PHYSICAL REVIEW D **93,** 106002 (2016)

Irregular vertex operators for irregular conformal blocks

Dimitri Polyakov[1,2,*] and Chaiho Rim[3,†]

[1]*Center for Theoretical Physics, College of Physical Science and Technology Sichuan University, Chengdu 6100064, China*

[2]*Institute of Information Transmission Problems (IITP) Bolshoi Karetny per. 19/1, Moscow 127994, Russia*

[3]*Department of Physics and Center for Quantum Spacetime Sogang University, Seoul 121-742, Korea*

(Received 24 February 2016; published 4 May 2016)

We construct the free field representation of irregular vertex operators of arbitrary rank which generates simultaneous eigenstates of positive modes of Virasoro and W symmetry generators. The irregular vertex operators turn out to be the exponentials of combinations of derivatives of Liouville or Toda fields, creating irregular coherent states. We compute examples of correlation functions of these operators and study their operator algebra.

DOI: 10.1103/PhysRevD.93.106002

I. INTRODUCTION

Primary vertex operators in two-dimensional conformal field theory are the objects playing a crucial role in the Alday-Gaiotto-Tachikawa (AGT) conjecture [1], connecting regular Liouville conformal blocks to Nekrasov's partition function [2] on the Coulomb branch of $N = 2$ supersymmetric gauge theories in four dimensions. Among the interplay between the four dimensional gauge theory and two dimensional conformal field theory (CFT), there appears a nontrivial IR fixed point, Argyres-Douglas type theory [3,4]. This class of theories does not allow marginal deformations and is described in terms of colliding limit of the primary vertex operators. The operator of rank q, obtained from the colliding limit [5,6], generates an irregular state of rank q when applied to the vacuum. The irregular state is annihilated by L_k with $k > 2q$ but becomes a simultaneous eigenstate of positive Virasoro generators L_k with $q \le k \le 2q$. This irregular state is called Gaiotto state [7] or Whittaker state [8]. The usual regular primary state corresponds to the rank 0 state

One obvious try to construct the irregular vertex operators (IVO) or the irregular conformal states was the construction of the state as the combination of the primary state and its descendents [7–11]. However, the attempt to find the irregular state beyond the rank 1 has met a serious difficulty to fix the coefficient if one uses the fact that the state is the simultaneous eigenstate of the positive Virasoro generators only. The state thus constructed has undetermined parameters which should be further fixed by the consistency condition with the lower mode $L_{k<q}$ [12].

In this paper we reconsider the irregular vertex operator directly in terms of free bosonic field representation. To get an idea, we notes that the two-point conformal block, one primary vertex operator at infinity and one irregular vertex operator at the origin, is given as the irregular matrix model (IMM) [5,13] which has the form of Penner-type matrix models

$$Z = \int \prod_{i=1}^{N} d\lambda_i \prod_{1\le j<k\le N} (\lambda_j - \lambda_k)^{-2b^2} e^{-2b\sum_i V(\lambda_i)} \tag{1.1}$$

where the potential has the logarithmic term together with the inverse powerlike contributions

$$V(\lambda_i) = c_0 \log(\lambda_i) - \sum_{j=1}^{q} \frac{c_j}{j(\lambda_i)^j}. \tag{1.2}$$

Seiberg-Witten curve obtained from the loop equation of IMM has the quadratic form and IMM is expected to reproduce the instanton contributions to the partition functions in the Argyres-Douglas theories according to AGT. The irregular conformal blocks (ICB) are in general not simple objects to explore, even though the IMM approach to the ICB provides a relatively simple procedure but needs tedious steps to find ICB working with loop equations. Therefore, it is desirable to find IVO directly from the eigenvalue constraints using the (Liouville) free fields and provide ICB in terms of IVO directly.

The general feature of the potential term of IMM is that IVO can be represented in terms of modified vertex operators which contains finite number of derivatives of the Liouville fields [14]. However, it is yet to be checked if the modified primary operator indeed represents the irregular vertex operator. In this paper we construct the free field representation of IVO explicitly without resorting to the ICB or IMM but only using the fact that IVO produces the simultaneous eigenstates of positive generators. For the Virasoro IVO, one has the conditions:

*polyakov@scu.edu.cn;polyakov@sogang.ac.kr
†rimpine@sogang.ac.kr

$$[L_k, I_q] = \rho_k I_q (q \le k \le 2q); \qquad [L_k, I_q] = 0 (k > 2q) \tag{1.3}$$

where I_q is the IVO of rank q and ρ_k is the eigenvalue of the positive mode Virasoro generator L_k.

In case of two or more copies of the Liouville fields as in the Toda field theories, IVO can have more constraints to incorporate the higher spin symmetry in addition to (1.3). For example, with two fields, IVO subjects to the $W^{(3)}$ symmetry constraints:

$$\begin{aligned}&[W_k^{(3)}, I_q] = \omega_k^{(3)} I_q (2q \le k \le 3q);\\ &[W_k^{(3)}, I_q] = 0 (k > 3q)\end{aligned} \tag{1.4}$$

where $W_k^{(3)}$ is the kth mode of the spin 3 W-current and $\omega_k^{(3)}$ is its eigenvalue. The corresponding irregular matrix models can be obtained from the colliding limit of the A_2 Toda field theory, whose loop equation provides the cubic form of the Seiberg-Witten curve and flow equations corresponding to $W^{(3)}$ symmetry [14,15]. It is generally expected that IMM obtained from the colliding limit of A_r Toda field theory results in the Seiberg-Witten curve with the $(r+1)$th power term and flow equations of $W^{(r+1)}$ symmetry. The corresponding IVO can be determined by the generalized constraints due to $W^{(r+1)}$ symmetry:

$$\begin{aligned}&[W_k^{(r+1)}, I_q] = \omega_k^{(3)} I_q (rq \le k \le (r+1)q);\\ &[W_k^{(r+1)}, I_q] = 0 (k > (r+1)q)\end{aligned}$$

This paper is organized as follows. In Sec. II, we consider the case with one free bosonic field which has Virasoro symmetry. We first develop the free field representation of the Virasoro IVO of rank 1 by solving the Virasoro constraint, reproducing the deformed Penner-type potential of the matrix model approach. We then extend this construction to higher ranks and present the general structure of IVO of arbitrary ranks. The explicit coordinate dependence of ICB constructed from N-point IVO correlator is given in free field formalism.

In Sec. III, we extend this construction to the system of two bosonic fields so that IVO obeys the $W^{(3)}$ symmetry. We explicitly check that IVO of lower rank has the similar free field representation as in the Virasoro case. The eigenvalues fix IVO with algebraic polynomial equations.

Section IV is the conclusion where IVO of arbitrary rank q with $W^{(r+1)}$-symmetry is given and its eigenvalues are presented explicitly in terms of the coefficients of IVO for $W^{(3)}$ case. In addition, some of physical implications of IVO are speculated.

II. IRREGULAR VERTEX OPERATOR WITH VIRASORO SYMMETRY

In this section we demonstrate the explicit construction for IVO in terms of one free bosonic field. Before we demonstrate the explicit ansatz, it is useful to comment on the structure of the answer that we expect and its relation to the colliding limit.

The irregular blocks of rank q essentially emerge as a result of the normal ordering $q+1$ Liouville vertex operators colliding at the same point. Let us consider the example of two vertex operators first. The operator product between two exponential operators at points z_1 and z_2 around z_2 is given by

$$e^{\alpha\phi}(z_1) e^{\beta\phi}(z_2) = (z_{12})^{-\alpha\beta} \sum_{n=0}^{\infty} (z_{12})^n :B_\alpha^{(n)}(\phi) e^{(\alpha+\beta)\phi}: (w) \tag{2.1}$$

where $z_{12} = z_1 - z_2$. $B_\alpha^{(n)}$ are the normalized Bell polynomial of the derivatives of ϕ and are defined as [16]

$$B_\alpha^{(n)} = \sum_{p=1}^{n} \alpha^p \sum_{n|k_1\ldots k_p} \frac{\partial^{k_1}\phi \ldots \partial^{k_p}\phi}{k_1! q_{k_1}! \ldots k_p! q_{k_p}!}. \tag{2.2}$$

Here the sum is taken over the ordered length p partitions of n $(1 \le p \le n)$: $n = k_1 + \cdots + k_p$; $k_1 \le k_2 \ldots \le k_p$ and q_{k_j} is the multiplicity of an element k_j in the partition. The operator product for three operators at z_1, z_2, z_3 colliding at z_1 is similarly given by

$$\begin{aligned}&e^{\alpha\phi}(z_1) e^{\beta\phi}(z_2) e^{\gamma\phi}(z_3)\\ &= (z_2 - z_1)^{-\alpha\beta} (z_3 - z_1)^{-\gamma(\alpha+\beta)}\\ &\quad\times \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{k=0}^{n_1} (z_2 - z_1)^{n_1} (z_3 - z_1)^{n_2-k}\\ &\quad\times \frac{\Gamma(-\beta\gamma+1)}{k!\Gamma(-\beta\gamma+1-k)} :B_\beta^{(n_1-k)} B_\gamma^{(n_2)} e^{\alpha+\beta+\gamma}: (z_1).\end{aligned} \tag{2.3}$$

One may have in general, for N vertices at $z_1, \ldots z_N$ around z_1

$$\begin{aligned}e^{\alpha_1\phi}(z_1) \ldots e^{\alpha_N}(z_N) &= \prod_{p=2}^{N} (z_{k1})^{-\alpha_p(\alpha_1+\cdots\alpha_{p-1})} \sum_{n_1,\ldots,n_{N-1}} \sum_{k_1,\ldots k_{N-2}} \sum_{q_1\ldots,q_{N-2}} (z_{21})^{n_1} (z_{31})^{n_2-k_1} \ldots (z_{N1})^{n_{N-1}-k_{N-2}}\\ &\quad\times \prod_{j=1}^{N-1} \lambda_{\{n,k,q\}} :B_{\alpha_{j+1}}^{(n_j - q_j(k_1,\ldots k_{N-1}))} e^{(\alpha_1+\cdots+\alpha_N)\phi}: (z_1)\end{aligned} \tag{2.4}$$

with the q-numbers satisfying

$$\sum_{j=1}^{N-2} k_j = \sum_{j=1}^{N-2} q_j \tag{2.5}$$

and $\lambda_{\{n,k,q\}}$ are some constants which are straightforward to evaluate but whose explicit form is of no importance to us. IVO is then obtained by taking the operator product (2.4) inside correlators and taking the simultaneous limits $z_j \to z_1; j = 2, \ldots, N$ maintaining $\sum \alpha_i z_i^k$ finite for $k = 0, 1, \ldots, n$. All the operators appearing on the right-hand side of the operator product (2.4) shall appear in the expression for IVO; thus, IVO of any rank q must contain infinite number of terms, having the general form

$$I_q \sim \sum_{n_1,\ldots,n_q=0}^{\infty} \lambda_{n_1\ldots n_q} e^{\sum_{j=1}^{q+1} \alpha_j \phi} B_{\alpha_1}^{(n_1)} \ldots B_{\alpha_q}^{(n_q)} \tag{2.6}$$

where the λ-coefficients must be determined from the Virasoro constraints (1.3). It is noteworthy that the objects similar to that type appear in string field theory as analytic solutions of the equations of motion, presumably describing the collective higher spin vacuum state [16].

However, taking the colliding limits directly for the expansion in (2.4) is obviously a tedious procedure and technically seems to be beyond control in an arbitrary case. In addition, the products of Bell polynomial operators, generally lead to tedious recursion relations, and are hard to solve analytically. Therefore, it shall be better to work in a different operator basis, namely, a field derivative basis so that we can apply the Virasoro constraints (1.3) directly to the operator basis.

To start, we look for the solution IVO of rank one in the form

$$I_1 = \sum_{N_1,N_2=0}^{\infty} \lambda_{N_1 N_2} \phi^{N_1} (\partial\phi)^{N_2} \tag{2.7}$$

Second and higher derivatives are not allowed since I_1 should be annihilated by all the L_k-generators with $k > 2$: Note that $L_k = \oint \frac{dz}{2i\pi} z^{k+1} T(z)$ and the stress-energy tensor with background charge Q

$$T(z) = -\frac{1}{2}(\partial\phi)^2 + \frac{Q}{2}\partial^2\phi \tag{2.8}$$

will have the leading operator product expansion (OPE) singularity of the order $\sim(z_1 - z_2)^{-3}$ with I_1 since we are using the free field normalization $\langle\phi(z)\phi(w)\rangle = -\log(z-w)$.

The solution is obtained if one finds the generating function $F(x, y)$ of two variables with the same $\lambda_{N_1 N_2}$:

$$F(x,y) = \sum_{N_1,N_2} \lambda_{N_1 N_2} x^{N_1} y^{N_2} \tag{2.9}$$

The eigenvalue constraint

$$[L_2, I_1] = \rho_2 I_1 \tag{2.10}$$

leads to the relation

$$\rho_2 \sum_{N_1,N_2=0}^{\infty} \lambda_{N_1 N_2} \phi^{N_1} (\partial\phi)^{N_2} = -\sum_{N_1=0,N_2=2}^{\infty} N_2(N_2-1)\lambda_{N_1 N_2} \phi^{N_1} (\partial\phi)^{N_2-2}. \tag{2.11}$$

It is easy to see that this equation is equivalent to a simple partial differential equation on the generating function $F(x, y)$:

$$\partial_y^2 F(x,y) = -\rho_2 F(x,y) \tag{2.12}$$

whose general solution is

$$F(x,y) = e^{i\sqrt{\rho_2}y} f(x). \tag{2.13}$$

Similarly, the second eigenvalue problem:

$$[L_1, I_1] = \rho_1 I_1 \tag{2.14}$$

leads to the second recursion relation for λ:

$$\rho_1 \sum_{N_1,N_2=0}^{\infty} \lambda_{N_1 N_2} \phi^{N_1} (\partial\phi)^{N_2} = -\sum_{N_1=1,N_2=1}^{\infty} N_1 N_2 \lambda_{N_1 N_2} \phi^{N_1-1} (\partial\phi)^{N_2-1} + Q \sum_{N_1=1,N_2=1}^{\infty} N_2 \lambda_{N_1 N_2} \phi^{N_1} (\partial\phi)^{N_2-1} \tag{2.15}$$

leading to the second order differential equation on F:

$$-\partial_x \partial_y F + Q\partial_y F = \rho_1 F. \tag{2.16}$$

Substituting the general solution of the first equation and identifying $f(x)$ we find the generating function to be given by

$$F(x,y) = e^{(i\frac{\rho_1}{\sqrt{\rho_2}} + Q)x + i\sqrt{\rho_2}y}. \tag{2.17}$$

Accordingly, substituting for $\lambda_{N_1 N_2}$ we find that the expression for IVO of the rank 1

$$I_1 =: e^{(i\frac{\rho_1}{\sqrt{\rho_2}} + Q)\phi + i\sqrt{\rho_2}\partial\phi} : \tag{2.18}$$

The contribution to the Penner type potential is given by the log of the leading order OPE term of $I_1(z_1)$ with a regular vertex $e^{\alpha\phi}(z_2)$. Expanding I_1 in terms of ϕ and $\partial\phi$ and exponentiating one easily finds

$$V(z_{12}) \sim \left(i\frac{\rho_1}{\sqrt{\rho_2}} + Q\right)\log(z_{12}) + i\frac{\sqrt{\rho_2}}{z_{12}} \tag{2.19}$$

reproducing the known result from the matrix model approach, which leads to the identification of the potential coefficients with the eigenvalues.

Next, let us consider the rank 2 case. The eigenvalue constraints for the rank 2 are

$$\begin{aligned} [L_k, I_2] &= \rho_k I_2; \qquad (k = 2, 3, 4) \\ [L_k, I_2] &= 0; \qquad (k > 4). \end{aligned} \tag{2.20}$$

Accordingly, the ansatz for the rank 2 will be

$$I_2 = \sum_{N_1,N_2,N_3} \lambda_{N_1N_2N_3}\phi^{N_1}(\partial\phi)^{N_2}(\partial^2\phi)^{N_3} \tag{2.21}$$

since I_2 is by construction annihilated by all L_k for $k \geq 5$. With the generating function

$$F(x, y, z) = \sum_{N_1,N_2,N_3} \lambda_{N_1N_2N_3} x^{N_1} y^{N_2} z^{N_3}.$$

the eigenvalue constraints for $k = 2$, 3 and 4 lead, in turn to the characteristic 3 PDE's

$$\begin{aligned} -4\partial_z^2 F &= \rho_4 F \\ -2\partial_y\partial_z F &= \rho_3 F \\ -2\partial_x\partial_z F - \partial_y^2 F + \frac{3}{2}\partial_z F &= \rho_2 F. \end{aligned} \tag{2.22}$$

The solution of the system is

$$F = e^{\frac{i}{\sqrt{\rho_4}}((\rho_2 - \frac{\rho_3^2}{\rho_4}) + \frac{3}{2}Q)x + i\frac{\rho_3}{\sqrt{\rho_4}}y + \frac{i}{2}\sqrt{\rho_4}z}, \tag{2.23}$$

leading to IVO:

$$I_2 = e^{\frac{i}{\sqrt{\rho_4}}((\rho_2 - \frac{\rho_3^2}{\rho_4}) + \frac{3}{2}Q)\phi + i\frac{\rho_3}{\sqrt{\rho_4}}\partial\phi + \frac{i}{2}\sqrt{\rho_4}\partial^2\phi} \tag{2.24}$$

with the corresponding contributions to the Penner's potential:

$$\begin{aligned} V(z_{12}) \sim{}& \frac{i}{\sqrt{\rho_4}}\left(\left(\rho_2 - \frac{\rho_3^2}{\rho_4}\right) + \frac{3}{2}Q\right)\log(z_{12}) \\ &+ i\frac{\rho_3}{\sqrt{\rho_4}}z_{12}{}^{-1} + \frac{i}{2}\sqrt{\rho_4}z_{12}{}^{-2}. \end{aligned} \tag{2.25}$$

It is not difficult to extend the same pattern to the higher ranks. For the rank 3, the IVO ansatz will include the third derivatives of the Liouville field:

$$I_2 = \sum_{N_1,N_2,N_3,N_4} \lambda_{N_1N_2N_3N_4}\phi^{N_1}(\partial\phi)^{N_2}(\partial^2\phi)^{N_3}(\partial^3\phi)^{N_4}. \tag{2.26}$$

The generating function will have 4 variables and satisfy the system of 4 linear second order differential equations. The computation similar to the above gives the answer for the rank 3 irregular block in terms of the irregular vertex operator:

$$I_3 = e^{\frac{i}{\sqrt{\rho_6}}\{(\rho_3 - \frac{\rho_4\rho_5}{\rho_6} + \frac{\rho_5^3}{\rho_6^2} - 2iQ\sqrt{\rho_6})\phi + (\rho_4 - \frac{\rho_5^2}{\rho_6})\partial\phi + \frac{\rho_5}{2}\partial^2\phi + \frac{\rho_6}{6}\partial^3\phi\}} \tag{2.27}$$

with the related contribution to the Penner's potential

$$\begin{aligned} V_3(z_{12}) \sim{}& \frac{i}{\sqrt{\rho_6}}\left\{\left(\rho_3 - \frac{\rho_4\rho_5}{\rho_6} + \frac{\rho_5^3}{\rho_6^2} - 2iQ\sqrt{\rho_6}\right)\log(z_{12})\right. \\ &\left. + \left(\rho_4 - \frac{\rho_5^2}{\rho_6}\right)z_{12}{}^{-1} + \frac{\rho_5}{2}z_{12}{}^{-2} + \frac{\rho_6}{6}z_{12}{}^{-3}\right\}. \end{aligned} \tag{2.28}$$

It is now not difficult to guess the general structure of the answer for an arbitrary rank q: IVO of the rank q is given by

$$I_q =: e^{\sum_{k=0}^{q}\alpha_k\partial^k\phi}: \tag{2.29}$$

with the related Penner type potential contribution

$$V_q(z_{12}) \sim \alpha_0\log(z_{12}) + \sum_{k=1}^{q}\alpha_q(z_{12})^{-k} \tag{2.30}$$

where

$$\begin{aligned} -i\sqrt{\rho_{2q}}\alpha_k &= \frac{\rho_{q+k}}{k!} + \sum_{m=1}^{q-k-1}\frac{(-1)^m}{\rho_{2q}^m}\sum_{j=1}\sum_{(2m+1)q+k|q_1\ldots q_j} \\ &\quad \times n_{q_1\ldots q_j}\rho_{q_1}\ldots\rho_{q_j} - \frac{i\sqrt{\rho_{2q}}(q+1)Q}{2}\delta_0^k \\ -i\sqrt{\rho_{2q}}\alpha_q &= \frac{\rho_{2q}}{q!} \end{aligned} \tag{2.31}$$

with $0 \leq k \leq q-1$. $n_{q_1\ldots q_j}$ are positive integers and the second sum in the expression $\sum_{(2m+1)q+k|q_1\ldots q_j}$ is taken over all possible length j ordered partitions of

$$(2m+1)q + k = q_1 + \cdots + q_j$$

such that $q \leq q_1 \leq \ldots \leq q_j \leq 2q$ with the subsequent summation over the lengths. Irregular conformal state obtained by IVO of the form (2.29) has the simultaneous eigenvalues ρ_k of $L_k(k = q, \ldots, 2q)$ whose relation with α-coefficients are given in terms of $q+1$ algebraic equations. The objects of the type (2.29) were also considered in [17] in a different context, as well as in [18].

In addition, IVO of the form (2.29) provides the N-point ICB:

$$\langle I_{q_1}(z_1)\ldots I_{q_N}(z_N)\rangle = \left\langle \prod_{l=1}^{N} :e^{\sum_{k_l=0}^{q_l}\alpha_{k_l}^{(q_l)}\partial^{k_l}\phi}(z_l): \right\rangle \tag{2.32}$$

where $q_l(l=1,\ldots,N)$ are the ranks of IVO. Below we shall compute this correlator in the limit of zero Liouville cosmological constant, i.e. in the free field limit. This calculation still holds at nonzero constant, as long as long as the neutrality condition $\sum_{1\le\ell\le N}\alpha_0^{(q_l)} = Q$ holds. Despite that, the free field calculation still makes sense even when the neutrality condition is not satisfied, since the irregular blocks are the objects essentially appearing in the process of perturbative expansion in the screening operator, and, as such, get inserted inside the free field correlators.

To compute the holomorphic correlator, consider the functional integral

$$\langle I_{q_1}(z_1)\ldots I_{q_N}(z_N)\rangle = Z^{-1}\int D\phi \prod_{l=1}^{N} e^{\sum_{k_l=0}^{q_l}\alpha_{k_l}^{(q_l)}\partial^{k_l}\phi}(z_l)e^{-\frac{1}{8\pi}\int d^2z\partial\phi\bar{\partial}\phi}. \tag{2.33}$$

This integral can be written as

$$\langle I_{q_1}(z_1)\ldots I_{q_N}(z_N)\rangle = \int D\phi e^{\int d^2z\frac{1}{8\pi}\partial\phi\bar{\partial}\phi+\phi\sum_{l=1}^{N}\sum_{k_l=0}^{q_l}\alpha_{k_l}^{(q_l)}(-1)^{k_l}\partial_z^{k_l}\delta^{(2)}(z-z_l)} \tag{2.34}$$

where we took the exponential insertions at z_l inside the z-integral by using the δ-functions $\delta^{(2)}(z-z_l)$ and integrated k_l times by parts for each derivative field $\partial^{k_l}\phi$. This integral is now the Gaussian integral with the linear source term

$$j(z,\bar{z}) = \sum_{l=1}^{N}\sum_{k_l=0}^{q_l}\alpha_{k_l}^{(q_l)}(-1)^{k_l}\partial_z^{k_l}\delta^{(2)}(z-z_l) \tag{2.35}$$

and its value simply equals to that of the generating functional

$$W(j) = e^{\int d^2z\int d^2w j(z,\bar{z})j(w,\bar{w})G(|z-w|)} \tag{2.36}$$

with $G(|z-w|) = -\log|z-w|^2$ Substituting for $j(z,\bar{z})$, integrating again by parts for each term in the sum to bring the derivatives of the delta-functions into the delta-functions—and finally integrating out the delta-functions, we obtain:

$$\begin{aligned}\langle I_{q_1}(z_1)\ldots I_{q_N}(z_N)\rangle &= e^{\frac{1}{2}\sum_{l_1=1}^{N}\sum_{l_2=1}^{N}\sum_{k_{l_1}=0}^{q_{l_1}}\sum_{k_{l_2}=0}^{q_{l_2}}\alpha_{k_{l_1}}^{(q_{l_1})}\alpha_{k_{l_2}}^{(q_{l_2})}\int d^2z\int d^2w\delta^{(2)}(z-z_{l_1})\delta^{(2)}(w-z_{l_2})\partial_z^{k_{l_1}}\partial_w^{k_{l_2}}G(|z-w|)} \\ &= \prod_{l_1,l_2=1;l_1\neq l_2}(z_{l_1}-z_{l_2})^{-\alpha_0^{q(l_1)}\alpha_0^{q(l_2)}} e^{\frac{1}{2}\sum_{l_1=1}^{N}\sum_{l_2=1;l_1\neq l_2}^{N}\sum_{k_{l_1}=0}^{q_{l_1}}\sum_{k_{l_2}=0}^{q_{l_2}}\frac{(-1)^{k_{l_1}}(k_{l_1}+k_{l_2}-1)!\alpha_{k_{l_1}}^{(q_{l_1})}\alpha_{k_{l_2}}^{(q_{l_2})}}{(z_{l_1}-z_{l_2})^{k_{l_1}+k_{l_2}}}}.\end{aligned} \tag{2.37}$$

This is the general answer. For example, applied to the three-point function of the rank 2 blocks, it gives:

$$\langle I_2(z_1)I_2(z_2)I_2(z_3)\rangle = [(z_1-z_2)(z_1-z_3)(z_2-z_3)]^{\frac{1}{\rho_4}(\rho_2-\frac{\rho_3^2}{\rho_4}+\frac{Q}{2})} e^{\frac{\rho_3^2}{\rho_4}(\frac{1}{(z_1-z_2)^2}+\frac{1}{(z_1-z_3)^2}+\frac{1}{(z_2-z_3)^2})+6\rho_4(\frac{1}{(z_1-z_2)^4}+\frac{1}{(z_1-z_3)^4}+\frac{1}{(z_1-z_2)^4})} \tag{2.38}$$

where $\rho_{2,3,4}$ are the eigenvalues of $L_{2,3,4}$ respectively. Note that the exponent only contains the *even* powers of the inverse z_{ij}, as it should be (otherwise the answer would have been unphysical since for close z_i and z_j interchanging points would e.g. make an infinitely large exponent out of infinitely small).

III. TODA GENERALIZATIONS AND W_n SYMMETRIES FOR IRREGULAR CONFORMAL BLOCKS

There was an insightful observation made in [10,14,15] that, when the Liouville theory is extended to A_2 Toda model containing two copies of the scalar field, the irregular conformal block of such a model possesses additional symmetries related to $W^{(3)}$ algebra. Therefore, irregular state will be the eigenstate not only of Virasoro generators L_n with $q\le n\le 2q$ but also of eigenvalues of the $W_n^{(3)}$ generators with $2q\le n\le 3q$. This property has been demonstrated explicitly in the random matrix model approach. However, its generalization to higher $W^{(n)}$ symmetry remain somewhat uncontrollable, As we shall demonstrate below, the whole construction and its generalizations become much more simple and transparent in the vertex operator formalism using the manifest free-field representation. The free-field representation for irregular rank q conformal blocks involving r scalar fields, leading to irregular vertex operators of the type (2.29), is given by

$$I_{q|r} = \sum_{N_1^{(1)}\ldots N_q^{(r)}=0}^{\infty} \lambda_{N_1^{(1)}\ldots N_q^{(1)}|N_1^{(2)}\ldots N_q^{(2)}|\ldots|N_1^{(r)}\ldots N_q^{(r)}} (\phi^{(1)})^{N_1^{(1)}} \times \ldots (\partial^q \phi^{(1)})^{N_q^{(1)}} \times \ldots \times (\phi^{(r)})^{N_1^{(r)}} \ldots (\partial^q \phi^{(r)})^{N_q^{(r)}}. \quad (3.1)$$

For simplicity, let us start from the most elementary nontrivial case $q = r = 2$, relevant to the $W^{(3)}$ IVO, whose emergence was already observed in the matrix model approach [14,15]. We shall look for the free field realization of this block in the form:

$$I_{2|2} = \sum_{N_1,N_2,N_3=0;P_1,P_2,P_3=0}^{\infty} \lambda_{N_1N_2N_3|P_1P_2P_3} : \phi_1^{N_1} (\partial\phi_1)^{N_2} (\partial^2\phi_1)^{N_3} \times \phi_2^{P_1} (\partial\phi_2)^{P_2} (\partial^2\phi_2)^{P_3} : \quad (3.2)$$

with the stress-energy tensor

$$T = \sum_{i=1}^{2} \left(-\frac{1}{2} \left(\partial\phi_i \right)^2 + \frac{1}{2} Q^i \partial^2 \phi_i \right) \quad (3.3)$$

and the W_3-current

$$j_w = \sum_{i,j,k=1}^{2} (\nu^i \partial^3 \phi_i + \nu^{ij} \partial^2 \phi_i \partial\phi_j + \nu_{ijk} \partial\phi_i \partial\phi_j \partial\phi_k) \quad (3.4)$$

where the ν-coefficients will be determined below from the condition that j_w is a dimension 3 primary field.

The generating function $F(x_1, x_2, x_3|y_1, y_2, y_3)$ for $I_{2|2}$ is thus the function of 6 variables that is to be determined from 3 Virasoro constraints and 3 $W^{(3)}$ constraints. We start from the Virasoro constraints first. As in the case of a single field, $I_{2|2}$ is the eigenvalue of L_2, L_3 and L_4 and, since it does not contain higher than second derivatives of the Toda fields, it is by construction annihilated by all higher L_n's. As before, consider the L_4-eigenvalue problem first. As before, by simple straightforward calculation the eigenvalue problem leads to the recursion.

The constraint $[L_4, I_{2|2}] = \rho_4 I_{2|2}$ results in the relation

$$-4 \sum_{N_1,N_2=0,N_3=2;P_1,P_2=0,P_3=2}^{\infty} (N_3(N_3-1) + P_3(P_3-1)) \lambda_{N_1N_2N_3|P_1P_2P_3} : \phi_1^{N_1} (\partial\phi_1)^{N_2} (\partial^2\phi_1)^{N_3-2} \phi_2^{P_1} (\partial\phi_2)^{P_2} (\partial^2\phi_2)^{P_3-2} : = \rho_4 \sum_{N_1,N_2,N_3=0;P_1,P_2,P_3=0} \lambda_{N_1N_2N_3|P_1P_2P_3} : \phi_1^{N_1} (\partial\phi_1)^{N_2} (\partial^2\phi_1)^{N_3} \phi_2^{P_1} (\partial\phi_2)^{P_2} (\partial^2\phi_2)^{P_3} : \quad (3.5)$$

which is equivalent to the second order PDE for the generating function:

$$(\partial_{x_3}^2 + \partial_{y_3}^2) F(x_1, x_2, x_3|y_1, y_2, y_3) = -\frac{\rho_4}{4} F(x_1, x_2, x_3|y_1, y_2, y_3). \quad (3.6)$$

The general solution is given as

$$F(x_1, x_2, x_3|y_1, y_2, y_3) = e^{i\alpha x_3 + i\beta y_3} F^{(2)}(x_1, x_2|y_1, y_2) \quad (3.7)$$

with α and β coefficients satisfying

$$\alpha^2 + \beta^2 = \frac{\rho_4}{4}. \quad (3.8)$$

Similarly, the second eigenvalue problem, $[L_3, I_{2|2}] = \rho_3 I_{2|2}$, leads to the second PDE

$$(\partial_{x_2}\partial_{x_3} + \partial_{y_2}\partial_{y_3}) F = -\frac{\rho_3}{2} F. \quad (3.9)$$

Finally, the third eigenvalue constraint, $[L_2, I_{2|2}] = \rho_2 I_{2|2}$ leads to the third PDE on F:

$$2(\partial_{x_1}\partial_{x_3} + \partial_{y_1}\partial_{y_3}) F + (\partial_{x_2}^2 + \partial_{y_2}^2) F - (Q_1 \partial_{x_3} + Q_2 \partial_{y_3}) F = -\rho_2 F. \quad (3.10)$$

The general solution of the three PDE's (3.6), (3.9), (3.10) is given in terms of the generating function

$$F(x_1, x_2, x_3|y_1, y_2, y_3) = \exp\left\{ \frac{i}{4} \left[\rho_2 - iQ_1\alpha - iQ_2\beta - \frac{(\frac{\rho_3}{4} + \lambda)^2}{\alpha^2} - \frac{(\frac{\rho_3}{4} - \lambda)^2}{\beta^2} + \frac{\xi}{\alpha} \right] x_1 + \frac{i}{4} \left[\rho_2 - iQ_1\alpha - iQ_2\beta - \frac{(\frac{\rho_3}{4} + \lambda)^2}{\alpha^2} - \frac{(\frac{\rho_3}{4} - \lambda)^2}{\beta^2} - \frac{\xi}{\beta} \right] y_1 + i \left[\frac{(\frac{\rho_3}{4} + \lambda) x_2}{\alpha} + \frac{(\frac{\rho_3}{4} - \lambda) y_2}{\beta} + \alpha x_3 + \beta y_3 \right] \right\} \quad (3.11)$$

and, accordingly, IVO is given as

$$I_{2|2} =: e^{\frac{i}{4}\{\rho_2 - iQ_1\alpha - iQ_2\beta - \frac{(\frac{\rho_3}{4}+\lambda)^2}{\alpha^2} - \frac{(\frac{\rho_3}{4}-\lambda)^2}{\beta^2} + \frac{\xi}{\alpha}\}\phi_1 + i\frac{(\frac{\rho_3}{4}+\lambda)\partial\phi_1}{\alpha} + i\alpha\partial^2\phi_1}: \times : e^{\frac{i}{4}\{\rho_2 - iQ_1\alpha - iQ_2\beta - \frac{(\frac{\rho_3}{4}+\lambda)^2}{\alpha^2} - \frac{(\frac{\rho_3}{4}-\lambda)^2}{\beta^2} - \frac{\xi}{\beta}\}\phi_2 + i\frac{(\frac{\rho_3}{4}-\lambda)\partial\phi_2}{\beta} + i\beta\partial^2\phi_2}: \qquad (3.12)$$

where 3 constants: λ, ξ and one of α or β [related by (3.8)] are not yet fixed and must be determined from the remaining W_3-current constraints.

To apply the W constraint, we need to fix the coefficients in j_w current (3.4) first. To make W_3, j_w the dimension 3 primary field, Generically, the OPE of $T(z_1)$ with $j_w(z_2)$ has the form:

$$\begin{aligned} T(z_1)j_w(z_2) \sim\ & z_{12}^{-5}(12\nu_j Q^j - 2\nu_j^j) + z_{12}^{-4}\partial\phi_j(-6\nu^j - 3\nu_i^{ij} + 3Q_i\nu^{ij}) \\ & + z_{12}^{-3}\{\partial^2\phi_j(-6\nu^j + \nu^{ji}Q_i) + \partial\phi_i\partial\phi_j(-2\nu^{ij} + 3\nu^{ijk}Q_k)\} \end{aligned} \qquad (3.13)$$

(all the upper and lower indices are equivalent, distinguished merely for the convenience of the notations) To make the W_3-current dimension 3 we have four relations

$$\begin{aligned} 6\nu_j Q^j - \nu_j^j &= 0 \\ -2\nu^j - \nu_i^{ij} + Q_i\nu^{ij} &= 0 \\ -6\nu^j + \nu^{ji}Q_i &= 0 \\ -2\nu^{ij} + 3\nu^{ijk}Q_k &= 0. \end{aligned} \qquad (3.14)$$

Note that ν_{ijk} is symmetric by construction; *a priori* ν_{ij} is not necessarily symmetric, however, the last equation in (3.14) imposes the symmetry condition on ν_{ij}. The system (3.14) is thus consistent, being the system of 8 linear equations for 9 variables (an extra variable corresponds to the overall normalization of j_w, that is fixed by the normalization of $W^{(3)}$-algebra). The j_w current (3.4) is thus completely fixed by (3.14).

The final step to construct the rank 2 IVO with $W^{(3)}$-symmetry is to solve the eigenvalue problems for $I_{2|2}$ with respect to j_w modes: $j_w(z) = \sum_n z^{-n-3}W_n^{(3)}$. Namely, $I_{2|2}$ must be the simultaneous eigenvector of $W_k^{(3)}$ with $k = 4$, 5, 6 and annihilated by higher modes. As in the Virasoro case, the annihilation constraint is automatically ensured by the manifest form of the ansatz (3.2). The W-constraints on $I_{2|2}$ lead to extra 3 linear partial differential equations of the third order on the generating function F, allowing us to fix the remaining unknown constants in (3.11). Namely, applying (3.4) to (3.2) and proceeding precisely as explained above, we obtain the system of 3 extra differential equations on F. For the eigenvalue problem

$$[W_6^{(3)}, I_{2|2}] = \omega_6 I_{2|2}; \qquad (3.15)$$

we have

$$\left(3\nu_{111}\partial_{x_3}^3 + 2\nu_{112}\partial_{x^3}^2\partial_{y_3} + 2\nu_{122}\partial_{x_3}\partial_{y_3}^2 + 3\nu_{222}\partial_{y_3}^3 + \frac{\omega_6}{8}\right)F(x_1, x_2, x_3|y_1, y_2, y_3) = 0. \qquad (3.16)$$

For the eigenvalue problem

$$[W_5^{(3)}, I_{2|2}] = \omega_5 I_{2|2}; \qquad (3.17)$$

we have

$$\begin{aligned} \Big\{&\nu_{111}(6\partial_{x_3}^2\partial_{x_1} + 3\partial_{x_2}^2\partial_{x_3}) + \nu_{112}(2\partial_{x_3}\partial_{y_3}\partial_{x_1} + \partial_{x_2}^2\partial_{y_3} + \partial_{x_2}\partial_{x_3}\partial_{y_2} + 2\partial_{x_3}^2\partial_{y_1}) \\ &+ \nu_{122}(2\partial_{x_3}\partial_{y_3}\partial_{y_1} + \partial_{y_2}^2\partial_{x_3} + \partial_{y_2}\partial_{y_3}\partial_{x_2} + 2\partial_{y_3}^2\partial_{x_1}) \\ &+ \nu_{222}(6\partial_{y_3}^2\partial_{y_1} + 3\partial_{y_2}^2\partial_{y_3}) + \frac{\omega_5}{4}\Big\}F(x_1, x_2, x_3|y_1, y_2, y_3) = 0. \end{aligned} \qquad (3.18)$$

And finally, for the eigenvalue problem

$$[W_4^{(3)}, I_{2|2}] = \omega_4 I_{2|2}; \qquad (3.19)$$

we have

$$\Big\{\nu_{111}(6\partial_{x_3}^2\partial_{x_1}+3\partial_{x_2}^2\partial_{x_3})+\nu_{112}(2\partial_{x_1}\partial_{x_3}\partial_{y_3}+\partial_{x_2}\partial_{x_3}\partial_{y_2}+2\partial_{x_3}^2\partial_{y_1})+\nu_{122}(2\partial_{x_3}\partial_{y_1}\partial_{y_3}+\partial_{x_3}\partial_{y_2}^2+2\partial_{x_1}\partial_{y_3}^2)$$
$$+\nu_{222}(6\partial_{y_3}^2\partial_{y_1}+3\partial_{y_2}^2\partial_{y_3})+3\nu_{11}\partial_{x_3}^2+6\nu_{12}\partial_{x_3}\partial_{y_3}+3\nu_{22}\partial_{y_3}^2+\frac{\omega_4}{4}\Big\}F(x_1,x_2,x_3|y_1,y_2,y_3)=0. \tag{3.20}$$

From the 3 PDE with the form F in (3.11) give the following algebraic constraints on the remaining constants:

$$3\nu_{111}\alpha^3+3\nu_{222}\beta^3+\nu_{112}\alpha^2\beta+\nu_{122}\alpha\beta^2+\frac{i\omega_6}{16}=0 \tag{3.21}$$

$$3\nu_{111}\alpha\left(\frac{\rho_3}{4}+\lambda\right)+3\nu_{222}\beta\left(\frac{\rho_3}{4}-\lambda\right)+\nu_{112}\left(\frac{\alpha^2}{\beta}\left(\frac{\rho_3}{4}-\lambda\right)+\beta\left(\frac{\rho_3}{4}+\lambda\right)\right)+\nu_{122}\left(\frac{\beta^2}{\alpha}\left(\frac{\rho_3}{4}+\lambda\right)+\alpha\left(\frac{\rho_3}{4}-\lambda\right)\right)+\frac{i\omega_5}{8}=0 \tag{3.22}$$

$$\nu_{111}\left(\frac{3}{2}\alpha^2\left(\rho_2-iQ_1\alpha-iQ_2\beta-\frac{1}{\alpha^2}\left(\frac{\rho_3}{4}+\lambda\right)^2+\frac{1}{\beta^2}\left(\frac{\rho_3}{4}-\lambda\right)^2\right)+3\beta\left(\frac{\rho_3}{4}+\lambda\right)+\frac{\xi}{\alpha}\right)$$
$$+\nu_{222}\left(\frac{3}{2}\beta^2\left(\rho_2-iQ_1\alpha-iQ_2\beta-\frac{1}{\alpha^2}\left(\frac{\rho_3}{4}+\lambda\right)^2+\frac{1}{\beta^2}\left(\frac{\rho_3}{4}-\lambda\right)^2\right)+3\alpha\left(\frac{\rho_3}{4}+\lambda\right)-\frac{\xi}{\beta}\right)$$
$$+\nu_{112}\left(\frac{1}{2}\alpha\beta\left(\rho_2-iQ_1\alpha-iQ_2\beta-\frac{1}{\alpha^2}\left(\frac{\rho_3}{4}+\lambda\right)^2+\frac{1}{\beta^2}\left(\frac{\rho_3}{4}-\lambda\right)^2+\frac{\xi}{\alpha}\right)+\frac{\beta}{\alpha^2}\left(\frac{\rho_3}{4}+\lambda\right)^2+\frac{1}{\beta}\left(\frac{\rho_3}{4}+\lambda\right)\left(\frac{\rho_3}{4}-\lambda\right)\right.$$
$$\left.+\frac{\alpha^2}{2}\left(\rho_2-iQ_1\alpha-iQ_2\beta-\frac{1}{\alpha^2}\left(\frac{\rho_3}{4}+\lambda\right)^2+\frac{1}{\beta^2}\left(\frac{\rho_3}{4}-\lambda\right)^2-\frac{\xi}{\beta}\right)\nu_{122}\left(\frac{1}{2}\alpha\beta\left(\rho_2-iQ_1\alpha-iQ_2\beta-\frac{1}{\alpha^2}\left(\frac{\rho_3}{4}+\lambda\right)^2\right.\right.\right.$$
$$\left.+\frac{1}{\beta^2}\left(\frac{\rho_3}{4}-\lambda\right)^2-\frac{\xi}{\beta}\right)+\frac{\alpha}{\beta^2}\left(\frac{\rho_3}{4}-\lambda\right)^2+\frac{1}{\alpha}\left(\frac{\rho_3}{4}+\lambda\right)\left(\frac{\rho_3}{4}-\lambda\right)+\frac{\beta^2}{2}\left(\rho_2-iQ_1\alpha-iQ_2\beta-\frac{1}{\alpha^2}\left(\frac{\rho_3}{4}+\lambda\right)^2\right.$$
$$\left.\left.+\frac{1}{\beta^2}\left(\frac{\rho_3}{4}-\lambda\right)^2+\frac{\xi}{\alpha}\right)\right)+3\nu_{11}\alpha^2+3\nu_{22}\beta^2+6\nu_{12}\alpha\beta+\frac{i\omega_4}{4}=0. \tag{3.23}$$

This system of cubic algebraic equations fixes the remaining coefficients and fully defines the Virasoro and W_3 irregular vertex operator.

IV. CONCLUSION

In this paper we have constructed an explicit form of the irregular vertex operator with Virasoro and W-symmetry. Given the irregular vertex operators, constructed in this work, it is straightforward to read off the associate Penner type potentials whose random matrix model has the Seiberg-Witten curves corresponding to the 4d gauge theories. Although in the text we limited the explicit examples to the Virasoro cases and to the W_3-case of rank 2, it is not difficult to see the pattern for the general $W^{(N)}$ with arbitrary rank q. The vertex operators would generally contain $N-1$ Toda fields and involve the derivatives of orders up to q:

$$I_{N|q}=:e^{\sum_{a=1}^{N-1}\sum_{k=0}^{q}\alpha_{a|k}\partial^k\phi^{(a)}}: \tag{4.1}$$

This IVO again generates the simultaneous eigenstate of L_n for $q\le n\le 2q$ with eigenvalues ρ_n (annihilated by the higher L_n's) and of the expansion modes $W_n^{(s)}$ for $(s-1)q\le n\le sq$ with eigenvalues $\lambda_n^{(s)}$ (annihilated by higher $W_n^{(s)}$). The W current with the integer spin $3\le s\le N$ has the form

$$j_w^{(s)}(z)\equiv\sum_n\frac{W_n^{(s)}}{z^{n+s}}=\sum_{r=1}^{N}\sum_{s|p_1\ldots p_r}\sum_{\{a_1,\ldots,a_r\}}\nu^{(s)}_{a_1\ldots a_r|p_1\ldots p_r}\partial^{p_1}\phi^{(a_1)}\ldots\partial^{p_r}\phi^{(a_r)} \tag{4.2}$$

where the sum is taken over the ordered partitions of $s=p_1+\cdots+p_r;1\le p_1\ldots\le p_r$ with the lengths $1\le r\le N$ and $1\le a_1\le a_2\ldots\le a_r\le N-1$. The coefficients $\nu^{(s)}_{a_1\ldots a_r|p_1\ldots p_r}$ are determined by $N-2$ systems of linear algebraic equations (one per each s) stemming from the primary field constraints for each $j_w^{(s)}$. Once the ν-coefficients are fixed, the $\alpha_{a|k}$ coefficients are related to $\nu^{(s)}_{a_1\ldots a_r|p_1\ldots p_r}$ and the eigenvalues ρ_n and $\lambda_n^{(s)}$ by the system $(N-1)(q+1)$ algebraic (nonlinear) equations, exactly matching the number of the coefficients.

These algebraic constraints altogether (for ν and for α) fully determine the W_N irregular blocks related to the degree N Seiberg-Witten curves in Argyres-Douglas theories. Given the coefficients in the irregular vertex operators, it is straightforward to establish their relation to eigenvalues of Virasoro generators and W generators. For simplicity, we shall demonstrate it for the $W^{(3)}$ irregular vertex operator of an arbitrary rank. However, the computation below is straightforward to establish for the arbitrary n case. Let us consider the irregular vertex operator (4.1) for the $W^{(3)}$-case and expand it in series of ϕ and its derivatives:

$$I_{3|q} =: e^{\sum_{a=1}^{2}\sum_{k=0}^{q}\alpha_{a|k}\partial^k\phi^{(a)}} = \prod_{a=1}^{2}\prod_{k_a=0}^{q}\sum_{N_{a|k_a}=0}^{\infty}\frac{(\alpha_{a|k_a}\partial^{k_a}\phi_a)^{N_{a|k_a}}}{N_{a|k_a}!}. \quad (4.3)$$

Applying the stress-energy tensor to (4.3) and reexponentiating we obtain for $[L_r, I_{3|q}] = \rho_r I_{3|q}$ with $q \le r \le 2q$ where

$$\rho_r = -\sum_{a=1}^{2}\sum_{p_a+q_a=r;0\le p_a,q_a\le q}(p_a)!(q_a)!\alpha_{a|p_a}\alpha_{a|q_a} + \frac{Q}{2}(p_a+1)!\alpha_{a|p_a}\delta_{p_a|q}. \quad (4.4)$$

Finally, applying $[W_r^{(3)}, I_{3|q}] = \omega_r I_{3|q}$ we have

$$\omega_r = -\sum_{i,j,k=1}^{2}\Bigg\{\sum_{m_i+p_j+q_k=r;0\le m_i,p_j,q_k\le q}\sigma^{ijk}\nu^{ijk}m_i!p_j!q_k!\alpha_{i|m_i}\alpha_{j|p_j}\alpha_{k|q_k} + \sum_{m_i+p_j=r;0\le m_i,p_j\le q-1}m_i!(p_j+1)!\alpha_{i|m_i}\alpha_{j|p_j} + (r+2)!\nu^i\alpha_{i|r}\Bigg\} \quad (4.5)$$

where σ^{ijk} is the symmetric factor, symmetric in the i, j, k and $\sigma^{111} = \sigma^{222} = 3!$, $\sigma^{112} = \sigma^{122} = 2!$. The relations (4.4) and (4.5) reproduce those obtained earlier in [18] using a different approach, by direct application of the operator product expansion to the colliding limit of regular vertex operators.

It is straightforward to check that the irregular states created by the irregular vertex operators are coherent, i.e. are the eigenstates of the spin 1 conserving current $\partial\phi$ (note that, just as spin 2 conserving current $T(z)$, conserving spin 1 is not a primary field if $Q \ne 0$). Indeed, expanding the general irregular operator (4.1) in series similarly to (4.3) and reexponentiating, it is easy to verify the OPE

$$\partial\phi^{(b)}(z)I_{N|q}(w) = -\sum_{k=0}^{q}\frac{\alpha_{b|k}k!}{(z-w)^k}I_{N|q}(w) + \text{regular} \quad (4.6)$$

from which the coherent state property follows.

The exponents for the irregular vertices of the type (4.3), whose explicit examples have been constructed in our work, can of course be expanded in powers of the derivatives of $\phi^{(a)}$, leading to combinations of these derivatives acting on *regular* vertex operators $e^{\sum_a\alpha_{a|0}\phi^{(a)}}$ in Toda theories. These terms can be classified according to total conformal dimensions h carried by the derivatives acting on the regular vertex. Each dimension h's contribution to the expansion can be cast as some combination of products of the negative Virasoro and W-current modes $\sim L_{-h_1}\ldots L_{-h_p}W^{(s_1)}_{-h_{p+1}}W^{(s_q)}_{-h_{p+q}}$ acting on the regular vertex. where $h_k; k = 1\ldots p+q$ are the elements of the length $p+q$ partitions of h.

This generalizes the expansion of the irregular states in terms of the Virasoro descendants of the primaries created by regular vertex operators, discussed in [10,14,15] to $W^{(N)}$-case. Note that, in this descendent expansion approach, the expansion coefficients were not completely fixed even in the rank 2 Virasoro case from the eigenvalue constraint. One needs further consistency conditions with the lower Virasoro mode [12]. As seen in this free field approach, the expansion coefficients for the irregular vertex operators should be determined completely without resorting to other conditions. The difficulty simply is related with the fact that if the descendent decomposition has more variables than the number of eigenvalue constraints.

The irregular vertex operators and the irregular blocks, studied in this paper, appear to be quite fascinating objects by themselves, and may be of interest far beyond AGT conjecture and Liouville/Toda theories. First of all, from the AdS/CFT point of view it seems plausible that the irregular blocks may be string-theoretic duals of some important classes of local composite operators on the gauge/CFT side, e.g. such as $\sim T_{\mu_1\nu_1}\ldots T_{\mu_n\nu_n}$. On the other hand, the operators of this sort must correspond to higher spin modes in AdS with mixed symmetries. As $T_{\mu\nu}$ is the CFT dual of the graviton vertex operator [19], the operators like $\sim T^n$ can be understood as the colliding limit of n gravitons, i.e. a rank n irregular block, generalized to string theory. Being non-primaries, these objects are of course not in the Becchi-Rouet-Stora-Tyutin (BRST) cohomology and therefore are essentially off-shell. On the other hand, they constitute a subclass of operators which is far richer than the subspace of primaries, but appear to have very nice and controllable behavior under global conformal transformations. As such, they may play an important role in string field theory (SFT), being crucial elements for finding new classes of analytic solutions. Given that SFT is currently our best hope to advance toward background independent formulation of string theory, and that analytic solutions constitute a crucial ingredient in such a formulation, one can anticipate that the irregular blocks may be of importance and interest in describing various nonperturbative backgrounds in string theory (such as collective higher spin vacuum states). Ultimately, the deeper understanding of the irregular blocks may be an important step toward understanding the interplays between two-dimensional and four-dimensional theories which at the moment still largely retain the status of conjectures.

ACKNOWLEDGMENTS

This work is partially supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP) (NRF-2014R1A2A2A01004951) and by the National Natural Science Foundation of China under Grant No. 11575119. We would like to thank Bruno Le Floch for useful remarks and comments, as well as for pointing out some typos in the initial version of the manuscript. We also thank B. Le Floch and H. Nagoya for pointing us out the references containing discussions and results related to those investigated in this work.

[1] L. F. Alday, D. Gaiotto, and Y. Tachikawa, Liouville correlation functions from four-dimensional gauge theories, Lett. Math. Phys. **91**, 167 (2010).

[2] N. A. Nekrasov, Seiberg-Witten prepotential from instanton counting, Adv. Theor. Math. Phys. **7**, 831 (2003).

[3] P. C. Argyres and M. R. Douglas, New phenomena in SU(3) supersymmetric gauge theory, Nucl. Phys. **B448**, 93 (1995).

[4] P. C. Argyres, M. R. Plesser, N. Seiberg, and E. Witten, New N = 2 superconformal field theories in four dimensions, Nucl. Phys. **B461**, 71 (1996).

[5] T. Eguchi and K. Maruyoshi, Penner type matrix model and Seiberg-Witten theory, J. High Energy Phys. 02 (2010) 022.

[6] D. Gaiotto and J. Teschner, Irregular singularities in Liouville theory, J. High Energy Phys. 12 (2012) 050.

[7] D. Gaiotto, Asymptotically free N = 2 theories and irregular conformal blocks, J. Phys. Conf. Ser. **462**, 012014 (2013).

[8] E. Felinska, Z. Jaskolski, and M. Kosztolowicz, Whittaker pairs for the Virasoro algebra and the Gaiotto—Bmt states, Math. Phys. Appl. Math. **53**, 033504 (2012).

[9] A. Marshakov, A. Mironov, and A. Morozov, On non-conformal limit of the AGT Relations, Phys. Lett. B **682**, 125 (2009).

[10] H. Kanno, K. Maruyoshi, S. Shiba, and M. Taki, W_3 irregular states and isolated N = 2 superconformal field theories, J. High Energy Phys. 03 (2013) 147.

[11] G. Bonelli, K. Maruyoshi, and A. Tanzini, Wild quiver gauge theories, J. High Energy Phys. 02 (2012) 031.

[12] S. K. Choi, C. Rim, and H. Zhang, Virasoro irregular conformal block and beta deformed random matrix model, Phys. Lett. B **742**, 50 (2015).

[13] T. Nishinaka and C. Rim, Matrix models for irregular conformal blocks and Argyres-Douglas theories, J. High Energy Phys. 10 (2012) 138.

[14] S. K. Choi, C. Rim, and H. Zhang, Irregular conformal block, spectral curve and flow equations, J. High Energy Phys. 03 (2016) 118.

[15] S. K. Choi and C. Rim, Irregular matrix model with $\mathcal{W}$ symmetry, J. Phys. A **49**, 075201 (2016).

[16] D. Polyakov, Solutions in bosonic string field theory and higher spin algebras in AdS, Phys. Rev. D **92**, 106008 (2015).

[17] H. Nagoya and J. Sun, Confluent primary fields in the conformal field theory, J. Phys. A **43**, 465203 (2010).

[18] J. Gomis, B. Le Floch, M2-brane surface operators and gauge theory dualities in Toda, arXiv:1407.1852.

[19] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from non-critical string theory, Phys. Lett. B **428**, 105 (1998).