Unitarity and vacuum deformation in QED with critical potential steps

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The present article can be considered as a complement to the work of Phys. Rev. D **93**, 045002 (2016), where an nonperturbative approach to QED with x-electric critical potential steps was developed. In the beginning, we study conditions when in and out spaces of the QED under consideration are unitarily equivalent. Then, we construct a general density operator with the vacuum initial condition. Such an operator describes a deformation of the initial vacuum state by x-electric critical potential steps. We construct reductions of the deformed state to electron and positron subsystems, calculating the loss of the information in these reductions. We illustrate the general consideration studying the deformation of the quantum vacuum between two capacitor plates. Finally, we calculate the entanglement measures of these reduced matrices as von Neumann entropies.

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I. INTRODUCTION

Problems of quantum field theory with external field violating the vacuum stability are already being studied systematically for a long time. Recently, they turned out to be of special attention due to new real possible applications in astrophysics and physics of nanostructures. A nonperturbative formulation of QED with the so-called t-potential electric steps (time-dependent potentials) was developed in Refs. [1–3] and applied to various model and realistic physical problems, see, e.g., [4–6]. In the recent work [7], Gavrilov and Gitman succeeded to construct a consistent version of QED with the so-called x-electric critical potential steps (time-independent nonuniform electric fields of constant direction that are concentrated in restricted space areas), for which a large area of new important applications opens, see reviews in [7,8]. However, many principle questions of the formulation still require detailed clarification. The present work is devoted to some of them. In the beginning, we study conditions when in and out spaces of the QED under consideration are unitarily equivalent. Then, we construct a general density operator with the vacuum initial condition. Such an operator describes a deformation of the initial vacuum state by x-electric critical potential steps. We construct reductions of the deformed state to electron and positron subsystems, calculating the loss of the information in these reductions. We illustrate the general consideration studying the deformation of the quantum vacuum between two capacitor plates. In this article, we generally adapt the notations of the paper [7], where the general theory of QED with x-electric critical potential steps was developed, and Ref. [8], where the particular case of a constant electric field between two capacitor plates was studied. In fact, the present article can be considered as a complement to the work [7].

II. UNITARITY IN QED WITH x-ELECTRIC POTENTIAL STEPS

It was shown in Ref. [7] that in the presence of x-electric potential steps the quantized Dirac field can be described in terms of in and out electrons and positrons. Such particles are characterized by quantum numbers n that can be divided in five ranges Ω_i, $i = 1, \ldots, 5$. We denote the corresponding quantum numbers by n_i, so that $n_i \in \Omega_i$. The manifold of all the quantum numbers n is denoted by Ω, so that $\Omega = \Omega_1 \cup \cdots \cup \Omega_5$. The in and out vacua can be factorized

$$|0,\mathrm{in}\rangle = \prod_{i=1}^{5}{}^{\otimes} |0,\mathrm{in}\rangle^{(i)}, \qquad |0,\mathrm{out}\rangle = \prod_{i=1}^{5}{}^{\otimes} |0,\mathrm{out}\rangle^{(i)}, \tag{1}$$

where $|0,\mathrm{in}\rangle^{(i)}$ and $|0,\mathrm{out}\rangle^{(i)}$ are the partial vacua in the ranges Ω_i. Note that in each range Ω_i it is also possible to factorize vacuum vectors in modes with fixed quantum number n so that

$$|0,\mathrm{in}\rangle^{(i)} = \prod_{n\in\Omega_i} |0,\mathrm{in}\rangle_n^{(i)}, \qquad |0,\mathrm{out}\rangle^{(i)} = \prod_{n\in\Omega_i} |0,\mathrm{out}\rangle_n^{(i)}. \tag{2}$$

It was shown that all in and out vacua, except the vacua in the range Ω_3 (in the so-called Klein zone) coincide,

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$$|0,\mathrm{out}\rangle^{(i)} = |0,\mathrm{in}\rangle^{(i)}, \qquad i = 1,2,4,5,$$
$$|0,\mathrm{out}\rangle^{(3)} \neq |0,\mathrm{in}\rangle^{(3)}. \tag{3}$$

In what follows, we use the subindex K to denote all the quantities from the Klein zone, e.g., $|0,\mathrm{in}\rangle^{(3)} = |0,\mathrm{in}\rangle^{(K)}$, $\Omega_3 = \Omega_K$, and so on.

The vacuum-to-vacuum transition amplitude $c_v = \langle 0,\mathrm{out}|0,\mathrm{in}\rangle$ coincides [due to Eq. (3)] with the vacuum-to-vacuum transition amplitude $c_v^{(K)}$ in the Klein zone,

$$c_v = \langle 0,\mathrm{out}|0,\mathrm{in}\rangle = c_v^{(K)} = {}^{(K)}\langle 0,\mathrm{out}|0,\mathrm{in}\rangle^{(K)}. \tag{4}$$

The linear canonical transformation between the in and out sets of creation and annihilation operators in the Klein zone (a and b operators are related to electrons and positrons, respectively) can be written in the following form:

$$ {}^{-}a_n(\mathrm{in}) = w_n(+|+)^{-1}[{}^{+}a_n(\mathrm{out}) + w_n(+-|0)_{+}b_n^{\dagger}(\mathrm{out})],$$
$$ {}_{-}b_n^{\dagger}(\mathrm{in}) = w_n(-|-)^{-1}[{}_{+}b_n^{\dagger}(\mathrm{out}) - w_n(+-|0)^{+}a_n(\mathrm{out})], \tag{5}$$

where

$$w(+|+)_{n'n} = c_v^{-1}\langle 0,\mathrm{out}|{}^{+}a_{n'}(\mathrm{out}){}^{-}a_n^{\dagger}(\mathrm{in})|0,\mathrm{in}\rangle,$$
$$w(-|-)_{n'n} = c_v^{-1}\langle 0,\mathrm{out}|{}_{+}b_{n'}(\mathrm{out}){}_{-}b_n^{\dagger}(\mathrm{in})|0,\mathrm{in}\rangle, \tag{6}$$

are relative scattering amplitudes of electrons and positrons, and

$$w(+-|0)_{n'n} = c_v^{-1}\langle 0,\mathrm{out}|{}^{+}a_{n'}(\mathrm{out}){}_{+}b_n(\mathrm{out})|0,\mathrm{in}\rangle,$$
$$w(0|-+)_{nn'} = c_v^{-1}\langle 0,\mathrm{out}|{}_{-}b_n^{\dagger}(\mathrm{in}){}^{-}a_{n'}^{\dagger}(\mathrm{in})|0,\mathrm{in}\rangle \tag{7}$$

are relative amplitudes of a pair creation and a pair annihilation, and

$$c_v = c_v^{(K)} = \prod_n w_n(-|-)^{-1}. \tag{8}$$

All the amplitudes can be expressed via the coefficients $g({}_{\zeta}|^{\zeta'})$, which, in turn, are calculated via corresponding solutions of the Dirac equation with x-electric potential steps.

An important question is whether in and out spaces are unitarily equivalent? The answer is positive if the linear canonical transformation (5) (together with its adjoint transformation) is proper one. In the latter case, there exists a unitary operator V, such that

$$V(a(\mathrm{out}), a^{\dagger}(\mathrm{out}), b(\mathrm{out}), b^{\dagger}(\mathrm{out}))V^{\dagger}$$
$$= (a(\mathrm{in}), a^{\dagger}(\mathrm{in}), b(\mathrm{in}), b^{\dagger}(\mathrm{in})),$$
$$|0,\mathrm{in}\rangle = V|0,\mathrm{out}\rangle, \qquad V^{\dagger} = V^{-1}. \tag{9}$$

Let us denote all the out operators via α and all the in operators via β. Then the linear uniform canonical transformation between these operators can be written as (we consider the only Fermi case here)

$$\beta = \Phi\alpha + \Psi\alpha^{+},$$
$$\Phi\Phi^{+} + \Psi\Psi^{+} = 1,$$
$$\Phi\Psi^{T} + \Psi\Phi^{T} = 0. \tag{10}$$

According to ([9,10]), transformation (10) is proper one if Ψ is a Hilbert-Schmidt operator, i.e., $\sum_{m,n}|\Psi_{mn}|^2 < \infty$. It is easily to see that Hilbert-Schmidt criterion for the transformation (5) reads

$$\sum_n \left[\left|\frac{w_n(+-|0)}{w_n(+|+)}\right|^2 + \left|\frac{w_n(+-|0)}{w_n(-|-)}\right|^2\right] < \infty. \tag{11}$$

As it was shown in Ref. [7],

$$\left|\frac{w_n(+-|0)}{w_n(+|+)}\right|^2 = N_n^a, \qquad \left|\frac{w_n(+-|0)}{w_n(-|-)}\right|^2 = N_n^b, \tag{12}$$

where N_n^a and N_n^b are differential mean numbers of electrons and positrons created from the vacuum by the potential step. Then, the left-hand side of Eq. (11) is the total number N of particles created from the vacuum, such that unitarity condition can be written as

$$\sum_n (N_n^a + N_n^b) = N < \infty. \tag{13}$$

Note that in- and out-spaces of the scalar QED in the presence of critical potential steps are unitarily equivalent under the same condition.

For realistic external field limited in space and time, this condition is obviously satisfied.

Inequality (11) derived for QED with x-electric potential steps can be considered as one more confirmation of the consistency of the latter theory and correct interpretation of in and out particles there. One should note that qualitatively similar result was established in Ref. [2] for QED with time-dependent electric potential steps.

III. DEFORMATION OF INITIAL VACUUM STATE

In this section we study deformation of initial vacuum state under the action of a x-electric potential step.

In the Heisenberg picture, the density operator of the system whose initial state is the vacuum, is given by equation

$$\hat{\rho} = |0,\mathrm{in}\rangle\langle 0,\mathrm{in}|. \tag{14}$$

The in and out Fock spaces are related by the unitary operator V, see (9). Then

$$\hat{\rho} = V|0,\text{out}\rangle\langle 0,\text{out}|V^{\dagger}. \tag{15}$$

In QED with x-electric potential steps the operator V was constructed in [7]. Since it can be factorized, the density operator (15) can be factorized as well,

$$V = \prod_{i=1}^{5} V^{(i)}, \qquad |0,\text{in}\rangle^{(i)} = V^{(i)}|0,\text{out}\rangle^{(i)},$$
$$\hat{\rho} = \prod_{i=1}^{5} V^{(i)}|0,\text{out}\rangle^{(i)(i)}\langle 0,\text{out}|V^{(i)\dagger}. \tag{16}$$

Due to the specific structure of the operator $V^{(i)}$, $i = 1, 2, 4, 5$, we have

$$\begin{aligned} &V^{(i)}|0,\text{out}\rangle^{(i)(i)}\langle 0,\text{out}|V^{(i)\dagger} \\ &\quad = |0,\text{out}\rangle^{(i)(i)}\langle 0,\text{out}| = |0,\text{in}\rangle^{(i)(i)}\langle 0,\text{in}|, \\ &\quad i = 1,2,4,5. \end{aligned}$$

The latter relation has clear physical meaning; vacuum states in the ranges Ω_1, Ω_2, Ω_4, and Ω_5 do not change with time. There is no particle creation there. Let us use the following notation:

$$P' = \prod_{i=1,2,4,5} |0,\text{out}\rangle^{(i)(i)}\langle 0,\text{out}| = \prod_{i=1,2,4,5} |0,\text{in}\rangle^{(i)(i)}\langle 0,\text{in}|,$$
$$\hat{\rho}_K = V^{(K)} P_K V^{(K)\dagger}, \qquad P_K = |0,\text{out}\rangle^{(K)(K)}\langle 0,\text{out}|, \tag{17}$$

then

$$\hat{\rho} = P'\hat{\rho}_K. \tag{18}$$

Using the following explicit form of the operator $V^{(K)} = V^{(3)}$ derived in Ref. [7],

$$\begin{aligned} V^{(K)} = &\exp\left[-\sum_{n\in\Omega_K} {}^{+}a_n^{\dagger}(\text{out}) w_n(+-|0)\, {}_{+}b_n^{\dagger}(\text{out})\right] \\ &\times \exp\left[-\sum_{n\in\Omega_K} {}_{+}b_n(\text{out}) \ln w_n(-|-)\, {}_{+}b_n^{\dagger}(\text{out})\right] \\ &\times \exp\left[\sum_{n\in\Omega_K} {}^{+}a_n^{\dagger}(\text{out}) \ln w_n(+|+)\, {}^{+}a_n(\text{out})\right] \\ &\times \exp\left[-\sum_{n\in\Omega_K} {}_{+}b_n(\text{out}) w_n(0|-+)\, {}^{+}a_n(\text{out})\right], \end{aligned}$$

one can derive two alternative expressions for the density operator $\hat{\rho}_K$.

The first one is a normal form exponential with respect to the out operators (denoted by $:\ldots:$),

$$\begin{aligned} \hat{\rho}_K|c_v|^{-2} =: \exp\Big\{-\sum_{n\in\Omega_K} [&{}^{+}a_n^{\dagger}(\text{out})\, {}^{+}a_n(\text{out}) \\ &+ {}_{+}b_n^{\dagger}(\text{out})\, {}_{+}b_n(\text{out}) \\ &+ {}^{+}a_n^{\dagger}(\text{out}) w_n(+-|0)\, {}_{+}b_n^{\dagger}(\text{out}) \\ &+ {}_{+}b_n(\text{out}) w_n(+-|0)^{*}\, {}^{+}a_n(\text{out})]\Big\}:. \end{aligned} \tag{19}$$

Representation (19) can be derived in the following way: Using (17) and the explicit form of $V^{(K)}$, we can write

$$\begin{aligned} \hat{\rho}_K|c_v|^{-2} = &\exp\left[-\sum_{n\in\Omega_K} {}^{+}a_n^{\dagger}(\text{out}) w_n(+-|0)\, {}_{+}b_n^{\dagger}(\text{out})\right] \\ &\times P_K \exp\left[-\sum_{n\in\Omega_K} {}_{+}b_n(\text{out}) w_n(+-|0)^{*}\, {}^{+}a_n(\text{out})\right]. \end{aligned} \tag{20}$$

Making use of well-known Berezin representation [9] for a projection operator P_K on the vacuum state,

$$\begin{aligned} P_K =: \exp\Big\{-\sum_{n\in\Omega_K} [&{}^{+}a_n^{\dagger}(\text{out})\, {}^{+}a_n(\text{out}) \\ &+ {}_{+}b_n^{\dagger}(\text{out})\, {}_{+}b_n(\text{out})]\Big\}: \end{aligned} \tag{21}$$

and taking into account that the left and the right exponents in Eq. (20) are already normal ordered, we easily obtain representation (19).

The second representation reads

$$\begin{aligned} \hat{\rho}_K|c_v|^{-2} = &\prod_{n\in\Omega_K} [1 - {}^{+}a_n^{\dagger}(\text{out}) w_n(+-|0)\, {}_{+}b_n^{\dagger}(\text{out})] \\ &\times P_{K,n}[1 - {}_{+}b_n(\text{out}) w_n(+-|0)^{*}\, {}^{+}a_n(\text{out})], \\ P_{K,n} = &|0,\text{out}\rangle_n^{(K)(K)}{}_n\langle 0,\text{out}|. \end{aligned} \tag{22}$$

Representation (22) can be derived as follows: Using the fact that operators with different quantum numbers n commute and using the relation, see, e.g., Ref. [4],

$$\exp[a^{\dagger} D a] =: \exp[a^{\dagger}(e^{D} - 1)a]:; \tag{23}$$

to transform exponents from $V^{(K)}$, we expand then the obtained expressions in the power series. Since the out-operators in $V^{(K)}$ are Fermi type, these series are reduced to finite term expressions. Their actions on the vacuum $|0,\text{out}\rangle^{(K)}$ can be easily calculated, and using of Eq. (2), we arrive at Eq. (22).

Finally, we consider the structure of the $|0,\text{in}\rangle$ state in terms of out operators. First of all, we use the fact that the state vector under discussion is factorized,

$$|0,\mathrm{in}\rangle = V|0,\mathrm{out}\rangle = |0,\mathrm{in}\rangle'|0,\mathrm{in}\rangle^{(K)},$$
$$|0,\mathrm{in}\rangle' = \prod_{i=1,2,4,5}|0,\mathrm{in}\rangle^{(i)},$$
$$|0,\mathrm{in}\rangle^{(K)} = V^{(K)}|0,\mathrm{out}\rangle^{(K)}. \quad (24)$$

Then, using the explicit form $V^{(K)}$, we obtain

$$|0,\mathrm{in}\rangle^{(K)} = c_v\prod_{n\in\Omega_K}[1-{}^{+}a_n^{\dagger}(\mathrm{out})w_n(+-|0)_{+}b_n^{\dagger}(\mathrm{out})]|0,\mathrm{out}\rangle^{(K)}. \quad (25)$$

In each fixed mode $n\in\Omega_K$, the state vector $|0,\mathrm{in}\rangle$ is a linear superposition of two terms: the vacuum vector in this mode and a state with an electron-positron pair.

IV. REDUCTIONS TO ELECTRON AND POSITRON SUBSYSTEMS

It should be stressed that the system under consideration can be considered as a composed from a subsystem of electrons and a subsystem of positrons. One can introduce the so-called two reduced density operators, $\hat{\rho}_+$ of the electron subsystem and $\hat{\rho}_-$ of the positron subsystem, averaging complete density operator (14) over all possible positron states or over all possible electron states, respectively,

$$\hat{\rho}_+ = \mathrm{tr}_-\hat{\rho} = \sum_{i=3}^{5}\sum_{M}\sum_{\{m\}\in\Omega_i} {}_b^{(i)}\langle M,\mathrm{out}|\hat{\rho}|M,\mathrm{out}\rangle_b^{(i)},$$
$$\hat{\rho}_- = \mathrm{tr}_+\hat{\rho} = \sum_{i=1}^{3}\sum_{M}\sum_{\{m\}\in\Omega_i} {}_a^{(i)}\langle M,\mathrm{out}|\hat{\rho}|M,\mathrm{out}\rangle_a^{(i)},$$
$$|M,\mathrm{out}\rangle_b^{(i)} = (M!)^{-1/2}b_{m_1}^{\dagger}(\mathrm{out})\ldots b_{m_M}^{\dagger}(\mathrm{out})|0,\mathrm{out}\rangle_b^{(i)},$$
$$|M,\mathrm{out}\rangle_a^{(i)} = (M!)^{-1/2}a_{m_1}^{\dagger}(\mathrm{out})\ldots a_{m_M}^{\dagger}(\mathrm{out})|0,\mathrm{out}\rangle_a^{(i)}. \quad (26)$$

Vectors $|0,\mathrm{out}\rangle_a^{(i)}$ and $|0,\mathrm{out}\rangle_b^{(i)}$ are the electron and positron vacua in the Ω_i range, defined by

$$a_n^{(i)}(\mathrm{out})|0,\mathrm{out}\rangle_a^{(i)} = 0,$$
$$b_n^{(i)}(\mathrm{out})|0,\mathrm{out}\rangle_b^{(i)} = 0, \quad (27)$$

where $a_n^{(i)}(\mathrm{out})$ and $b_n^{(i)}(\mathrm{out})$ are corresponding annihilation operators of electrons and positrons in this range, respectively. Of course, these electron and positron vacua can be factorized in quantum modes, as was mentioned already above. One can see that

$$|0,\mathrm{out}\rangle^{(1,2)} = |0,\mathrm{out}\rangle_a^{(1,2)} = \prod_{n\in\Omega_{1,2}}|0,\mathrm{out}\rangle_{n,a}^{(1,2)},$$
$$|0,\mathrm{out}\rangle^{(4,5)} = |0,\mathrm{out}\rangle_b^{(4,5)} = \prod_{n\in\Omega_{4,5}}|0,\mathrm{out}\rangle_{n,b}^{(4,5)},$$
$$|0,\mathrm{out}\rangle^{(3)} = |0,\mathrm{out}\rangle^{(K)} = |0,\mathrm{out}\rangle_a^{(K)}\otimes|0,\mathrm{out}\rangle_b^{(K)},$$
$$|0,\mathrm{out}\rangle_a^{(K)} = \prod_{n\in\Omega_K}|0,\mathrm{out}\rangle_{n,a}^{(K)},$$
$$|0,\mathrm{out}\rangle_b^{(K)} = \prod_{n\in\Omega_K}|0,\mathrm{out}\rangle_{n,b}^{(K)}. \quad (28)$$

Using Eq. (18) and representation (22) for $\hat{\rho}_K$, it is easy to calculate traces in Eqs. (26) and to obtain thus explicit forms of the reduced operators $\hat{\rho}_\pm$:

$$\hat{\rho}_+|c_v|^{-2} = \prod_{i=1,2}|0,\mathrm{out}\rangle^{(i)(i)}\langle 0,\mathrm{out}|\otimes\prod_{n\in\Omega_K}[P_{K,a,n}+|w_n(+-|0)|^{2+}a_n^{\dagger}(\mathrm{out})P_{K,a,n}{}^{+}a_n(\mathrm{out})],$$
$$\hat{\rho}_-|c_v|^{-2} = \prod_{i=4,5}|0,\mathrm{out}\rangle^{(i)(i)}\langle 0,\mathrm{out}|\otimes\prod_{n\in\Omega_K}[P_{K,b,n}+|w_n(+-|0)|^{2}{}_{+}b_n^{\dagger}(\mathrm{out})P_{K,b,n+}b_n(\mathrm{out})],$$
$$P_{K,a,n} = |0,\mathrm{out}\rangle_{n,a}^{(K)}{}_{n,a}^{(K)}\langle 0,\mathrm{out}|, \qquad P_{K,b,n} = |0,\mathrm{out}\rangle_{n,b}^{(K)}{}_{n,b}^{(K)}\langle 0,\mathrm{out}|. \quad (29)$$

We can also consider a reduction of density operator (18), which occurs due to measurement of a physical quantity by some classical tool or, in other words, due to decoherence. Suppose that we are measuring the number of particles $N(\mathrm{out})$ in the state $\hat{\rho}$ of the system under consideration. The operator corresponding to this physical quantity is $\hat{N}(\mathrm{out}) = \sum_{i=1}^{5}\hat{N}_i(\mathrm{out})$, where

$$\hat{N}_1(\mathrm{out}) = \sum_{n\in\Omega_1} [{}^+a_n^\dagger(\mathrm{out})^+a_n(\mathrm{out}) + {}_-a_n^\dagger(\mathrm{out})_-a_n(\mathrm{out})],$$
$$\hat{N}_2(\mathrm{out}) = \sum_{n\in\Omega_2} a_n^\dagger a_n,$$
$$\hat{N}_4(\mathrm{out}) = \sum_{n\in\Omega_4} b_n^\dagger b_n,$$
$$\hat{N}_3(\mathrm{out}) = \sum_{n\in\Omega_K} [{}^+a_n^\dagger(\mathrm{out})^+a_n^\dagger(\mathrm{out}) + {}_+b_n^\dagger(\mathrm{out})_+b_n(\mathrm{out})],$$
$$\hat{N}_5(\mathrm{out}) = \sum_{n\in\Omega_5} [{}_+b_n^\dagger(\mathrm{out})_+b_n(\mathrm{out}) + {}^-b_n^\dagger(\mathrm{out})^-b_n(\mathrm{out})]. \tag{30}$$

According to von Neumann [11], the density operator $\hat{\rho}$ after such a measurement is reduced to the operator $\hat{\rho}_N$ of a form

$$\hat{\rho}_N = \sum_s \langle s,\mathrm{out}|\hat{\rho}|s,\mathrm{out}\rangle \hat{P}_s,$$
$$\hat{P}_s = |s,\mathrm{out}\rangle\langle s,\mathrm{out}|, \tag{31}$$

where $|s,\mathrm{out}\rangle$ are eigenstates of the operator $\hat{N}(\mathrm{out})$ with the eigenvalues s that represent the total number of electrons and positrons in the state $|s,\mathrm{out}\rangle$,

$$\hat{N}(\mathrm{out})|s,\mathrm{out}\rangle = s|s,\mathrm{out}\rangle,$$
$$|s,\mathrm{out}\rangle = \prod_{n\in\Omega_1}[{}^+a_n^\dagger(\mathrm{out})]^{l_{n,1}}[{}_-a_n^\dagger(\mathrm{out})]^{k_{n,1}}\prod_{n\in\Omega_2}(a_n^\dagger)^{l_{n,2}}\prod_{n\in\Omega_4}(b_n^\dagger)^{l_{n,4}}$$
$$\times\prod_{n\in\Omega_5}[{}_+b_n^\dagger(\mathrm{out})]^{l_{n,5}}[{}^-b_n^\dagger(\mathrm{out})]^{k_{n,5}}\prod_{n\in\Omega_K}[{}^+a_n^\dagger(\mathrm{out})]^{l_{n,3}}[{}_+b_n^\dagger(\mathrm{out})]^{k_{n,3}}|0,\mathrm{out}\rangle,$$
$$s = \sum_{n\in\Omega_1}(l_{n,1}+k_{n,1}) + \sum_{n\in\Omega_2}(l_{n,2}) + \sum_{n\in\Omega_4}(l_{n,4}) + \sum_{n\in\Omega_5}(l_{n,5}+k_{n,5}) + \sum_{n\in\Omega_K}(l_{n,3}+k_{n,3}).$$

Note that $l_{n,i}, k_{n,i} = (0,1)$ due to the fact that we deal with fermions.

Due to the structure of the operator $\hat{\rho}$, the weights $\langle s,\mathrm{out}|\hat{\rho}|s,\mathrm{out}\rangle$ are nonzero only for pure states $|s,\mathrm{out}\rangle$ with an integer number of pairs in Ω_K (since the initial state of the system was a vacuum, and there is no particle creation outside of the Klein zone). Thus, the operator $\hat{\rho}_N$ takes the form

$$\hat{\rho}_N|c_v|^{-2} = P'\prod_{n\in\Omega_K}[P_{K,n} + |w_n(+-|0)|^2 {}^+a_n^\dagger(\mathrm{out})_+b_n^\dagger(\mathrm{out})$$
$$\times P_{K,n+}b_n(\mathrm{out})^+a_n(\mathrm{out})], \tag{32}$$

where operators $P_{K,n}$ and P' were defined in the previous section, see Eq. (22). Note that the measurement destroys nondiagonal terms of the density operator (22).

Since the operator V is unitary and the initial state of the system under consideration is a pure state (the vacuum state) the density operator (18) describes a pure state as well. Therefore, its von Neumann entropy is zero. However, the reduced density operators $\hat{\rho}_\pm$ (29) describe already mixed states, and their entropies $S(\hat{\rho}_\pm)$ are not zero,

$$S(\hat{\rho}_\pm) = -k_B \mathrm{tr}\hat{\rho}_\pm \ln\hat{\rho}_\pm. \tag{33}$$

It is known that this entropy can be treated as a measure of the quantum entanglement of the electron and positron subsystems and can be treated as the measure of the information loss.

Using the normalization condition for the reduced density operators, $\mathrm{tr}\hat{\rho}_\pm = 1$, the relation (23), definitions for differential mean numbers of particles N_n^a and antiparticles N_n^b created from vacuum

$$N_n^a = \mathrm{tr}\hat{\rho}_+ a_n^\dagger(\mathrm{out})a_n(\mathrm{out}),$$
$$N_n^b = \mathrm{tr}\hat{\rho}_- b_n^\dagger(\mathrm{out})b_n(\mathrm{out}), \tag{34}$$

and the fact that

$$N_n^a = N_n^b = N_n^{\mathrm{cr}},$$
$$|w_n(+-|0)|^2 = N_n^{\mathrm{cr}}(1-N_n^{\mathrm{cr}})^{-1}. \tag{35}$$

We can calculate traces in Eqs. (33) and rewrite RHS in these equations as

$$S(\hat{\rho}_\pm) = \sum_{n\in\Omega_K} S_n,$$
$$S_n = -k_B[(1-N_n^{\mathrm{cr}})\ln(1-N_n^{\mathrm{cr}}) + N_n^{\mathrm{cr}}\ln N_n^{\mathrm{cr}}]. \tag{36}$$

The von Neumann-reduced density operator (32) also describes the mixed state; making use of the fact that the pure states $|0,\mathrm{out}\rangle_n^{(K)}$ and ${}^+a_n^\dagger(\mathrm{out})_+b_n^\dagger(\mathrm{out})|0,\mathrm{out}\rangle_n^{(K)}$ are orthogonal and normalized, it is not difficult to show that the von Neumann entropy $S(\hat{\rho}_N)$ of the mixed state (32) coincides with the entropies $S(\hat{\rho}_{\pm})$ of the reduced density operators $\hat{\rho}_{\pm}$.

The differential mean number of fermions created N_n^{cr} can vary only within the range (0,1). The partial entropy S_n for given n in Eq. (36) is symmetric with respect to value of N_n^{cr}. It reaches maximum at $N_n^{\mathrm{cr}} = 1/2$ and turns to zero at $N_n^{\mathrm{cr}} = 1$ and $N_n^{\mathrm{cr}} = 0$. This fact can be interpreted as follows: In the case of $N_n^{\mathrm{cr}} = 0$, there are no particles created by the external field, and the initial vacuum state in the mode remains unchanged. The case $N_n^{\mathrm{cr}} = 1$ corresponds to the situation when a particle is created with certainty. The maximum of S_n, corresponding to $N_n^{\mathrm{cr}} = 1/2$, is associated with the state with the maximum amount of uncertainty.

V. DEFORMATION OF THE QUANTUM VACUUM BETWEEN TWO CAPACITOR PLATES

Here, we illustrate the general consideration considering the deformation of the quantum vacuum between two infinite capacitor plates separated by a finite distance L. Some aspects of particle creation by the constant electric field between such plates (this field is also called L-constant electric field) were studied in Ref. [8]. The latter field is a particular case of x-electric potential step. Thus, we consider the L-constant electric field in $d = D + 1$ dimensions. We chose $\mathbf{E}(x) = (E^i, i = 1, \ldots, D)$, $E^1 = E_x(x)$, $E^{2,\ldots,D} = 0$,

$$E_x(x) = \begin{cases} 0, & x \in (-\infty, -L/2] \\ E = \mathrm{const} > 0, & x \in (-L/2, L/2) \\ 0, & x \in [L/2, \infty) \end{cases}.$$

The potential energy of an electron in the L-electric field under consideration is

$$U(x) = \begin{cases} U_{\mathrm{L}} = -eEL/2, & x \in (-\infty, -L/2] \\ eEx, & x \in (-L/2, L/2) \\ U_{\mathrm{R}} = eEL/2, & x \in [L/2, \infty) \end{cases}. \quad (37)$$

The magnitude of the corresponding x-electric is $\mathbb{U} = eEL$. We are interested in the critical steps, for which

$$\mathbb{U} = eEL > 2m \quad (38)$$

and the vacuum is unstable in the Klein zone.

We consider a particular case with a sufficiently large length L between the capacitor plates,

$$\sqrt{eE}L \gg \max\{1, E_c/E\}. \quad (39)$$

Here, $E_c = m^2/e$ is the critical Schwinger field. In what follows, we conditionally call this approximation a large work approximation. Such a kind of x-electric step represents a regularization for a constant uniform electric field and is suitable for imitating a small-gradient field.

It was shown in Ref. [8] that the main particle production occurs in an inner subrange $\tilde{\Omega}_K$ of the Klein zone, $\tilde{\Omega}_K \subset \Omega_K$,

$$\tilde{\Omega}_K : |p_0|/\sqrt{eE} < \sqrt{eE}L/2 - K, \qquad \lambda < K_\perp^2,$$
$$\lambda = \frac{\mathbf{p}_\perp^2 + m^2}{eE}, \qquad \sqrt{eE}L \gg K \gg K_\perp^2 \gg \max\{1, E_c/E\}, \quad (40)$$

where K and $K_\perp$ are any given positive numbers satisfying the condition (40).

The differential number of particles with quantum numbers $n \in \tilde{\Omega}_K$ created from the vacuum reads

$$N_n^{\mathrm{cr}} = e^{-\pi\lambda}[1 + O(|\xi_1|^{-3}) + O(|\xi_2|^{-3})],$$
$$\xi_1 = \frac{-eEL/2 - p_0}{\sqrt{eE}},$$
$$\xi_2 = \frac{eEL/2 - p_0}{\sqrt{eE}}. \quad (41)$$

We recall that, in fact, the quantum numbers n that label electron and positron states in general formulas gather several quantum numbers,

$$n = (p_0, \mathbf{p}_\perp, \sigma), \qquad \mathbf{p}_\perp = (p_2, \ldots, p_D), \quad (42)$$

where for an electron p_0 is its energy, for a positron $-p_0$ is its energy, and for an electron $\mathbf{p}_\perp$ denotes its transversal components of the momentum, whereas for a positron $-\mathbf{p}_\perp$ denotes its transversal components of the momentum. For an electron σ is its spin polarization, and for a positron $-\sigma$ is its spin polarization. Note that the electron and positron in a pair created by an external field have the same quantum numbers n.

The quantity (41) is almost constant over a wide range of energy p_0 for any given $\lambda < K_\perp^2$, for these quantum numbers we can assume $N_n^{\mathrm{cr}} \approx e^{-\pi\lambda}$. In the limiting case of the large work approximation, $\sqrt{eE}L \to \infty$, one obtains the well-known result for particle creation by a constant uniform electric field $N_n^{\mathrm{cr}} = e^{-\pi\lambda}$, see Refs. [12–14].

In the approximation under the consideration, the total number of particles created from the vacuum is given by a sum (integral) over $n \in \tilde{\Omega}_K$,

$$N^{\mathrm{cr}} = \sum_{n\in\Omega_K} N_n^{\mathrm{cr}} \approx \sum_{\mathbf{p}_\perp, p_0 \in \tilde{\Omega}_K} \sum_{\sigma} N_n^{\mathrm{cr}}$$
$$= \frac{J_{(d)} T V_\perp}{(2\pi)^{d-1}} \int_{\tilde{\Omega}_K} dp_0 d\mathbf{p}_\perp N_n^{\mathrm{cr}}, \quad (43)$$

where $J_{(d)} = 2^{[d/2]-1}$ is a spin summation factor, $V_\perp$ is the $(d-2)$-dimensional spatial volume in hypersurface orthogonal to the electric field direction, and T is the time duration of the electric field. The integration over p_0 results in

$$N^{\rm cr} = \frac{J_{(d)} T V_\perp L e E}{(2\pi)^{d-1}} \int_{\tilde{\Omega}_K} d\mathbf{p}_\perp e^{-\pi\lambda}. \quad (44)$$

Integrating Eq. (44) over $p_\perp$, we obtain that the total number of created from the vacuum particles in the large work approximation has the form

$$N^{\rm cr} = \frac{J_{(d)} T V (eE)^{d/2}}{(2\pi)^{d-1}} \exp\left(-\pi \frac{E_c}{E}\right), \quad (45)$$

where $V = LV_\perp$ is the volume inside of the capacitor (the volume occupied by the electric field).

It is obvious that $N^{\rm cr} < \infty$, when the values V and T are finite or, in other words, when regularization of the finite volume and finite time of the field action is used. Looking on the condition (13), we see that the x-electric potential step, which represents the electric field inside of the capacitor, does not violate the unitarity in QED.

Let us estimate the information loss of the reduced states of the deformed vacuum, which can be calculated as entropies (36) of these states,. Using the same summation rule as in (43), one can write

$$S(\hat{\rho}_\pm) = -k_B \frac{J_{(d)} T V_\perp}{(2\pi)^{d-1}} \int_{\Omega_K} dp_0 d\mathbf{p}_\perp [N_n^{\rm cr} \ln N_n^{\rm cr} + (1 - N_n^{\rm cr}) \ln(1 - N_n^{\rm cr})]. \quad (46)$$

For Fermi particles under the consideration, $N_n^{\rm cr} \le 1$. This allows us to expand the logarithm in the rhs of Eq. (46) in powers of $N_n^{\rm cr}$. Thus, we represent the term $(1 - N_n^{\rm cr}) \ln(1 - N_n^{\rm cr})$ as follows:

$$(1 - N_n^{\rm cr}) \ln(1 - N_n^{\rm cr}) = -(1 - N_n^{\rm cr}) \sum_{l=1}^{\infty} l^{-1} (N_n^{\rm cr})^l. \quad (47)$$

Using (47) in Eq. (46), we obtain the following intermediate result:

$$S(\hat{\rho}_\pm) = k_B \frac{J_{(d)} T V_\perp}{(2\pi)^{d-1}} \int_{\Omega_K} dp_0 d\mathbf{p}_\perp \left[-N_n^{\rm cr} \ln N_n^{\rm cr} + (1 - N_n^{\rm cr}) \sum_{l=1}^{\infty} l^{-1} (N_n^{\rm cr})^l \right]. \quad (48)$$

As we have mentioned before, the considerable amount of particles is created only in the subrange $\tilde{\Omega}_K \in \Omega_K$, where terms proportional to $|\xi_{1,2}|^{-3}$ are small and can be neglected, allowing us to use the leading-order approximation $N_n^{\rm cr} \approx e^{-\pi\lambda}$ in the rhs of Eq. (48). Then, we obtain

$$S(\hat{\rho}_\pm) \approx k_B \frac{J_{(d)} T V e E}{(2\pi)^{d-1}} \int_{\tilde{\Omega}_K} d\mathbf{p}_\perp \left[\pi\lambda e^{-\pi\lambda} + (1 - e^{-\pi\lambda}) \sum_{l=1}^{\infty} l^{-1} e^{-\pi\lambda l} \right] \quad \text{if } d > 2,$$
$$S(\hat{\rho}_\pm) \approx k_B \frac{T V e E}{2\pi} A(2, E_c/E) \quad \text{if } d = 2,$$
$$A(2, E_c/E) = \{\pi E_c/E \exp(-\pi E_c/E) - [1 - \exp(-\pi E_c/E)] \ln[1 - \exp(-\pi E_c/E)]\}. \quad (49)$$

In the dimensions $d > 2$ the integration over the transversal components of the momentum can be easily performed. Outside of the subrange $\tilde{\Omega}_K$, the integrand is very small, so that we can extend the integration limits of $p_\perp$ to the infinity. Thus, we finally get

$$S(\hat{\rho}_\pm) \approx k_B \frac{J_{(d)} T V (eE)^{d/2}}{(2\pi)^{d-1}} A(d, E_c/E) \quad \text{if } d > 2, \quad (50)$$

where the factor $A(d, E_c/E)$ has the form

$$A(d, E_c/E) = (\pi E_c/E + d/2 - 1) \exp(-\pi E_c/E) + \sum_{l=1}^{\infty} [l^{-d/2} - l^{-1}(l+1)^{(2-d)/2} \times \exp(-\pi E_c/E)] \exp(-\pi l E_c/E). \quad (51)$$

For example, estimations of this factor for strong field $E_c/E \ll 1$ and critical field $E_c/E = 1$ with $d = 4$, 3 are $A(4,0) = \pi^2/6$, $A(4,1) \approx 0,22$, $A(3,0) \approx 0,93$, and $A(3,1) \approx 0,20$. In the case of a weak field, $E_c/E \gg 1$, the entropy is exponentially small for any d,

$$A(d, E_c/E) \approx (\pi E_c/E + d/2) \exp(-\pi E_c/E).$$

One can note that the large work approximation (50) obtained for $S(\hat{\rho}_\pm)$ in the case of the x-electric step under consideration coincides with the same approximation for $S(\hat{\rho}_\pm)$ in the case of the t-electric step with an uniform electric field that is acting during a finite time interval T (the so called T-constant field) obtained in Ref. [6]. This observation confirms the fact that the T-constant and L-constant fields produce equal physical effects in the large work approximation (or as $T \to \infty$ and $L \to \infty$), such that it is possible to consider these fields as regularizations of a constant uniform electric field given by two distinct gauge conditions for electromagnetic potentials. Obviously, exact expressions for the entropies $S(\hat{\rho}_{\pm})$ differ in the general case.

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