Massless Lüscher terms and the limitations of the AdS_3 asymptotic Bethe ansatz

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(Received 29 February 2016; published 25 May 2016)

In AdS_5/CFT_4 integrability the Bethe ansatz gives the spectrum of long strings, accurate up to exponentially small corrections. This is no longer true in three-dimensional anti–de Sitter (AdS_3) space, as we demonstrate here by studying Lüscher F-terms with a massless particle running in the loop. We apply this to the classic test of Hernández and López, in which the $su(2)$ sector Bethe equations (including the one-loop dressing phase) should match the semiclassical string theory result for a circular spinning string. These calculations do not agree in $AdS_3 \times S^3 \times T^4$, and we show that the sum of all massless Lüscher F-terms can reproduce the difference.

DOI: 10.1103/PhysRevD.93.106006

I. INTRODUCTION

There has been much recent work on extending what we have learned about the integrability of strings in $AdS_5 \times S^5$ [1,2] to the less than maximally supersymmetric background $AdS_3 \times S^3 \times T^4$ [3,4], which arises from a D1- and D5-brane intersection. Some results for the massive sector can be adapted quite simply from the five-dimensional anti–de Sitter (AdS_5) case, or even generalized to every $AdS_n \times S^n$. What is completely new is the presence of a massless sector, corresponding to the T^4 directions and their superpartners.

In all results to date it has been possible to ignore the massless excitations when studying the massive sector. This is true for the calculation of the exact S matrix from centrally extended symmetries [5] and the corresponding Bethe equations, for coset methods [6] and semiclassical energy corrections to spinning strings [7,8], for near-Berenstein-Maldacena-Nastase (-BMN) diagrammatic calculations of two- and four-point functions [9–11][1] and the one-loop S matrix via unitarity methods [13,14], and for the various calculations of the massive dressing phase [15–17].

But not everything works perfectly in this decoupled picture of the massive sector and, in particular, the comparison of the one-loop energy correction to circular spinning strings in S^3 to the expansion of the $su(2)$ sector Bethe equations fails. This is precisely the comparison from which Hernández and López constructed the complete one-loop dressing phase in $AdS_5 \times S^5$ [18,19], and a similar construction in $AdS_3 \times S^3 \times T^4$ was done in [7]. However the dressing phase constructed this way does not solve the crossing equations [16], nor does it agree with a construction from semiclassical magnon scattering [17], nor give the amplitude seen in near-BMN scattering [9]. Conversely, using the correct dressing phase (as agreed on by the authors of [16,17], and [9]) breaks the circular spinning string comparison. The resulting mismatch is of the same order as the one-loop energy.

Our paper offers a solution to this problem. Lüscher F-terms involving massive modes running in the loop give exponentially small finite-size corrections in $AdS_5 \times S^5$ [20,21]. But massless modes are in some sense "infinite range," and thus give rise to Lüscher F-terms which are not exponentially suppressed. In fact they are of the same order as the prediction from the asymptotic Bethe equations, and thus cannot be ignored, even when L is large. This is the first time it is necessary to include the massless particles in order to understand the massive sector.

In general one can consider Lüscher corrections wrapping the space any number of times [22], and with a massless virtual particle these all contribute at the same order. For multiparticle physical states (such as the circular string, with order $\sqrt{\lambda}$ excitations) however only the singly wrapped Lüscher terms are well understood [23]. Combining the results of [22] and [23], we write down a simple formula for a multiparticle correction wrapping n times. While we believe this omits some multiple wrapping effects which should contribute at the same order, when applied to the circular string (and summed over all n) it gives the correct functional form of the mismatch, up to a factor of 2.

II. CONFLICTING RESULTS ABOUT $su(2)$ CIRCULAR STRINGS

The $su(2)$ sector at strong coupling concerns strings in $\mathbb{R} \times S^3$. For classical string theory this is a consistent

[1]Earlier papers had to include massless modes on the internal legs of diagrams [12], but better ways to handle divergent integrals eventually removed this [9,10].

truncation of both $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$, thus we expect an identical integrable description. However at one loop the string should feel the entire space: this is seen explicitly in the list of modes needed for semiclassical analysis, and is encoded in the dressing phase of integrability.

The circular spinning string in S^3 which we study is given by [24]

$$t = \kappa\tau, \quad Z_1 = \frac{1}{\sqrt{2}} e^{i(\mathcal{J}\tau + m\sigma)}, \quad Z_2 = \frac{1}{\sqrt{2}} e^{i(\mathcal{J}\tau - m\sigma)}, \quad (1)$$

where t is AdS time, $S^3 \subset \mathbb{C}^2$ has coordinates Z_i, and we take $\sigma \in [0, 2\pi]$. Clearly $m \in \mathbb{Z}$ is a winding number. The Virasoro constraints impose $\kappa = \sqrt{\mathcal{J}^2 + m^2}$. This solution has two equal angular momenta, $J_1 = J_2 = R^2 \mathcal{J}/2$;[2] and energy $\Delta = R^2 \kappa$. We define the 't Hooft coupling as in AdS_5 by $R^2 = \sqrt{\lambda}$.

This solution is particularly simple in string theory because it is homogeneous, in the sense that a translation along σ maps to an isometry of the target space. It is also particularly simple in integrability because all the Bethe roots lie on one connected curve. Below we present the calculation of its energy at one loop in both of these pictures.

A. World sheet semiclassical calculation

The bosonic modes of the same solution in S^5 were calculated in [25]. While it is simple to repeat their calculation, there is no need to do so, as we can safely just keep the modes lying in S^3, and discard the other two. They have frequencies

$$w_n^{S\pm} = \sqrt{n^2 + 2\mathcal{J}^2 \pm 2\sqrt{n^2(\mathcal{J}^2 + m^2) + \mathcal{J}^4}}.$$

The bosonic modes in AdS directions of course have mass $s = \kappa$, and those in torus directions are massless:

$$w_n^A = \sqrt{n^2 + \kappa^2}, \qquad w_n^T = |n|.$$

For the fermionic modes, the calculation was performed in [7], and we briefly sketch it here. The equations of motion are as usual given by $\rho_- D_+ \Theta^1 = 0 = \rho_+ D_- \Theta^2$, where $D_\mu \Theta^I = (\partial_\mu + t_\mu)\Theta^I + F\rho_\mu \Theta^{\text{not}I}$ with $\mu = 0, 1$,

$$\rho_\mu = \partial_\mu X^M E_M^A \Gamma_A$$
$$t_\mu = \frac{1}{4}\partial_\mu X^M \omega_M^{AB} \Gamma_{AB}$$
$$F = \frac{1}{4}F_{(3)} = \frac{1}{4}(\Gamma^{012} + \Gamma^{345}),$$

and $\partial_\pm = \partial_0 \pm \partial_1$, $\rho_\pm = \rho_0 \pm \rho_1$, etc. If we adopt the κ gauge $\Theta^1 = \Theta^2$ then the equations of motion simplify to

$$(\rho_\mp \partial_\pm + \rho_\mp t_\pm + \rho_\mp F \rho_\pm)\Theta(\sigma, \tau) = 0.$$

Taking the sum of these equations, and using a plane wave ansatz for the modes $\Theta(\sigma, \tau) = e^{iw_n\tau + in\sigma}\Theta_0$ (where Θ_0 is a constant Majorana-Weyl spinor), we get

$$(\rho_0\partial_0 - \rho_1\partial_1 + \rho_0 t_0 - \rho_1 t_1 + \rho_0 F \rho_0 - \rho_1 F \rho_1) \times e^{iw_n\tau + in\sigma}\Theta_0 = 0.$$

Solving this equation for the mode frequencies, we find equally many massive and massless fermions:

$$w_n^F = \begin{cases} |n| & 4\,\text{massless} \\ \sqrt{n^2 + \mathcal{J}^2} & 4\,\text{of mass}\ \mathcal{J}. \end{cases}$$

With these mode frequencies we can now calculate the semiclassical one-loop energy correction

$$\delta E = \sum_n e(n), \qquad e(n) = \sum_b^{8+8} (-1)^F \frac{1}{2\kappa} w_n^b.$$

Clearly the 4 massless fermionic and 4 massless bosonic modes cancel, and we need only the $4+4$ massive mode frequencies. Using the resummation procedure of [18], the nonanalytic term δE^{int} was determined in [7] to be the following:

$$\delta E_{\text{BLMT}} = \frac{m^4}{2\mathcal{J}^3} - \frac{7m^6}{12\mathcal{J}^5} + \frac{29m^8}{48\mathcal{J}^7} - \frac{97m^{10}}{160\mathcal{J}^9} + \frac{2309m^{12}}{3840\mathcal{J}^{11}} + \cdots. \quad (2)$$

It is this nonanalytic term that should be directly compared to the Bethe ansatz calculation, presented next.

B. Bethe ansatz calculation

The relevant Bethe equations were found in [5], and the $su(2)$ sector is exactly the same as the $su(2)$ sector Bethe equation from $AdS_5 \times S^5$, as it must be. This is the case $\eta = +1$ of

$$\left(\frac{x_i^+}{x_i^-}\right)^L = \prod_{j\neq i}^K \left[\frac{x_i^+ - x_j^-}{x_i^- - x_j^+}\right]^\eta \left(\frac{1 - 1/x_i^+ x_j^-}{1 - 1/x_i^- x_j^+}\right) \sigma^{\bullet\bullet}(x_i, x_j)^2. \quad (3)$$

Here we use the following expansion of the dressing phase:

$$\sigma^{\bullet\bullet}(x, y) = \exp\left[\frac{i}{4\pi}\sum_{r,s} c_{r,s} Q_r(x) Q_s(y)\right], \quad (4)$$

[2] Note the factor of 2. If we write $J = J_1 + J_2$ then $\mathcal{J} = J/\sqrt{\lambda}$. Later we use $J = L$, for instance in (6).

where $c_{r,s} = hc^{(0)}_{r,s} + c^{(1)}_{r,s} + \mathcal{O}(1/h)$ with $c^{(0)}_{r,s}$ being the usual Arutyunov-Frolov-Staudacher (AFS) phase [26]. We normalize the one-loop phase $c^{(1)}_{r,s}$ as in [19], and note that in AdS_3 we are interested in the left-left phase.[3]

The analysis we need is identical to that of [19], and we briefly review the procedure. Assuming that there is just one cut (with one mode number k), in the thermodynamic limit we can replace the product by an integral. Multiplying by the relevant weight and integrating over the cut, [19] obtained this expression in terms of the resolvents[4]:

$$\begin{aligned} G^2 - 2\pi k G - \eta G^{(1)} &= \tilde{g}^2[G^{(1)2} - 2\pi k G^{(2)}] + \tilde{g}^2(1+\eta)[G^{(1)}\tilde{Q}_2 - G^{(2)}\tilde{Q}_1] \\ &\quad + \sum_{r,s} - 2\eta c^{(1)}_{r,s}\frac{1}{\sqrt{\lambda}}\tilde{g}^{r+s-1}[G^{(r)}\tilde{Q}_s - G^{(s)}\tilde{Q}_r] + \mathcal{O}\left(\frac{1}{\lambda}\right), \end{aligned} \tag{5}$$

where

$$G(\tilde{x}) = -\sum_{n=0}^{\infty}\tilde{Q}_{n+1}\tilde{x}^n, \qquad G^{(r)} = -\sum_{m=0}^{\infty}\tilde{Q}_{m+r+1}\tilde{x}^m,$$

and

$$\tilde{g} = \frac{1}{4\pi\mathcal{J}} = \frac{h}{2L}. \tag{6}$$

The first two lines of (5) are the classical Bethe equation. To calculate the effect of the dressing phase (on the second line), we perturb all the charges:

$$\tilde{Q}_n = \tilde{Q}^0_n + \frac{1}{\sqrt{\lambda}}\delta\tilde{Q}_n, \qquad \delta\tilde{Q}_1 = 0,$$

and expand to order $1/\sqrt{\lambda}$.[5] We are interested in the simplest case with $k = 2m$. This gives a cancellation such that $\delta\tilde{Q}_2$ can be found alone, and the result is

$$\delta E = \frac{\tilde{g}^2}{2\pi}\delta\tilde{Q}_2 = \sum_{r,s} c^{(1)}_{r,s}\tilde{g}^{r+s}\frac{\tilde{Q}^0_{r+1}\tilde{Q}^0_s - \tilde{Q}^0_{s+1}\tilde{Q}^0_r}{\pi(1+2\tilde{g}^2\tilde{Q}^0_2)}.$$

Expanding the one-cut resolvent

$$G^0(\tilde{x}) = 2\pi m + \frac{\sqrt{1+(4\pi m\tilde{g})^2} - \sqrt{1+(4\pi m\tilde{x})^2}}{2(\tilde{x} - \tilde{g}^2/\tilde{x})} \tag{7}$$

we obtain classical charges starting with

$$\tilde{Q}^0_1 = -2\pi m,$$
$$\tilde{Q}^0_2 = \frac{1}{2\tilde{g}^2}[-1 + \sqrt{1+16\pi^2\tilde{g}^2m^2}].$$

Substituting these into δE and expanding leads to

$$\begin{aligned} \delta E &= \frac{m^4 c_{1,2}}{4\mathcal{J}^3} + \frac{m^6(-4c_{1,2} - c_{1,4} + c_{2,3})}{16\mathcal{J}^5} \\ &\quad + \frac{m^8(15c_{1,2} + 5c_{1,4} + 2c_{1,6} - 5c_{2,3} - 2c_{2,5} + c_{3,4})}{64\mathcal{J}^7} \\ &\quad + \cdots. \end{aligned} \tag{8}$$

Then using the coefficients from [19] (i.e. $c^{\mathrm{HL}}_{r,s} = -8\frac{(r-1)(s-1)}{(r+s-2)(s-r)}$ for $r, s \geq 2$) naturally gives us the AdS_5 answer:

$$\delta E_{\mathrm{HL}} = -\frac{m^6}{3\mathcal{J}^5} + \frac{m^8}{3\mathcal{J}^7} - \frac{49m^{10}}{120\mathcal{J}^9} + \frac{2m^{12}}{5\mathcal{J}^{11}} + \mathcal{O}\left(\frac{1}{\mathcal{J}^{13}}\right).$$

In fact the only change from the $AdS_5 \times S^5$ derivation made here is that we have allowed for the possibility of $c_{1,s} \neq 0$ in (8).

To use this result (8) in $AdS_3 \times S^3 \times T^4$ we simply need to substitute in different coefficients. The first set $c^{\mathrm{BLMT}}_{r,s}$ proposed in [7] is

$$c^{\mathrm{BLMT}}_{r,s} = 2\frac{s-r}{r+s-2}, \qquad r+s \text{ odd}, r, s \geq 1 \tag{9}$$

and using this, we recover δE_{BLMT} of (2) above, by design: [7] performed exactly this comparison.

The second set of coefficients $c^{\mathrm{BOSST}}_{r,s}$ was found by solving symmetry conditions on the S matrix including crossing symmetry [16], as well as from a semiclassical calculation involving giant magnon scattering [17].[6] These two techniques agree perfectly, and also agree with the near-BMN scattering amplitude [9]. They give

$$c^{\mathrm{BOSST}}_{r,s} = \left[2\frac{s-r}{r+s-2} - \delta_{r,1} + \delta_{1,s}\right], \qquad r+s \text{ odd}, r, s \geq 1 \tag{10}$$

[3]The spectrum of $AdS_3 \times S^3 \times T^4$ divides massive particles into left and right sectors. The product of the left-left phase $\sigma^{\bullet\bullet}$ and the left-right phase $\tilde{\sigma}^{\bullet\bullet}$ is in fact the dressing phase of AdS_5. We assume that the coefficients $c_{r,s}$ are antisymmetric in $r \leftrightarrow s$, and zero when $r+s$ even.

[4]We write tildes on quantities defined to match [19], which differ from those in the S matrix and dressing phase papers [5,16,17,27] by powers of $\tilde{g}$. The spectral parameters are related as $\tilde{x} \equiv \tilde{g}x$. The charges are related as $\tilde{Q}_n = \tilde{g}^{n-1}Q_n$ but we always use $\tilde{Q}_n$ for the total, and $Q_n(x^\pm_k)$ for the constituents.

[5]The Bethe equations give us an expansion in a coupling h which in general may be a nontrivial function of the 't Hooft coupling $\sqrt{\lambda}$. However this does not happen in $AdS_5 \times S^5$, nor (at least to one loop) in $AdS_3 \times S^3 \times T^4$, where we have $\sqrt{\lambda} = 2\pi h + \mathcal{O}(1/h)$.

[6]This calculation was also done in [7], which omitted a crucial twist in the algebraic curve. The correct integral was calculated earlier in [15], however they did not express it in terms of the charges Q_n. The analogous calculations in $AdS_5 \times S^5$ are in [28,29], in that case done after both [19] and [30,31].

and using this in (8) we obtain

$$\delta E_{\rm BOSST} = \frac{m^4}{4\mathcal{J}^3} - \frac{13m^6}{48\mathcal{J}^5} + \frac{25m^8}{96\mathcal{J}^7} - \frac{311m^{10}}{1280\mathcal{J}^9} + \frac{1723m^{12}}{7680\mathcal{J}^{11}} + \cdots$$

thus

$$\delta E_{\rm BOSST} - \delta E_{\rm BLMT} = -\frac{m^4}{4\mathcal{J}^3} + \frac{5m^6}{16\mathcal{J}^5} - \frac{11m^8}{32\mathcal{J}^7} + \frac{93m^{10}}{256\mathcal{J}^9} - \frac{193m^{12}}{512\mathcal{J}^{11}} + \cdots = +\frac{m^2(\mathcal{J}-\kappa)}{2\kappa^2}. \quad (11)$$

This difference is the mismatch we seek to explain. (The closed form on the last line is guessed from the series, and checked to order $1/\mathcal{J}^{15}$.)

III. THE CONTRIBUTION FROM MASSLESS LÜSCHER TERMS

When calculating quantum corrections to the mass of a particle, by drawing Feynman diagrams for the self-energy, Lüscher terms are the effect of the new diagrams possible in finite volume, namely those in which a loop wraps the space. The original context was relativistic theories [32,33], for which (in $1+1$ dimensions) the effect is

$$\delta E^F = -s\int \frac{d\theta}{2\pi} e^{-sL\cosh\theta}\cosh\theta \left[S^{ab}_{ab}\left(\theta + i\frac{\pi}{2}\right) - 1\right]. \quad (12)$$

In this case the dispersion relation is $\varepsilon(p) = \sqrt{p^2+s^2} = s\cosh\theta$ and the S matrix is a function only of the difference of the rapidities $S(p_a,p_b) = S(\theta_a - \theta_b)$. The reason the S matrix appears is that taking a large L puts the particle circling the space on shell, leaving just one integral over the loop momentum. This has made the formula very useful for integrable theories, where the same S matrix is what defines the Bethe equations. The derivation of δE^F can be done allowing arbitrary dispersion relations [34], including the magnon dispersion relation of AdS/CFT integrability. This has provided various tests at strong coupling [20,34–38] including some in AdS_3 [39].

Lüscher corrections are usually exponentially suppressed in a large volume L. The crucial observation for this paper is that the exponential $\sim e^{-sL}$ in (12) contains the mass s of the virtual particle, and thus in a system with massless particles, we no longer expect this suppression. There are similar terms in which the particle wraps the space n times, typically $\sim e^{-nsL}$ and thus subleading. But with a massless virtual particle, we expect these to all be of the same order.[7]

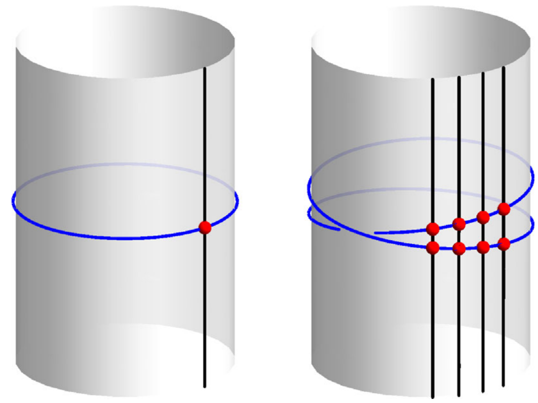

FIG. 1. (Left panel) The simplest F-term Feynman diagram. (Right panel) The generalizations we consider allow a multiparticle physical state ($K=4$ shown) and multiple wrappings by the virtual particle ($n=2$ shown).

Thus we need a generalization of the simplest formula in two directions: to treat a multiparticle physical state, and to allow multiple wrappings. (Both are drawn in Fig. 1.) These have been studied separately in the literature, using techniques other than the original Feynman diagrams, and were reviewed in [21].

A. Multiply wrapped and multiparticle formulas

The derivation of [22] is a one-loop semiclassical correction, treating the physical particle as a soliton. They extended this to include the multiple wrappings appearing at one-loop level (by picturing a cylinder of twice the radius with two physical solitons, and so on), and obtained the following sum over n:

$$\delta E_{\rm HJL} = -\sum_b (-1)^{F_b} \int_{-\infty}^{\infty} \frac{dq}{2\pi}\left(1 - \frac{\varepsilon_a'(p)}{\varepsilon_b'(q_\star)}\right) \times \sum_{n=1}^{\infty}\frac{1}{n} e^{-inq_\star L}[S^{ba}_{ba}(q_\star,p)^n - 1]. \quad (13)$$

The $n=1$ term here agrees with [34], and reduces to (12) in the relativistic case. Pausing to fix our notation, this is an energy correction to a particle of type a (and momentum p, dispersion relation ε_a) due to virtual particles of all types b (and ε_b) circling the cylinder of size L. The momentum $q_\star$ is defined as a function of q by the on-shell condition $q^2 + \varepsilon_b^2(q_\star) = 0$. The integration contour we use has the Euclidean energy q real, and thus $q_\star$ is imaginary. In this notation the Lorentzian two-momenta of the real and virtual particles are

[7] For our nonrelativistic system the exponent is not proportional to the mass, but the conclusions of this paragraph still hold. See (17) for the form: $q_\star \propto |q|$ means that the contribution from near $q=0$ in the integral is not exponentially suppressed.

$$p_\mu = (\varepsilon_a(p), p)$$
$$q_\mu = (iq, q_\star) = (\varepsilon_b(q_\star), q_\star).$$

Another derivation is possible from the thermodynamic Bethe ansatz (TBA); in fact this was the first approach in AdS/CFT [40]. A Lüscher formula for multiparticle physical states was derived in this way in [23]:

$$\delta E_{\mathrm{BJ}} = -\sum_{j,k} \varepsilon'_a(p_k) \left(\frac{\delta \mathrm{BY}_k}{\delta p_j} \right)^{-1} \delta\Phi_j - \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{b_1 \cdots b_K} (-1)^{F_{b_1}} [S^{b_2 a}_{b_1 a}(q_\star, p_1) \times S^{b_3 a}_{b_2 a}(q_\star, p_2) \cdots S^{b_1 a}_{b_K a}(q_\star, p_K)] e^{-iq_\star L}. \quad (14)$$

The pieces of the first term come from writing the Bethe equations as $2\pi n_k = \mathrm{BY}_k + \delta\Phi_k$ with

$$\mathrm{BY}_k = p_k L - i \sum_{j \neq k} \log S^{aa}_{aa}(p_k, p_j)$$
$$\delta\Phi_j = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{b_1 \cdots b_K} (-1)^{F_{b_1}} [S^{b_2 a}_{b_1 a}(q_\star, p_1) \cdots \cdots S^{b_1 a}_{b_K a}(q_\star, p_K)] e^{-iq_\star L} \frac{\partial}{\partial q} \log S^{b_{j+1} a}_{b_j\ a}(q_\star, p_j).$$

When $K = 1$ this reduces to the $n = 1$ term of (13). For the case of interest here, we will see that the first line of (14) will vanish, and the sum over various internal choices in $S^{b_2 a}_{b_1 a} S^{b_3 a}_{b_2 a} \cdots S^{b_1 a}_{b_K a}$ will turn out to be trivial, as the structure of the S matrix forces $b_j = b$ always.

Combining features of (13) and (14), we will also consider the following formula:

$$\delta E = -\int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-inq_\star L} \sum_b^{4+4} (-1)^{F_b} \left[\prod_{k=1}^{K} S^{ba}_{ba}(q_\star, p_k) \right]^n. \quad (15)$$

All terms n here will contribute at the same order. As was pointed out in [22] about (13), at $n > 1$ this omits other multiply wrapped effects which would be expected to contribute at the same order.

After some preliminaries, we apply the singly wrapped formula (14) to the circular string in Sec. III D, and then look at multiple wrappings using (15) in Sec. III E.

B. Variables and S matrix

In order to use the known S matrix [27,41] we must describe both particles with Zukhovski variables. Allowing a generic mass s these are defined by[8]

$$p_z = -i \log \frac{z^+}{z^-}, \qquad s_z = \frac{h}{2i} \left(z^+ + \frac{1}{z^+} - z^- - \frac{1}{z^-} \right)$$

and describe dispersion relation

$$\varepsilon_z(p_z) = \frac{h}{2i} \left(z^+ - \frac{1}{z^+} - z^- + \frac{1}{z^-} \right) = \sqrt{s_z^2 + 4h^2 \sin^2 \frac{p}{2}}.$$

It is often useful to define the spectral parameter z by

$$z^\pm + \frac{1}{z^\pm} = z + \frac{1}{z} \pm i \frac{s_z}{h}.$$

For the real particles, we will write $x^\pm$ (or rather $x_k^\pm$) in terms of x defined like this, see (21) below.

For massive virtual particles $y^\pm$, usually one transforms the integral on q into an integral on y along the upper half unit circle. But when this is massless, the variable y is not useful as it approaches 1 in the limit $s_y \to 0$. Instead, we write everything in terms of $q = -i\varepsilon(q_\star)$, finding

$$y^\pm = 1 \pm \frac{|q|}{2h} + \frac{|q|^2}{8h^2} + \mathcal{O}\left(\frac{1}{h^3} \right) \quad (16)$$

and

$$q_\star = -\frac{i}{h} |q| + \frac{i|q|^3}{24h^3} + \cdots$$
$$\Rightarrow e^{-iq_\star L} = e^{-L|q|/h} + \mathcal{O}\left(\frac{1}{h^3} \right). \quad (17)$$

The relevant matrix components of the S matrix for massless-massive scattering are given in Appendix N of [27]. Taking the physical particle to be $a = Y^L_p$ (a left-sector sphere boson), we are interested in the following matrix components:

$$\hat{S}_{q_\star, p} = \hat{S}^{ba}_{ba}(y^\pm, x^\pm) = \hat{S}^{-1}_{p, q_\star} = (N^{\bullet\circ}_{p, q_\star})^{-1} \begin{cases} (A^{LL}_{p,q_\star} B^{LL}_{p,q_\star})^{-1} & 4\,\text{bosons},\ b = T^{\dot{a}a}_{q_\star} \\ (A^{LL}_{p,q_\star} A^{LL}_{p,q_\star})^{-1} & 2\,\text{fermions},\ b = \chi^a_{q_\star}, \\ (B^{LL}_{p,q_\star} B^{LL}_{p,q_\star})^{-1} & 2\,\text{fermions},\ b = \tilde{\chi}^a_{q_\star} \end{cases} \quad (18)$$

where in (M.1) of [27] we find

$$A^{LL}_{p,q_\star} \equiv 1, \qquad B^{LL}_{p,q_\star} \equiv \sqrt{\frac{x^-}{x^+}} \frac{x^+ - y^+}{x^- - y^+}.$$

We will also need the normalization factor from (O.2) of [27]:

[8]The mass is normalized so that $s = 1$ in $\mathrm{AdS}_5 \times S^5$, rather than κ as in Sec. II A. Note also that this equation fixes the definition of h to match [17,27,42] but not [5,7,16].

$$N^{\bullet\circ}_{p,q_\star} \equiv \sqrt{\frac{x^-}{x^+}}\sqrt{\frac{1-1/x^-y^+}{1-1/x^+y^-}}\sqrt{\frac{1-1/x^-y^-}{1-1/x^+y^+}}.$$

We have defined this without the dressing phase, so in full we want $S_{p,q_\star} = \hat{S}_{p,q_\star}(\sigma^{\bullet\circ}_{p,q_\star})^2$.

Our calculation will be sensitive to the classical part of the mixed-mass dressing phase $\sigma^{\bullet\circ}$, which we take to be[9]

$$\sigma^{\bullet\circ}_{\rm AFS}(x^\pm,y^\pm) = \exp\left\{ i\frac{h}{W}\sum_{r=2}^{\infty}[Q_r(x^\pm)Q_{r+1}(y^\pm) - Q_{r+1}(x^\pm)Q_r(y^\pm)]\right\} \quad (19)$$

in terms of charges defined as $Q_1(z) \equiv p_z = -i\log(z^+/z^-)$ and, for $n>1$,

$$Q_n(z) \equiv \frac{i}{n-1}\left[\frac{1}{(z^+)^{n-1}} - \frac{1}{(z^-)^{n-1}}\right].$$

For now we leave the coefficient W unfixed.[10]

C. Ingredients for the $su(2)$ circular string

The S-matrix part of (15) involves a product over all of the physical particles $x_k^\pm$. We can rewrite this as an integral over the cut in the resolvent, in exactly the same way as is done for the Bethe equations to derive (5) above:

$$\begin{aligned} S_b &\equiv \prod_{k=1}^{K} S_{ba}(y^\pm, x_k^\pm) \\ &= \exp\left[\sum_{k=1}^{K}\log S_{ba}(y^\pm,x_k^\pm)\right] \\ &= \exp\left[L\int_C d\tilde{x}\,\frac{\tilde{x}^2-\tilde{g}^2}{\tilde{x}^2}\rho(\tilde{x})\log S_{ba}(y^\pm, x^\pm(\tilde{x}))\right]. \end{aligned} \quad (20)$$

The density ρ encodes the resolvent as

$$G(\tilde{x}) = \int_C d\tilde{z}\frac{\rho(\tilde{z})}{\tilde{x}-\tilde{z}} = -\sum_{n=0}^{\infty}\tilde{Q}_{n+1}\tilde{x}^n$$

and the integration is over the single connected cut C defining our solution. It is easy to work out ρ from (7), but to do the integral over $\tilde{x}$ it is much better to use identities

$$\tilde{Q}_n = \int_C d\tilde{z}\frac{\rho(\tilde{z})}{\tilde{z}^n}.$$

The spectral parameters $\tilde{x}$, $\tilde{z}$ appearing here are scaled by $\tilde{g}$ relative to the $x^\pm$, $y^\pm$ used in the S matrix. This follows the convention in [19], and is convenient for taking the limit $L\to\infty$ at a fixed $\tilde{g}$. This expansion gives $x_k^\pm$ as follows:

$$x^\pm = \frac{1}{\tilde{g}}\left[\tilde{x} \pm \frac{i}{2L}\frac{\tilde{x}^2}{\tilde{x}^2-\tilde{g}^2} + \mathcal{O}\left(\frac{1}{L^2}\right)\right]. \quad (21)$$

The variables $y^\pm$ are still given by (16) above. We have an expansion in h, but using $h = 2\tilde{g}L$ we regard this as an expansion in L, i.e. $y^\pm = 1 \pm |q|/4\tilde{g}L + \mathcal{O}(1/L^2)$.

With these expansions we can now write the leading contribution for the S-matrix terms

$$-\log B^{LL}_{p_k,q_\star} = +\log N^{\bullet\circ}_{p_k,q_\star} = -\frac{i}{2L}\frac{\tilde{x}}{(\tilde{x}-\tilde{g})^2} + \mathcal{O}\left(\frac{1}{L^2}\right).$$

Expanding in $\tilde{g} \ll 1$ and integrating as in (20), we get (using $\tilde{Q}_n = 0$ for every odd $n \geq 3$)

$$\begin{aligned} 2i\theta &\equiv \sum_{k=1}^{K} -\log B^{LL}_{p_k,q_\star} \\ &= -\frac{i}{2}\left(\tilde{Q}_1 + 2\sum_{n=1}^{\infty} g^n\tilde{Q}_{n+1}\right) \\ &= -i\pi m + iG(\tilde{g}). \end{aligned} \quad (22)$$

We must perform a similar sum for the AFS phase, since (19) refers to the constituent particles. The charges are

$$\begin{aligned} Q_n(x_k^\pm) &= \frac{\tilde{g}^{n-1}}{L\tilde{x}^{n-2}}\frac{1}{\tilde{x}^2-\tilde{g}^2} + \mathcal{O}\left(\frac{1}{L^2}\right) \\ Q_n(y^\pm) &= -\frac{i|q|}{h} + \cdots \end{aligned}$$

and thus the total phase is

$$\begin{aligned} \prod_{k=1}^{K}\sigma^{\circ\bullet}_{\rm AFS}(y^\pm,x_k^\pm) &= \prod_{k=1}^{K}\exp\left[-\frac{|q|}{W}Q_2(x_k^\pm)\right] \\ &= \exp\left[-\frac{|q|\tilde{g}\tilde{Q}_2}{W}\right]. \end{aligned}$$

Let us combine this with the $e^{-iq_\star L}$ factor as follows:

$$e^{-iq_\star L}(\sigma^{\circ\bullet}_{\rm AFS})^2 = \exp(-|q|\phi), \qquad \phi = \frac{1}{2\tilde{g}} + \frac{\tilde{g}\tilde{Q}_2}{W}. \quad (23)$$

Then finally putting all of this into the S matrix (18), we have

[9] The massive sector phase $\sigma^{\bullet\bullet}$ contains an AFS phase of this form, with $W=2$. This is $c^{(0)}_{r,s}$ in (4).

[10] In [42] we used the same form (19) for the AFS phase for particles of mass α and $1-\alpha$ (in $\mathrm{AdS}_3\times S^3\times S^3\times S^1$). The coefficient there was $W_{xy} = 4s_xs_y/(s_x+s_y)$, which goes to zero if $s_x = 1$, $s_y = 0$.

$$e^{-iq_\star L}S_b = \begin{cases} e^{-|q|\phi} & b = T \text{ massless bosons} \\ e^{-|q|\phi}e^{-2i\theta} & b = \chi \text{ massless fermions} \\ e^{-|q|\phi}e^{+2i\theta} & b = \tilde{\chi}. \end{cases}$$

D. Singly wrapped term

Using all the pieces we have calculated, it is now straightforward to work out the singly wrapped Lüscher F-term (14), or equivalently, the $n = 1$ term of (15). We set $W = 2$, and assume m is even:

$$\begin{aligned} \delta E &= -\int_{-\infty}^{\infty}\frac{dq}{2\pi}\sum_b^{4+4}(-1)^{F_b}e^{-iq_\star L}S_b \\ &= -\int_{-\infty}^{\infty}\frac{dq}{2\pi}8e^{-|q|\phi}\sin^2\theta \\ &= -\frac{8}{\pi}\frac{\sin^2\theta}{\phi}. \end{aligned}$$

Then expanding in $\tilde{g} = 1/4\pi\mathcal{J} \ll 1$,

$$\begin{aligned} \delta E = \frac{-m^4}{2\mathcal{J}^3} + \frac{15m^6 + \pi^2 m^8}{24\mathcal{J}^5} \\ - \frac{990m^8 + 135\pi^2 m^{10} + 2\pi^4 m^{12}}{1440\mathcal{J}^7} + \cdots. \end{aligned} \tag{24}$$

This is clearly of the same order as (11), although all the coefficients are wrong (and there are unwanted powers of m). These will be changed by including the $n > 1$ terms below. But first we check our claim above that this $n = 1$ term of (15) agrees exactly with the multiparticle formula (14):

(i) We can see that the first line of (14) vanishes by calculating $\delta\Phi_j$. The constituent S-matrix elements are independent of q, and thus $\frac{\partial}{\partial q}\log S_{b_j a}^{b_{j+1}a}(q_\star, p_j) = 0$.

(ii) We have simplified $S_{b_1 a}^{b_2 a}S_{b_2 a}^{b_3 a}\cdots S_{b_K a}^{b_1 a}$ by setting $b_k = b$ for all k. Looking at the S matrix as given in (N.3) of [27] again, notice that acting on $|Y^L T^{\dot{a}a}\rangle$ this never gives $|X\,Y^L\rangle$ with $X \neq T^{\dot{a}a}$. The same is true for the massless fermions, thus no diagrams with $b_k \neq b_{k+1}$ survive.

E. Sum over all wrappings

Doubly wrapped Lüscher F-terms would usually (with massive particles) be suppressed by the exponential factor squared. But this is not true with massless virtual particles, and we find that the n-wrapped terms all contribute at order $1/\mathcal{J}^3$, the same as the singly wrapped term (24). Thus we ought to sum all of them.

We can do this using (15), which attempts to add multiple wrapping corrections along the lines of (13). Using the same ingredients as above, the steps are as follows:

$$\begin{aligned} \delta E &= -\int_{-\infty}^{\infty}\frac{dq}{2\pi}\sum_{n=1}^{\infty}\sum_b^{4+4}(-1)^{F_b}\frac{1}{n}[e^{-iq_\star L}S_b]^n \\ &= -\int_{-\infty}^{\infty}\frac{dq}{2\pi}\sum_{n=1}^{\infty}\frac{8}{n}e^{-n|q|\phi}\sin^2 n\theta \\ &= -\int_{-\infty}^{\infty}\frac{dq}{2\pi}2\log\frac{(1-e^{-|q|\phi+2i\theta})(1-e^{-|q|\phi-2i\theta})}{(1-e^{-|q|\phi})^2} \\ &= \frac{2}{\phi}\left[-\frac{\pi}{3} + \frac{1}{\pi}\mathrm{Li}_2(e^{2i\theta}) + \frac{1}{\pi}\mathrm{Li}_2(e^{-2i\theta})\right]. \end{aligned} \tag{25}$$

At this point we can expand in $\tilde{g}$ by brute force, but we can also obtain the closed form of (11) more elegantly. Begin by observing that (for an even m)[11] the term $\frac{-1}{4}Q_1 = -\frac{1}{2}\pi m$ in θ (22) does not contribute to $\sin^2 n\theta$. Thus we may replace θ with the $\bar{\theta}$ given by

$$\bar{\theta} = -\frac{2\pi^2 m^2\tilde{g}}{\sqrt{1+16\pi^2\tilde{g}^2 m^2}} = -\frac{\pi m^2}{2\kappa}$$

recalling $\kappa = \sqrt{\mathcal{J}^2 + m^2}$. We also write

$$\phi = \frac{2\pi}{W}[\mathcal{J}(W-1) + \kappa].$$

Now we can use some properties of the dilogarithm to simplify (25). For $0 < |\gamma| < 2\pi$ we have [43]

$$\mathrm{Li}_2(e^{\pm i\gamma}) = \sum_{n=1}^{\infty}\frac{\cos(\pm n\gamma)}{n^2} + i\sum_{n=1}^{\infty}\frac{\sin(\pm n\gamma)}{n^2}.$$

Then using the elementary sum $\sum_{n=1}^{\infty}\frac{\cos(\pm n\gamma)}{n^2} = \frac{\pi^2}{6} \mp \frac{\pi\gamma}{2} + \frac{\gamma^2}{4}$, we obtain

$$\mathrm{Li}_2(e^{+i\gamma}) + \mathrm{Li}_2(e^{-i\gamma}) = \frac{\pi^2}{3} + \frac{\gamma^2}{2}.$$

For our calculation $\gamma = 2\bar{\theta}$, and when $\tilde{g}$ is small (κ is large) this is in the required range. Thus (25) becomes simply

$$\begin{aligned} \delta E &= \frac{1}{\pi\phi}(2\bar{\theta})^2 \\ &= -\frac{Wm^2[\mathcal{J}(W-1)-\kappa]}{2\kappa^2[1-(W-2)W\mathcal{J}^2/m^2]} \\ &= -\frac{m^2(\mathcal{J}-\kappa)}{\kappa^2} \quad \text{if } W = 2. \end{aligned} \tag{26}$$

We obtain the mismatch (11) up to a factor of 2. Setting $W = 2$ makes (19) precisely the usual AFS phase. Expanding, we get

[11] The case of an odd m should give the same physics, but one will need to be more careful about branches of functions.

$$\delta E = -2\left[-\frac{m^4}{4\mathcal{J}^3}+\frac{5m^6}{16\mathcal{J}^5}-\frac{11m^8}{32\mathcal{J}^7}+\frac{93m^8}{256\mathcal{J}^9}+\cdots\right].$$

The formula (15) we used here is a rather crude attempt to write down the appropriate multiply wrapped multiparticle Lüscher term. It is very encouraging that it almost works, but the real answer is probably much more complicated. Some comments in this direction follow:

(i) There is another derivation of Lüscher terms from the TBA in [44,45]. The leading order (i.e. $n=1$) term there, $\delta E^{(1)}$, is (14), but they also derive two "next to leading order" (NLO) terms. One of these, $\delta E^{(2,1)}$, is similar to our $n=2$ term but with a different trace structure. The other, $\delta E^{(2,2)}$, has two momentum integrals, and a factor of the virtual-virtual S matrix.

The original application of these was at weak coupling. Naively attempting to evaluate these terms at strong coupling, we do not see how to obtain the correct results. For instance for giant magnons, we can compare them to [46], where we computed $n=2$ terms using (13) and saw agreement with string theory.

(ii) However it seems likely that a better derivation for terms with $n\geq 2$ wrappings may include similar features: several loop momentum integrals, and factors of the virtual-virtual S matrix. While the $n=2$ term may not be too difficult (from either Feynman diagrams or from TBA), it seems clear that we will need to sum all wrappings for the effects studied here.

We stress that our conclusion that the Bethe ansatz is not complete without massless Lüscher corrections does not depend on these details. The singly wrapped term (14) alone produces a correction (24) of the same order as the mismatch (11) which we set out to explain.

IV. CONCLUSION

The problem we aimed to solve was that the correct one-loop dressing phase $\sigma^{\bullet\bullet}$ (as deduced from crossing symmetry, and direct semiclassical calculations) does not produce the correct one-loop energy in the Bethe ansatz, for circular strings in S^3. The solution we found is that Lüscher F-terms with a massless virtual particle circling the space contribute without an exponential suppression factor, at precisely the right orders in $1/\mathcal{J}$ to repair this disagreement. By summing over all wrappings, we are able to recover the difference almost exactly.

From this we conclude that the $su(2)$ Bethe equations are not sufficient to describe the spectrum in this sector: they must be supplemented by the effect of massless wrapping terms. This effect is the first place in which the massless excitations of $AdS_3\times S^3\times T^4$ do not decouple from the massive sector.

Our calculation involves the mixed-mass dressing phase $\sigma^{\bullet\circ}$, and we show that its classical term has the AFS form. We observe that this phase is not the limit $\alpha\to 1$ of the one needed for the scattering of mass α and mass $1-\alpha$ particles in the $AdS_3\times S^3\times S^3\times S^1$ case [42], and in particular cannot have the Q_1Q_2 term which seems to be necessary there.[12] We interpret this as more evidence that the matching of the variables used for that integrable system to those for $AdS_3\times S^3\times T^4$ is not simple.

Massless Lüscher terms will also be needed for macroscopic solutions in $AdS_3\times S^3\times S^3\times S^1$, where the situation is very similar: by placing the same resolvent in each S^3 one finds an equally simple $su(2)$ sector of the Bethe equations, and in string theory one can likewise place the same solution in each S^3 [7]. The correct massive dressing phase there differs only by a factor of $\frac{1}{2}$ [17], and the mixed-mass S matrix is now also known [47]. Further ahead, similar effects are surely going to be important in learning to treat "macroscopic massless" solutions in integrability, as we proposed in [48].

ACKNOWLEDGMENTS

We thank Romuald Janik for many insightful comments and suggestions. We have also benefited from conversations with Diego Bombardelli, Dmitri Sorokin, Per Sundin, Alessandro Torrielli, and Arkady Tseytlin at various stages. M. C. A. is supported by a NRF Innovation Fellowship. I. A. was supported by NCN Grant No. 2012/06/A/ST2/00396.

[12]The effect of using a classical phase starting with Q_1Q_2 as in [42] in place of (19) is that the final answer has a term $\delta E^F\sim 1/\mathcal{J}^4$, which is undesirable. In addition the coefficient W_{xy} needed in [42] goes to zero as $s_y\to 0$.

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