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Supersymmetric electric-magnetic duality in $D=3+3$ and $D=5+5$ dimensions as foundation of self-dual supersymmetric Yang-Mills theory

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We present electric-magnetic (EM)-duality formulations for non-Abelian gauge groups with $N=1$ supersymmetry in $D=3+3$ and $5+5$ space-time dimensions. We show that these systems generate self-dual $N=1$ supersymmetric Yang-Mills (SDSYM) theory in $D=2+2$. For a $N=2$ supersymmetric EM-dual system in $D=3+3$, we have the Yang-Mills multiplet $(A_\mu{}^I, \lambda_A{}^I)$ and a Hodge-dual multiplet $(B_{\mu\nu\rho}{}^I, \chi_A{}^I)$, with an auxiliary tensors $C_{\mu\nu\rho\sigma}{}^I$ and $K_{\mu\nu}$. Here, I is the adjoint index, while A is for the doublet of $Sp(1)$. The EM-duality conditions are $F_{\mu\nu}{}^I = (1/4!)\epsilon_{\mu\nu}{}^{\rho\sigma\tau\lambda} G_{\rho\sigma\tau\lambda}{}^I$ with its superpartner duality condition $\lambda_A{}^I = -\chi_A{}^I$. Upon appropriate dimensional reduction, this system generates SDSYM in $D=2+2$. This system is further generalized to $D=5+5$ with the EM-duality condition $F_{\mu\nu}{}^I = (1/8!)\epsilon_{\mu\nu}{}^{\rho_1\cdots\rho_8} G_{\rho_1\cdots\rho_8}{}^I$ with its superpartner condition $\lambda^I = -\chi^I$. Upon appropriate dimensional reduction, this theory also generates SDSYM in $D=2+2$. As long as we maintain Lorentz covariance, $D=5+5$ dimensions seems to be the maximal space-time dimensions that generate SDSYM in $D=2+2$. Namely, EM-dual system in $D=5+5$ serves as the Master Theory of all supersymmetric integrable models in dimensions $1 \le D \le 3$.

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I. INTRODUCTION

In our recent paper [1], we have presented electric-magnetic (EM) duality for non-Abelian gauge groups in $D=3+1$ and $D=9+1$ space-time dimensions. These formulations are based on the recently developed "tensor-hierarchy" formulation for non-Abelian tensors [2–4]. The EM-duality conditions are such that $F_{\mu\nu}{}^I = (1/2)\epsilon_{\mu\nu}{}^{\rho\sigma} G_{\rho\sigma}{}^I$ in $D=3+1$ or $F_{\mu\nu}{}^I = (1/8!)\epsilon_{\mu\nu}{}^{\rho_1\cdots\rho_8} G_{\rho_1\cdots\rho_8}{}^I$ in $D=9+1$, where $G_{\rho\sigma}{}^I$ or $G_{\rho_1\cdots\rho_8}{}^I$ is a new field strength *dual* to the original Yang-Mills (YM)-field strength $F_{\mu\nu}{}^I$, with the adjoint-representation index I. Before the discovery of tensor hierarchy [2] and its elaborations [3,4], such formulations with non-Abelian tensors were problematic because the naive definition of the field strengths $G_{\mu\nu}{}^I \equiv 2D_{[\mu}B_{\nu]}{}^I$ or $G_{\mu_1\cdots\mu_8}{}^I \equiv 8D_{[\mu_1}B_{\mu_2\cdots\mu_8]}{}^I$ with the adjoint index I led to inconsistencies [5].

The tensor-hierarchy formulation was first discovered in Ref. [2], as the generalization of $E_{6(6)}$ symmetry for $D=4+1$ maximal supergravity. Afterward, tensor hierarchies more elaborated in Ref. [3]. For our purpose, the approach of our recent paper [4] in four dimensions is of special relevance. New ingredients in Ref. [4] compared with the previous works in Refs. [2] and [3] can be summarized as follows. First, the tensor-hierarchy formations originally presented in Refs. [2] and [3] were either for *nonsupersymmetric* general bosonic systems, or supersymmetric theories in dimensions such as $D=2+1$, $D=4+1$ or $D=5+1$, different from $D=3+1$ of Ref. [4]. Second, in our paper [4], we presented an explicit formulation for the supersymmetric non-Abelian tensor in four dimensions, with the sophisticated combination of three multiplets: (i) vector multiplet $(A_\mu{}^I, \lambda^I)$, (ii) non-Abelian tensor multiplet $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$, and (iii) compensator vector multiplet $(C_\mu{}^I, \rho^I)$. Even though our system in Ref. [4] is covered as a special case of more general (but *not* necessarily supersymmetric) formulations in Refs. [2] and [3], we stress that the nontrivial feature of our system emerges for its supersymmetrization. This is because fixing an actually working supersymmetric system is a highly nontrivial procedure in practice. We have carried it out in Ref. [4] by an explicit supersymmetric non-Abelian system in four dimensions with the combination of the aforementioned nontrivial three multiplets.

We emphasize the importance of *non-Abelian* tensors in these duality relationships. There has already been considerable research since the 1990s on *Abelian* duality symmetries in the case that the relevant tensors carry no adjoint indices. (cf. Refs. [6,7], and [8]) For example, Ref. [6] discusses the duality symmetries between the conventional flux tensor fields and its Hodge duals in supergravities in $D=9+1$ and $D=10+1$. In Ref. [7], the so-called democratic formulation of equal treatments among tensor fields of different ranks in D8-O8 domain walls is presented. Reference [8] deals with Hodge dualities of various forms in (half-)maximal supergravities in diverse

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TABLE I. Field content of EM-duality formulations. The symbol $[n]$ is for the totally antisymmetric space-time indices to save space. The symbol $\overset{*}{=}$ is for an equality associated with duality.

Space-time	SYM	HDM	Auxiliary tensors	Bosonic EM duality
$D = 3+3$	$(\hat{A}_{\hat{\mu}}{}^I, \hat{\lambda}_A{}^I)$	$(\hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I, \hat{\chi}_A{}^I)$	$\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I$, $\hat{K}_{\hat{\mu}\hat{\nu}}$	$\hat{F}_{\hat{\mu}\hat{\nu}}{}^I \overset{*}{=} +\frac{1}{4!}\hat{\epsilon}_{\hat{\mu}\hat{\nu}}{}^{\widehat{[4]}}\hat{G}_{\widehat{[4]}}{}^I$
$D = 5+5$	$(\hat{A}_{\hat{\mu}}{}^I, \hat{\lambda}^I)$	$(\hat{B}_{\widehat{[7]}}{}^I, \hat{\chi}^I)$	$\hat{C}_{\widehat{[8]}}{}^I$, $\hat{K}_{\widehat{[6]}}$	$\hat{F}_{\hat{\mu}\hat{\nu}}{}^I \overset{*}{=} +\frac{1}{8!}\epsilon_{\hat{\mu}\hat{\nu}}{}^{\widehat{[8]}}\hat{G}_{\widehat{[8]}}{}^I$

dimensions. However, the works [6,7], and [8] are all based on Abelian-type duality symmetries between tensors *without adjoint indices*. Note that the realization of duality symmetries among non-Abelian tensors became possible, only after the consistent formulation of tensor hierarchy was discovered in Ref. [2] and elaborated upon in Refs. [3] and [4]. In other words, tensor-hierarchy formulation in Refs. [2,3], and [4] motivates us to study the duality symmetries among non-Abelian tensors, as the next natural and important step.

Independent of tensor-hierarchy formulations [2–4], considerable development for self-dual supersymmetric Yang-Mills (SDSYM) theories in $D = 2+2$ space-time dimensions was accomplished in Refs. [9,10], and [11]. This research direction is traced back to the original conjecture by M. Atiyah [12] that self-dual Yang-Mills theory in $D = 2+2$ generates *all* bosonic integrable models in $1 \le D \le 3$ as the Master Theory. This original conjecture for purely *bosonic* systems [12] was further *supersymmetrized* in $D = 2+2$ in the mid-1990s [9,10].

SDSYM in $D = 2+2$ was also investigated from the viewpoint that SDSYM in $D = 2+2$ is nothing but the consistent background [13] for the $N = 2$ superstring [14]. Thus, SDSYM in $D = 2+2$ [9,10] was motivated by two important concepts: (i) Master Theory of all supersymmetric integrable models in $1 \le D \le 3$ and (ii) consistent backgrounds for $N = 2$ superstring theory.

The self-duality $F_{\mu\nu}{}^I = (1/2)\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}{}^I$ in $D = 2+2$ was further generalized to higher space-time dimensions, such as six, seven, and eight dimensions with *reduced holonomies* $SU(3), G_2$, and $SO(7)$ [15,16]. These reduced holonomies are required, due to the absence of the ϵ tensor with only *four* indices in these higher space-time dimensions. By introducing reduced holonomies, self-dual conditions are modified to $F_{\mu\nu}{}^I = (1/2)\psi_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}{}^I$ in eight dimensions, where the ϵ tensor in $D = 2+2$ is replaced by the octonion structure constant $\psi_{\mu\nu}{}^{\rho\sigma}$ with only four indices. These formulations were further supersymmetrized in our paper [17], as SDSYM multiplets in $6 \le D \le 8$. It is not surprising that the quest for the Master Theory of supersymmetric integrable models is reaching out to higher and higher space-time dimensions.

On the other hand, our recent paper [1] has shown that the non-Abelian EM duality works not only in $D = 3+1$ but also in $D = 9+1$ space-time dimensions [1]. Considering these new developments in the last 20 years, the next natural step is to consider EM duality in higher dimensions such as $D = 5+5$ that generates SDSYM in $D = 2+2$. In other words, it is imperative to seek EM-duality formulations in higher dimensions with the aim of establishing the more fundamental Master Theory yielding SDSYM in $D = 2+2$ or supersymmetric integrable models in $1 \le D \le 3$. The important point here is that the space-time signature $D = 5+5$ is different from $D = 9+1$, which has been already studied in Ref. [1].

In this paper, we take the initial first step in the direction of supersymmetric EM duality in diverse dimensions with space-time with *nonconventional signatures*, such as $D = 3+3$ or $D = 5+5$. We first construct $N = (1,0)$ supersymmetric EM-duality formulation in $D = 3+3$ space-time dimensions. Our field content consists of the usual supersymmetric YM multiplet (SYM) $(\hat{A}_{\hat{\mu}}{}^I, \hat{\lambda}_A{}^I)$,[1] a Hodge-dual multiplet (HDM) $(\hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I, \hat{\chi}_A{}^I)$, in addition to auxiliary tensors $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I$ and $\hat{K}_{\hat{\mu}\hat{\nu}}$. The (bosonic) EM-duality condition is $\hat{F}_{\hat{\mu}\hat{\nu}}{}^I \overset{*}{=} +(1/4!)\hat{\epsilon}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}}\hat{G}_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}}{}^I$,[2] with its superpartner condition $\hat{\lambda}_A{}^I \overset{*}{=} -\hat{\chi}_A{}^I$. We show that by imposing a set of dimensional-reduction conditions [18] relating the SYM and HDM the usual $N = 1$ SYM in $D = 2+2$ space-time dimensions emerges.

We next generalize this result to EM-duality formulation in $D{=}5{+}5{=}10$. Our field content is a SYM $(\hat{A}_{\hat{\mu}}{}^I, \hat{\lambda}^I)$, a HDM $(\hat{B}_{\hat{\mu}_1\cdots\hat{\mu}_7}{}^I, \hat{\chi}^I)$, in addition to the auxiliary tensors $\hat{C}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I$ and $\hat{K}_{\hat{\mu}_1\cdots\hat{\mu}_6}$. The (bosonic) EM-duality condition is $\hat{F}_{\hat{\mu}\hat{\nu}}{}^I \overset{*}{=} +(1/8!)\hat{\epsilon}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}_1\cdots\hat{\rho}_8}\hat{G}_{\hat{\rho}_1\cdots\hat{\rho}_8}{}^I$ with its superpartner condition $\hat{\lambda}^I \overset{*}{=} -\hat{\chi}^I$. Upon imposing appropriate dimensional-reduction rules, this system again yields $N = 1$ supersymmetric SDSYM in $D = 2+2$. The field contents in $D = 3+3$ and $D = 5+5$ space-time dimensions are summarized in Table 1 below.

This paper is organized as follows. In the next section, we present the EM-duality formulation in $D = 3+3$. In Sec. III, we perform the dimensional reduction of this EM-duality formulation from $D = 3+3$ into SDSYM in $D = 2+2$. In Sec. IV, we present the EM-duality

[1]We use the hat symbols for fields and indices associated with higher space-time dimensions, in order to distinguish them from the corresponding ones in $D = 2+2$. This is the same convention as in Ref. [18].

[2]We use the symbol $\overset{*}{=}$ for an equality related to dualities.

formulation in $D = 5 + 5$. In Sec. V, we perform its dimensional reduction into SDSYM in $D = 2 + 2$. The concluding remarks are given in Sec. VI. Appendix A is devoted to the notational clarifications in $D = 2 + 2$, $D = 3 + 3$ and $D = 5 + 5$ space-time dimensions. Appendix B is presented for establishing the notation for $D = 4 + 2$ and $D = 8 + 2$.

II. EM-DUALITY FORMULATION IN $D = 3 + 3$ SPACE-TIME DIMENSIONS

We first present our EM duality in $D = 3 + 3$ space-time dimensions. Our field content is the SYM $(\hat{A}_{\hat{\mu}}{}^I, \hat{\lambda}_A{}^I)$ and HDM $(\hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I, \hat{\chi}_A{}^I)$, with auxiliary tensors $\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I$ and $\hat{K}_{\hat{\mu}\hat{\nu}}$. These auxiliary tensor fields are needed for the tensor-hierarchy formulation [2–4]. The "hat" symbols are used for fields and indices in $D = 3 + 3$, distinguished from those in $D = 2 + 2$ space-time dimensions.[3] The indices $I, J, ... = 1, 2, ..., \dim G$ are for the adjoint representation of the YM gauge group G. The indices $A, B, ... = (1), (2)$ are for the **2** of $Sp(1)$, and they are contracted by the $Sp(1)$ metric ϵ_{AB}, where $\epsilon_{(1)(2)} = -\epsilon_{(2)(1)} = \epsilon^{(1)(2)} = -\epsilon^{(2)(1)} = +1$. The fermions $\hat{\lambda}_A{}^I$ and $\hat{\chi}_A{}^I$ are both Majorana-Weyl spinors with the chiralities $\hat{\gamma}_7(\hat{\lambda}_A{}^I, \hat{\chi}_A{}^I) = (+\hat{\lambda}_A{}^I, +\hat{\chi}_A{}^I)$.

Our $N = 2$ supersymmetry transformation rule is[4]

$$\delta_Q \hat{A}_{\hat{\mu}}{}^I = +(\bar{\hat{\epsilon}}^A \hat{\gamma}_{\hat{\mu}} \hat{\lambda}_A{}^I) \equiv +(\bar{\hat{\epsilon}} \hat{\gamma}_{\hat{\mu}} \hat{\lambda}^I), \tag{2.1a}$$

$$\delta_Q \hat{\lambda}_A{}^I = -\frac{1}{2}(\hat{\gamma}^{\hat{\mu}\hat{\nu}} \hat{\epsilon}_A) \hat{F}_{\hat{\mu}\hat{\nu}}{}^I, \tag{2.1b}$$

$$\delta_Q \hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I = +(\bar{\hat{\epsilon}} \hat{\gamma}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{\chi}^I) + 3\hat{K}_{[\hat{\mu}\hat{\nu}}(\bar{\hat{\epsilon}} \hat{\gamma}_{\hat{\rho}]} \hat{\lambda}^I), \tag{2.1c}$$

$$\delta_Q \hat{\chi}_A{}^I = +\frac{1}{24}(\hat{\gamma}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{\epsilon}_A) \hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I, \tag{2.1d}$$

$$\delta_Q \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I = -4f^{IJK}(\bar{\hat{\epsilon}} \hat{\gamma}_{[\hat{\mu}} \hat{\lambda}^J) \hat{B}_{\hat{\nu}\hat{\rho}\hat{\sigma}]}{}^K, \tag{2.1e}$$

$$\delta_Q \hat{K}_{\hat{\mu}\hat{\nu}} = 0. \tag{2.1f}$$

As in Eq. (2.1a), the contracted $Sp(1)$ indices are omitted for simplicity. Our field strengths $\hat{F}, \hat{G}, \hat{H}$, and $\hat{L}$ are defined by

$$\hat{F}_{\hat{\mu}\hat{\nu}}{}^I \equiv +2\hat{\partial}_{[\hat{\mu}} \hat{A}_{\hat{\nu}]}{}^I + mf^{IJK} \hat{A}_{\hat{\mu}}{}^J \hat{A}_{\hat{\nu}}{}^K \tag{2.2a}$$

$$\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I \equiv +4\hat{D}_{[\hat{\mu}} \hat{B}_{\hat{\nu}\hat{\rho}\hat{\sigma}]}{}^I + m\hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I - 6\hat{K}_{[\hat{\mu}\hat{\nu}} \hat{F}_{\hat{\rho}\hat{\sigma}]}{}^I, \tag{2.2b}$$

$$\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}{}^I \equiv +5\hat{D}_{[\hat{\mu}} \hat{C}_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}]}{}^I + 10f^{IJK} \hat{F}_{[\hat{\mu}\hat{\nu}}{}^J \hat{B}_{\hat{\rho}\hat{\sigma}\hat{\tau}]}{}^K, \tag{2.2c}$$

$$\hat{L}_{\hat{\mu}\hat{\nu}\hat{\rho}} \equiv +3\hat{\partial}_{[\hat{\mu}} \hat{K}_{\hat{\nu}\hat{\rho}]}. \tag{2.2d}$$

These field strengths satisfy their proper Bianchi identities (BIds):

$$\hat{D}_{[\hat{\mu}} \hat{F}_{\hat{\nu}\hat{\rho}]}{}^I \equiv 0, \tag{2.3a}$$

$$\hat{D}_{[\hat{\mu}} \hat{G}_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}]}{}^I \equiv +\frac{1}{5} m\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}{}^I - 2\hat{L}_{[\hat{\mu}\hat{\nu}\hat{\rho}} \hat{F}_{\hat{\sigma}\hat{\tau}]}{}^I \tag{2.3b}$$

$$\hat{D}_{[\hat{\mu}} \hat{H}_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}]}{}^I \equiv +\frac{5}{2} f^{IJK} \hat{F}_{[\hat{\mu}\hat{\nu}}{}^J \hat{G}_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}]}{}^K. \tag{2.3c}$$

Note that the right side of Eq. (2.3c) vanishes upon the use of EM duality in Eq. (2.9a) below. The arbitrary variations of our field strengths are

$$\delta \hat{F}_{\hat{\mu}\hat{\nu}}{}^I = +2\hat{D}_{[\hat{\mu}}(\delta \hat{A}_{\hat{\nu}]}{}^I), \tag{2.4a}$$

$$\delta \hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I = +4\hat{D}_{[\hat{\mu}}(\tilde{\delta} \hat{B}_{\hat{\nu}\hat{\rho}\hat{\sigma}]}{}^I) + m(\tilde{\delta} \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I) - 6(\delta \hat{K}_{[\hat{\mu}\hat{\nu}}) \hat{F}_{\hat{\rho}\hat{\sigma}]}{}^I - 4(\delta \hat{A}_{[\hat{\mu}}{}^I) \hat{L}_{\hat{\nu}\hat{\rho}\hat{\sigma}]}, \tag{2.4b}$$

$$\delta \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}{}^I = +5\hat{D}_{[\hat{\mu}}(\tilde{\delta} \hat{C}_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}]}{}^I) - 10f^{IJK}(\tilde{\delta} \hat{B}_{[\hat{\mu}\hat{\nu}\hat{\rho}}{}^J) \hat{F}_{\hat{\sigma}\hat{\tau}]}{}^K + 5f^{IJK}(\delta \hat{A}_{[\hat{\mu}}{}^J) \hat{G}_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}]}{}^K, \tag{2.4c}$$

$$\delta \hat{L}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I = +3\hat{\partial}_{[\hat{\mu}}(\delta \hat{K}_{\hat{\nu}\hat{\rho}]}), \tag{2.4d}$$

where

$$\tilde{\delta} \hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I \equiv \delta \hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I - 3(\delta \hat{A}_{[\hat{\mu}}{}^I) \hat{K}_{\hat{\nu}\hat{\rho}]}, \tag{2.5a}$$

$$\tilde{\delta} \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I \equiv \delta \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I + 4f^{IJK}(\delta \hat{A}_{[\hat{\mu}}{}^J) \hat{B}_{\hat{\nu}\hat{\rho}\hat{\sigma}]}{}^K. \tag{2.5b}$$

Accordingly, the δ_Q transformations of our field strengths are

$$\delta_Q \hat{F}_{\hat{\mu}\hat{\nu}}{}^I = -2(\bar{\hat{\epsilon}} \hat{\gamma}_{[\hat{\mu}} \hat{D}_{\hat{\nu}]} \hat{\lambda}^I), \tag{2.6a}$$

$$\delta_Q \hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I \stackrel{*}{=} -4(\bar{\hat{\epsilon}} \hat{\gamma}_{[\hat{\mu}\hat{\nu}\hat{\rho}} \hat{D}_{\hat{\sigma}]} \hat{\chi}^I), \tag{2.6b}$$

$$\delta_Q \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}{}^I = -10f^{IJK}(\bar{\hat{\epsilon}} \hat{\gamma}_{[\hat{\mu}\hat{\nu}\hat{\rho}} \hat{\lambda}^J) \hat{F}_{\hat{\sigma}\hat{\tau}]}{}^K + 5f^{IJK}(\bar{\hat{\epsilon}} \hat{\gamma}_{[\hat{\mu}} \hat{\lambda}^J) \hat{G}_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}]}{}^K. \tag{2.6c}$$

Note that Eq. (2.1e) leads to $\tilde{\delta}_Q \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I = 0$ so that the m-linear term in $\delta_Q \hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I$ is absent. Also, the $(\delta_Q \hat{K})\hat{F}$ and $(\delta_Q \hat{A})\hat{L}$ terms in Eq. (2.6b) vanish, when the duality-related

[3] We repeat the same hat symbols also in $D = 5 + 5$ in Sec. IV.

[4] Our conventions are $(\hat{\eta}_{\hat{\mu}\hat{\nu}}) = \mathrm{diag}(+, +, -, -, +, -)$, where $\hat{\mu}, \hat{\nu}, ... = 1, 2, ..., 6$. For other relationships, see Appendix A.

equations (2.9) below are used. Eventually, only one term with $\hat{D}\hat{\chi}$ remains.

Following the general tensor-hierarchy formulations [2–4], the YM-gauge and proper gauge transformations for $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{K}$ fields are

$$\delta_T \hat{A}_{\hat{\mu}}{}^I = \hat{D}_{\hat{\mu}} \hat{\alpha}^I,$$
$$\delta_T(\hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I, \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I, \hat{K}_{\hat{\mu}\hat{\nu}}) = -m f^{IJK} \hat{\alpha}^J (\hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^K, \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^K, 0), \tag{2.7a}$$

$$\delta_U(\hat{A}_{\hat{\mu}}{}^I, \hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I, \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I, \hat{K}_{\hat{\mu}\hat{\nu}}) = (0, 3\hat{D}_{[\hat{\mu}}\hat{\beta}_{\hat{\nu}\hat{\rho}]}{}^I, -6 f^{IJK} \hat{F}_{[\hat{\mu}\hat{\nu}}{}^J \hat{\beta}_{\hat{\rho}\hat{\sigma}]}{}^K, 0), \tag{2.7b}$$

$$\delta_V(\hat{A}_{\hat{\mu}}{}^I, \hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I, \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I, \hat{K}_{\hat{\mu}\hat{\nu}}) = (0, -m\hat{\gamma}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I, 4\hat{D}_{[\hat{\mu}}\hat{\gamma}_{\hat{\nu}\hat{\rho}\hat{\sigma}]}{}^I, 0), \tag{2.7c}$$

$$\delta_K(\hat{A}_{\hat{\mu}}{}^I, \hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I, \hat{C}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I, \hat{K}_{\hat{\mu}\hat{\nu}}) = (0, 3\hat{\kappa}_{[\hat{\mu}}\hat{F}_{\hat{\nu}\hat{\rho}]}{}^I, 0, 2\hat{\partial}_{[\hat{\mu}}\hat{\kappa}_{\hat{\nu}]}). \tag{2.7d}$$

Note that $\delta_K \hat{B}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I \neq 0$. It is straightforward to confirm by Eq. (2.4) that all the field strengths $\hat{F}$, $\hat{G}$, $\hat{H}$, and $\hat{L}$ are invariant under δ_U, δ_V, and δ_K.

The closure among δ_Q, δ_T, δ_U, δ_V, and δ_K are confirmed as

$$[\delta_{Q_1}, \delta_{Q_2}] = \delta_{P_3} + \delta_{T_3} + \delta_{U_3} + \delta_{V_3} + \delta_{K_3},$$
$$\xi_3^\mu \equiv +2(\bar{\hat{\epsilon}}_2 \hat{\gamma}^{\hat{\mu}} \hat{\epsilon}_1), \qquad \alpha_3^I \equiv -\hat{\xi}_3^\mu \hat{A}_{\hat{\mu}}{}^I,$$
$$\hat{\beta}^3_{\hat{\mu}\hat{\nu}}{}^I \equiv -\hat{\xi}_3^{\hat{\rho}} \hat{B}_{\hat{\rho}\hat{\mu}\hat{\nu}}{}^I, \qquad \hat{\gamma}^3_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I \equiv -\hat{\xi}_3^{\hat{\sigma}} \hat{C}_{\hat{\sigma}\hat{\mu}\hat{\nu}\hat{\rho}}{}^I,$$
$$\hat{\kappa}^3_{\hat{\mu}} \equiv -\hat{\xi}_3^\nu \hat{K}_{\hat{\nu}\hat{\mu}} + \hat{\xi}_3^\mu, \tag{2.8a}$$

$$[\delta_Q, \delta_U] = \delta_V, \qquad \hat{\gamma}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I \equiv -3 f^{IJK} (\bar{\hat{\epsilon}} \hat{\gamma}_{[\hat{\mu}} \hat{\lambda}^J) \hat{\beta}_{\hat{\nu}\hat{\rho}]}{}^K, \tag{2.8b}$$

$$[\delta_Q, \delta_K] = \delta_U, \qquad \hat{\beta}_{\hat{\mu}\hat{\nu}}{}^I \equiv +2(\bar{\hat{\epsilon}} \gamma_{[\hat{\mu}} \hat{\lambda}^I) \hat{\kappa}_{\hat{\nu}]}, \tag{2.8c}$$

$$[\delta_{T_1}, \delta_{T_2}] = \delta_{T_3}, \qquad \hat{\alpha}_3^I \equiv -f^{IJK} \hat{\alpha}_1^J \hat{\alpha}_2^K, \tag{2.8d}$$

$$[\delta_Q, \delta_T] = [\delta_Q, \delta_V] = [\delta_T, \delta_U] = [\delta_T, \delta_V] = [\delta_U, \delta_V]$$
$$= [\delta_{U_1}, \delta_{U_2}] = [\delta_{V_1}, \delta_{V_2}] = [\delta_{K_1}, \delta_{K_2}]$$
$$= [\delta_T, \delta_K] = [\delta_U, \delta_K] = [\delta_V, \delta_K] = 0. \tag{2.8e}$$

These are just parallel to the EM duality in $D = 3+1$ and $D = 9+1$ in Ref. [1].

The most important supersymmetric EM-duality conditions and field equations in $D = 3+3$ are[5]

$$\hat{F}_{\hat{\mu}\hat{\nu}}{}^I \stackrel{*}{=} +\frac{1}{4!} \hat{\epsilon}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}} \hat{G}_{\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\lambda}}{}^I,$$
$$\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I \stackrel{*}{=} -\frac{1}{2} \hat{\epsilon}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^{\hat{\rho}\hat{\sigma}} \hat{F}_{\hat{\rho}\hat{\sigma}}{}^I, \tag{2.9a}$$

$$\hat{\lambda}_A{}^I \stackrel{*}{=} -\hat{\chi}_A{}^I, \tag{2.9b}$$

$$\hat{D}_{\hat{\nu}} \hat{F}_{\hat{\mu}}{}^{\hat{\nu} I} \doteq +\frac{1}{2} m f^{IJK} (\bar{\hat{\lambda}}^J \hat{\gamma}_{\hat{\mu}} \hat{\lambda}^K), \tag{2.9c}$$

$$\hat{\not{D}} \hat{\lambda}^I \doteq 0, \qquad \hat{\not{D}} \hat{\chi}^I \doteq 0, \tag{2.9d}$$

$$\hat{H}_{\mu\nu\rho\sigma\tau}{}^I \stackrel{*}{=} +\frac{1}{2} f^{IJK} (\bar{\hat{\lambda}}^J \hat{\gamma}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}} \hat{\lambda}^K), \tag{2.9e}$$

$$\hat{L}_{\mu\nu\rho} \stackrel{*}{=} 0. \tag{2.9f}$$

The consistency of Eq. (2.9) with $N = 2$ supersymmetry (2.1) is easily confirmed by varying the former under the latter transformation. A typical nontrivial case is Eq. (2.9e):

$$0 \stackrel{?}{=} \delta_Q \left[\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}{}^I - \frac{1}{2} f^{IJK} (\bar{\hat{\lambda}}^J \hat{\gamma}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}} \hat{\lambda}^K) \right]$$
$$= +10 f^{IJK} (\bar{\hat{\epsilon}} \hat{\gamma}_{[\hat{\mu}\hat{\nu}\hat{\rho}} \hat{\lambda}^J) \hat{F}_{\hat{\sigma}\hat{\tau}]}{}^K + 5 f^{IJK} (\bar{\hat{\epsilon}} \hat{\gamma}_{[\hat{\mu}} \hat{\lambda}^J) G_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}]}{}^{\hat{K}}$$
$$- f^{IJK} \left[\frac{1}{2} (\bar{\hat{\epsilon}} \hat{\gamma}^{\hat{\lambda}\hat{\omega}}) \hat{F}_{\hat{\lambda}\hat{\omega}}{}^J \right] \gamma_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}} \hat{\lambda}^K$$
$$= +10 f^{IJK} (\bar{\hat{\epsilon}} \hat{\gamma}_{[\hat{\mu}\hat{\nu}\hat{\rho}} \hat{\lambda}^J) \hat{F}_{\hat{\sigma}\hat{\tau}]}{}^K + 5 f^{IJK} (\bar{\hat{\epsilon}} \hat{\gamma}^{\hat{\lambda}}{}_{[\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{\lambda}^J) \hat{F}_{\hat{\tau}]\hat{\lambda}}{}^I$$
$$+ [-5 f^{IJK} (\bar{\hat{\epsilon}} \hat{\gamma}^{\hat{\lambda}}{}_{[\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{\lambda}^J) \hat{F}_{\hat{\tau}]\hat{\lambda}}{}^K - 10 f^{IJK} (\bar{\hat{\epsilon}} \hat{\gamma}_{[\hat{\mu}\hat{\nu}\hat{\rho}} \hat{\lambda}^J) \hat{F}_{\hat{\sigma}\hat{\tau}]}{}^K]$$
$$= 0 \quad (Q.E.D.). \tag{2.10}$$

As is easily seen, the duality-associated relations and field equations in Eq. (2.9) are similar to our EM duality in Minkowskian $D = 5+1$ dimensions [1].

Some of the duality-related equations in Eq. (2.9) are also used for the closure of gauge algebra. For example, in the commutator $[\delta_{Q_1}, \delta_{Q_2}] C_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I$, we need a sophisticated relationship,

$$[-4 f^{IJK} (\bar{\hat{\epsilon}}_2 \hat{\gamma}_{[\hat{\mu}} \hat{\lambda}^J)(\bar{\hat{\epsilon}}_1 \gamma_{\hat{\nu}\hat{\rho}\hat{\sigma}]} \hat{\chi}^K)] - (1 \leftrightarrow 2)$$
$$\stackrel{*}{=} +\frac{1}{2} f^{IJK} \hat{\xi}^\tau (\bar{\hat{\lambda}}^J \hat{\gamma}_{\hat{\tau}\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{\lambda}^K) \stackrel{*}{=} +\hat{\xi}^\tau \hat{H}_{\hat{\tau}\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I, \tag{2.11}$$

where use is made of the duality-related Eqs. (2.9b) and (2.9e). Relevantly, we have also used the nontrivial gamma identities,

$$(\hat{\gamma}^{\hat{\mu}\hat{\nu}[\hat{\rho}})_{\hat{a}\hat{b}} (\hat{\gamma}_{\hat{\mu}\hat{\nu}}{}^{\hat{\sigma}]})_{\hat{c}\hat{d}} \equiv 0, \tag{2.12}$$

where the spinorial indices $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ carry the same chirality in $D = 3+3$. Equation (2.12) is confirmed by multiplying it by two independent matrices for the indices $\hat{b}\,\hat{c}$, namely, $(\hat{\gamma}^{\hat{\lambda}})^{\hat{b}\hat{c}}$ and $(\hat{\gamma}^{\hat{\lambda}\hat{\omega}\hat{\psi}})^{\hat{b}\hat{c}}$.

[5]We use the symbol $\doteq$ for a field equation.

III. DIMENSIONAL REDUCTION FROM $D = 3 + 3$ TO $D = 2 + 2$

Our dimensional-reduction rule from $D = 3 + 3$ into $D = 2 + 2$ is summarized as

$$\hat{A}_\mu{}^I \overset{*}{=} A_\mu{}^I, \qquad \hat{A}_\alpha{}^I = 0 \quad (\alpha = 5, 6), \tag{3.1a}$$

$$\hat{\lambda}_A{}^I \overset{*}{=} \begin{pmatrix} \lambda_{\uparrow A}{}^I \\ \lambda_{\downarrow A}{}^I \end{pmatrix}, \qquad \hat{\chi}_A{}^I \overset{*}{=} \begin{pmatrix} \chi_{\uparrow A}{}^I \\ \chi_{\downarrow A}{}^I \end{pmatrix},$$
$$\hat{\epsilon}^A \overset{*}{=} \begin{pmatrix} \epsilon_{\uparrow A} \\ \epsilon_{\downarrow A} \end{pmatrix}, \qquad (A, B, \cdots = (1), (2)), \tag{3.1b}$$

$$\hat{\lambda}_{\uparrow(1)}{}^I \overset{*}{=} \hat{\lambda}_{\downarrow(2)}{}^I \equiv \frac{1}{\sqrt{2}} \lambda^I,$$
$$\chi_{\uparrow(1)}{}^I \overset{*}{=} \chi_{\downarrow(2)}{}^I \equiv \frac{1}{\sqrt{2}} \chi^I \overset{*}{=} -\frac{1}{\sqrt{2}} \lambda^I, \tag{3.1c}$$

$$\epsilon_{\uparrow(1)} \overset{*}{=} \epsilon_{\downarrow(2)} \overset{*}{=} \frac{1}{\sqrt{2}} \epsilon, \quad \gamma_5(\lambda^I, \chi^I) = (-\lambda^I, -\chi^I), \tag{3.1d}$$

$$\lambda_{\uparrow(2)}{}^I \overset{*}{=} \lambda_{\downarrow(1)}{}^I \overset{*}{=} \chi_{\uparrow(2)}{}^I \overset{*}{=} \chi_{\downarrow(1)}{}^I \overset{*}{=} \epsilon_{\uparrow(2)} \overset{*}{=} \epsilon_{\downarrow(1)} \overset{*}{=} 0, \tag{3.1e}$$

$$(\hat{\gamma}_{\hat{\mu}})_{\hat{a}}{}^{\hat{b}} \overset{*}{=} \begin{cases} (\hat{\gamma}_\mu)_{\hat{a}}{}^{\hat{b}} = (\gamma_\mu)_a{}^b \delta_i{}^j \quad (i, j = \uparrow, \downarrow) \\ (\hat{\gamma}_5)_{\hat{a}}{}^{\hat{b}} = (\gamma_5)_a{}^b (\sigma_1)_i{}^j, \\ (\hat{\gamma}_6)_{\hat{a}}{}^{\hat{b}} = i(\gamma_5)_a{}^b (\sigma_2)_i{}^j, \end{cases} \tag{3.1f}$$

$$(\hat{\gamma}_7)_{\hat{a}}{}^{\hat{b}} \equiv (\hat{\gamma}_{123456})_{\hat{a}}{}^{\hat{b}} = -(\gamma_5)_a{}^b (\sigma_3)_i{}^j,$$
$$\hat{C}_{\hat{a}\hat{b}} = C_{ab} (\sigma_2)_{ij} = +\hat{C}_{\hat{b}\hat{a}}, \tag{3.1g}$$

$$(\hat{\gamma}_{\hat{\mu}})_{\hat{a}\hat{b}} \equiv (\hat{\gamma}_{\hat{\mu}})_{\hat{a}}{}^{\hat{c}} \hat{C}_{\hat{c}\hat{b}} \overset{*}{=} \begin{cases} (\hat{\gamma}_\mu)_{\hat{a}\hat{b}} = (\gamma_\mu)_{ab} (\sigma_2)_{ij} = -(\hat{\gamma}_{\hat{\mu}})_{\hat{b}\hat{a}}, \\ (\hat{\gamma}_5)_{\hat{a}\hat{b}} = -i(\gamma_5)_{ab} (\sigma_3)_{ij} = -(\hat{\gamma}_5)_{\hat{b}\hat{a}}, \\ (\hat{\gamma}_6)_{\hat{a}\hat{b}} = -i(\gamma_5)_{ab} \delta_{ij} = -(\hat{\gamma}_6)_{\hat{b}\hat{a}}, \end{cases} \tag{3.1h}$$

$$\hat{B}_{\mu 56}{}^I \overset{*}{=} A_\mu{}^I, \qquad \hat{B}_{\mu\nu\alpha}{}^I \overset{*}{=} 0, \qquad \hat{B}_{\mu\nu\rho}{}^I \overset{*}{=} B_{\mu\nu\rho}{}^I, \tag{3.1i}$$

$$\hat{C}_{\mu\nu 56}{}^I \overset{*}{=} 2m^{-1} \partial_{[\mu} A_{\nu]}{}^I, \qquad \hat{C}_{\mu\nu\rho\alpha}{}^I \overset{*}{=} \hat{C}_{\mu\nu\rho\sigma}{}^I \overset{*}{=} 0, \tag{3.1j}$$

$$\hat{K}_{\mu\nu} \overset{*}{=} \hat{K}_{\mu\alpha} \overset{*}{=} 0, \qquad \hat{K}_{56} \overset{*}{=} +1, \tag{3.1k}$$

$$\delta_Q B_{\mu\nu\rho}{}^I \overset{*}{=} +i(\bar{\epsilon} \gamma_{\mu\nu\rho} \chi^I) \overset{*}{=} +\epsilon_{\mu\nu\rho}{}^\sigma (\delta_Q A_\sigma{}^I). \tag{3.1l}$$

For space-time indices, we use $\hat{\mu} = (\mu, \alpha), \hat{\nu} = (\nu, \beta), \cdots$, where $\mu, \nu, \cdots = 1, 2, 3, 4$, and $\alpha, \beta, \cdots = 5, 6$. For fermionic indices, we use $\hat{a} \equiv (a, i), \hat{b} \equiv (b, j), \cdots$, $a, b, \cdots = 1, 2, \ldots, 4$, and $i, j, \cdots = \uparrow, \downarrow$. The equality with $\overset{*}{=}$ is associated with dimensional reductions, EM, or Hodge dualities. The charge-conjugation matrix $\hat{C}_{\hat{a}\hat{b}}$ in $D = 3 + 3$ is *symmetric*, consistent with the flipping property (A4a). The charge-conjugation matrix in $D = 2 + 2$ is antisymmetric: $C_{ab} = -C_{ba}$. Accordingly, we also have[6]

$$\hat{G}_{\mu\nu 56}{}^I \overset{*}{=} +\frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}{}^I \overset{*}{=} F_{\mu\nu}{}^I, \tag{3.2a}$$

$$\hat{F}_{\mu\alpha}{}^I \overset{*}{=} \hat{F}_{\alpha\beta}{}^I \overset{*}{=} 0, \tag{3.2b}$$

$$\hat{G}_{\mu\nu\rho\sigma}{}^I \overset{*}{=} 0, \qquad \hat{H}_{\mu\nu\rho\sigma\alpha}{}^I \overset{*}{=} \hat{H}_{\mu\nu\rho 56}{}^I \overset{*}{=} 0, \tag{3.2c}$$

$$\hat{L}_{\mu\nu\rho} \overset{*}{=} \hat{L}_{\mu\nu\alpha} \overset{*}{=} \hat{L}_{\mu 56} \overset{*}{=} 0, \tag{3.2d}$$

$$\not{D}\lambda^I \doteq 0, \qquad \not{D}\chi^I \doteq 0. \tag{3.2e}$$

In particular, the SDSYM conditions in $D = 2 + 2$ [10] are satisfied,

$$F_{\mu\nu}{}^I \overset{*}{=} +\frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}{}^I, \quad \gamma_5(\lambda^I, \chi^I) \overset{*}{=} -(\lambda^I, \chi^I), \tag{3.3}$$

with $N = 1$ supersymmetry

$$\delta_Q A_\mu{}^I = -i(\bar{\epsilon} \gamma_\mu \lambda^I), \tag{3.4a}$$

$$\delta_Q \lambda^I = -\frac{1}{2} (\gamma^{\mu\nu} \epsilon) F_{\mu\nu}{}^I. \tag{3.4b}$$

As a typical example of our dimensional reduction, consider $\delta_Q \hat{A}_\mu{}^I$[7]:

$$\begin{aligned} \delta_Q A_\mu{}^I &\overset{?}{=} \delta_Q \hat{A}_\mu{}^I = (\bar{\hat{\epsilon}}^A \hat{\gamma}_{\hat{\mu}} \hat{\lambda}_A{}^I) = -\hat{\epsilon}^{\hat{a}A} (\hat{\gamma}_\mu)_{\hat{a}\hat{b}} \hat{\lambda}^{\hat{b}}{}_A{}^I \\ &\overset{*}{=} -(\hat{\epsilon}_\uparrow{}^A, \hat{\epsilon}_\downarrow{}^A) \gamma_\mu \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \begin{pmatrix} \lambda_{\uparrow A}{}^I \\ \lambda_{\downarrow A}{}^I \end{pmatrix} \\ &= +i(\bar{\epsilon}_\uparrow{}^A \gamma_\mu \lambda_{\downarrow A}{}^I) - i(\bar{\epsilon}_\downarrow{}^A \gamma_\mu \lambda_{\uparrow A}{}^I) \\ &\overset{*}{=} -i(\bar{\epsilon}_{\uparrow(1)} \gamma_\mu \lambda_{\downarrow(2)}{}^I) - i(\bar{\epsilon}_{\downarrow(2)} \gamma_\mu \lambda_{\uparrow(1)}{}^I) \\ &= -i\left(\frac{1}{\sqrt{2}} \bar{\epsilon}\right) \gamma_\mu \left(\frac{1}{\sqrt{2}} \lambda^I\right) - i\left(\frac{1}{\sqrt{2}} \bar{\epsilon}\right) \gamma_\mu \left(\frac{1}{\sqrt{2}} \lambda^I\right) \\ &\overset{*}{=} -i(\bar{\epsilon} \gamma_\mu \lambda^I) \quad (Q.E.D.). \end{aligned} \tag{3.5}$$

A nontrivial example is Eq (3.1i) confirmed as

[6]The symbol $\doteq$ is used for field equations.
[7]The symbol $\overset{?}{=}$ is used for equalities under question.

$$\begin{aligned}\delta_Q \hat{B}_{\mu\nu\rho}{}^I &= +(\bar{\hat{\epsilon}}^A \hat{\gamma}_{\mu\nu\rho} \hat{\chi}_A{}^I) + 3\hat{K}_{[\mu\nu|}(\bar{\hat{\epsilon}}^A \hat{\gamma}_{|\rho]} \hat{\lambda}_A{}^I) \\ &\overset{*}{=} +(\bar{\epsilon}_\uparrow{}^A, \bar{\epsilon}_\downarrow{}^A)\gamma_{\mu\nu\rho} \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \begin{pmatrix} \chi_{\uparrow A}{}^I \\ \chi_{\downarrow A}{}^I \end{pmatrix} \\ &= +i(\bar{\epsilon}_{\uparrow(1)} \gamma_{\mu\nu\rho} \chi_{\downarrow(2)}{}^I) + i(\bar{\epsilon}_{\downarrow(2)} \gamma_{\mu\nu\rho} \chi_{\uparrow(1)}{}^I) \\ &\overset{*}{=} +i\left(\frac{1}{\sqrt{2}}\bar{\epsilon}\right)\gamma_{\mu\nu\rho}\left(\frac{1}{\sqrt{2}}\chi^I\right) + i\left(\frac{1}{\sqrt{2}}\bar{\epsilon}\right)\gamma_{\mu\nu\rho}\left(\frac{1}{\sqrt{2}}\chi^I\right) \\ &= +i(\bar{\epsilon}\gamma_{\mu\nu\rho}\chi^I) = +\epsilon_{\mu\nu\rho}{}^\sigma (\delta_Q A_\sigma{}^I) \ (Q.E.D.). \end{aligned} \tag{3.6}$$

The consistency of this with $\hat{G}_{\mu\nu\rho\sigma}{}^I \overset{*}{=} 0$ in Eq. (3.2c) is also confirmed as

$$\begin{aligned} 0 \overset{?}{=} \delta_Q \hat{G}_{\mu\nu\rho\sigma}{}^I &= +4\hat{D}_{[\mu|}(\tilde{\delta}_Q \hat{B}_{|\nu\rho\sigma]}{}^I) + m(\tilde{\delta}_Q \hat{C}_{\mu\nu\rho\sigma}{}^I) \\ &\quad - 6(\delta_Q \hat{K}_{[\mu\nu})\hat{F}_{\rho\sigma]}{}^I - 4(\bar{\hat{\epsilon}}\gamma_{[\mu|}\hat{\lambda}^I)\hat{L}_{|\nu\rho\sigma]} \\ &= +4D_{[\mu}[+i(\bar{\epsilon}\gamma_{\nu\rho\sigma]}\chi^I)] \\ &= +i\epsilon_{\mu\nu\rho\sigma}(\bar{\epsilon}\not{D}\chi^I) \doteq 0 \quad (Q.E.D.)., \end{aligned} \tag{3.7}$$

where the $\hat{\chi}$-field equation (3.2e) has been used.

The crucial role by the nonzero value $\hat{K}_{56} = +1$ (3.1k) is played in $\hat{G}_{\mu\nu 56}{}^I$:

$$\begin{aligned} F_{\mu\nu}{}^I \overset{?}{=} \hat{G}_{\mu\nu 56}{}^I \overset{*}{=} &+ 2\partial_{[\mu}\hat{B}_{\nu]56}{}^I + 2mf^{IJK}\hat{A}_\mu{}^J \hat{A}_\nu{}^K \\ &+ m\hat{C}_{\mu\nu 56}{}^I - \hat{K}_{56}\hat{F}_{\mu\nu}{}^I \end{aligned} \tag{3.8a}$$

$$\begin{aligned} &= +2\partial_{[\mu}A_{\nu]}{}^I + 2mf^{IJK}A_\mu{}^J A_\nu{}^K + 2\partial_{[\mu}A_{\nu]}{}^I - F_{\mu\nu}{}^I \\ &= F_{\mu\nu}{}^I \quad (Q.E.D.). \end{aligned} \tag{3.8b}$$

In particular, $\hat{K}_{56} = +1$ contributes to the last term in Eq. (3.8a). It is also interesting that the rank 2 of $\hat{K}_{\hat{\mu}\hat{\nu}}$ coincides with that of the two extra dimensions accommodating $\hat{K}_{\alpha\beta} = \epsilon_{\alpha\beta}$.

As for $\hat{H}_{\hat{\mu}_1\cdots\hat{\mu}_9}{}^I$, all of its components become zero as in Eq. (3.2c), despite the $\hat{\lambda}$-bilinear term (2.9e) in the original $D = 3 + 3$. This is because there is *no* λ-bilinear term sandwiching an *odd* number of γ matrices by λ^I with the negative chirality in $D = 2 + 2$.

The dimensional reduction for the fermionic fields $\hat{\lambda}^I$ and $\hat{\chi}^I \overset{*}{=} -\hat{\chi}^I$ also works,

$$\begin{aligned} \delta_Q \hat{\lambda}^I &= \delta_Q \begin{pmatrix} \lambda_{\uparrow A}{}^I \\ \lambda_{\downarrow A}{}^I \end{pmatrix} \\ &= -\frac{1}{2}(\hat{\gamma}^{\mu\nu}\hat{\epsilon}_A)\hat{F}_{\mu\nu}{}^I - (\hat{\gamma}^{\mu\alpha}\hat{\epsilon}_A)\hat{F}_{\mu\alpha}{}^I - (\hat{\gamma}^{56}\hat{\epsilon}_A)\hat{F}_{\mu\alpha}{}^I \\ &= -\frac{1}{2}\gamma^{\mu\nu}\begin{pmatrix} \epsilon_{\uparrow A} \\ \epsilon_{\downarrow A} \end{pmatrix} F_{\mu\nu}{}^I = \begin{pmatrix} -\frac{1}{2}(\gamma^{\mu\nu}\epsilon_{\uparrow A})F_{\mu\nu}{}^I \\ -\frac{1}{2}(\gamma^{\mu\nu}\epsilon_{\downarrow A})F_{\mu\nu}{}^I \end{pmatrix}. \end{aligned} \tag{3.9}$$

The nonzero components are $\delta_Q \lambda_{\uparrow(1)}{}^I = (1/\sqrt{2})\delta_Q \lambda^I$ and $\delta_Q \lambda_{\downarrow(2)}{}^I = (1/\sqrt{2})\delta_Q \lambda^I$, so that both of them consistently yield (3.4b), as desired. As for $\delta_Q \hat{\chi}^I$, it is just parallel to $\delta_Q \hat{\lambda}^I$ because $\delta_Q \hat{\chi}^I$ in Eq. (2.1d) already is equivalent to $-\delta_Q \hat{\lambda}^I$ in Eq. (2.1b) by the use of the duality (2.9a). So, its confirmation is skipped here.

IV. EM-DUALITY FORMULATION IN $D = 5 + 5$

We can generalize the field-content pattern in $D = 3 + 3$ to $D = 5 + 5$ space-time dimensions, based on general tensor-hierarchy formulation [2–4]. Our field content is the SYM $(\hat{A}_{\hat{\mu}}{}^I, \hat{\lambda}^I)$ and HDM $(\hat{B}_{\widehat{[7]}}{}^I, \hat{\chi}^I)$, with auxiliary tensors $\hat{C}_{\widehat{[8]}}{}^I$ and $\hat{K}_{\widehat{[6]}}$. The fermions have the positive chirality: $\gamma_{11}(\lambda^I, \chi^I) = +(\lambda^I, \chi^I)$. Note that the rank 6 of the $\hat{K}$ field is the same as the number of extra dimensions, six out of ten dimensions.

Our supersymmetry transformation rule is

$$\delta_Q \hat{A}_{\hat{\mu}}{}^I = +(\bar{\hat{\epsilon}}\hat{\gamma}_{\hat{\mu}}\hat{\lambda}^I), \tag{4.1a}$$

$$\delta_Q \hat{\lambda}^I = +\frac{1}{2}(\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\epsilon})\hat{F}_{\mu\nu}{}^I, \tag{4.1b}$$

$$\delta_Q \hat{B}_{\hat{\mu}_1\cdots\hat{\mu}_7}{}^I = +(\bar{\hat{\epsilon}}\hat{\gamma}_{\hat{\mu}_1\cdots\hat{\mu}_7}\hat{\chi}^I) + 7\hat{K}_{[\hat{\mu}_1\cdots\hat{\mu}_6}(\bar{\hat{\epsilon}}\hat{\gamma}_{\hat{\mu}_7]}\hat{\lambda}^I), \tag{4.1c}$$

$$\delta_Q \hat{\chi}^I = -\frac{1}{8!}(\hat{\gamma}^{\widehat{[8]}}\hat{\epsilon})\hat{G}_{\widehat{[8]}}{}^I \overset{*}{=} -\frac{1}{2}(\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\epsilon})\hat{F}_{\hat{\mu}\hat{\nu}}{}^I, \tag{4.1d}$$

$$\delta_Q \hat{C}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I = -8f^{IJK}(\bar{\hat{\epsilon}}\hat{\gamma}_{[\hat{\mu}_1}\hat{\lambda}^J)\hat{B}_{\hat{\mu}_2\cdots\hat{\mu}_8]}{}^K, \quad (\tilde{\delta}_Q \hat{C}_{\widehat{[8]}}{}^I = 0), \tag{4.1e}$$

$$\delta_Q \hat{K}_{\hat{\mu}_1\cdots\hat{\mu}_6} = 0. \tag{4.1f}$$

Similar to $D = 3 + 3$, the field strengths $\hat{F}$, $\hat{G}$, $\hat{H}$, and $\hat{L}$ are defined by

$$\hat{F}_{\hat{\mu}\hat{\nu}}{}^I \equiv +2\hat{\partial}_{[\hat{\mu}}\hat{A}_{\hat{\nu}]}{}^I + mf^{IJK}\hat{A}_{\hat{\mu}}{}^J\hat{A}_{\hat{\nu}}{}^K, \tag{4.2a}$$

$$\hat{G}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I \equiv +8\hat{D}_{[\mu_1}\hat{B}_{\hat{\mu}_2\cdots\hat{\mu}_8]}{}^I + m\hat{C}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I - 28\hat{K}_{[\hat{\mu}_1\cdots\hat{\mu}_6}\hat{F}_{\hat{\mu}_7\hat{\mu}_8]}{}^I, \tag{4.2b}$$

$$\hat{H}_{\hat{\mu}_1\cdots\hat{\mu}_9}{}^I \equiv +9\hat{D}_{[\hat{\mu}_1}\hat{C}_{\hat{\mu}_2\cdots\hat{\mu}_9]}{}^I + 36f^{IJK}\hat{F}_{[\hat{\mu}_1\hat{\mu}_2}{}^I\hat{B}_{\hat{\mu}_3\cdots\hat{\mu}_9]}{}^K, \tag{4.2c}$$

$$\hat{L}_{\hat{\mu}_1\cdots\hat{\mu}_7} \equiv +7\hat{\partial}_{[\hat{\mu}_1}\hat{K}_{\hat{\mu}_2\cdots\hat{\mu}_7]}. \tag{4.2d}$$

These field strengths satisfy their proper BIds:

$$\hat{D}_{[\hat{\mu}}\hat{F}_{\hat{\nu}\hat{\rho}]}{}^I \equiv 0, \tag{4.3a}$$

$$\hat{D}_{[\hat{\mu}_1}\hat{G}_{\hat{\mu}_2\cdots\hat{\mu}_9]}{}^I \equiv +\frac{1}{9}m\hat{H}_{\hat{\mu}_1\cdots\hat{\mu}_9}{}^I - 4\hat{L}_{[\hat{\mu}_1\cdots\hat{\mu}_7}\hat{F}_{\hat{\mu}_8\hat{\mu}_9]}{}^I, \tag{4.3b}$$

$$\hat{D}_{[\hat{\mu}_1}\hat{H}_{\hat{\mu}_2\cdots\hat{\mu}_{10}]}{}^I \equiv +\frac{9}{2}f^{IJK}\hat{F}_{[\hat{\mu}_1\hat{\mu}_2}{}^J\hat{G}_{\hat{\mu}_3\cdots\hat{\mu}_{10}]}{}^K. \quad (4.3c)$$

Note that the right side of Eq. (4.3c) vanishes under the EM duality (4.9a) below.

Compared with $D = 3 + 3$, there are differences as well as similarities. Similarities are such as the number of multiplets: SYM and HDM, in addition to the auxiliary fields C and K. The difference is that the rank of the $\hat{B}$, $\hat{C}$, and $\hat{K}$ fields are increased. Other features are parallel to the $D = 3 + 3$ case, such as the general variations of our field strengths,

$$\delta\hat{F}_{\hat{\mu}\hat{\nu}}{}^I = +2\hat{D}_{[\hat{\mu}}(\delta\hat{A}_{\hat{\nu}]}{}^I), \quad (4.4a)$$

$$\delta\hat{G}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I = +8\hat{D}_{[\hat{\mu}_1}(\tilde{\delta}\hat{B}_{\hat{\mu}_2\cdots\hat{\mu}_8]}{}^I) + m(\tilde{\delta}\hat{C}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I) - 28(\delta\hat{K}_{[\hat{\mu}_1\cdots\hat{\mu}_6})\hat{F}_{\hat{\mu}_7\hat{\mu}_8]}{}^I - 8(\delta\hat{A}_{[\hat{\mu}_1}{}^I)\hat{L}_{\hat{\mu}_2\cdots\hat{\mu}_8]}, \quad (4.4b)$$

$$\delta\hat{H}_{\hat{\mu}_1\cdots\hat{\mu}_9}{}^I = +9\hat{D}_{[\hat{\mu}_1}(\tilde{\delta}\hat{C}_{\hat{\mu}_2\cdots\hat{\mu}_9]}{}^I) - 36f^{IJK}(\tilde{\delta}\hat{B}_{[\hat{\mu}_1\cdots\hat{\mu}_7}{}^J)\hat{F}_{\hat{\mu}_8\hat{\mu}_9]}{}^K + 9f^{IJK}(\delta\hat{A}_{[\hat{\mu}_1}{}^J)\hat{G}_{\hat{\mu}_2\cdots\hat{\mu}_9]}{}^K, \quad (4.4c)$$

$$\delta\hat{L}_{\hat{\mu}_1\cdots\hat{\mu}_7} = +7\hat{\partial}_{[\hat{\mu}_1}(\delta\hat{K}_{\hat{\mu}_2\cdots\hat{\mu}_7]}), \quad (4.4d)$$

where

$$\tilde{\delta}\hat{B}_{\hat{\mu}_1\cdots\hat{\mu}_7}{}^I \equiv \delta\hat{B}_{\hat{\mu}_1\cdots\hat{\mu}_7}{}^I - 7(\delta\hat{A}_{[\hat{\mu}_1}{}^I)\hat{K}_{\hat{\mu}_2\cdots\hat{\mu}_7]}, \quad (4.5a)$$

$$\tilde{\delta}\hat{C}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I \equiv \delta\hat{C}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I + 8f^{IJK}(\delta\hat{A}_{[\hat{\mu}_1}{}^J)\hat{B}_{\hat{\mu}_2\cdots\hat{\mu}_8]}{}^K. \quad (4.5b)$$

Their δ_Q transformations are

$$\delta_Q\hat{F}_{\hat{\mu}\hat{\nu}}{}^I = -2(\bar{\hat{\epsilon}}\hat{\gamma}_{[\hat{\mu}}\hat{D}_{\hat{\nu}]}\hat{\lambda}^I), \quad (4.6a)$$

$$\delta_Q\hat{G}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I \overset{*}{=} -8(\bar{\hat{\epsilon}}\hat{\gamma}_{[\hat{\mu}_1\cdots\hat{\mu}_7}\hat{D}_{\hat{\mu}_8]}\hat{\lambda}^I), \quad (4.6b)$$

$$\delta_Q\hat{H}_{\hat{\mu}_1\cdots\hat{\mu}_9}{}^I = -36f^{IJK}(\bar{\hat{\epsilon}}\hat{\gamma}_{[\hat{\mu}_1\cdots\hat{\mu}_7}\hat{\chi}^J)\hat{F}_{\hat{\mu}_8\hat{\mu}_9]}{}^K + 9f^{IJK}(\bar{\hat{\epsilon}}\gamma_{[\hat{\mu}_1}\hat{\lambda}^J)\hat{G}_{\hat{\mu}_2\cdots\hat{\mu}_9]}{}^K,$$
$$\delta_Q\hat{L}_{\widehat{[7]}} = 0. \quad (4.6c)$$

As in $D = 3 + 3$, the $m\delta_Q\hat{C}$, $(\delta_Q\hat{K})\hat{F}$, and $(\delta_Q\hat{A})\hat{L}$ terms in Eq. (4.6b) all vanish, when the duality-related equations (4.9) below have been used, leaving only one term.

Similarly, the YM-gauge and proper gauge transformations for $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{K}$ fields are

$$\delta_T\hat{A}_{\hat{\mu}}{}^I = \hat{D}_{\hat{\mu}}\alpha^I,$$
$$\delta_T(\hat{B}_{\widehat{[7]}}{}^I, \hat{C}_{\widehat{[8]}}{}^I, \hat{K}_{\widehat{[6]}}) = -mf^{IJK}\hat{\alpha}^J(\hat{B}_{\widehat{[7]}}{}^K, \hat{C}_{\widehat{[8]}}{}^K, 0), \quad (4.7a)$$

$$\delta_U\hat{B}_{\hat{\mu}_1\cdots\hat{\mu}_7}{}^I = +7\hat{D}_{[\hat{\mu}_1}\hat{\beta}_{\hat{\mu}_2\cdots\hat{\mu}_7]}{}^I,$$
$$\delta_U\hat{C}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I = -28f^{IJK}\hat{F}_{[\hat{\mu}_1\hat{\mu}_2}{}^J\hat{\beta}_{\hat{\mu}_3\cdots\hat{\mu}_8]}{}^K, \quad (4.7b)$$

$$\delta_V\hat{C}_{\hat{\mu}_1\cdots\hat{\mu}_8}{}^I = +8\hat{D}_{[\hat{\mu}_1}\gamma_{\hat{\mu}_2\cdots\hat{\mu}_8]}{}^I,$$
$$\delta_V\hat{B}_{\hat{\mu}_1\cdots\hat{\mu}_7}{}^I = -m\hat{\gamma}_{\hat{\mu}_1\cdots\hat{\mu}_7}{}^I, \quad (4.7c)$$

$$\delta_K\hat{K}_{\hat{\mu}_1\cdots\hat{\mu}_6} = +6\hat{\partial}_{[\hat{\mu}_1}\hat{\kappa}_{\hat{\mu}_2\cdots\hat{\mu}_6]},$$
$$\delta_K\hat{B}_{\hat{\mu}_1\cdots\hat{\mu}_7}{}^I = +21\hat{\kappa}_{[\hat{\mu}_1\cdots\hat{\mu}_5}\hat{F}_{\hat{\mu}_6\hat{\mu}_7]}{}^I. \quad (4.7d)$$

To save space, we skipped other invariants, such as $\delta_U\hat{A}_{\hat{\mu}}{}^I = 0$. As in $D = 3 + 3$, it is straightforward to confirm that all of our field strengths are invariant under δ_U, δ_V, and δ_K.

As in the previous $D = 3 + 3$ case, the closure of gauge algebra is confirmed as

$$[\delta_{Q_1}, \delta_{Q_2}] = \delta_{P_3} + \delta_{T_3} + \delta_{U_3} + \delta_{V_3} + \delta_{K_3},$$
$$\hat{\xi}_3^{\mu} \equiv +2(\bar{\hat{\epsilon}}_1\hat{\gamma}^{\hat{\mu}}\hat{\epsilon}_2), \qquad \hat{\alpha}_3^I \equiv -\hat{\xi}_3^{\hat{\mu}}\hat{A}_{\hat{\mu}}{}^I,$$
$$\hat{\beta}^3_{\hat{\mu}_1\cdots\hat{\mu}_6}{}^I \equiv -\hat{\xi}_3^{\hat{\nu}}\hat{B}_{\hat{\nu}\hat{\mu}_1\cdots\hat{\mu}_6}{}^I, \qquad \hat{\gamma}^3_{\hat{\mu}_1\cdots\hat{\mu}_7}{}^I \equiv -\hat{\xi}_3^{\hat{\nu}}\hat{C}_{\hat{\nu}\hat{\mu}_1\cdots\hat{\mu}_7}{}^I,$$
$$\hat{\kappa}^3_{\hat{\mu}_1\cdots\hat{\mu}_5} \equiv -\hat{\xi}_3^{\hat{\nu}}\hat{K}_{\hat{\nu}\hat{\mu}_1\cdots\hat{\mu}_5} + 2(\bar{\hat{\epsilon}}_1\hat{\gamma}_{\hat{\mu}_1\cdots\hat{\mu}_5}\hat{\epsilon}_2), \quad (4.8a)$$

$$[\delta_Q, \delta_U] = \delta_V, \qquad \hat{\gamma}_{\hat{\mu}_1\cdots\hat{\mu}_7}{}^I \equiv -7f^{IJK}(\bar{\hat{\epsilon}}\hat{\gamma}_{[\hat{\mu}_1}\hat{\lambda}^J)\hat{\beta}_{\hat{\mu}_2\cdots\hat{\mu}_7]}{}^K, \quad (4.8b)$$

$$[\delta_Q, \delta_K] = \delta_U, \quad \hat{\beta}_{\hat{\mu}_1\cdots\hat{\mu}_6}{}^I \equiv -6\hat{\kappa}_{[\hat{\mu}_1\cdots\hat{\mu}_5}(\bar{\hat{\epsilon}}\hat{\gamma}_{\hat{\mu}_6]}\hat{\lambda}^I), \quad (4.8c)$$

$$[\delta_{T_1}, \delta_{T_2}] = \delta_{T_3}, \qquad \hat{\alpha}_3^I \equiv -f^{IJK}\hat{\alpha}_1^J\hat{\alpha}_2^K, \quad (4.8d)$$

$$[\delta_Q, \delta_T] = [\delta_Q, \delta_V] = [\delta_T, \delta_U] = [\delta_T, \delta_V] = [\delta_U, \delta_V]$$
$$= [\delta_{U_1}, \delta_{U_2}] = [\delta_{V_1}, \delta_{V_2}] = [\delta_{K_1}, \delta_{K_2}]$$
$$= [\delta_T, \delta_K] = [\delta_U, \delta_K] = [\delta_V, \delta_K] = 0. \quad (4.8e)$$

These are just parallel to the EM duality in $D = 3 + 1$ and $D = 9 + 1$ in Ref. [1].

The most crucial supersymmetric EM-duality conditions and field equations in $D = 5 + 5$ are

$$\hat{F}_{\hat{\mu}\hat{\nu}}{}^I \overset{*}{=} +\frac{1}{8!}\hat{\epsilon}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}_1\cdots\hat{\rho}_8}\hat{G}_{\hat{\rho}_1\cdots\hat{\rho}_8}{}^I, \qquad \hat{G}_{\widehat{[8]}}{}^I \overset{*}{=} -\frac{1}{2}\hat{\epsilon}_{\widehat{[8]}}{}^{\hat{\mu}\hat{\nu}}\hat{F}_{\hat{\mu}\hat{\nu}}{}^I, \quad (4.9a)$$

$$\hat{\lambda}^I \overset{*}{=} -\hat{\chi}^I, \quad (4.9b)$$

$$\hat{D}_{\hat{\nu}}\hat{F}_{\hat{\mu}}{}^{\hat{\nu}I} \doteq +\frac{1}{2}mf^{IJK}(\bar{\hat{\lambda}}^J\hat{\gamma}_{\hat{\mu}}\hat{\lambda}^K), \quad (4.9c)$$

$$\hat{\not{D}}\hat{\lambda}^I \doteq 0, \qquad \hat{\not{D}}\hat{\chi}^I \doteq 0, \quad (4.9d)$$

$$\hat{H}_{\widehat{[9]}}{}^I \overset{*}{=} +\frac{1}{2}f^{IJK}(\bar{\hat{\lambda}}^J\hat{\gamma}_{\widehat{[9]}}\hat{\lambda}^K), \quad (4.9e)$$

$$\hat{L}_{\hat{\mu}_1\cdots\hat{\mu}_7} \stackrel{*}{=} 0. \qquad (4.9f)$$

The consistency of these equations with $N = (1,0)$ supersymmetry is confirmed, just as in $D = 3+3$. Especially, the variation of Eq. (4.9e) under Eq. (4.1) is parallel to $D = 3+3$, which we skip here. The duality and field equations in Eq. (4.9) are just parallel to the Minkowskian case in $D = 9+1$ [1].

V. DIMENSIONAL REDUCTION FROM D = 5 + 5 TO D = 2 + 2

Our dimensional reduction and truncation rule from $D = 5+5$ into $D = 2+2$ is summarized as

$$\hat{A}_\mu{}^I \stackrel{*}{=} A_\mu{}^I, \qquad \hat{A}_\alpha{}^I = 0 \quad (\alpha = 5, \ldots, 10), \qquad (5.1a)$$

$$\hat{\lambda}^I \equiv \begin{pmatrix} \lambda_\uparrow{}^I \\ \lambda_\downarrow{}^I \end{pmatrix} \stackrel{*}{=} \begin{pmatrix} \lambda_\uparrow{}^I \\ \beta_3 \lambda_\uparrow{}^I \end{pmatrix}, \qquad \hat{\chi}^I \equiv \begin{pmatrix} \chi_\uparrow{}^I \\ \chi_\downarrow{}^I \end{pmatrix} \stackrel{*}{=} \begin{pmatrix} \chi_\uparrow{}^I \\ \beta_3 \chi_\uparrow{}^I \end{pmatrix}, \qquad (5.1b)$$

$$\alpha_1 \equiv i\begin{pmatrix} \sigma_1 & O \\ O & \sigma_1 \end{pmatrix}, \qquad \alpha_2 \equiv i\begin{pmatrix} \sigma_2 & O \\ O & \sigma_2 \end{pmatrix}, \qquad \alpha_3 \equiv i\begin{pmatrix} \sigma_3 & O \\ O & \sigma_3 \end{pmatrix}, \qquad (5.1c)$$

$$\beta_1 \equiv i\begin{pmatrix} O & I_2 \\ I_2 & O \end{pmatrix}, \qquad \beta_2 \equiv i\begin{pmatrix} O & -iI_2 \\ iI_2 & O \end{pmatrix}, \qquad \beta_3 \equiv i\begin{pmatrix} I_2 & O \\ O & -I_2 \end{pmatrix}, \qquad (5.1d)$$

$$\alpha_p \alpha_q = -\delta_{pq} I_4 - \epsilon_{pqr}\alpha_r, \qquad \beta_p \beta_q = -\delta_{pq} I_4 - \epsilon_{pqr}\beta_r, \qquad (p,q,r = 1,2,3), \qquad (5.1e)$$

$$\lambda_\uparrow{}^I \stackrel{*}{=} \frac{1}{2}\begin{pmatrix} \lambda^I \\ 0 \\ 0 \\ \lambda^I \end{pmatrix} \stackrel{*}{=} -\chi_\uparrow{}^I \stackrel{*}{=} -\frac{1}{2}\begin{pmatrix} \chi^I \\ 0 \\ 0 \\ \chi^I \end{pmatrix}, \qquad \epsilon_\uparrow \stackrel{*}{=} \frac{1}{2}\begin{pmatrix} \epsilon \\ 0 \\ 0 \\ \epsilon \end{pmatrix}, \qquad (5.1f)$$

$$\bar{\epsilon}_\uparrow \stackrel{*}{=} \frac{1}{2}(\bar{\epsilon}, 0, 0, \bar{\epsilon}), \qquad \gamma_5(\lambda^I, \chi^I) = -(\lambda^I, \chi^I), \qquad \bar{\epsilon}\gamma_5 = +\bar{\epsilon}, \qquad \gamma_5 \epsilon = -\epsilon, \qquad (5.1g)$$

$$(\hat{\gamma}_{\hat{\mu}})_{\hat{a}}{}^{\hat{b}} \stackrel{*}{=} \begin{cases} (\hat{\gamma}_\mu)_{\hat{a}}{}^{\hat{b}} = \gamma_\mu \otimes \sigma_3 \otimes I_4, \\ (\hat{\gamma}_{4+m})_{\hat{a}}{}^{\hat{b}} = I_4 \otimes \sigma_1 \otimes \beta_3 \alpha_m, \quad (m = 1,2,3) \\ (\hat{\gamma}_{7+n})_{\hat{a}}{}^{\hat{b}} = \gamma_5 \otimes I_2 \otimes \beta_n, \quad (n = 1,2) \\ (\hat{\gamma}_{10})_{\hat{a}}{}^{\hat{b}} = -\gamma_5 \otimes \sigma_3 \otimes \beta_3, \end{cases} \qquad (5.1h)$$

$$(\hat{\gamma}_{11})_{\hat{a}}{}^{\hat{b}} \equiv \hat{\gamma}_{123456789,10} = -iI_4 \otimes \sigma_2 \otimes \beta_3, \qquad \hat{C}_{\hat{a}\hat{b}} = C \otimes \sigma_3 \otimes \alpha_2 \beta_2, \qquad (5.1i)$$

$$(\hat{\gamma}_{\hat{\mu}})^{\hat{a}\hat{b}} \equiv \hat{C}^{\hat{a}\hat{c}}(\hat{\gamma}_{\hat{\mu}})_{\hat{c}}{}^{\hat{a}} = \begin{cases} (\hat{\gamma}_\mu)^{\hat{a}\hat{b}} = \gamma_\mu \otimes I_2 \otimes \alpha_2\beta_2, \\ (\hat{\gamma}_5)^{\hat{a}\hat{b}} = iC \otimes \sigma_2 \otimes \alpha_3\beta_1, \\ (\hat{\gamma}_6)^{\hat{a}\hat{b}} = -iC \otimes \sigma_2 \otimes I_4, \\ (\hat{\gamma}_7)^{\hat{a}\hat{b}} = -iC \otimes \sigma_2 \otimes \alpha_1\beta_1, \\ (\hat{\gamma}_8)^{\hat{a}\hat{b}} = \gamma_5 \otimes \sigma_3 \otimes \alpha_2\beta_3, \\ (\hat{\gamma}_9)^{\hat{a}\hat{b}} = -\gamma_5 \otimes \sigma_3 \otimes \alpha_2, \\ (\hat{\gamma}_{10})^{\hat{a}\hat{b}} = \gamma_5 \otimes I_2 \otimes \alpha_2\beta_1, \end{cases} \qquad (5.1j)$$

$$\hat{B}_{\mu 56789,10}{}^I \stackrel{*}{=} A_\mu{}^I, \qquad \tilde{\hat{B}}_{\mu\nu\alpha}{}^I \stackrel{*}{=} 0, \qquad \tilde{\hat{B}}_{\mu\nu\rho}{}^I \stackrel{*}{=} \tilde{B}_{\mu\nu\rho}{}^I \equiv -\epsilon_{\mu\nu\rho}{}^\sigma A_\sigma{}^I, \qquad (5.1k)$$

$$\hat{C}_{\mu\nu 56789,10}{}^I \stackrel{*}{=} 2m^{-1}\partial_{[\mu}A_{\nu]}{}^I, \qquad \tilde{\hat{C}}_{\mu\alpha}{}^I \stackrel{*}{=} \tilde{\hat{C}}_{\alpha\beta}{}^I \stackrel{*}{=} 0, \qquad (5.1l)$$

$$\hat{K}_{56789,10} \stackrel{*}{=} +1, \qquad (5.1m)$$

$$\delta_Q \tilde{\hat{B}}_{\mu\nu\rho}{}^I \stackrel{*}{=} -\epsilon_{\mu\nu\rho}{}^\sigma(\bar{\epsilon}\gamma_\sigma\lambda^I), \qquad \delta_Q \tilde{\hat{B}}_{\mu 58}{}^I \stackrel{*}{=} -\delta_Q \tilde{\hat{B}}_{\mu 69}{}^I \stackrel{*}{=} -(\bar{\epsilon}\gamma_\mu\lambda^I), \qquad (5.1n)$$

while all other components of $\hat{K}_{\hat{\mu}_1\cdots\hat{\mu}_6}$ and $\delta_Q \tilde{\hat{B}}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I$ are zero. The $\tilde{\hat{B}}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^I \equiv (1/7!)\hat{\epsilon}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{\widehat{[7]}}\hat{B}_{\widehat{[7]}}{}^I$ and $\tilde{\hat{C}}_{\hat{\mu}\hat{\nu}}{}^I \equiv (1/8!)\hat{\epsilon}_{\hat{\mu}\hat{\nu}}{}^{\widehat{[8]}}\hat{C}_{\widehat{[8]}}{}^I$ are the Hodge duals of $\hat{B}_{\widehat{[7]}}{}^I$ and $\hat{C}_{\widehat{[8]}}{}^I$, respectively. For space-time indices, we use $\hat{\mu} = (\mu, \alpha)$, $\hat{\nu} = (\nu, \beta), \cdots$, where $\mu, \nu, \cdots = 1, 2, 3, 4$, and $\alpha, \beta, \cdots = 5, 6, 7, 8, 9, 10$. The matrix I_n stands for an $n \times n$ unit matrix. The charge-conjugation matrix $\hat{C}_{\hat{a}\hat{b}}$ in $D = 5+5$ is *antisymmetric*, consistent with the flipping property (A9a). Relevantly, all the matrix components in Eq. (5.1j) are symmetric, consistent with Eq. (A9a). The matrices α_p and β_p $(p = 1, 2, 3)$ are 4×4 matrices. The specification of $\hat{\lambda}^I$ and $\hat{\chi}^I$ in Eq. (5.1b) is to satisfy the Weyl-spinor conditions $\hat{\gamma}_{11}(\hat{\lambda}^I, \hat{\chi}^I) = (+\hat{\lambda}^I, +\hat{\chi}^I)$.

Our objective now is to reproduce the $N = 1$ SDSYM system in $D = 2+2$ [10],

$$F_{\mu\nu}{}^I \overset{*}{=} +\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}{}^I, \quad \gamma_5(\lambda^I,\chi^I) \overset{*}{=} -(\lambda^I,\chi^I), \tag{5.2a}$$

$$\delta_Q A_\mu{}^I = -(\bar\epsilon\gamma_\mu\lambda^I), \tag{5.2b}$$

$$\delta_Q\lambda^I = +\frac{1}{2}(\gamma^{\mu\nu}\epsilon)F_{\mu\nu}{}^I. \tag{5.2c}$$

As a representative example of our dimensional reduction, we consider $\delta_Q\hat A_\mu{}^I$:

$$\begin{aligned}\delta_Q A_\mu{}^I \overset{?}{=} \delta_Q\hat A_\mu{}^I &= +(\bar{\hat\epsilon}\hat\gamma_\mu\hat\lambda^I)\\ &= -(\bar\epsilon_\uparrow,\bar\epsilon_\uparrow\beta_3)(\gamma_\mu\sigma_3)(\sigma_3\alpha_2\beta_2)\begin{pmatrix}\lambda_\uparrow{}^I\\ \beta_3\lambda_\uparrow{}^I\end{pmatrix}\\ &= -(\bar\epsilon_\uparrow\gamma_\mu\alpha_2\beta_2\lambda_\uparrow{}^I)-(\bar\epsilon_\uparrow\gamma_\mu\beta_3\alpha_2\beta_2\beta_3\lambda_\uparrow{}^I)\\ &= +2\cdot\frac{1}{2}(\bar\epsilon,0,0,\bar\epsilon)\gamma_\mu\begin{pmatrix}O & -i\sigma_2\\ +i\sigma_2 & O\end{pmatrix}\frac{1}{2}\begin{pmatrix}\lambda^I\\0\\0\\\lambda^I\end{pmatrix}\\ &= -\frac{1}{2}(\bar\epsilon,0,0,\bar\epsilon)\gamma_\mu\lambda^I-\frac{1}{2}(\bar\epsilon,0,0,\bar\epsilon)\gamma_\mu\lambda^I\\ &= -(\bar\epsilon\gamma_\mu\lambda^I)\ \ (Q.E.D.).\end{aligned} \tag{5.3}$$

Similar to $D=3+3$, the $\delta_Q\tilde{\hat B}_{\mu\nu\rho}{}^I$ in Eq. (5.1n) is nontrivial:

$$\begin{aligned}\delta_Q\tilde{\hat B}_{\mu\nu\rho}{}^I &= +(\bar{\hat\epsilon}\hat\gamma_{\mu\nu\rho}\hat\chi^I)+\hat K_{\mu\nu\rho}{}^{\hat\sigma}(\bar{\hat\epsilon}\hat\gamma_{\hat\sigma}\hat\lambda^I)\\ &\overset{*}{=} -[\bar{\hat\epsilon}(\gamma_{\mu\nu\rho}\sigma_3)(\sigma_3\alpha_2\beta_2)\hat\chi^I]\\ &= -(\bar\epsilon_\uparrow,\bar\epsilon_\uparrow\beta_3)\gamma_{\mu\nu\rho}\alpha_2\beta_2\begin{pmatrix}\chi_\uparrow{}^I\\ \beta_3\chi_\uparrow{}^I\end{pmatrix}\\ &= -2\frac{1}{2}(\bar\epsilon,0,0,\bar\epsilon)\gamma_{\mu\nu\rho}\begin{pmatrix}o & -i\sigma_2\\ i\sigma_2 & O\end{pmatrix}\frac{1}{2}\begin{pmatrix}\chi^I\\0\\0\\\chi^I\end{pmatrix}\\ &= +(\bar\epsilon\gamma_{\mu\nu\rho}\chi^I)\overset{*}{=}-\epsilon_{\mu\nu\rho}{}^\sigma(\bar\epsilon\gamma_\sigma\lambda^I).\end{aligned} \tag{5.4}$$

Needless to say, this is Hodge dual to

$$\delta_Q\hat B_{\mu 56789,10}{}^I = -(\bar\epsilon\gamma_\mu\lambda^I), \tag{5.5}$$

consistent with $\delta_Q A_\mu{}^I = \delta_Q\hat B_{\mu 56789,10}{}^I$ in Eq. (5.1k).

As the next nontrivial example, we compute $\delta_Q\tilde{\hat B}_{\mu 58}{}^I$[8]:

$$\begin{aligned}\delta_Q\tilde{\hat B}_{\mu 58}{}^I &= +(\bar{\hat\epsilon}\hat\gamma_\mu\hat\gamma_5\hat\gamma_8\hat\chi^I)\\ &= -[\bar{\hat\epsilon}(\gamma_\mu\sigma_3)(\sigma_1\beta_3\alpha_1)(\gamma_5\beta_1)(\sigma_3\alpha_2\beta_2)\hat\chi^I]\\ &= -(\bar\epsilon_\uparrow,\bar\epsilon_\uparrow\beta_3)\gamma_\mu\begin{pmatrix}0&1\\1&0\end{pmatrix}\alpha_3\begin{pmatrix}\chi_\uparrow{}^I\\ \beta_3\chi_\uparrow{}^I\end{pmatrix}\\ &= -2(\bar\epsilon_\uparrow\gamma_\mu\beta_3\alpha_3\chi_\uparrow{}^I)\\ &= +\frac{1}{2}(\bar\epsilon,0,0,\bar\epsilon)\gamma_\mu\begin{pmatrix}\sigma_3 & O\\ O & -\sigma_3\end{pmatrix}\begin{pmatrix}\chi^I\\0\\0\\\chi^I\end{pmatrix} = +(\bar\epsilon\gamma_\mu\chi^I).\end{aligned} \tag{5.6}$$

We skip the parallel case for $\delta_Q\tilde{\hat B}_{\mu 69}{}^I$. The $\delta_Q\hat B_{\mu\nu\rho 679,10}{}^I$ is obtained by the Hodge duality:

$$\delta_Q\hat B_{\mu\nu\rho 679,10}{}^I = -\hat\epsilon_{\mu\nu\rho 679,10}{}^{\sigma 58}\delta_Q\tilde{\hat B}_{\sigma 58}{}^I = +(\bar\epsilon\gamma_{\mu\nu\rho}\chi^I). \tag{5.7}$$

This is used to confirm $\delta_Q\hat G_{\mu\nu\rho\sigma 679,10}{}^I \overset{*}{=} 0$:

$$\begin{aligned}0 \overset{?}{=} \delta_Q\hat G_{\mu\nu\rho\sigma 679,10}{}^I &= +4\hat D_{[\mu|}(\tilde\delta_Q\hat B_{|\nu\rho\sigma]679,10}{}^I)\\ &\quad + m(\tilde\delta\hat C_{\mu\nu\rho\sigma 679,10}{}^I)\\ &\overset{*}{=} +4\hat D_{[\mu|}(\bar\epsilon\gamma_{|\nu\rho\sigma]}\chi^I)\\ &= +\epsilon_{\mu\nu\rho\sigma}(\bar\epsilon \not{D}\chi^I) \doteq 0\ \ (Q.E.D.).\end{aligned} \tag{5.8}$$

As in $D=3+3$, we perform the dimensional reduction on $\hat G_{\mu\nu 56789,10}{}^I$, as the last crucial confirmation:

$$\begin{aligned}F_{\mu\nu}{}^I \overset{?}{=} \hat G_{\mu\nu 56789,10}{}^I &= 2\partial_{[\mu}\hat B_{\nu]56789,10}{}^I\\ &\quad + 2mf^{IJK}\hat A_{[\mu}{}^J\hat B_{\nu]56789,10}{}^K\\ &\quad - \hat K_{56789,10}\hat F_{\mu\nu}{}^K + m\hat C_{\mu\nu 56789,10}{}^I\\ &\overset{*}{=} 2\partial_{[\mu}A_{\nu]}{}^I + 2mf^{IJK}A_\mu{}^JA_\nu{}^K\\ &\quad - (2\partial_{[\mu}A_{\nu]}{}^I + mf^{IJK}A_\mu{}^JA_\nu{}^K)\\ &\quad + m(2m^{-1}\partial_{[\mu}A_{\nu]}{}^I)\\ &\overset{*}{=} F_{\mu\nu}{}^I\quad (Q.E.D.).\end{aligned} \tag{5.9}$$

As for the dimensional reduction for fermion $\hat\lambda^I$, it is just parallel to the $D=3+3$ case:

[8]Since there are many indices in $\hat B_{\hat\mu_1\cdots\hat\mu_7}{}^I$, we use its Hodge-dual components.

$$\delta_Q\hat{\lambda}^I = \delta_Q\begin{pmatrix} \hat{\lambda}_\uparrow{}^I \\ \beta_3\hat{\lambda}_\uparrow{}^I \end{pmatrix} = \frac{1}{2}(\hat{\gamma}^{\hat{\mu}\hat{\nu}}\hat{\epsilon})\hat{F}_{\hat{\mu}\hat{\nu}}{}^I = \frac{1}{2}(\hat{\gamma}^{\mu\nu}\hat{\epsilon})F_{\mu\nu}{}^I$$
$$= \frac{1}{2}(\gamma^\mu\sigma_3)(\gamma^\nu\sigma_3)\begin{pmatrix} \epsilon_\uparrow \\ \beta_3\epsilon_\uparrow \end{pmatrix}F_{\mu\nu}{}^I = \frac{1}{2}\begin{pmatrix} \gamma^{\mu\nu}\epsilon_\uparrow F_{\mu\nu}{}^I \\ \gamma^{\mu\nu}\beta_3\epsilon_\uparrow F_{\mu\nu}{}^I \end{pmatrix}. \tag{5.10}$$

As desired, this yields the λ-transformation rule,

VI. CONCLUDING REMARKS

In this paper, we have presented supersymmetric EM-duality formulations in $D = 3 + 3$ and $D = 5 + 5$. We have shown that both these formulations generate $N = 1$ SDSYM in $D = 2 + 2$, after appropriate dimensional reductions.

Our results indicate that self-dual YM and SDSYM systems or supersymmetric integrable models in $1 \leq D \leq 3$ form merely the subsets of EM-duality systems in $D = 3 + 3$ or $D = 5 + 5$. Furthermore, if we maintain Lorentz covariance,[9] $D = 10$ is the maximal space-time dimension for the SYM multiplet. Therefore, $D = 5 + 5$ seems to be the maximal space-time dimension for EM duality. In this sense, the $D = 5 + 5$ EM-duality system may well serve as the Master Theory of SDSYM in $D = 2 + 2$ [9,10], generating all supersymmetric integrable models in $1 \leq D \leq 3$.

All our results are based on the recently developed tensor-hierarchy formulation [2–4]. Before this formulation, the formulation of the dual tensor in the adjoint representations, such as our $\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I$ or $\hat{G}_{\widehat{[8]}}{}^I$ had inconsistencies [5]. Because of this problem, one way to proceed was to use the so-called reduced holonomies, such as $SO(7)$, G_2, $SU(3)$, and $SU(2)$, respectively, in eight, seven, six, and four dimensions [15–17]. The self-duality relationships in these formulations are of the type $\hat{F}_{\hat{\mu}\hat{\nu}}{}^I \overset{*}{=} (1/2)\hat{\psi}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}}\hat{F}_{\hat{\rho}\hat{\sigma}}{}^I$, where $\hat{\psi}_{\hat{\mu}\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}}$ is a certain tensor invariant under the reduced holonomies, such as the octonion structure constant in the case of $SO(7)$ or G_2.

After the discovery [2] of tensor-hierarchy and subsequent elaborations [3,4], it is *no longer* necessary to restrict oneself only to the second-rank field strength $F_{\mu\nu}{}^I$, as one can use more general tensor field strength with adjoint indices, such as $\hat{G}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}{}^I$ or $\hat{G}_{\widehat{[8]}}{}^I$, so that the EM duality is $\hat{F}_{\hat{\mu}\hat{\nu}}{}^I = (1/4!)\hat{\epsilon}_{\hat{\mu}\hat{\nu}}{}^{\widehat{[4]}}\hat{G}_{\widehat{[4]}}{}^I$ in $D = 3 + 3$ or $\hat{F}_{\hat{\mu}\hat{\nu}}{}^I = (1/8!)\hat{\epsilon}_{\hat{\mu}\hat{\nu}}{}^{\widehat{[8]}}\hat{G}_{\widehat{[8]}}{}^I$ in $D = 5 + 5$. Also, one does not need reduced holonomies or sophisticated octonion structure constants [20] any longer. Our EM dualities in higher dimensions respect full Lorentz covariance, which can be easily established with straightforward manipulations.

Some readers might wonder why we have to go to higher space-time dimensions, by regarding the $D = 5 + 5$ theory as more fundamental theory than the $D = 2 + 2$ SDSYM theory. To answer this question, we stress the following points. First, in conventional SYM theories, it has already been well known that the $D = 9 + 1$ SYM is more fundamental than SYM in $D = 3 + 1$. Second, we note the parallel structures between $D = 9 + 1$ and $D = 5 + 5$ or between $D = 3 + 1$ and $D = 2 + 2$. By combining the first and second points, it is natural to regard the $D = 5 + 5$ EM-duality-symmetric theory as the more fundamental theory than the $D = 2 + 2$ SDSYM theory. Even though we have not presented any "new" integrable models in dimensions $4 \leq D \leq 9$ generated by our theory in $D = 5 + 5$, examples such as these may well help us find some integrable sectors of the hypothetical non-Abelian $D = 5 + 1$ superconformal field theory [21] describing multiple M5 brane. From these viewpoints, we claim our theory may well be the Grand Master Theory in $D = 5 + 5$, which generates the Master Theory in $D = 2 + 2$. In other words, Grand Master Theory in $D = 5 + 5$ may well be more fundamental than the Master Theory in $D = 2 + 2$.

When Atiyah and Ward presented their first works in Ref. [12], there was no SDSYM known even in $D = 2 + 2$, but only nonsupersymmetric self-dual YM theory was known. Therefore, there was no motivation to consider going to higher dimensions such as $D = 5 + 5$. It is the parallel structure between the $D = 9 + 1$ (or $D = 3 + 1$) and $D = 5 + 5$ (or $D = 2 + 2$) combined with supersymmetries that strongly motivates the studies of these higher-dimensional duality-symmetric and supersymmetric theories.

As careful readers may have noticed, it is not mere coincidence that the ranks of our $\hat{K}$-tensor fields are the same as the numbers of extra space-time dimensions, i.e, 2 (or 6) for $D = 3 + 3$ (or $D = 5 + 5$). This is because the nonzero value for $\hat{K}_{56}$ (or $\hat{K}_{56789,10}$) plays an important role for establishing the consistency of our dimensional reductions in the resulting SDSYM in $D = 2 + 2$ dimensions. These features may well be closely related to supergravity [22], or a superstring, such as the $N = 2$ superstring [23], extended objects, and M theory [24]. From these viewpoints, we expect deeper significance of our EM formulations in these higher dimensions associated with SDSYM in $D = 2 + 2$.

APPENDIX A: NOTATIONS IN $D = 2 + 2$, $D = 3 + 3$, AND $D = 5 + 5$

In this Appendix, we clarify our notations in $D = 2 + 2$, $D = 3 + 3$, and $D = 5 + 5$.

[9] If we give up Lorentz covariance, we can go even to $D \geq 11$ by introducing null vectors [19].

1. Notation in $D=2+2$

Our metric for $D=2+2$ is $(\eta_{\mu\nu})=\mathrm{diag}(+,+,-,-)$ so that $\epsilon^{1234}=+1$ and $\gamma_5\equiv\gamma_1\gamma_2\gamma_3\gamma_4$ [10]. Relevantly, we have

$$\frac{1}{n!}\epsilon_{\mu_1\cdots\mu_{4-n}}{}^{[n]}\epsilon_{[n]}{}^{\nu_1\cdots\nu_{4-n}}=+(-1)^n(4-n)!\delta_{[\mu_1}{}^{\nu_1}\cdots\delta_{\mu_{4-n}]}{}^{\nu_{4-n}}, \tag{A1a}$$

$$\gamma_{[n]}=+\frac{(-1)^{n(n-1)/2}}{(4-n)!}\epsilon_{[n]}{}^{[4-n]}\gamma_{[4-n]}\gamma_5. \tag{A1b}$$

The flipping and Hermitian-conjugate properties for two Majorana-Weyl spinors ψ and ρ in $D=2+2$ are

$$(\bar\psi\gamma^{[n]}\rho)=+(-1)^{n(n+1)/2}(\bar\rho\gamma^{[n]}\psi)=-(-1)^{(n-1)(n-2)/2}(\bar\rho\gamma^{[n]}\psi), \tag{A2a}$$

$$(\bar\psi\gamma^{[n]}\rho)^\dagger=+(\bar\psi\gamma^{[n]}\rho). \tag{A2b}$$

Typical examples are $(\bar\psi\rho)=+(\bar\rho\psi)=+(\bar\psi\rho)^\dagger$, $(\bar\psi\gamma^\mu\rho)=-(\bar\rho\gamma^\mu\psi)=+(\bar\psi\gamma^\mu\rho)^\dagger$, etc. Especially, any fermion bilinear is *Hermitian* needing no imaginary unit i in its front.

2. Notation in $D=3+3$

Our convention in $D=3+3$ with the metric $(\hat\eta_{\hat\mu\hat\nu})=\mathrm{diag}(+,+,-,-,+,-)$ is like $\hat\epsilon^{123456}=+1$ with $\hat\gamma_6\equiv\hat\gamma_1\hat\gamma_2\hat\gamma_3\hat\gamma_4\hat\gamma_5\hat\gamma_6\equiv\hat\gamma_{123456}$ so that

$$\frac{1}{n!}\hat\epsilon_{\hat\mu_1\cdots\hat\mu_{6-n}}{}^{[n]}\hat\epsilon_{[n]}{}^{\hat\nu_1\cdots\hat\nu_{6-n}}=-(-1)^n(6-n)!\hat\delta_{[\hat\mu_1}{}^{\hat\nu_1}\cdots\hat\delta_{\hat\mu_{6-n}]}{}^{\hat\nu_{6-n}}, \tag{A3a}$$

$$\hat\gamma_{\widehat{[n]}}=\frac{(-1)^{n(n-1)/2}}{(6-n)!}\hat\epsilon_{\widehat{[n]}}{}^{\widehat{[6-n]}}\hat\gamma_{\widehat{[6-n]}}\hat\gamma_7. \tag{A3b}$$

According to the general categorizations in arbitrary space-time dimensions (s space and t time coordinates) [25,26], the spinors in $s=t=3\Rightarrow s-t=8\ (\mathrm{mod}\,8)$ are either Majorana (or pseudo-Majorana) spinors with the parameters $\epsilon=+1$, $\eta=+1$ (or $\epsilon=+1$, $\eta=-1$), following the notation in Ref. [26]. For our objective in this paper, we choose Majorana spinors with $\epsilon=+1$, $\eta=+1$. Accordingly, the flipping and Hermitian-conjugate properties for two Majorana spinors ψ and ρ are

$$(\bar{\hat\psi}\gamma^{[\hat n]}\rho)=-\epsilon\eta^{t+n}(-1)^{(t-n)(t-n+1)/2}(\bar{\hat\rho}\hat\gamma^{\widehat{[n]}}\hat\psi)=-(-1)^{n(n+1)/2}(\bar{\hat\rho}\hat\gamma^{\widehat{[n]}}\hat\psi), \tag{A4a}$$

$$(\bar{\hat\psi}\hat\gamma^{\widehat{[n]}}\hat\rho)^\dagger=+\epsilon\eta^{t+n}(\bar{\hat\psi}\hat\gamma^{\widehat{[n]}}\hat\rho)=+(\bar{\hat\psi}\hat\gamma^{\widehat{[n]}}\hat\rho), \tag{A4b}$$

where the middle sides are the expressions for the general ϵ, η, t, and n. The typical examples of Eq (A4a) are $(\bar{\hat\psi}\,\hat\rho)=-(\bar{\hat\rho}\,\hat\psi)$ and $(\bar{\hat\psi}\hat\gamma^{\hat\mu}\hat\rho)=+(\bar{\hat\rho}\hat\gamma^{\hat\mu}\hat\psi)$, while Eq. (A4b) means that any fermionic bilinears are Hermitian so that we need no imaginary unit i in front.

For our system of $N=(2,0)$ supersymmetry, we need to impose the Weyl conditions on our fermions,

$$\hat\gamma_7(\hat\lambda^I,\hat\chi^I)=(+\hat\lambda^I,+\hat\chi^I), \tag{A5}$$

and *double* the number of fermions. This is very similar to the conventional case of Yang-Mills multiplet in $D=5+1$ [27,28]. Equation (A4a) yields the flipping property $(\bar{\hat\epsilon}_1\hat\gamma^{\hat\mu}\hat\epsilon_2)=+(\bar{\hat\epsilon}_2\hat\gamma^{\hat\mu}\hat\epsilon_1)$ which is not acceptable for the closure of supersymmetry because it should be antisymmetric under $\epsilon_1\leftrightarrow\epsilon_2$. This is accomplished by introducing the $Sp(1)$ indices $A,B,\cdots$ such that

$$(\bar{\hat\psi}^A\hat\gamma^{\widehat{[n]}}\hat\rho_A)\equiv\epsilon^{AB}(\bar{\hat\psi}_B\hat\gamma^{\widehat{[n]}}\hat\rho_A), \tag{A6}$$

where ϵ^{AB} is the $Sp(1)$ metric: $\epsilon^{(1)(2)}=-\epsilon^{(2)(1)}=+\epsilon_{(1)(2)}=-\epsilon_{(2)(1)}=+1$. In this paper, we omit the contracted $Sp(1)$ indices, e.g., $(\bar{\hat\psi}\hat\gamma^{[n]}\hat\rho)\equiv(\bar{\hat\psi}^A\hat\gamma^{[n]}\hat\rho_A)$ to save space. In other words, after the $Sp(1)$ contractions, Eq. (A4) is modified to

$$(\bar{\hat\psi}^A\gamma^{\widehat{[n]}}\hat\rho_A)=+\epsilon\eta^{t+n}(-1)^{(t-n)(t-n+1)/2}(\bar{\hat\rho}^A\hat\gamma^{\widehat{[n]}}\hat\psi_A)=+(-1)^{n(n+1)/2}(\bar{\hat\rho}^A\hat\gamma^{\widehat{[n]}}\hat\psi_A), \tag{A7a}$$

$$(\bar{\hat\psi}^A\hat\gamma^{\widehat{[n]}}\hat\rho_A)^\dagger=+\epsilon\eta^{t+n}(\bar{\hat\psi}^A\hat\gamma^{\widehat{[n]}}\hat\rho_A)=+(\bar{\hat\psi}^A\hat\gamma^{\widehat{[n]}}\hat\rho_A). \tag{A7b}$$

In particular, Eq. (A7a) guarantees the antisymmetry $(\bar{\hat\epsilon}_1{}^A\hat\gamma_{\hat\mu}\hat\epsilon_{2A})=-(\bar{\hat\epsilon}_2{}^A\hat\gamma_{\hat\mu}\hat\epsilon_{1A})$ for the closure of supersymmetry, as desired.

3. Notation in $D=5+5$

Our $D=5+5$ has the metric $(\hat\eta_{\hat\mu\hat\nu})=\mathrm{diag}(+,+,-,-,+,+,+,-,-,-)$, with $\hat\epsilon^{123456789,10}=+1$, and $\hat\gamma_{11}\equiv\hat\gamma_1\hat\gamma_2\hat\gamma_3\hat\gamma_4\hat\gamma_5\hat\gamma_6\hat\gamma_7\hat\gamma_8\hat\gamma_9\hat\gamma_{10}\equiv\hat\gamma_{123456789,10}$, so that

$$\frac{1}{n!}\hat\epsilon_{\hat\mu_1\cdots\hat\mu_{10-n}}{}^{\widehat{[n]}}\hat\epsilon_{\widehat{[n]}}{}^{\hat\nu_1\cdots\hat\nu_{10-n}}=-(10-n)!(-1)^n\hat\delta_{[\hat\mu_1}{}^{\hat\nu_1}\cdots\hat\delta_{\hat\mu_{10-n}]}{}^{\hat\nu_{10-n}}, \tag{A8a}$$

$$\hat\gamma_{[n]}=\frac{(-1)^{n(n-1)/2}}{(10-n)!}\hat\epsilon_{\widehat{[n]}}{}^{\widehat{[10-n]}}\hat\gamma_{\widehat{[10-n]}}\hat\gamma_{11}. \tag{A8b}$$

In $D=5+5$, we have $s=t=5\Rightarrow s-t=8\ (\mathrm{mod}\,8)\Rightarrow\epsilon=\eta=+1$ (or $\epsilon=+1$, $\eta=-1$) for Majorana (or pseudo-Majorana) spinors. For our purpose, we choose Majorana spinors with $\epsilon=\eta=+1$. Accordingly, the flipping and Hermitian-conjugate properties are similar to Eq. (A4),

$$(\bar{\hat{\psi}}\hat{\gamma}^{\widehat{[n]}}\hat{\rho}) = -\epsilon\eta^{t+n}(-1)^{(t-n)(t-n+1)/2}(\bar{\hat{\rho}}\hat{\gamma}^{\widehat{[n]}}\hat{\psi}) = +(-1)^{n(n+1)/2}(\bar{\hat{\rho}}\hat{\gamma}^{\widehat{[n]}}\hat{\psi}), \tag{A9a}$$

$$(\bar{\hat{\psi}}\hat{\gamma}^{\widehat{[n]}}\hat{\rho})^\dagger = +\epsilon\eta^{t+n}(\bar{\hat{\psi}}\hat{\gamma}^{\widehat{[n]}}\hat{\rho}) = +(\bar{\hat{\psi}}\hat{\gamma}^{\widehat{[n]}}\hat{\rho}), \tag{A9b}$$

for Majorana spinors. The difference in signature between Eqs. (A9a) and (A4a) is caused by the difference in t. The typical examples of Eq. (A9a) are $(\bar{\hat{\psi}}\hat{\rho}) = +(\bar{\hat{\rho}}\hat{\psi})$ and $(\bar{\hat{\psi}}\hat{\gamma}^{\hat{\mu}}\hat{\rho}) = -(\bar{\hat{\rho}}\hat{\gamma}^{\hat{\mu}}\hat{\psi})$, while Eq. (A9b) necessitates no imaginary unit in front of fermionic bilinears. For our purpose of $N = (1,0)$ supersymmetry in $D = 5 + 5$, we impose the Weyl conditions on our fermions:

$$\hat{\gamma}_{11}(\hat{\lambda}^I, \hat{\chi}^I) = (+\hat{\lambda}^I, +\hat{\chi}^I). \tag{A10}$$

APPENDIX B: OTHER SIGNATURES $D = 4 + 2$ AND $D = 8 + 2$

In the main text, we have studied the cases of $D = 3 + 3$ and $D = 5 + 5$ with the same number of the + and − signatures ($s = t$). However, we mention other possible options for space-time signatures for EM dualities, in particular, in $D = 4 + 2$ and $D = 8 + 2$.

1. Example of $D = 4 + 2$

For example, the case of $D = 4 + 2$ gives $s = 4$, $t = 2 \Rightarrow s - t = 2 \Rightarrow \epsilon = \eta = -1$ for pseudo-symplectic Majorana spinors [26]. The flipping and Hermitian properties are

$$(\bar{\hat{\psi}}^A\gamma^{\widehat{[n]}}\hat{\rho}_B) = -\epsilon\eta^{t+n}(-1)^{(t-n)(t-n+1)/2}(\bar{\hat{\rho}}_B\hat{\gamma}^{\widehat{[n]}}\hat{\psi}^A) = -(-1)^{n(n+1)/2}(\bar{\hat{\rho}}_B\hat{\gamma}^{\widehat{[n]}}\hat{\psi}^A), \tag{B1a}$$

$$(\bar{\hat{\psi}}^A\hat{\gamma}^{\widehat{[n]}}\hat{\rho}_B)^\dagger = +\epsilon\eta^{t+n}(\bar{\hat{\psi}}^A\hat{\gamma}^{\widehat{[n]}}\hat{\rho}_B) = -(-1)^n(\bar{\hat{\psi}}^A\hat{\gamma}^{\widehat{[n]}}\hat{\rho}_B). \tag{B1b}$$

Therefore, compared with the $D = 3 + 3$ case in Eq. (A7), we get

$$(\bar{\hat{\psi}}^A\hat{\gamma}^{\widehat{[n]}}\hat{\rho}_A) = +\epsilon\eta^{t+n}(-1)^{(t-n)(t-n+1)/2}(\bar{\hat{\rho}}^A\hat{\gamma}^{\widehat{[n]}}\hat{\psi}_A) = +(-1)^{n(n+1)/2}(\bar{\hat{\rho}}^A\hat{\gamma}^{\widehat{[n]}}\hat{\psi}_A), \tag{B2a}$$

$$(\bar{\hat{\psi}}^A\hat{\gamma}^{\widehat{[n]}}\hat{\rho}_A)^\dagger = +\epsilon\eta^{t+n}(\bar{\hat{\psi}}^A\hat{\gamma}^{\widehat{[n]}}\hat{\rho}_A) = -(-1)^n(\bar{\hat{\psi}}^A\hat{\gamma}^{\widehat{[n]}}\hat{\rho}_A). \tag{B2b}$$

In other words, only the Hermitian conjugate has different signs. Since the flipping property is equally valid for the closure $(\bar{\hat{\epsilon}}_1{}^A\hat{\gamma}^{\hat{\mu}}\hat{\epsilon}_{2A}) = -(\bar{\hat{\epsilon}}_2{}^A\hat{\gamma}^{\hat{\mu}}\hat{\epsilon}_{1A})$, the EM duality in $D = 4 + 2$ is an alternative possible Master Theory for SDSYM in $D = 2 + 2$ dimensions.

2. Example of $D = 8 + 2$

Another example is $D = 8 + 2$ with $s = 8$, $t = 2 \Rightarrow s - t = 6 \Rightarrow \epsilon = +1$, $\eta = -1$ for pseudo-Majorana spinors [26]. Accordingly, we get

$$(\bar{\hat{\psi}}\hat{\gamma}^{\widehat{[n]}}\hat{\rho}) = -\epsilon\eta^{t+n}(-1)^{(t-n)(t-n+1)/2}(\bar{\hat{\rho}}\hat{\gamma}^{\widehat{[n]}}\hat{\psi}) = +(-1)^{n(n+1)/2}(\bar{\hat{\rho}}\hat{\gamma}^{\widehat{[n]}}\hat{\psi}), \tag{B3a}$$

$$(\bar{\hat{\psi}}\hat{\gamma}^{\widehat{[n]}}\hat{\rho})^\dagger = +\epsilon\eta^{t+n}(\bar{\hat{\psi}}\hat{\gamma}^{\widehat{[n]}}\hat{\rho}) = +(-1)^n(\bar{\hat{\psi}}\hat{\gamma}^{\widehat{[n]}}\hat{\rho}). \tag{B3b}$$

Equation (B3a) is exactly the same as Eq. (A9a), while Eq. (B3b) is equivalent to Eq. (A9b) by replacing γ^μ by $i\gamma^\mu$. This is simply equivalent to the switch from the signature $8 + 2$ to $2 + 8$. Therefore, the EM duality in $D = 8 + 2$ is equally important as the possible Master Theory for supersymmetric integrable models in $1 \leq D \leq 3$.

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[1] H. Nishino and S. Rajpoot, Phys. Rev. D **92**, 085014 (2015).
[2] B. de Wit and H. Samtleben, Fortschr. Phys. **53**, 442 (2005).
[3] B. de Wit, H. Nicolai, and H. Samtleben, J. High Energy Phys. 02 (2008) 044; C.-S. Chu, arXiv:1108.5131; H. Samtleben, E. Sezgin, and R. Wimmer, J. High Energy Phys. 12 (2011) 062.
[4] H. Nishino and S. Rajpoot, Phys. Rev. D **85**, 105017 (2012).
[5] H. Nishino and S. Rajpoot, J. Phys. Conf. Ser. **485**, 012049 (2014).
[6] A. Candiello and K. Lechner, Nucl. Phys. **B412**, 479 (1994).
[7] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, and A. Van Proeyen, Classical Quantum Gravity **18**, 3359 (2001).
[8] P. Howe and J. Palmkvist, J. High Energy Phys. 05 (2015) 032 and references therein.
[9] W. Siegel, Phys. Rev. D **46**, R3235 (1992).
[10] S. V. Ketov, H. Nishino, and S. J. Gates, Jr., Phys. Lett. B **307**, 323 (1993); **307**, 331 (1993); **297**, 99 (1992); Nucl. Phys. **B393**, 149 (1993).
[11] E. Bergshoeff and E. Sezgin, Phys. Lett. B **292**, 87 (1992).
[12] M. F. Atiyah (unpublished); R. S. Ward and M. Tabor, Phil. Trans. R. Soc. A **315**, 451 (1985); Lect. Notes Phys. **280**, 106 (1987); N. J. Hitchin, Proc. London Math. Soc. **s3-55**, 59 (1987); T. A. Ivanova and A. D. Popov, Phys. Lett. A **205**, 158 (1995).

[13] H. Ooguri and C. Vafa, Mod. Phys. Lett. A **05**, 1389 (1990); Nucl. Phys. **B361**, 469 (1991); **B367**, 83 (1991); H. Nishino and S. J. Gates, Mod. Phys. Lett. A **07**, 2543 (1992).

[14] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, and R. Pettorino, Phys. Lett. **62B**, 105 (1976); M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, R. Pettorino, and J. H. Schwarz, Nucl. Phys. **B111**, 77 (1976); L. Brink and J. H. Schwarz, Nucl. Phys. **B121**, 283 (1977).

[15] B. S. Acharya and M. O'Loughlin, Phys. Rev. D **55**, R4521 (1997); M. Günaydin and H. Nicolai, Phys. Lett. B **351**, 169 (1995); **376**, 329 (1996); I. Bakas, E. G. Floratos, and A. Kehagias, Phys. Lett. B **445**, 69 (1998); E. G. Floratos and A. Kehagias, Phys. Lett. B **427**, 283 (1998); N. Hitchin, arXiv: math/0107101; M. Cvetič, G. W. Gibbons, H. Lu, and C. N. Pope, Nucl. Phys. **617**, 151 (2001); Phys. Rev. D **65**, 106004 (2002); B. Acharya and E. WittenarXiv:hep-th/0109152; A. Brandhuber, Nucl. Phys. **B629**, 393 (2002); S. Gukov and J. Sparks, Nucl. Phys. **B625**, 3 (2002); A. Bilal, J.-P. Derendinger, and K. Sfetsos, Nucl. Phys. **B628**, 112 (2002).

[16] For reviews, see, e.g., M. Atiyah and E. Witten, arXiv:hep-th/0107177; M. J. Duff, Supergravity at 25, Stony Brook, New York, 2001 (unpublished); L. Anguelova and C. I. Lazaroiu, J. High Energy Phys. 01 (2003) 066, and references therein.

[17] H. Nishino and S. Rajpoot, J. High Energy Phys. 04 (2004) 020.

[18] J. Scherk and J. H. Schwarz, Nucl. Phys. **B153**, 61 (1979).

[19] H. Nishino and E. Sezgin, Phys. Lett. B **388**, 569 (1996); H. Nishino, Nucl. Phys. **B523**, 450 (1998).

[20] See, e.g., H. Nishino and S. Rajpoot, J. High Energy Phys. 04 (2004) 020.

[21] E. Bergshoeff, E. Sezgin, and A. van Proyen, Nucl. Phys. **B264**, 653 (1986).

[22] See, e.g., J. Wess and J. Bagger, *Superspace and Supergravity* (Princeton University, Princeton, NJ, 1992).

[23] H. Ooguri and C. Vafa, Mod. Phys. Lett. A **05**, 1389 (1990); Nucl. Phys. **B361**, 469 (1991); **B367**, 83 (1991); H. Nishino and S. J. Gates, Mod. Phys. Lett. A **07**, 2543 (1992).

[24] See, e.g., K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory: A Modern Introduction* (Cambridge University Press, Cambridge, England, 2007) and references therein.

[25] T. Kugo and P. K. Townsend, Nucl. Phys. **B221**, 357 (1983).

[26] A. Salam and E. Sezgin, *Supergravities in Diverse Dimensions* (North-Holland/World Scientific, Amsterdam/Singapore, 1989), Vol. 1, p. 5.

[27] N. Marcus and J. H. Schwarz, Phys. Lett. **115B**, 111 (1982).

[28] H. Nishino and E. Sezgin, Phys. Lett. **144B**, 187 (1984); Nucl. Phys. **B278**, 353 (1986); **B505**, 497 (1997).