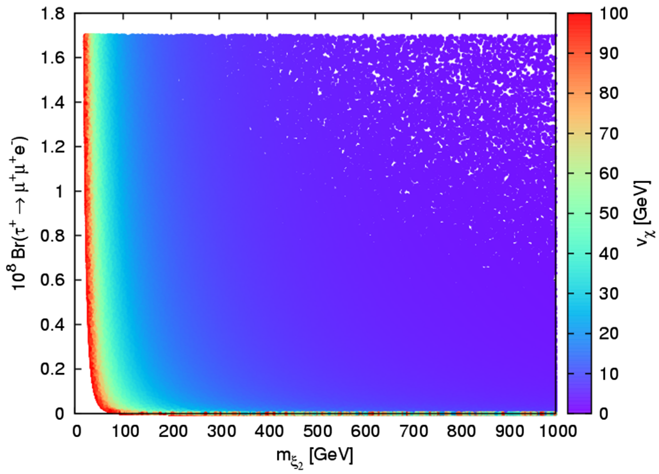

FIG. 4. $\mathrm{Br}(\tau^+ \to \mu^+\mu^+e^-)$ as a function of m_{ξ_2} with υ_χ shown in the palette on the right.

Eq. (2.25), by giving a lower bound on the ratio of the flavon VEV with respect to the cutoff scale (namely, $\frac{\upsilon_\chi}{\Lambda} > 0.004$) and taking $m_{\xi_2} \simeq \upsilon_\chi$, we find an upper bound for the cutoff scale given by

$$\Lambda \lesssim 1.14 \times 10^4 \text{ GeV}. \tag{5.17}$$

Notice that in Eq. (5.9) if the flavon trilinear coupling $A_\chi \geq 0$, the mass of the flavon ξ_3 could be heavier than $m_{\xi_2} = m_{\xi_1}$. However, the lower bound of the flavon mass in Eq. (5.16) depends on υ_χ and is specific for our model; in general, such a constraint is model dependent. To illustrate the relationship between the mass m_{ξ_2} and the VEV υ_χ, we plot in Fig. 4 the branching ratio $\mathrm{Br}(\tau^+ \to \mu^+\mu^+e^-)$ as a function of m_{ξ_2} for $\upsilon_\chi < 10^2$ GeV represented by the color palette on the right of the figure. We observe that for $\upsilon_\chi \in [40\text{–}100]$ GeV the mass m_{ξ_2} is less than 100 GeV including the value we find above for $m_{\xi_2} \simeq \upsilon_\chi$; on the other hand, when the value of υ_χ goes down to 40 GeV, m_{ξ_2} rises up until 1 TeV which corresponds to $\upsilon_\chi \simeq 10$ GeV and to an upper bound of the cutoff scale of the order $\Lambda \lesssim 2.5 \times 10^3$ GeV. Hence, as m_{ξ_2} increases both Λ and υ_χ decrease.

As a general comment, since the four flavon superfields we added in our model are all gauge singlets, they do not contribute to the mass of $W^\pm$ and Z^0 bosons. However, in the scalar potential (3.26) we notice that the flavon χ' mixes with the Higgs doublets H_u and H_d; thus, they might contribute to the so-called S and T oblique parameters [47]. Moreover, because some of the flavons could be lighter than the Higgs or the Z^0 boson, they will open new decay channels for these particles; as these two final points require examining the collider phenomenology of the flavons, we leave detailed investigations to future work.

VI. CONCLUSION AND DISCUSSION

In this paper, we have constructed a supersymmetric neutrino model based on $A_4 \times A_3$ discrete symmetry. In this model, neutrinos acquire a Majorana mass via the type II seesaw mechanism, and TBM acquires an appropriate deviation with $\theta_{13} \neq 0$.

First, we showed that it is possible to obtain the TBM pattern with only one A_4 triplet; however, we found that the physical observable $\Delta m_{31}^2 = 0$, which is in conflict with the present data. We then allowed for the presence of an extra A_4 scalar singlet $\Phi \sim 1_{1,1}$ which successfully reproduced the TBM matrix with $\Delta m_{31}^2 \neq 0$; see Eq. (3.20). We have studied the scalar potential of the supersymmetric model where we allowed the addition of an extra A_3 discrete symmetry, which is necessary to forbid the terms coming from the interchange between the TBM A_4 triplet and the one involved in the charged lepton sector, and also to avoid the sequestering problem.

We next studied the perturbation of the neutrino mass matrix that induces a deviation from the TBM matrix, leading therefore to a nonzero θ_{13} as proved by many experiments recently. This deviation is made with the help of a nontrivial A_4 singlet Φ' which transforms under it as $1_{1,\omega}$. In the beginning, we gave the resulting neutrino mass matrix (4.3) which received a new contribution from the VEV singlet Φ'. Then, we gave the deformed TBM matrix where the reactor angle $\theta_{13} \neq 0$ [Eq. (4.4)]. Next, we showed numerically by means of scatter plots the allowed regions of the parameters of the model which we have constrained by using the 3σ ranges of the neutrino oscillation parameters $\sin\theta_{31}$, $\sin\theta_{23}$, and Δm_{31}^2. Moreover, we gave the allowed regions of the parameter c where we found that the normal and inverted hierarchies are both permitted in our model. Finally, after discussing how the VEV alignment of the flavon triplet in the charged lepton sector breaks A_4 to Z_3, we studied the LFV in this sector and we found that only the three-body decays $\tau \to ee\mu$ and $\tau \to \mu\mu e$ are possible under the residual symmetry Z_3. We also found that these decays are mediated by the flavons ξ_2 and ξ_3; therefore, we calculated the lower bound of the flavon mass m_{ξ_2} by using the experimental branching ratio of the decay $\tau \to \mu\mu e$ where we found that m_{ξ_2} is very light ($m_{\xi_2} \gtrsim 45.6$ GeV) if we assume $m_{\xi_2} \simeq \upsilon_\chi$. We then used the relation between the cutoff scale Λ and υ_χ (namely, $\frac{\upsilon_\chi}{\Lambda} > 0.004$) to get an estimation on the upper bound of the cutoff scale, which we found to be of the order of 1.14×10^4 GeV. Nevertheless, we showed in Fig. 4 that the bound of m_{ξ_2} increases when υ_χ decreases, and therefore the cutoff scale also decreases, giving its relation with υ_χ.

We end this conclusion by making a comment on the TBM deviation using the other non-A_4 singlet $\mathbf{1}_{(1,\omega^2)} \sim \Phi''$ instead of $\mathbf{1}_{(1,\omega)} \sim \Phi'$. The new contributions added to the superpotential (3.2) are given by

$$\delta W_\nu = \frac{\Phi''}{\Lambda}(L_e\Delta_d L_\tau + L_\mu\Delta_d L_\mu + L_\tau\Delta_d L_e), \tag{6.1}$$

where the cutoff Λ is the same as before. The invariance of the above δW_ν under A_4 may be exhibited explicitly by using

$$\begin{aligned} L_e\Delta_d L_\tau\frac{\Phi''}{\Lambda} &\sim 1_{(1,1)}\otimes 1_{(1,1)}\otimes 1_{(1,\omega)}\otimes 1_{(1,\omega^2)},\\ L_\mu\Delta_d L_\mu\frac{\Phi''}{\Lambda} &\sim 1_{(1,\omega^2)}\otimes 1_{(1,1)}\otimes 1_{(1,\omega^2)}\otimes 1_{(1,\omega^2)}. \end{aligned} \tag{6.2}$$

With this Φ' correction, the previous neutrino mass matrix M'_υ gets deformed as

$$\hat{M}_\upsilon = \upsilon_{\Delta_d}\begin{pmatrix} 1 & b+c & b+c+\varepsilon \\ b+c & b+\varepsilon & 1+c \\ b+c+\varepsilon & 1+c & b \end{pmatrix}. \tag{6.3}$$

We repeat the same study as in the case of the singlet Φ'. We find that the eigenvectors at first order of ε are as follows:

$$\tilde{U}' = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & \frac{\varepsilon}{2\sqrt{2}(b-1)} \\ \frac{1}{\sqrt{6}} - \frac{\sqrt{3}\varepsilon}{4\sqrt{2}(b-1)} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} - \frac{\varepsilon}{4\sqrt{2}(b-1)} \\ \frac{1}{\sqrt{6}} + \frac{\sqrt{3}\varepsilon}{4\sqrt{2}(b-1)} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} - \frac{\varepsilon}{4\sqrt{2}(b-1)} \end{pmatrix} + O(\varepsilon^2), \tag{6.4}$$

where after diagonalizing $\hat{M}_\upsilon$ by the transformation $M_{\mathrm{diag}} = \tilde{U}'^T\hat{M}_\upsilon\tilde{U}'$, we obtain the same mass eigenvalues as in the case of the singlet Φ' [Eq. (4.4)] and therefore the same neutrino mass-squared differences Δm^2_{ij} as in Eq. (4.9). The mixing angles in the case of Φ'' are given by

$$\begin{aligned} \sin\theta_{13} &= \left|\frac{\varepsilon}{2\sqrt{2}(b-1)}\right|,\\ \sin\theta_{23} &= \left|-\frac{1}{\sqrt{2}} - \frac{\varepsilon}{4\sqrt{2}(b-1)}\right|. \end{aligned} \tag{6.5}$$

The deviation of the atmospheric angle θ_{23} from its TBM value can be seen as

$$\sin^2\theta_{23} = \frac{1}{2} + \frac{\varepsilon}{4(b-1)} + O(\varepsilon^2), \tag{6.6}$$

where the sign in front of $\frac{\varepsilon}{4(b-1)}$ is changed compared to the case of the singlet Φ'. Therefore, the signs of its intervals are reversed as follows:

$$\begin{aligned} -0.108 \le \frac{\varepsilon}{4(b-1)} \le 0.143 \quad \text{for NH.}\\ -0.097 \le \frac{\varepsilon}{4(b-1)} \le 0.14 \quad \text{for IH.} \end{aligned} \tag{6.7}$$

APPENDIX A: DISCRETE ALTERNATING A_4

We here provide three appendices. Appendix A contains useful aspects of the alternating A_4. Appendix B concerns the explicit derivation of the vacuum alignment property. Appendix C concerns properties of the tensor algebra of flavon superfield triplets used in the computation of the scalar potential.

The alternating A_4 group has 12 elements that can be generated by two noncommuting basic ones that we denote by S and T, satisfying the periodicity relations $S^2 = I_{id} \equiv e$ and $T^3 = I_{id}$. In terms of these generators, we have [16]

$$\begin{aligned} &a_1 = e, \quad a_2 = S, \quad a_3 = TST^2,\\ &a_4 = T^2ST, \quad b_1 = Ty, \quad b_2 = ST,\\ &b_3 = TS, \quad b_4 = STS, \quad c_1 = T^2,\\ &c_2 = ST^2, \quad c_3 = TST, \quad c_4 = T^2S. \end{aligned} \tag{A1}$$

This discrete group has four irreducible representations; three of them have one dimension, while the nontrivial fourth one has three dimensions. A realization of these elements in terms of 3×3 matrices is given by

$$\begin{aligned} a_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\\ a_4 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \end{aligned} \tag{A2}$$

and

$$b_3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad b_4 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad c_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$
$$c_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad c_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad c_4 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{A3}$$

Recall that A_4 is a subgroup of the symmetric S_4 consisting of only even permutations; a canonical representation of A_4 elements is naturally obtained by considering 4×4 matrices acting on four elements x_i and we the generators as $S = (12)$ (34), $T = (123)$ (4), with matrix representations as follows:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \\ x_4 \end{pmatrix}. \tag{A4}$$

Recall also that the discrete group A_4 has four irreducible representations $\boldsymbol{R}_i$ with properties encoded in the orthogonality character relations; in particular, in the formula $12 = 1^2 + 1^2 + 1^2 + 3^2$. It also has four conjugacy classes $\mathcal{C}_i$ given by

$$\mathcal{C}_1 = \{e\}, \qquad \mathcal{C}_3 = \{S, TST^2, T^2ST\}, \qquad \mathcal{C}_4 = \{T, TS, ST, STS\}, \qquad \mathcal{C}_{4'} = \{T^2, ST^2, T^2S, TST\}, \tag{A5}$$

and it is used in building the character table χ_{ij} which reads as follows:

$\chi_{ij}(A_4)$	$\boldsymbol{R}_1$	$\boldsymbol{R}_{1'}$	$\boldsymbol{R}_{1''}$	$\boldsymbol{R}_3$
$\mathcal{C}_1$	1	1	1	3
$\mathcal{C}_2$	1	1	1	−1
$\mathcal{C}_3$	1	ω	ω^2	0
$\mathcal{C}_4$	1	ω^2	ω	0

(A6)

APPENDIX B: VACUUM ALIGNMENT

The scalar potential (3.26) is derived from the usual F, D and soft terms of the supersymmetric minimal standard model and its extensions. The F terms are given by

$$\begin{aligned} &|F_u|^2 = |\mu H_d + \lambda_u \Delta_u H_u + h_\zeta \Phi H_d|^2, \qquad |F_d|^2 = |\mu H_u + \lambda_d \Delta_d H_d + h_\zeta H_u \Phi|^2, \\ &|F_{\Delta_u}|^2 = |\mu_\Delta \Delta_d + \lambda_u H_u H_u + \delta_\zeta \Phi \Delta_d|^2, \qquad |F_{\Delta_d}|^2 = |\mu_\Delta \Delta_u + \lambda_d H_d H_d + \delta_\zeta \Delta_u \Phi|^2, \qquad |F_\chi|^2 = |3\lambda\chi^2|^2, \\ &|F_{\chi'}|^2 = |2\mu_\chi \chi' + 2\lambda_{\zeta\chi}\chi'\Phi + 3\lambda'\chi'^2|^2, \qquad |F_\Phi|^2 = |h_\zeta H_u H_d + \delta_\zeta \mathrm{Tr}(\Delta_u \Delta_d) + 2\mu_\zeta \Phi + k_\zeta + \lambda_{\zeta\chi}\chi'^2 + 3\lambda_\zeta \Phi^2|^2. \end{aligned} \tag{B1}$$

The D terms are

$$D^2 = \frac{g_1^2}{2}\left[\frac{1}{2}(H_u^\dagger H_u - H_d^\dagger H_d) + \mathrm{Tr}(\Delta_d^\dagger \Delta_d) - \mathrm{Tr}(\Delta_u^\dagger \Delta_u)\right]^2,$$
$$\vec{D}^2 = \frac{g_2^2}{2}\sum_{a=1}^{3}\left[\frac{1}{2}(H_u^\dagger \sigma^a H_u + H_d^\dagger \sigma^a H_d) + \frac{1}{2}\mathrm{Tr}(\Delta_d^\dagger[\sigma^a, \Delta_d]) + \frac{1}{2}\mathrm{Tr}(\Delta_u^\dagger[\sigma^a, \Delta_u])\right]^2,$$

and for the soft terms we have

$$V_{\text{soft}} = m_{H_d}^2|H_d|^2 + m_{H_u}^2|H_u|^2 + m_{\Delta_d}^2|\Delta_d|^2 + m_{\Delta_u}^2|\Delta_u|^2 + m_\chi^2|\chi|^2 + m_{\chi'}^2|\chi'|^2 + m_\zeta^2|\Phi|^2 + (b_H H_u H_d + \text{H.c.}) \\ + (b_\Delta \text{Tr}(\Delta_u\Delta_d) + \text{H.c.}) + (b_{\chi'}\chi'^2 + \text{H.c.}) + (b_\zeta \Phi^2 + \text{H.c.}) + [(A_u H_u \Delta_u H_u + A_d H_d \Delta_d H_d + A_{H\zeta} H_u \Phi H_d) + \text{H.c.}] \\ + (A_{\Delta\zeta}\Phi \text{Tr}(\Delta_u\Delta_d) + \text{H.c.}) + (A_{\zeta\chi'}\chi'^2\Phi + A_\chi \chi^3 + A_{\chi'}\chi'^3 + A_\zeta \Phi^3 + \text{H.c.}). \tag{B2}$$

To break the flavor and electroweak symmetries, we give nonzero VEVs to the neutral fields of the Higgs doublets, the triplets, and the flavons. Focusing on the A_4 triplets χ and χ', and denoting by

$$\langle\chi\rangle = (v_{\chi_1}, v_{\chi_2}, v_{\chi_3}), \qquad \langle\chi'\rangle = (v_{\chi'_1}, v_{\chi'_2}, v_{\chi'_3}),$$

the VEVs solve the minimum conditions

$$\frac{\partial\mathcal{V}}{\partial\chi_i} = 0, \qquad \frac{\partial\mathcal{V}}{\partial\chi'_i} = 0 \tag{B3}$$

with $\mathcal{V}$ as in Eq. (3.26) and the VEVs of the triplets are as in Eqs. (2.20) and (3.12). To get these VEVs, we should take into account all possible A_4-invariant contributions coming from the tensor products of three and four triplets of A_4 as they appear in the computation of $|\chi|^4$ and $|\chi|^3$; see also Appendix C for more details. By using the fusion operator algebra of A_4, we have for the tensor product $(3_{-1,0})^{\otimes 4}$ the following expression:

$$(3_{-1,0}\otimes 3_{-1,0})^{\otimes 2} \to (1_{1,1}\otimes 1_{1,1}) \oplus (1_{1,\omega}\otimes 1_{1,\omega^2}) \oplus (1_{1,\omega^2}\otimes 1_{1,\omega}) \oplus (3^s_{-1,0}\otimes 3^s_{-1,0}) \\ \oplus (3^s_{-1,0}\otimes 3^a_{-1,0}) \oplus (3^a_{-1,0}\otimes 3^s_{-1,0}) \oplus (3^a_{-1,0}\otimes 3^a_{-1,0}),$$

which can be reduced further. Using the method of Ref. [24], we can approach the solution of the minimum conditions $\mathcal{V}$ for the A_4 triplet χ through the relations

$$v_{\chi_2}\frac{\partial\mathcal{V}}{\partial v_{\chi_1}} - v_{\chi_1}\frac{\partial\mathcal{V}}{\partial v_{\chi_2}} = 0, \qquad v_{\chi_3}\frac{\partial\mathcal{V}}{\partial v_{\chi_1}} - v_{\chi_1}\frac{\partial\mathcal{V}}{\partial v_{\chi_3}} = 0, \qquad v_{\chi_3}\frac{\partial\mathcal{V}}{\partial v_{\chi_2}} - v_{\chi_2}\frac{\partial\mathcal{V}}{\partial v_{\chi_3}} = 0, \tag{B4}$$

they read explicitly as

$$0 = 36\lambda^2 v_{\chi_1}v_{\chi_2}(v_{\chi_1}^2 - v_{\chi_2}^2) + 12A_\chi v_{\chi_3}(v_{\chi_2}^2 - v_{\chi_1}^2), \qquad 0 = 36\lambda^2 v_{\chi_1}v_{\chi_3}(v_{\chi_1}^2 - v_{\chi_3}^2) + 12A_\chi v_{\chi_2}(v_{\chi_3}^2 - v_{\chi_1}^2), \\ 0 = 36\lambda^2 v_{\chi_2}v_{\chi_3}(v_{\chi_2}^2 - v_{\chi_3}^2) + 12A_\chi v_{\chi_1}(v_{\chi_3}^2 - v_{\chi_2}^2). \tag{B5}$$

Clearly, the solution for the last three equations is given by

$$v_{\chi_1} = v_{\chi_2} = v_{\chi_3} = v_\chi. \tag{B6}$$

It is precisely the VEV structure we choose in Eq. (3.12) to produce the TBM matrix pattern. The same method applies for the minimum conditions coming from the triplet χ'; we have

$$v_{\chi'_2}\frac{\partial\mathcal{V}}{\partial v_{\chi'_1}} - v_{\chi'_1}\frac{\partial\mathcal{V}}{\partial v_{\chi'_2}} = 0, \qquad v_{\chi'_3}\frac{\partial\mathcal{V}}{\partial v_{\boldsymbol{\chi}'_1}} - v_{\chi'_1}\frac{\partial\mathcal{V}}{\partial v_{\chi'_3}} = 0, \qquad v_{\chi'_3}\frac{\partial\mathcal{V}}{\partial v_{\chi'_2}} - v_{\chi'_2}\frac{\partial\mathcal{V}}{\partial v_{\chi'_3}} = 0.$$

Explicitly,

$$0 = 36\lambda'^2 v_{\chi'_1}v_{\chi'_2}(v_{\chi'_1}^2 - v_{\chi'_2}^2) + 72\lambda' v_{\chi'_3}(v_{\chi'_2}^2 - v_{\chi'_1}^2)(\mu_\chi + \lambda_{\zeta\chi}v_\Phi) + 4\lambda_{\zeta\chi}^2 v_{\chi'_1}v_{\chi'_2}(v_{\chi'_1}^2 - v_{\chi'_2}^2) + 12A_{\chi'}v_{\chi'_3}(v_{\chi'_2}^2 - v_{\chi'_1}^2) \tag{B7}$$

and

$$0 = 36\lambda'^2 v_{\boldsymbol{\chi}'_1}v_{\boldsymbol{\chi}'_3}(v_{\boldsymbol{\chi}'_1}^2 - v_{\boldsymbol{\chi}'_3}^2) + 72\lambda' v_{\chi'_2}(v_{\boldsymbol{\chi}'_3}^2 - v_{\boldsymbol{\chi}'_1}^2)(\mu_\chi + \lambda_{\zeta\chi}v_\Phi) + 4\lambda_{\zeta\chi}^2 v_{\boldsymbol{\chi}'_1}v_{\boldsymbol{\chi}'_3}(v_{\boldsymbol{\chi}'_1}^2 - v_{\boldsymbol{\chi}'_3}^2) + 12A_{\chi'}v_{\boldsymbol{\chi}'_2}(v_{\boldsymbol{\chi}'_3}^2 - v_{\boldsymbol{\chi}'_1}^2),$$

as well as

$$0 = 36\lambda'^2 v_{\boldsymbol{\chi}'_2}v_{\boldsymbol{\chi}'_3}(v_{\boldsymbol{\chi}'_2}^2 - v_{\boldsymbol{\chi}'_3}^2) + 72\lambda' v_{\chi'_1}(v_{\boldsymbol{\chi}'_3}^2 - v_{\boldsymbol{\chi}'_2}^2)(\mu_\chi + \lambda_{\zeta\chi}v_\Phi) + 4\lambda_{\zeta\chi}^2 v_{\boldsymbol{\chi}'_2}v_{\boldsymbol{\chi}'_3}(v_{\boldsymbol{\chi}'_2}^2 - v_{\boldsymbol{\chi}'_3}^2) + 12A_{\chi'}v_{\boldsymbol{\chi}'_1}(v_{\boldsymbol{\chi}'_3}^2 - v_{\boldsymbol{\chi}'_2}^2).$$

These equations have three solutions: we choose one to produce the neutrino mass matrix $\langle\chi'\rangle = (\upsilon_{\chi'_1}, 0, 0)$, and the other two possibilities are $\langle\chi'\rangle = (0, \upsilon_{\chi'_2}, 0)$ and $\langle\chi'\rangle = (0, 0, \upsilon_{\chi'_3})$.

APPENDIX C: TENSOR PRODUCT OF A_4 TRIPLETS

Here we give useful tools for the computation of the tensor product of A_4 triplets. For the case of two A_4 triplets taken as $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, their tensor product is reducible with irreducible components given by the following decomposition relation:

$$3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3_S \oplus 3_A. \tag{C1}$$

Expressing this product as

$$\mathbf{a} \otimes \mathbf{b} = \oplus_i\, ((\mathbf{a} \otimes \mathbf{b})|_{R_i}), \tag{C2}$$

the irreducible components are given by

$$\begin{aligned}
(\mathbf{a} \otimes \mathbf{b})|_1 &= a_1 b_1 + a_2 b_2 + a_3 b_3,\\
(\mathbf{a} \otimes \mathbf{b})|_{1'} &= a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3,\\
(\mathbf{a} \otimes \mathbf{b})|_{1''} &= a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3,\\
(\mathbf{a} \otimes \mathbf{b})|_{3_S} &= (a_2 b_3 + a_3 b_2, a_3 b_1 + a_1 b_3, a_1 b_2 + a_2 b_1),\\
(\mathbf{a} \otimes \mathbf{b})|_{3_A} &= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).
\end{aligned} \tag{C3}$$

As an application, we present all possible A_4-invariant terms for the monomials χ^2, χ^3, and χ^4 which we encounter in the scalar potential (3.26) by using Eq. (C3). For the case χ^2, the previous $\mathbf{a}$ and $\mathbf{b}$ are identical, so we have

$$(\chi \otimes \chi)|_1 = \chi_1^2 + \chi_2^2 + \chi_3^2. \tag{C4}$$

The other $(\chi \otimes \chi)|_{R_i}$ are directly obtained from Eq. (C3). For χ^3, we have for the example of $(\chi \otimes \chi \otimes \chi)|_1$ the following expression:

$$\begin{aligned}
(\chi \otimes \chi \otimes \chi)|_1 &= \left.\left[\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix} \otimes \begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix}\right]_3 \otimes \begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix}_3\right|_1\\
&= \left.\left[\begin{pmatrix}2\chi_2\chi_3\\ 2\chi_1\chi_3\\ 2\chi_1\chi_2\end{pmatrix}_S + \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}_A\right] \otimes \begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix}\right|_1,
\end{aligned} \tag{C5}$$

leading to

$$(\chi \otimes \chi \otimes \chi)|_1 = 6\chi_1\chi_2\chi_3. \tag{C6}$$

Similar expressions can be written down for the other $(\chi \otimes \chi \otimes \chi)|_{R_i}$; they are not relevant for our study. To determine $(\chi \otimes \chi \otimes \chi \otimes \chi)|_1$, we start from

$$(\chi \otimes \chi \otimes \chi \otimes \chi)|_1 = \left.\left[\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix} \otimes \begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix}\right] \otimes \left[\begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix} \otimes \begin{pmatrix}\chi_1\\ \chi_2\\ \chi_3\end{pmatrix}\right]\right|_1. \tag{C7}$$

Then, using

$$(3 \otimes 3 \otimes 3 \otimes 3)|_1 = [1 \oplus 1' \oplus 1'' \oplus 3_S \oplus 3_A] \otimes [1 \oplus 1' \oplus 1'' \oplus 3_S \oplus 3_A]|_1 \tag{C8}$$

and by setting

$$1 \times 1 = X, \qquad 1' \times 1'' = Y, \qquad 1'' \times 1' = Z, \tag{C9}$$

we have

$$\begin{aligned}
X &= [(\chi_1)^2 + (\chi_2)^2 + (\chi_3)^2]_1 \times [(\chi_1)^2 + (\chi_2)^2 + (\chi_3)^2]_1,\\
Y &= [(\chi_1)^2 + \omega(\chi_2)^2 + \omega^2(\chi_3)^2]_{1'} \times [(\chi_1)^2 + \omega^2(\chi_2)^2 + \omega(\chi_3)^2]_{1''},\\
Z &= [(\chi_1)^2 + \omega^2(\chi_2)^2 + \omega(\chi_3)^2]_{1''} \times [(\chi_1)^2 + \omega(\chi_2)^2 + \omega^2(\chi_3)^2]_{1'}.
\end{aligned} \tag{C10}$$

We also have

$$3_S \times 3_S = \begin{pmatrix} 2\chi_2\chi_3 \\ 2\chi_1\chi_3 \\ 2\chi_1\chi_2 \end{pmatrix}_S \times \begin{pmatrix} 2\chi_2\chi_3 \\ 2\chi_1\chi_3 \\ 2\chi_1\chi_2 \end{pmatrix}_S, \qquad 3_S \times 3_A = \begin{pmatrix} 2\chi_2\chi_3 \\ 2\chi_1\chi_3 \\ 2\chi_1\chi_2 \end{pmatrix}_S \times \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_A, \qquad 3_A \times 3_A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_A \times \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_A,$$
$$3_A \times 3_S = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_A \times \begin{pmatrix} 2\chi_2\chi_3 \\ 2\chi_1\chi_3 \\ 2\chi_1\chi_2 \end{pmatrix}_S. \tag{C11}$$

We end with

$$\chi^4|_1 = 3[(\chi_1)^4 + (\chi_2)^4 + (\chi_3)^4] + 4[(\chi_1)^2(\chi_2)^2 + (\chi_1)^2(\chi_3)^2 + (\chi_2)^2(\chi_3)^2]. \tag{C12}$$

Analogously, the exact calculations for the triplet χ' lead to

$$\begin{aligned} &\chi'^2|_1 = \chi_1'^2 + \chi_2'^2 + \chi_3'^2, \qquad \chi'^3|_1 = 6\chi_1'\chi_2'\chi_3', \\ &\chi'^4|_1 = 3[(\chi_1')^4 + (\chi_2')^4 + (\chi_3')^4] + 4[(\chi_1')^2(\chi_2')^2 + (\chi_1')^2(\chi_3')^2 + (\chi_2')^2(\chi_3')^2]. \end{aligned} \tag{C13}$$

After substituting the above results into the scalar potential (3.26), the minimum conditions (B3) are as follows:

$$\frac{\partial \mathcal{V}}{\partial \chi_1}\Big|_{\langle\chi_i\rangle = v_{\chi_i}} = 0, \qquad \frac{\partial \mathcal{V}}{\partial \chi_2}\Big|_{\langle\chi_i\rangle = v_{\chi_i}} = 0, \qquad \frac{\partial \mathcal{V}}{\partial \chi_3}\Big|_{\langle\chi_i\rangle = v_{\chi_i}} = 0, \tag{C14}$$

leading to

$$\begin{aligned} &108\lambda^2 v_{\chi_1}^3 + 72\lambda^2 v_{\chi_1} v_{\chi_2}^2 + 72\lambda^2 v_{\chi_1} v_{\chi_3}^2 + 2m_\chi^2 v_{\chi_1} + 12A_\chi v_{\chi_2} v_{\chi_3} = 0, \\ &108\lambda^2 v_{\chi_2}^3 + 72\lambda^2 v_{\chi_2} v_{\chi_1}^2 + 72\lambda^2 v_{\chi_2} v_{\chi_3}^2 + 2m_\chi^2 v_{\chi_2} + 12A_\chi v_{\chi_1} v_{\chi_3} = 0, \\ &108\lambda^2 v_{\chi_3}^3 + 72\lambda^2 v_{\chi_3} v_{\chi_1}^2 + 72\lambda^2 v_{\chi_3} v_{\chi_2}^2 + 2m_\chi^2 v_{\chi_3} + 12A_\chi v_{\chi_1} v_{\chi_2} = 0. \end{aligned} \tag{C15}$$

We also have

$$\frac{\partial \mathcal{V}}{\partial \chi_1'}\Big|_{\langle\chi_i'\rangle = v_{\chi_i'}} = 0, \qquad \frac{\partial \mathcal{V}}{\partial \chi_2'}\Big|_{\langle\chi_i'\rangle = v_{\chi_i'}} = 0, \qquad \frac{\partial \mathcal{V}}{\partial \chi_3'}\Big|_{\langle\chi_i'\rangle = v_{\chi_i'}} = 0, \tag{C16}$$

giving

$$\begin{aligned} 0 = {}& 8|\mu_\chi|^2 v_{\chi_1'} + 8\lambda'^2 v_{\chi_1'} v_\Phi^2 + 108\lambda'^2 v_{\chi_1'}^3 + 72\lambda'^2 v_{\chi_1'} v_{\chi_2'}^2 + 72\lambda'^2 v_{\chi_1'} v_{\chi_3'}^2 + 16\mu_\chi \lambda_{\zeta\chi} v_{\chi_1'} \\ &+ 72\mu_\chi \lambda' v_{\chi_2'} v_{\chi_3'} + 72\lambda_{\zeta\chi}\lambda' v_\Phi v_{\chi_2'} v_{\chi_3'} + 12\lambda_{\zeta\chi}^2 v_{\chi_1'}^3 + 8\lambda_{\zeta\chi}^2 v_{\chi_1'} v_{\chi_2'}^2 + 8\lambda_{\zeta\chi}^2 v_{\chi_1'} v_{\chi_3'}^2 \\ &+ 4k_\zeta \lambda_{\zeta\chi} v_{\chi_1'} + 12\lambda_{\zeta\chi}\lambda_\zeta v_{\chi_1'} v_\Phi^2 + 4h_\zeta \lambda_{\zeta\chi} v_u v_d v_{\chi_1'} + 4\delta_\zeta \lambda_{\zeta\chi} v_{\Delta_u} v_{\Delta_d} v_{\chi_1'} \\ &+ 8\mu_\chi \lambda_{\zeta\chi} v_\Phi v_{\chi_1'} + 2m_{\chi'}^2 v_{\chi_1'} + 4b_{\chi'} v_{\chi_1'} + 4A_{\zeta\chi'} v_\Phi v_{\chi_1'} + 12A_{\chi'} v_{\chi_3'} v_{\chi_2'} \end{aligned} \tag{C17}$$

and

$$\begin{aligned} 0 = {}& 8|\mu_\chi|^2 v_{\chi_2'} + 8\lambda'^2 v_{\chi_2'} v_\Phi^2 + 108\lambda'^2 v_{\chi_2'}^3 + 72\lambda'^2 v_{\chi_1'}^2 v_{\chi_2'} + 72\lambda'^2 v_{\chi_2'} v_{\chi_3'}^2 + 16\mu_\chi \lambda_{\zeta\chi} v_{\chi_2'} \\ &+ 72\mu_\chi \lambda' v_{\chi_1'} v_{\chi_3'} + 72\lambda_{\zeta\chi}\lambda' v_\Phi v_{\chi_1'} v_{\chi_3'} + 12\lambda_{\zeta\chi}^2 v_{\chi_2'}^3 + 8\lambda_{\zeta\chi}^2 v_{\chi_1'}^2 v_{\chi_2'} + 8\lambda_{\zeta\chi}^2 v_{\chi_2'} v_{\chi_3'}^2 \\ &+ 4k_\zeta \lambda_{\zeta\chi} v_{\chi_2'} + 12\lambda_{\zeta\chi}\lambda_\zeta v_{\chi_2'} v_\Phi^2 + 4h_\zeta \lambda_{\zeta\chi} v_u v_d v_{\chi_2'} + 4\delta_\zeta \lambda_{\zeta\chi} v_{\Delta_u} v_{\Delta_d} v_{\chi_2'} \\ &+ 8\mu_\chi \lambda_{\zeta\chi} v_\Phi v_{\chi_2'} + 2m_{\chi'}^2 v_{\chi_2'} + 4b_{\chi'} v_{\chi_2'} + 4A_{\zeta\chi'} v_\Phi v_{\chi_2'} + 12A_{\chi'} v_{\chi_3'} v_{\chi_1'}, \end{aligned} \tag{C18}$$

as well as

$$
\begin{aligned}
0 = {} & 8|\mu_\chi|^2 \upsilon_{\chi'_3} + 8\lambda'^2 \upsilon_{\chi'_3} \upsilon_\Phi^2 + 108\lambda'^2 \upsilon_{\chi'_3}^3 + 72\lambda'^2 \upsilon_{\chi'_3} \upsilon_{\chi'_1}^2 + 72\lambda'^2 \upsilon_{\chi'_3} \upsilon_{\chi'_2}^2 + 16\mu_\chi \lambda_{\zeta\chi} \upsilon_{\chi'_3} \\
& + 72\mu_\chi \lambda' \upsilon_{\chi'_1} \upsilon_{\chi'_2} + 72\lambda_{\zeta\chi} \lambda' \upsilon_\Phi \upsilon_{\chi'_1} \upsilon_{\chi'_2} + 12\lambda_{\zeta\chi}^2 \upsilon_{\chi'_3}^3 + 8\lambda_{\zeta\chi}^2 \upsilon_{\chi'_3} \upsilon_{\chi'_1}^2 + 8\lambda_{\zeta\chi}^2 \upsilon_{\chi'_3} \upsilon_{\chi'_2}^2 \\
& + 4k_\zeta \lambda_{\zeta\chi} \upsilon_{\chi'_3} + 12\lambda_{\zeta\chi} \lambda_\zeta \upsilon_{\chi'_3} \upsilon_\Phi^2 + 4h_\zeta \lambda_{\zeta\chi} \upsilon_u \upsilon_d \upsilon_{\chi'_3} + 4\delta_\zeta \lambda_{\zeta\chi} \upsilon_{\Delta_u} \upsilon_{\Delta_d} \upsilon_{\chi'_3} \\
& + 8\mu_\chi \lambda_{\zeta\chi} \upsilon_\Phi \upsilon_{\chi'_3} + 2m_{\chi'}^2 \upsilon_{\chi'_3} + 4b_{\chi'} \upsilon_{\chi'_3} + 4A_{\zeta\chi'} \upsilon_\Phi \upsilon_{\chi'_3} + 12A_{\chi'} \upsilon_{\chi'_1} \upsilon_{\chi'_2}.
\end{aligned}
\tag{C19}
$$

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