

Quantum and thermal fluctuations in quantum mechanics and field theories from a new version of semiclassical theory

M. A. Escobar-Ruiz,^{1,3,*} E. Shuryak,^{2,†} and A. V. Turbiner^{1,2,‡}

¹*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,
Apartado Postal 70-543, 04510 México, D.F., México*

²*Department of Physics and Astronomy, Stony Brook University, Stony Brook, New York 11794-3800, USA*

³*School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA*

(Received 22 February 2016; published 24 May 2016)

We develop a new semiclassical approach, which starts with the density matrix given by the Euclidean time path integral with fixed coinciding end points, and proceed by identifying classical (minimal Euclidean action) path, to be referred to as a *flucton*, which passes through this end point. Fluctuations around a flucton path are included, by standard Feynman diagrams, previously developed for instantons. We calculate the Green function and evaluate the one loop determinant both by direct diagonalization of the fluctation equation and also via the trick with the Green functions. The two-loop corrections are evaluated by explicit Feynman diagrams, and some curious cancellation of logarithmic and polylog terms is observed. The results are fully consistent with large-distance asymptotics obtained in quantum mechanics. Two classic examples—quartic double-well and sine-Gordon potentials—are discussed in detail, while powerlike potential and quartic anharmonic oscillator are discussed in brief. Unlike other semiclassical methods, like WKB, we do not use the Schrödinger equation, and all the steps generalize to multidimensional or quantum fields cases straightforwardly.

DOI: [10.1103/PhysRevD.93.105039](https://doi.org/10.1103/PhysRevD.93.105039)

I. INTRODUCTION

Semiclassical approximations are well-known tools, both in quantum mechanical and quantum field theory applications.

Quantum mechanics itself originated from Bohr-Sommerfeld quantization conditions, and semiclassical approximations for the wave function—the WKB and its extensions—has been developed already in the early days of its development and are since a standard part of quantum mechanics textbooks. Unfortunately, extending such methods beyond one-dimensional cases or those with separable variables proved to be difficult.

Semiclassical approximations in quantum field theory developed differently: their starting point is the Feynman path integrals [1,2], which is infinitely dimensional anyway, and thus the dimension of quantum mechanics coordinates or number of quantum fields is of secondary importance. So, their main advantage over the semiclassical approaches based on the Schrödinger equation (such as WKB) is that it can be used in multidimensional cases. Applications of such methods range from that by Rossi and Testa [3] in Quantum Field Theory (QFT) to recent studies of protein folding [4] in statistical mechanics.

Another general advantage of the latter approach is that path integrals lead to systematic perturbative series, in the form of Feynman diagrams, with clear rules for each order. Textbook perturbative approaches for the wave functions do not have that, and basically are never used beyond say first and second orders.

Of course, the higher level of generality comes with a heavy price. While classical part is relatively simple, already at a one-loop level one needs to calculate determinants of certain differential operators. At two and more loops Feynman diagrams need to be evaluated on top of space-time dependent backgrounds: therefore those can be done in a space-time representation rather than in the energy-momentum one mostly used in QFT applications. Most content of this paper is the explicit demonstration of how one can do all that, in analytic form, for two classic examples—quartic double-well and sine-Gordon potentials.

Let us now outline briefly the history of semiclassical evaluation of the path integrals in Euclidian time. Polyakov [5] used the example of symmetric double-well potential to demonstrate the physical meaning of the celebrated “instanton” solution in the non-Abelian gauge theories (he and collaborators discovered shortly before that). For pedagogical presentation of this material, including the one-loop corrections, see [6]. Feynman diagrams and two-loop corrections have been calculated by F. Wöhler and E. Shuryak [7] for the double-well potential, extended to three-loops in our recent two papers [8,9] for both the double-well and sine-Gordon potentials.

*mauricio.escobar@nucleares.unam.mx

†edward.shuryak@stonybrook.edu

‡turbiner@nucleares.unam.mx,alexander.turbiner@stonybrook.edu

All the development was focused on the phenomenon of *quantum tunneling* through the barrier for degenerate minima. Polyakov's instanton is the classical path, coming from one minimum of the potential to the other. The instanton amplitude, evaluated in the above mentioned papers in higher orders, are approximations for the path integral with the end points of the path corresponding to this arrangement, corresponding physically to a “spectral gap”, the splitting between the lowest states of opposite parity for the double-well potential case.

From the theoretical point of view, the instanton amplitudes and perturbative series around them are parts of more general construction nowadays known as *trans-series* which include series at small coupling constant g : powerlike terms $\sim g^n$, exponentially small terms $\sim e^{-\frac{\text{const}}{g^2}}$, and logarithms of the coupling multiplied by such exponents, $\sim (\log g)^k e^{-\frac{\text{const}}{g^2}}$. The issue of a unique definition of such series is related with the so-called *resurgence* theory, which provide certain relations between series near different extrema. Specific issues related to interplay between the perturbative series for trivial $x = 0$ path and instanton-antiinstanton contributions are extensively discussed e.g. in [10].

Even more general question—whether these trans-series do define uniquely the whole function, representing the path integral dependence on its parameters—is the central issue in rigorous mathematical definition of the QFT's. Related to it is the generalized definition of the path integrals, recently discussed by Witten [11]. No question, still there remain many open questions related even with finite-dimensional integrals. Furthermore, even $(1 + 0)$ dimensional path integrals—quantum mechanical examples under consideration—still include certain open theoretical problems which continue to attract attention of physicists (and mathematicians) today.

In this paper we move from the well-trodden path of tunneling theory into a somewhat different direction. Instead of probability to go through the barrier, we evaluate the *probability* to find a quantum system at a certain position x_0 inside a classically forbidden region. It measures a “strength” of quantum effects, a quantum nature of the problem. In general, this probability is given by path integral in which the end points of the path coincide and are fixed. We will develop a semiclassical theory for this case. The corresponding classical solutions for it we will call *fluctons*, following the old paper of one of us [12] where it was introduced. Another early paper devoted to the subject was that by Rossi and Testa [3].

The paper is organized as follows. In Sec. II the general setting of the problem is explained, and the corresponding classical solutions, the fluctons, are derived in the Sec. III. The next Sec. IV treats quantum oscillations around the classical path to quadratic order, resulting in defining the corresponding determinant in Sec. V and the Green function in Sec. VI.

Somewhat unexpectedly, we found that the quantum-mechanical potential for fluctuations around the flucton in the double-well problem allows an exact analytic solution in elementary functions. Therefore we were able to find analytic expression for the scattering phase and evaluate the determinant via standard integral over its derivative.

Alternative derivation of the determinant is described in Sec. VII, in which its derivative over the coupling is related to a certain Feynman diagram, which is evaluated using the (closed loop) Green function. Agreement of those results shows consistency of the determinant and the Green function. Since this correspondence has never been used in the instanton problem, we discuss the nontrivial sum rule for the Green function following from the determinant value: as shown in Appendix B, the Green function used in our previous works has passed this test.

In Sec. VIII we evaluate two-loop corrections by direct evaluation of the diagrams over the flucton background, with subtracted similar “vacuum diagrams”, fluctuations around the trivial $x(\tau) = 0$ vacuum. Surprisingly, all diagrams yield analytic answers. While the individual diagrams contain logs and polylogs, they all cancel in sum, leading to a rather simple analytic answer [13]. Would this property be true in two- and higher loop contributions: it is interesting open question. Expansion of the results obtained for large displacement x_0 is compared with the known asymptotic expansion of the ground state wave function in Appendix A.

The final Sec. X contains discussion of possible applications to other problems, in quantum mechanics with several variables, statistical mechanics and quantum field theories.

II. GENERAL SETTING

By definition the Feynman path integral gives the density matrix in quantum mechanics [1]

$$\rho(x_i, x_f, t_{\text{tot}}) = \int_{x(0)=x_i}^{x(t_{\text{tot}})=x_f} Dx(t) e^{iS[x(t)]/\hbar}. \quad (1)$$

Here S is the usual classical action of the problem, e.g.

$$S = \int_0^{t_{\text{tot}}} dt \left[\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - V(x) \right],$$

for a particle of mass m in a static potential $V(x)$ provides the weight of the paths in (1). Now let us move from quantum mechanics to statistic mechanics, from quantum system to thermal system, from density matrix to probability. Step one is to rotate time into its Euclidean version $\tau = it$. Step two is to define τ on a circle with circumference $\beta = \tau_{\text{tot}}$. Such periodic time is known as the Matsubara time, and the density matrix of quantum system is related to probability for thermal system with temperature

$$T = \hbar/\beta. \tag{2}$$

Periodicity of the path implies that there is only one end parameter $x_i = x_f = x_0$. The ensemble of such paths represent equilibrium quantum statistical mechanics at temperature T or at $T \rightarrow 0$, the ground state of the quantum system. See details of such setting in [2] and many other sources on statistical field theory.

The main object we will be studying in this paper is the diagonal matrix elements of the density matrix in coordinate representation, giving the probability for the specific coordinate value x_0 (of the field ϕ_0) to be found in this ensemble. The basic expression for it we will use below is a path integral with end points fixed and they coincided

$$P(x_0, \beta) = \int_{x(0)=x_0}^{x(\beta)=x_0} Dx(\tau) e^{-S_E[x(\tau)]/\hbar}; \tag{3}$$

thus, we will consider all (closed) trajectories starting and ending at x_0 , where $S_E = \int_0^\beta d\tau [\frac{m}{2} (\frac{dx}{d\tau})^2 + V(x)]$. There are two basic limits of this expression (3). One is at large β , or low T . Using standard definition of the density matrix in terms of states with definite energy

$$P(x_0, \beta) = \sum_n |\psi_n(x_0)|^2 e^{-E_n \beta}, \tag{4}$$

one sees that this limit P corresponds to the lowest—the ground state

$$P(x_0, \beta \rightarrow \infty) \sim |\psi_0(x_0)|^2. \tag{5}$$

In the opposite case of small β the circle is small, and one can ignore time dependence of the paths. In this limit

$$P(x_0, \beta) \sim e^{-\frac{V(x_0)}{T}}, \tag{6}$$

corresponding to classical thermal distribution in a potential V . Needless to say, the expression is correct for any T .

III. THE CLASSICAL PATHS: FLUCTONS

For pedagogical reasons, we will proceed using particular examples, for which expressions can be simple enough to allow analytic evaluation of all quantities. The main idea is that in Euclidean time the effective potential flips and the classical minimum becomes a maximum. Therefore classical paths with $E = 0$ “slipping down” from a maximum to any point exist.

A. It is hard not to start with the harmonic oscillator, as the first example. One can always select units in which the particle mass $m = 1$ and the oscillator frequency $\omega = 1$, so that our Lagrangian is written as

$$L_E = \frac{\dot{x}(\tau)^2}{2} + \frac{x(\tau)^2}{2}. \tag{7}$$

Note that, for positivity, the Euclidean sign change we apply not to the kinetic, but to the potential term. Anyway, in time τ the oscillator does not oscillate but relaxes, the classical equation of motion (EOM) produces solutions of the kind $e^\tau, e^{-\tau}$. The flucton solution at $E = 0$ on a circle with circumference β can be easily found as their superposition satisfying

$$x(0) = x(\beta) = x_0, \tag{8}$$

namely,

$$x_{\text{flucton}}(\tau) = x_0 \frac{(e^{\beta-\tau} + e^\tau)}{e^\beta + 1} \tag{9}$$

defined for $\tau \in [0, \beta]$. At low T (or large β) it is convenient, due to periodicity in τ , to shift its range to $\tau \in [-\beta/2, \beta/2]$. At zero $T = 1/\beta$ the range becomes infinitely large, and the solution becomes simply $x_0 e^{-|\tau|}$. At high T , on the other hand, the “thermal circle” gets small $\beta \rightarrow 0$, it can be just approximated by x_0 .

The classical action of such a path is

$$S_{\text{flucton}} = x_0^2 \tanh\left(\frac{\beta}{2}\right), \tag{10}$$

it tells us that the particle distribution,

$$P(x_0) \sim \exp\left(-\frac{x_0^2}{\coth(\frac{\beta}{2})}\right), \tag{11}$$

is Gaussian at any temperature. Note furthermore, that the width of the distribution,

$$\langle x^2 \rangle = \frac{1}{2} \coth\left(\frac{\beta}{2}\right) = \frac{1}{2} + \frac{1}{e^\beta - 1}, \tag{12}$$

can be recognized as the ground state energy plus one due to thermal excitation. These results are, of course, very well-known, see e.g. Feynman’s statistical mechanics [2].

B. Our next example is the symmetric powerlike potential

$$V = \frac{g^2}{2} x^{2N}, \quad N = 1, 2, 3, \dots, \tag{13}$$

for which we discuss only the zero temperature $\beta = 1/T \rightarrow \infty$ case. The (Euclidean) classical equation at zero energy $\frac{\dot{x}^2}{2} = V(x)$ has the following solution:

$$x_{\text{fluct}}(\tau) = \frac{x_0}{(1 + g(N-1)x_0^{N-1}|\tau|)^{N-1}}, \quad x_0 > 0, \tag{14}$$

with the action

$$S[x_{\text{fluct}}] = \frac{2gx_0^{N+1}}{N+1}, \quad (15)$$

hence

$$P(x_0) \sim \exp\left(-\frac{2gx_0^{N+1}}{N+1}\right), \quad (16)$$

which is in a complete agreement with WKB asymptotics at $x_0 \rightarrow \infty$ [14].

C. The third example is the anharmonic oscillator potential

$$V = \frac{1}{2}x^2(1 + gx^2), \quad g > 0, \quad (17)$$

at zero temperature $\beta = 1/T \rightarrow \infty$. The classical flucton solution with the energy $E = 0$ is given by

$$x_{\text{fluct}}(\tau) = \frac{\sqrt{g}x_0}{\cosh(\tau) + \sqrt{1 + gx_0^2} \sinh(\tau)}, \quad (18)$$

which leads to the flucton action

$$S[x_0] = \frac{2(1 + gx_0^2)^{\frac{3}{2}} - 1}{3g}. \quad (19)$$

In the limit $g \rightarrow 0$ we recover the action of the harmonic oscillator and at $x_0 \rightarrow \infty$ we obtain

$$S[x_{\text{fluct}}(\tau)] = \frac{2\sqrt{g}}{3}x_0^3 + \frac{1}{\sqrt{g}}x_0 - \frac{2}{3g} + \mathcal{O}\left(\frac{1}{x_0}\right) \quad (20)$$

in complete agreement with the asymptotic expansion of the ground state wave function squared (see Appendix A) [15].

However, for the most detailed studies we select two other examples.

D. One is the *quartic one-dimensional potential*

$$V(x) = \lambda(x^2 - \eta^2)^2, \quad (21)$$

with two degenerate minima. Tunneling between them is described by a well-known instanton solution

$$x_{\text{inst}}(\tau) = \eta \tanh\left(\frac{1}{2}\omega(\tau - \tau_c)\right), \quad (22)$$

assuming that $\omega^2 = 8\lambda\eta^2$. Note that the instanton has an arbitrary time location τ_c , while the flucton does not.

We will discuss both the *weak coupling* limit of small λ , and the *strong coupling* limit of large λ . In fact, the transition between them happens when the instanton action $S[x_{\text{inst}}(\tau)] = 1/12\lambda$ is larger or smaller than one, respectively.

Standard steps are selecting units for η such that $\omega = 1$ and shifting the coordinate by it,

$$x(\tau) = y(\tau) + \eta, \quad (23)$$

so that the potential (21) takes the form

$$V = \frac{y(\tau)^2}{2}(1 + \sqrt{2\lambda}y(\tau))^2, \quad (24)$$

corresponding to harmonic oscillator well at small y .

The *flucton* solution, the minimal action path for the path integral (3), in which the path is forced to pass through the point x_0 at $\tau = 0$ now takes the form

$$y_{\text{fluct}}(\tau) = \frac{x_0}{e^{|\tau|(1 + \sqrt{2\lambda}x_0)} - \sqrt{2\lambda}x_0}. \quad (25)$$

We remind that in zero T case, or infinite circle $\beta \rightarrow \infty$, $\tau \in (-\infty, \infty)$, and the solution exponentially decreases to both infinities, see Fig. 1. Its generalization to finite T is straightforward.

The action of this solution is

$$S[y_{\text{fluct}}] = x_0^2 \left(1 + \frac{2\sqrt{2\lambda}x_0}{3}\right), \quad (26)$$

and thus in the leading semiclassical approximation the probability to find the particle at x_0 takes the form

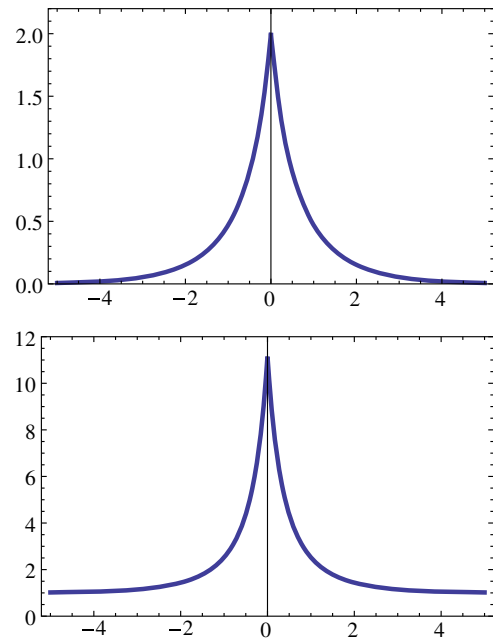


FIG. 1. Time dependence of the classical flucton solution $y_{\text{fluct}}(\tau)$, see (25) (upper plot) and the corresponding potential $(1 + W)$, see (37) of the fluctuations (lower plot), both for $x_0 = 2$, $\lambda = 0.1$.

$$P(x_0) \sim \exp\left(-x_0^2 - \frac{2\sqrt{2\lambda}}{3}x_0^3\right). \quad (27)$$

In the weak coupling limit only the first term remains, corresponding to the Gaussian ground state wave function of the harmonic oscillator. In the strong coupling limit the second term is dominant, and the distribution then corresponds to a well-known cubic dependence on the coordinate. These classical-order results are of course the same as one gets from a standard WKB approximation.

When $|x_0| < \eta$ the classical flucton solution can be constructed from the pieces of the instanton and anti-instanton solutions. In this region, fluctons, instantons and anti-instantons are distinct classical paths, all contributing to the path integral (3).

E. Our last example is the *sine-Gordon potential*

$$V = \frac{1}{g^2}(1 - \cos(gx)), \quad (28)$$

with infinite number of degenerate vacua. Tunneling between adjacent vacua is described by the well-known instanton solution

$$x_{\text{inst}}(\tau) = \frac{4}{g} \arctan(e^\tau). \quad (29)$$

In the zero temperature case, or very large circle $\beta \rightarrow \infty$, the flucton solution has a very simple form

$$x_{\text{fluct}}(\tau) = \frac{4\text{arccot}[e^\tau \cot(\frac{gx_0}{4})]}{g}. \quad (30)$$

The classical action for this solution is

$$S[x_{\text{fluct}}] = \frac{16 \sin^2(\frac{gx_0}{4})}{g^2}, \quad (31)$$

and, thus, in the leading semiclassical approximation the probability to find the particle at x_0 takes the form

$$P(x_0) \sim \exp\left[-\frac{16 \sin^2(\frac{gx_0}{4})}{g^2}\right]. \quad (32)$$

IV. FLUCTUATIONS AROUND THE CLASSICAL PATH

Now we turn to quantum fluctuations around the classical path

$$y(\tau) = y_{\text{fluct}}(\tau) + f(\tau). \quad (33)$$

Let us put this expression into the action and expand it to the needed order in f . But before we do so, let us remind the reader that, by the definition, all paths should pass

through the same point at $\tau = 0$ and, thus, there is an important condition

$$f(0) = 0, \quad (34)$$

which is absent in the instanton case. Since the classical path is a local minimum of the action, therefore there is no term $O(f^1)$. Small fluctuations are described by the Lagrangian

$$L = \frac{\dot{f}(\tau)^2}{2} + V''(y_{\text{fluct}}) \frac{f(\tau)^2}{2} + O(f^3), \quad (35)$$

where we used a short hand notations $V''(y_{\text{fluct}}) = \partial^2 V(y)/\partial y^2|_{y=y_{\text{fluct}}}$. Its variation leads to a Schrödinger-like equation with the potential V'' .

For a harmonic oscillator this potential V'' is just a constant, so in this case the fluctuations do not depend on the classical path. Higher order derivatives of V all vanish, hence, in this case all fluctuations are just Gaussian.

For quartic double-well potential for the famous classical solution $x_{\text{inst}}(t)$ (22), the *instanton*, the potential entering in (35) has the well-known form

$$V''(y_{\text{fluct}}) = \omega^2 \left(1 - \frac{3}{2 \cosh^2(\omega\tau/2)}\right). \quad (36)$$

This potential is one of few exactly solvable quantum mechanical problems. There are two bound states, the famous zero mode with eigenvalue zero and another state with eigenvalue $\frac{3}{4\omega^2}$, as well as the continuum of unbound states with eigenvalue above ω^2 . Since one has the analytic expression for the scattering phase δ_p , the determinant has been evaluated so to say "by definition", using a complete set of states, for a review see e.g. [6]. A new relation between the determinant and the Green function for the instanton we will discuss in Appendix B.

In the case of the flucton classical solution (25) the potential of the fluctuations we put into the form

$$V''(y_{\text{fluct}}) = 1 + W,$$

where

$$W = \frac{6X(1+X)e^{|\tau|}}{(e^{|\tau|} - X + Xe^{|\tau|})^2}. \quad (37)$$

The classical path depends on three parameters of the problem, λ , x_0 and ω (which we already put to 1): but in W the first two appear in one combination only

$$X \equiv x_0 \sqrt{2\lambda}. \quad (38)$$

This observation will be important in Sec. V.

An example of $(W + 1)$ is shown in Fig. 1 (lower plot). Note that W exponentially decreases at large τ .

In the sine-Gordon case the potential of the fluctuations has the following form:

$$V''(y_{\text{fluct}}) = \frac{1}{(1 + e^{-2\tau} \tan^2(\tilde{X}))^2} \times [1 + e^{-4\tau} \tan^4(\tilde{X}) - 6e^{-2\tau} \tan^2(\tilde{X})], \quad \tau > 0, \quad (39)$$

where the relevant combination of parameters is

$$\tilde{X} \equiv \frac{gx_0}{4}, \quad (40)$$

cf. (38).

V. THE FLUCTON DETERMINANT

The operator governing quadratic around flucton is

$$Of \equiv -\ddot{f}(\tau) + V''(y_{\text{fluct}})f(\tau), \quad (41)$$

where the derivative has already been described above (37). At large $|\tau|$ the nontrivial part of the potential disappears and solutions have a generic form

$$\psi_p(\tau) \sim \sin(p\tau + \delta_p), \quad (42)$$

where for momentum p , only the scattering phase δ_p depends on the potential. The eigenvalues of the operator O are, for the double well example (37), simply,

$$\lambda_p = 1 + p^2, \quad (43)$$

and the determinant $\text{Det}O$ is their infinite product. Its logarithm is the sum

$$\log \text{Det}O = \sum_n \log(1 + p_n^2), \quad (44)$$

where the sum is taken over all states satisfying zero boundary condition on the boundary of some large box.

Taking the path integral over fluctuations around the classical path, in the Gaussian approximation, leads to the following standard expression:

$$P(x_0) = \frac{\exp(-S[x_{\text{flucton}}])}{\sqrt{\text{Det}(O_{\text{flucton}})}} \times [1 + O(\text{two and more loops})], \quad (45)$$

with $O_{\text{flucton}} = O$ defined in (41). In this section we discuss numerical evaluation of the determinant: another method will be discussed in the Sec. V, after we will derive the corresponding Green function for the fluctuations in Sec. VI. Calculation of two and more loop corrections via Feynman diagrams will be discussed in Sec. VIII.

As it is well-known, the nontrivial part of the problem is not in the eigenvalues themselves, but in the counting of levels. Standard method (see e.g. § 77 of [16]) vanishing boundary conditions at the boundary of some large box, at $\tau = L$, leads to

$$p_n L + \delta_{p_n} = \pi n, \quad n = 1, 2, \dots \quad (46)$$

At large L and n one can replace summation to an integral, resulting in the generic expression

$$\log \text{Det}O = \sum_n \log(1 + p_n^2) = \int_0^\infty \frac{dp}{\pi} \frac{d\delta_p}{dp} \log(1 + p^2). \quad (47)$$

After using a few different numerical methods for particular values of the parameter X , we discovered that there exists an *exact* (non-normalized) analytic solution for the eigenfunctions of the operator (41) in the form

$$\psi_p(\tau) \sim \sin(p\tau + \Delta(p, \tau))F(p, \tau), \quad (48)$$

with the following two functions

$$\Delta(p, \tau) = \arctan \left[\frac{-3p(1 + 2X)}{1 - 2p^2 + 6X + 6X^2} \right] + \arctan \left[\frac{N}{D} \right],$$

where

$$\begin{aligned} N &= 3p[1 + 2X + X^2 - X^2 e^{-2\tau}], \\ D &= (2p^2 - 1)(1 + X^2) - 2e^{-\tau}(2(1 + p^2) - e^{-\tau}(2p^2 - 1))X + (2p^2 - 1)e^{-2\tau} - 4e^{-\tau}(1 + p^2), \\ F(p, \tau) &= \frac{1}{(e^\tau - X + e^\tau X)^2} \times [e^{4\tau}(1 + 5p^2 + 4p^4) + 4e^{3\tau}(1 + p^2)(2 - 4p^2 + e^\tau(1 + 4p^2))X \\ &\quad + 6e^{2\tau}(3 + p^2 + 4p^4 + 4e^\tau(1 - p^2 - 2p^4) + e^{2\tau}(1 + 5p^2 + 4p^4))X^2 + 4e^\tau(2(1 - p^2 - 2p^4) \\ &\quad + 6e^{2\tau}(1 - p^2 - 2p^4) + 3e^\tau(3 + p^2 + 4p^4) + e^{3\tau}(1 + 5p^2 + 4p^4))X^3 + (1 + 5p^2 + 4p^4 \\ &\quad + 8e^\tau(1 - p^2 - 2p^4) + 8e^{3\tau}(1 - p^2 - 2p^4) + 6e^{2\tau}(3 + p^2 + 4p^4) + e^{4\tau}(1 + 5p^2 + 4p^4))X^4]^{1/2}. \end{aligned}$$

It is important that at $\tau = 0$ the solution (48) goes to zero: according to the flucton definition, all fluctuations at this point must vanish (34). It is the condition which fixes the scattering phase.

At large time, where all terms with decreasing exponents in $\Delta(p, \tau)$ disappear and the remaining constant terms define the scattering phase, we need

$$\delta_p = \arctan \left[\frac{3p(1+2X)}{1-2p^2+6X+6X^2} \right] - \arctan \left[\frac{3p}{1-2p^2} \right]. \quad (49)$$

Comments:

- (i) the scattering phase is $O(p)$ at small p ;
- (ii) it is $O(1/p)$ at large p and, thus, there must be a maximum at some p ;
- (iii) for $X = 0$ two terms in (49) cancel out. This needs to be the case since in this limit the nontrivial potential W of the operator (41) also disappears;
- (iv) at large time the amplitude F (48) goes to a constant, as it should.

The arctan-function provides an angle, defined modulo the period, and thus it experiences jumps by π . Fortunately, its derivative $d\delta_p/dp$ entering the determinant (47) is single-valued and smooth. The momentum dependence of the integrand of this expression for $X = 4$ is shown in Fig. 2(a). Analytic evaluation of the integral (47) was not successful, the results of the numerical evaluation are shown by points in Fig. 2(b). However, the *guess* $2 \log(1+X)$, shown by the curve in Fig. 2(b) happens to be accurate to numerical accuracy, and thus it must be correct. We will demonstrate that it is exact below.

Since the calculation above includes only a half of the time line, $\tau > 0$, and the other half is symmetric, the complete result for the $\log \text{Det} O$ should be doubled. Substituting (49) to (47) we obtain a (surprisingly simple) exact result

$$\text{Det}(O) = (1+X)^4. \quad (50)$$

Note that at $X = 0$ we return to the harmonic oscillator case.

VI. THE GREEN FUNCTION OF THE FLUCTUATIONS AROUND THE FLUCTON SOLUTION

The general procedure for the inversion of the operator (41), leading to a Green function, is different for the instanton

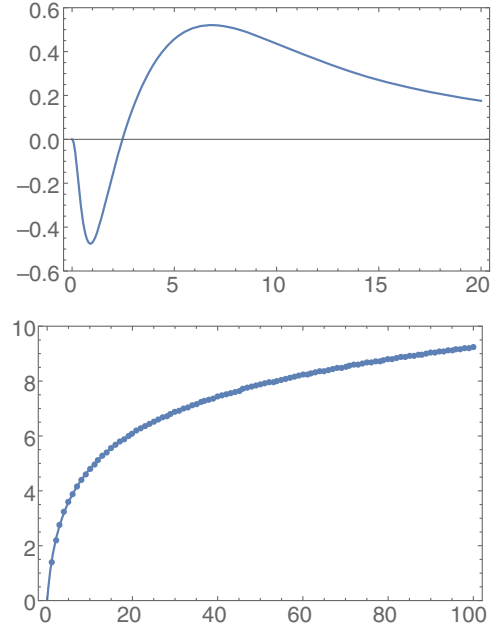


FIG. 2. The integrand of (47), $\log(1+p^2)d\delta_p/dp$, versus p , for $X = 4$, see upper plot. The integral (47) vs parameter X : points are numerical evaluation, line is defined in the text, see lower plot.

and flucton cases. In the instanton case the inversion is only possible in the subspace normal to the zero mode, leading to specific difficulties. The flucton problem we discuss now has no shift symmetry (no translation invariance) and thus no zero modes. Needless to say that this symmetry is killed by the boundary condition at the fixed moment, $f(\tau = 0) = 0$.

The corresponding equation to be solved thus is

$$-\frac{\partial^2 G(\tau_1, \tau_2)}{\partial \tau_1^2} + V''(y_{\text{fluct}}(\tau_1))G(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2). \quad (51)$$

The homogeneous equation (with zero rhs) has two solutions

$$f_0(\tau) = \frac{e^\tau}{(e^\tau(1+X) - X)^2}, \quad (52)$$

and

$$f_1(\tau) = \frac{e^{-\tau}}{2(X - e^\tau(1+X))^2} (8X^3(1+X)e^\tau + 12X^2(1+X)^2\tau e^{2\tau} - 8X(1+X)^3e^{3\tau} + (1+X)^4e^{4\tau} - X^4). \quad (53)$$

(Hereafter we only discuss the half line $\tau > 0$). The first solution—would be zero mode if shift be allowed—is exponentially decreasing at large time, the second one is increasing in time. Standard construction immediately yields the following Green function

$$\begin{aligned}
 G(\tau_1, \tau_2) = & \frac{e^{-|\tau_1 - \tau_2|}}{2(e^{\tau_1}(1+X) - X)^2(e^{\tau_2}(1+X) - X)^2} [8e^{\frac{1}{2}(\tau_1 + \tau_2 + 3|\tau_1 - \tau_2|)} X^3(1+X) \\
 & - 8e^{\frac{1}{2}(3\tau_1 + 3\tau_2 + |\tau_1 - \tau_2|)} X(1+X)^3 + e^{2(\tau_1 + \tau_2)}(1+X)^4 - 6e^{(\tau_1 + \tau_2 + |\tau_1 - \tau_2|)} X^2(1+X)^2 |\tau_1 - \tau_2| \\
 & + e^{(\tau_1 + \tau_2 + |\tau_1 - \tau_2|)} (6X^4(\tau_1 + \tau_2) + 12X^3(1 + \tau_1 + \tau_2) + 6X^2(3 + \tau_1 + \tau_2) + 4X - 1) - e^{2|\tau_1 - \tau_2|} X^4], \quad (54)
 \end{aligned}$$

for $\tau_1, \tau_2 > 0$. Similarly, in the sine-Gordon problem the same standard construction yields the following Green function:

$$\begin{aligned}
 G(\tau_1, \tau_2) = & \frac{1}{8(\cosh(\tau_1) + \cos(2\tilde{X}) \sinh(\tau_1))} \times \frac{1}{(\cosh(\tau_2) + \cos(2\tilde{X}) \sinh(\tau_2))} \\
 & \times \left[2(\tau_1 + \tau_2 - |\tau_2 - \tau_1|) \sin^2(2\tilde{X}) + 8 \cos(2\tilde{X}) \sinh^2\left(\frac{1}{2}(\tau_1 + \tau_2 - |\tau_2 - \tau_1|)\right) \right. \\
 & \left. + (3 + \cos(4\tilde{X})) \sinh(\tau_1 + \tau_2 - |\tau_2 - \tau_1|) \right], \quad (55)
 \end{aligned}$$

for $\tau_1, \tau_2 > 0$.

VII. RELATING THE DETERMINANT AND THE GREEN FUNCTION

The method we will use in this section relies on the following observation. When the fluctuation potential depends on some parameter, it can be varied. In the case at hand (37), the potential we write as

$$V_{\text{flucton}} = 1 + W(X, \tau)$$

depends on the combination (38). Its variation resulting in extra potential

$$\delta V_{\text{flucton}} = \frac{\partial W}{\partial X} \delta X \quad (56)$$

is a perturbation: its effect can be evaluated by the following Feynman diagram:

$$\frac{\partial \log \text{Det}(O_{\text{flucton}})}{\partial X} = \int d\tau G(\tau, \tau) \frac{\partial V_{\text{flucton}}(\tau)}{\partial X}, \quad (57)$$

containing derivative of the potential as a vertex and the “loop”, the same point Green function, at $\tau_1 = \tau_2 = \tau$, see Fig. 3. This relates the determinant and the Green function [17]: if the rhs of it can be calculated, the derivative over X can be integrated back.

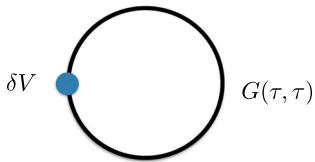


FIG. 3. Symbolic one-loop diagram, including variation of the fluctuation potential δV and the simplified “single-loop” Green function $G(\tau, \tau)$.

In the quartic double-well problem the “Green function loop” propagator is

$$\begin{aligned}
 G(\tau, \tau) = & \frac{1}{2(X - e^\tau(1+X))^4} \\
 & \times (-X^4 + 8e^\tau X^3(1+X) - 8e^{3\tau} X(1+X)^3 \\
 & + e^{4\tau}(1+X)^4 + e^{2\tau}(-1 + 4X + 18X^2 + 12X^3 \\
 & + 12X^2(1+X)^2\tau), \quad (58)
 \end{aligned}$$

and the “vertex”

$$\frac{\partial V_{\text{flucton}}(\tau)}{\partial X} = \frac{6e^\tau(X + e^\tau(1+X))}{(-X + e^\tau(1+X))^3}. \quad (59)$$

With these expressions one can evaluate the rhs of the relation (57), and adding the same expression for negative time, one gets the result

$$\frac{\partial \log \text{Det}(O_{\text{flucton}})}{\partial X} = \frac{4}{1+X}, \quad (60)$$

which exactly agrees with the result (50) from the direct evaluation of the determinant using the phase shift. So, the Green function has passed a very nontrivial test, and we conclude that it is ready to be used for evaluation of two and higher loop diagrams.

In the sine-Gordon problem the corresponding simplified expression for $G(\tau, \tau)$ is

$$\begin{aligned}
 G(\tau, \tau) = & \frac{1}{4(1 + e^{2\tau} + (e^{2\tau} - 1) \cos(2\tilde{X}))^2} \\
 & \times [4(e^{2\tau} - 1)^2 \cos(2\tilde{X}) - \cos(4\tilde{X}) \\
 & + e^{4\tau}(3 + \cos(4\tilde{X})) + 8e^{2\tau}\tau \sin^2(2\tilde{X}) - 3]. \quad (61)
 \end{aligned}$$

Evaluating the one-loop diagram Fig. 3, we arrive at the result

$$\log \text{Det}(O_{\text{flucton}}) = 4 \tan[\tilde{X}]. \quad (62)$$

For the powerlike potential (13) the ‘‘Green function loop’’ takes the form

$$G(\tau, \tau) = \frac{-1 + (N - 1)X_1\tau + (1 + X_1(1 - N)\tau)^{\frac{2N}{1-N}}}{(3N - 1)X_1}, \quad (63)$$

where

$$X_1 = gx_0^{N-1},$$

and the ‘‘vertex’’ reads

$$\frac{\partial V_{\text{flucton}}(\tau)}{\partial X_1} = \frac{2(2N - 1)NX_1}{((N - 1)\tau X_1 - 1)^3}, \quad (64)$$

($\tau < 0$). Hence, we obtain the result

$$\log \text{Det}(O_{\text{flucton}}) = \frac{2N}{N - 1} \log X_1.$$

In the case of the anharmonic oscillator (17) the ‘‘Green function loop’’ is

$$G(\tau, \tau) = \frac{(\sinh(\tau) + \cosh(\tau)X_2)}{4X_2(\cosh(\tau) + \sinh(\tau)X_2)^4} \times [-6\tau X_2(\sinh(\tau) + \cosh(\tau)X_2)(-1 + X_2^2) + \sinh(\tau)(4 + X_2[\sinh(2\tau) + 3(-3 + \cosh(2\tau))X_2 + 3 \sinh(2\tau)X_2^2 + (5 + \cosh(2\tau))X_2^3])], \quad (65)$$

where

$$X_2 = \sqrt{1 + gx_0^2},$$

while the ‘‘vertex’’ is given by

$$\frac{\partial V_{\text{flucton}}(\tau)}{\partial X_2} = \frac{12(\sinh(\tau) + \cosh(\tau)X_2)}{(\cosh(\tau) + \sinh(\tau)X_2)^3}, \quad (66)$$

($\tau > 0$). Thus,

$$\log \text{Det}(O_{\text{flucton}}) = 2 \log[X_2(1 + X_2)].$$

VIII. HIGHER ORDER FEYNMAN DIAGRAMS

Now, using only the tools from quantum field theory, the Feynman diagrams in the flucton background, we compute the two-loop correction to the density matrix (45) for the

double-well potential. In principle, the higher order diagrams are evaluated by standard rules.

Unlike the calculations near the instanton solution [7,8], in the case of flucton there are no zero modes and related Jacobian, so all diagrams follow from the Lagrangian. In the quartic double-well potential, the flucton-based Green function (54) was determined above, and the only vertices are triple and quartic ones

$$v_3(\tau) = \frac{6\sqrt{2\lambda}(X + e^\tau(1 + X))}{-X + e^\tau(1 + X)}, \quad (67)$$

$$v_4 = 24\lambda. \quad (68)$$

The loop corrections in (45) are written in the form

$$[1 + O(\text{two and more loops})] = 2 \sum_{n=0}^{\infty} B_n \lambda^n, \quad B_0 = \frac{1}{2},$$

where $B_n = B_n(X)$. Like in the calculations near the instanton solution, we need to separate the finite flucton-related contribution for each diagram from the infinite (time-divergent) contribution without it. This is done by subtracting the same expression with ‘‘vacuum vertices’’

$$v_{3,0} = 6\sqrt{2\lambda}, \quad (69)$$

$$v_{4,0} = 24\lambda, \quad (70)$$

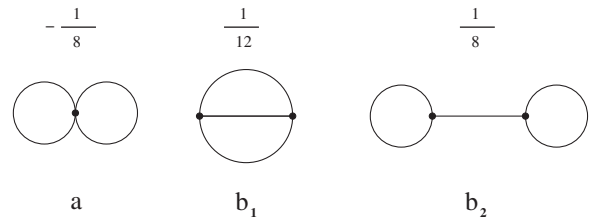


FIG. 4. Diagrams contributing to the two-loop correction $B_1 = a + b_1 + b_2$. The signs of contributions and symmetry factors are indicated.

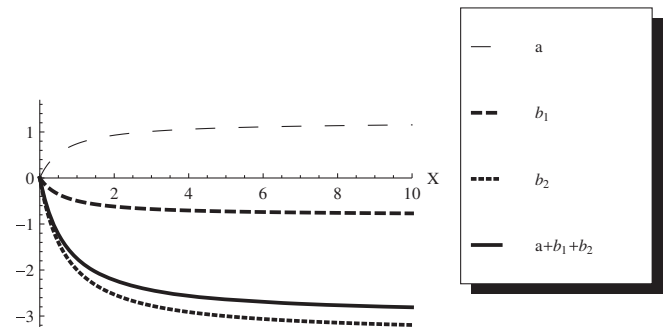


FIG. 5. The two-loop diagrams a, b_1, b_2 and the two-loop correction $B_1 \equiv a + b_1 + b_2$ as a function of variable X (38).

and the ‘‘vacuum propagator’’

$$G_0 = G(\tau_1, \tau_2) \Big|_{X \rightarrow 0} = \frac{e^{-|\tau_1 - \tau_2|}}{2} - \frac{e^{-\tau_1 - \tau_2}}{2}. \quad (71)$$

[Note that (71) differs from the vacuum propagator in the instanton problem where the second term in the rhs is

$$\begin{aligned} a \equiv & -\frac{1}{8\lambda} v_4 \int_0^\infty [G^2(\tau, \tau) - G_0^2(\tau, \tau)] d\tau = \frac{3}{560X^2(1+X)^4} \\ & \times \left(24X - 60X^2 - 520X^3 - 1024X^4 - 832X^5 - 245X^6 + 24(1+X)^2(1+2X)(-1+6X(1+X)) \log(1+X) \right. \\ & \left. + 288X^2(1+X)^4 \text{PolyLog} \left[2, \frac{X}{1+X} \right] \right), \end{aligned} \quad (72)$$

here $\text{PolyLog}[n, z] = \sum_{k=1}^\infty z^k/k^n$ is the polylogarithm function and

$$\begin{aligned} b_1 \equiv & \frac{1}{12\lambda} \int_0^\infty \int_0^\infty [v_3(\tau_1)v_3(\tau_2)G^3(\tau_1, \tau_2) - v_{3,0}v_{3,0}G_0^3(\tau_1, \tau_2)] d\tau_1 d\tau_2 \\ & = \frac{1}{280X^2(1+X)^4} \times \left(-24X + 60X^2 + 520X^3 + 1024X^4 + 832X^5 + 245X^6 \right. \\ & \left. + 24(1+X)^2(1-4X-18X^2-12X^3) \log(1+X) - 288X^2(1+X)^4 \text{PolyLog} \left[2, \frac{X}{1+X} \right] \right), \end{aligned} \quad (73)$$

$$\begin{aligned} b_2 \equiv & \frac{1}{8\lambda} \int_0^\infty \int_0^\infty [v_3(\tau_1)v_3(\tau_2)G(\tau_1, \tau_1)G(\tau_1, \tau_2)G(\tau_2, \tau_2) - v_{3,0}v_{3,0}G_0(\tau_1, \tau_1)G_0(\tau_1, \tau_2)G_0(\tau_2, \tau_2)] d\tau_1 d\tau_2 \\ & = -\frac{1}{560X^2(1+X)^4} \times \left(24X - 60X^2 + 1720X^3 + 5136X^4 + 4768X^5 + 1435X^6 \right. \\ & \left. + 24(1+X)^2(-1+4X+18X^2+12X^3) \log(1+X) + 288X^2(1+X)^4 \text{PolyLog} \left[2, \frac{X}{1+X} \right] \right). \end{aligned} \quad (74)$$

Eventually, the two-loop correction takes an amazingly simple form,

$$B_1 \equiv a + b_1 + b_2 = -\frac{X(4+3X)}{(1+X)^2}, \quad (75)$$

all log and PolyLog terms disappear. The results of calculations are shown on Fig. 5.

IX. SUMMARY

The combined results for the probability to find a particle at point x_0 in the quartic double-well potential at zero temperature is

$$P(x_0) \sim \frac{e^{-\frac{\lambda^2}{2\lambda} - \frac{\lambda^3}{3\lambda}}}{(1+X)^2} \left(1 - \lambda \frac{X(4+3X)}{(1+X)^2} + O(\lambda^2) \right), \quad (76)$$

absent. In particular, it is no longer translational invariant because of the extra boundary condition at $\tau_1 = \tau_2 = 0$ for fluctuations at the fixed point x_0 .]

The two-loop correction B_1 we are interested in can be written as the sum of three diagrams, see Fig. 4, diagram a which is a one-dimensional integral and diagrams b_1 and b_2 corresponding to two-dimensional ones.

Explicitly, we have

where, we remind $X = \sqrt{2\lambda}x_0$. Note that $X = -1$ is indeed a singularity of the potential in the unphysical domain.

The x_0 dependence of (76) is plotted in Fig. 6 by the thick line. The thin line is asymptotics derived in Appendix A: since x_0 -independent constant remained unknown we

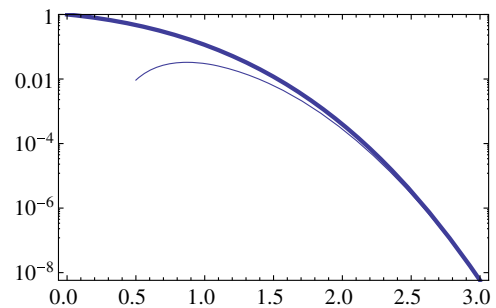


FIG. 6. The probability $P(x_0)$ to find particle at location x_0 for $\lambda = 0.1$. The thick line is our result (76), thin line is asymptotics derived in Appendix A.

normalized it to our curve at large distances. Their comparison shows good agreement for $x_0 > 1$.

Although derived semiclassically, and thus formally valid for large flucton action only, our answer is also obviously correct at small x_0 , where it merges with the answer for a harmonic oscillator.

For brevity of the paper we only calculated $P(x_0)$ at coordinates outside the two minima. If the barrier is sufficiently large, semiclassical calculation of this probability can also be extended to the region between the minima. In this case, as noted already in [12], there are four distinct classical trajectories going through a point: instanton, anti-instanton, and two different fluctons, relaxing to the left and right minima. Since contributions of those are additive and have different actions, the probability should be then written as their respective sums.

For completeness we present the corresponding probability $P(x_0)$ in the case of powerlike potential (13),

$$P(x_0) \sim \frac{\exp[-\frac{2g|x_0|^{N+1}}{N+1}]}{|x_0|^N}, \quad (77)$$

and the anharmonic oscillator case (17)

$$P(x_0) \sim \frac{\exp[-\frac{2}{3}\frac{X_2^3-1}{g}]}{X_2(1+X_2)}, \quad X_2 = \sqrt{1+gx_0^2}, \quad (78)$$

and the sine-Gordon potential

$$P(x_0) \sim \frac{\exp[-16\frac{\sin^2(\tilde{X})}{g^2}]}{\exp[2\tan(\tilde{X})]}, \quad \tilde{X} = \frac{gx_0}{4}. \quad (79)$$

X. DISCUSSION

In this section we discuss, without a detailed solution, several potential approximations of the developed method.

The simplest extension of what was already done is to QM/SM multidimensional problems in the cases in which the variables can be separated. For example, multidimensional motion with a spherically symmetric potential $V(r)$, $r^2 = \sum_i x_i^2$, the flucton is a classical path along the radial direction, easily calculated from energy conservation (as in the 1D case). However, in this case also one can use WKB or similar approximations as well.

The next problem can be multidimensional anharmonic oscillators, perturbed say by certain cubic and quartic terms. Physical problems of this type are e.g. vibrational states of the multiatomic molecules. Let us further note that the additional appeal of our approach is the fact that in a realistic case the temperature T and oscillation quanta $\hbar\omega_i$ are often comparable: so a generalization to finite- T fluctons is fully needed. Of course, in this case energy conservation is not enough, and a flucton solution should be found from the second-order equations of motion (in

Euclidean time) numerically. This solution should start at a given point of interest x_0 and relax to the bottom of the potential (the classical vacuum).

An example in which there are additional integrals of motion can be interesting to study as well. For example, an electron in a field of two static Coulomb centers has nontrivial integral of motion containing the “velocities” (first order time derivatives) of coordinates. Since the motion is in a plane, using the energy and this integral should allow us to use the first order equations and perhaps find the flucton solution.

One would also like to promote the semiclassical method to QFT. In this case the path integral in d -dimensions has a “boundary value” at $\tau = 0$ in the form of some $(d-1)$ -dimensional field configurations $\phi(0, \vec{x})$.

To give a simple example, consider a scalar field with $\lambda\phi^4/4!$ theory in four dimensions. A flucton can be made of two pieces of Fubini-Lipatov “instantons” [22,23], shifted away from the $\tau = 0$ plane by some equal distance $T/2$

$$\phi_{\text{inst}} = 4\sqrt{3}\lambda \frac{\rho}{x^2 + \rho^2}, \quad (80)$$

$$\phi_{\text{flucton}}(x) = \theta(\tau)\phi_{\text{inst}}(\tau + T/2) + \theta(-\tau)\phi_{\text{inst}}(\tau - T/2). \quad (81)$$

A configuration at the plane in a certain spherically symmetric “bump”, with a width ρ and a height $1/\rho$, and the corresponding action is $O(1/\lambda)$.

The fluctuations around classical fluctons are described by the quadratic form operator

$$O_D\phi \equiv -\partial^m\partial_m\phi + V''(y_{\text{fluct}})\phi(x), \quad (82)$$

which includes a Laplacian in all d dimensions. In cases (like the one just mentioned) when the flucton is locally made of instanton solutions, the operator is locally the same, but one should keep in mind that the corresponding Green functions are very different. Indeed, instantons have zero modes and issues related to the orthogonality to those, while in the flucton case the configuration is fixed at $\tau = 0$ and thus no zero modes are present. (Therefore, one should not use known instanton Green functions.)

Finally, let us remind the main flucton idea, applied for any theory. At $\tau = 0$ one may put any $d-1$ dimensional field configuration, and then complement it by a d -dimensional classical solution, leading to it from the classical vacuum. The exponent of its action provides the probability of the configuration to appear in a quantum (or thermal) ensemble. Of course, in many cases, the flucton can be constructed from pieces of known classical solutions, for example out of pieces of an instanton and an anti-instanton.

In gauge theories the coordinates are gauge fields $A_\mu(x)$, and the $(d-1)$ -dimensional field configuration in question should better possess some special properties, which would

be gauge independent. An example of a topologically important $(d-1)$ -dimensional property can be the famous Chern-Simons number N_{CS} . Specifically, the so-called *sphalerons* [24] are solutions with $N_{CS} = 1/2$: there is a large literature devoted to the calculations of the probability of its occurrence in quantum or thermal ensembles. We think its calculation by the flucton method would be of interest.

The fluctons should not be confused with other d -dimensional paths leading to sphalerons. In particular, in [25] such paths have been constructed via some four-dimensional instanton-antiinstanton “streamline” configurations. This construction is *different* from fluctons, because the four-dimensional instanton-antiinstanton configuration used satisfies a “streamline equation” different from classical Yang-Mills equations of motion. It can be called “forced tunneling”, and its objective is to produce $d-1$ configurations with zero kinetic energy, called the “turning points” configurations, ready to be continued into a Minkowski time.

We hope to be able to address some of these problems in detail in our subsequent publications.

APPENDIX A: ASYMPTOTICS AT LARGE X

While in the rest of the paper we only apply tools available in QFT settings, that is path integrals, in this appendix we still return to the Schrödinger equation, which has the form (here as in the text $\hbar = 1, m = 1$)

$$\left(-\frac{1}{2}\partial_y^2 + V(y) - E\right)\Psi = 0, \quad \partial_y \equiv \frac{d}{dy}, \quad (\text{A1})$$

where the double-well potential in shifted coordinates we use is

$$V(y) = \frac{y^2}{2} + \sqrt{2\lambda}y^3 + \lambda y^4. \quad (\text{A2})$$

Note that it smoothly goes to the harmonic oscillator at $\lambda \rightarrow 0$. Introducing the phase $\phi(y) = -\log \Psi(y)$ we move to the Riccati equation,

$$\partial_y^2 \phi - (\partial_y \phi)^2 = 2E - 2V(y), \quad (\text{A3})$$

to which one can plug the asymptotic expansion at $|y| \rightarrow \infty$ and obtain all the coefficients (cf. [26])

$$\begin{aligned} \phi &= \frac{1}{3}\sqrt{2}\sqrt{\lambda}|y|^2 + \frac{1}{2}y^2 - d \log |y|^2 \\ &+ \frac{1+2E}{2\sqrt{2\lambda}}\frac{1}{|y|} - \frac{1}{8\lambda y^2} + \dots, \end{aligned} \quad (\text{A4})$$

where $d = 1/2$. The first two terms in the expansion are classical coming from classical Hamilton-Jacobi equation, the log-term reflects an intrinsic property of the Laplacian:

y is zero mode or kernel, this term comes from the determinant, asymptotically the determinant behaves like $|y|^2$, where d is degree with which it enters to the wave function. Note that a constant, $O(x_0^0)$ term is absent: it can not be obtained from the Riccati equation containing derivatives only. Note also that so far the energy remains undefined: to find it one needs to solve the equation to all x . The last terms are true quantum corrections, decreasing at large distances. Intrinsically, this expansion corresponds to the ground state: it implies that the eigenphase ϕ has no logarithmic singularities at real y . Quantization for the Riccati equation implies a search for solutions growing at large y with a finite number of logarithmic singularities at a real finite y . For the n th excited state the first two growing terms in (A4) remains unchanged while the log-term gets an integer coefficient, $(n+1)\log|y|$, see [26].

Multiplying by 2 (the path integral is for the density matrix, or wave function squared) one finds, as expected, that the first two terms coincide with the classic action of the flucton. For the determinant one needs to expand at large x_0

$$\log(1 + \sqrt{2\lambda}x_0) = \log(x_0) + \log(\sqrt{2\lambda}) + \frac{1}{\sqrt{2\lambda}x_0} + \dots, \quad (\text{A5})$$

and observe that the leading term agrees with the $\log|y|$ term in the asymptotic expansion (A4).

The two-loop correction $B_1\lambda$ found in the text (75) expands in inverse powers of x_0 as follows

$$-\lambda \frac{X(4+3X)}{(1+X)^2} = -3\lambda + \frac{\sqrt{2\lambda}}{x_0} + \dots, \quad (\text{A6})$$

where $X = \sqrt{2\lambda}x_0$, see (38).

In order to compare the $1/x_0$ terms in the last two expressions one needs to substitute the ground state energy to $O(\lambda)$ accuracy

$$E = \frac{1}{2} - 2\lambda + \dots, \quad (\text{A7})$$

to the $O(\frac{1}{x_0})$ term in (A4). After that one finds agreement with both $O(\frac{1}{x_0})$ terms given in (A5) and (A6).

Finally, let us add a comment about the WKB expression, in which the semiclassical wave function has in front $1/\sqrt{p(x)}$ where p is momentum. While at large x its leading asymptotics is correct, as well as that of our determinant, the WKB one has an unphysical singularity at the turning point. Our determinant, on the other hand, is a smooth function of x_0 , and it correctly reproduces the fluctuations till small x_0 , where it joins with the harmonic oscillator behavior.

APPENDIX B: THE SUM RULE FOR THE INSTANTON GREEN FUNCTION

While the instanton path (22) depends on both parameters of the quartic potential, λ and ω , in the corresponding fluctuation potential (36) the coupling constant λ drops out. As a result, the spectrum and thus the determinant does not depend on λ . Therefore, the method we used in the main text to calculate the determinant via its derivative over λ cannot be used.

Nevertheless one can still differentiate the determinant over the remaining parameter ω . As we will show below, it produces a nontrivial sum rule for the Green function.

The basis for the sum rule is the relation

$$\frac{\partial}{\partial \omega^2} \log \text{Det}'(O_{\text{instanton}}) = - \int d\tau G(\tau, \tau) \frac{\partial V_{\text{instanton}}}{\partial \omega^2}, \quad (\text{B1})$$

where Det' stands for determinant with the zero mode excluded. Its value, normalized to that of the oscillator, is known [6]

$$\frac{\text{Det}' O_{\text{instanton}}}{\text{Det} O_{\text{osc}}} = \frac{1}{12\omega^2}, \quad (\text{B2})$$

and so the lhs of the relation above is $-1/\omega^2$. Unfortunately, this derivative does not depend on the numerical coefficient $1/12$, which we would like to calculate, so this sum rule is less useful than the one we used for fluctons in the main text.

The derivative of the potential (36) over ω is calculated directly. The Green's function $G(x, y)$ on top of the instanton solution [7,27] is

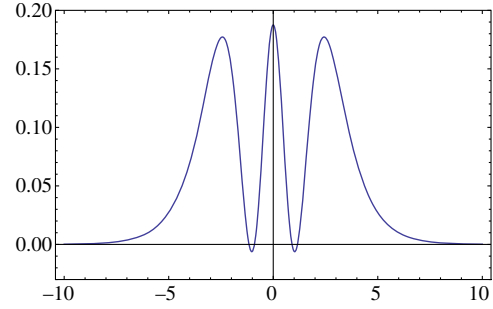


FIG. 7. The integrand of the sum rule, at $\omega = 1$.

$$G(x, y) = G^0(x, y) \left[2 - xy + \frac{1}{4} |x - y| (11 - 3xy) + (x - y)^2 \right] + \frac{3}{8\omega} (1 - x^2)(1 - y^2) \left[\log(2G^0(x, y)) - \frac{11}{3} \right], \quad (\text{B3})$$

expressed in variables $x = \tanh(\frac{\omega\tau_1}{2})$, $y = \tanh(\frac{\omega\tau_2}{2})$, in which the familiar Green function $G^0 = \frac{1}{2\omega} e^{-\omega|\tau_1 - \tau_2|}$ of the harmonic oscillator looks as follows:

$$G^0(x, y) = \frac{1}{2\omega} \frac{1 - |x - y| - xy}{1 + |x - y| - xy}. \quad (\text{B4})$$

We only need it at the equal arguments $\tau_1 = \tau_2$, so it simplifies. Also one needs to regularize the Green function, by subtracting that of the oscillator, resulting in

$$G_{\text{inst}}(\tau, \tau) - G^0(\tau, \tau) = \frac{-7 + 4 \cosh(\tau\omega)}{8\omega \cosh^4(\tau\omega/2)}. \quad (\text{B5})$$

But even with the simplification, the integrand of the rhs of the sum rule (B1) is rather complicated, see Fig. 7. And yet it integrates to the unit answer, as the sum rule requires, providing an additional test to the Green function.

-
- [1] R. P. Feynman and H. R. Hibbs, *Quantum Mechanics and Path Integrals* (Mcgraw-Hill, New York, 1965), Chap. 10.
 - [2] R. P. Feynman, *Statistical Mechanics: A Set of Lectures* (W. A. Benjamin Inc., Reading, MA, 1972).
 - [3] G. C. Rossi and M. Testa, *Ann. Phys. (N.Y.)* **148**, 144 (1983).
 - [4] P. Faccioli, *J. Phys. Conf. Ser.* **336**, 012030 (2011).
 - [5] A. M. Polyakov, *Nucl. Phys.* **B120**, 429 (1977).
 - [6] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, *Sov. Phys. Usp.* **25**, 195 (1982) [*Usp. Fiz. Nauk* **136**, 553 (1982)].
 - [7] F. Wöhler and E. Shuryak, *Phys. Lett. B* **333**, 467 (1994).
 - [8] M. A. Escobar-Ruiz, E. Shuryak, and A. V. Turbiner, *Phys. Rev. D* **92**, 025046 (2015); **92**, 089902(E) (2015).
 - [9] M. A. Escobar-Ruiz, E. Shuryak, and A. V. Turbiner, *Phys. Rev. D* **92**, 025047 (2015).
 - [10] G. V. Dunne and M. Ünsal, *Phys. Rev. D* **89**, 105009 (2014).
 - [11] E. Witten, [arXiv:1009.6032](https://arxiv.org/abs/1009.6032).
 - [12] E. V. Shuryak, *Nucl. Phys.* **B302**, 621 (1988).
 - [13] Note that in the case of two-loop diagrams on the top of the instanton solutions, no such terms are present.
 - [14] It is worth noting that $\sqrt{P(x_0)}$ in (16) taken as parameterless variational trial function provides very accurate ground state energy for the potential (13), see [28].
 - [15] It is worth noting that $\exp(-S[x_0]/2)$, see (19), taken as parameterless, g -dependent, variational trial function provides very accurate ground state energy for the potential (17) for any value of $g \geq 0$, see [28].

- [16] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, Vol. 5—Statistical Physics, Part 1, 3rd Edition, 1980.
- [17] The historical origin of this idea goes back to Brown and Cremer [18], see also [19], for gauge theory instanton. Zarembo [20] applied it for the monopole and Diakonov *et al.*, [21] for the calorons at nonzero holonomy. The reasons it has not been used for quantum mechanical instantons, and the related sum rule are discussed in Appendix B.
- [18] L. S. Brown and D. B. Cremer, *Phys. Rev. D* **18**, 3695 (1978).
- [19] E. Corrigan, P. Goddard, H. Osborn, and S. Templeton, *Nucl. Phys.* **B159**, 469 (1979).
- [20] K. Zarembo, *Nucl. Phys.* **B463**, 73 (1996).
- [21] D. Diakonov, N. Gromov, V. Petrov, and S. Slizovskiy, *Phys. Rev. D* **70**, 036003 (2004).
- [22] S. Fubini, *Nuovo Cimento Soc. Ital. Fis.* **34A**, 521 (1976).
- [23] L. N. Lipatov, *Zh. Eksp. Teor. Fiz.* **72**, 411 (1977) [*Sov. Phys. JETP* **45**, 216 (1977)].
- [24] F. R. Klinkhamer and N. S. Manton, *Phys. Rev. D* **30**, 2212 (1984).
- [25] D. M. Ostrovsky, G. W. Carter, and E. V. Shuryak, *Phys. Rev. D* **66**, 036004 (2002).
- [26] A. V. Turbiner, *Int. J. Mod. Phys. A* **25**, 647 (2010).
- [27] Š. Olejník, *Phys. Lett. B* **221**, 372 (1989).
- [28] A. V. Turbiner, *Pisma ZhETF* **30**, 379 (1979) [*Soviet Phys. JETP Lett.* **30**, 352 (1979) (English Translation)].