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Radiative corrections and the Palatini action

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By using the Faddeev-Popov quantization procedure, we demonstrate that the radiative effects computed using the first-order and second-order Einstein-Hilbert action for general relativity are the same, provided one can discard tadpoles. In addition, we show that the first-order form of this action can be used to obtain a set of Feynman rules that involves just two propagating fields and three three-point vertices; using these rules is considerably simpler than employing the infinite number of vertices that occur in the second-order form. We demonstrate this by computing the one-loop, two-point function.

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I. INTRODUCTION

In the Einstein-Hilbert (EH) action

$$S = \int d^d x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma), \tag{1.1}$$

where

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \tag{1.2}$$

and

$$R_{\mu\nu}(\Gamma) = \Gamma^{\rho}_{\mu\rho,\nu} - \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} \tag{1.3}$$

it is usual to take the metric $g_{\mu\nu}$ to be the independent variable and the affine connection $\Gamma^{\lambda}_{\mu\nu}$ to be dependent; this is the second-order Einstein-Hilbert action. Classically, it is possible to treat both $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ as independent; the equation of motion for $\Gamma^{\lambda}_{\mu\nu}$ in this first-order action yields Eq. (1.2). It was Einstein who first noted this, though the first-order Einstein-Hilbert (1EH) action is often attributed to Palatini [1].

Although the 1EH and second-order Einstein-Hilbert (2EH) actions are equivalent at the classical level, it has as yet not been established that the two forms of the EH action result in the same quantum effects. We first show this quantum equivalence of the 1EH and 2EH actions when using the Faddeev-Popov procedure in conjunction with the quantum mechanical path integral, provided that tadpole integrals can be set equal to zero. This is of some consequence, as it has been noted [2,3] that the first-order form of gauge theory actions is considerably simpler than the second-order form. This is true both in Yang-Mills theory (where two complicated vertices are replaced by a simple one that is independent of momentum) and in general relativity (where a single momentum independent vertex replaces an infinite series of momentum dependent vertices). The only disadvantage of using the first-order action is that there are now two propagating fields; in the 1EH case, these two fields have a rather involved mixed propagator.

Our second result is that it is possible to shift variables of integration in the 1EH action within the path integral to eliminate this mixed propagator. We are then left with a relatively simple set of Feynman rules; there are now just two propagating fields (that do not mix) and three vertices. This is an improvement over the situation that occurs in the 2EH action where there is one propagating field and an infinite number of vertices with an arbitrary number of external fields.

We then demonstrate the utility of our result by computing the two-point function to one-loop order using an arbitrary gauge fixing parameter. In the limiting case in which this parameter equals one, we reproduce the result of Ref. [4].

The first-order formalism has also been used for doing loop calculations in gravity in Ref. [5], though in the models considered there it is not clear if the first- and second-order formalisms are equivalent.

We begin by considering the first-order Yang-Mills (1YM) action.

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II. THE FIRST-ORDER YANG-MILLS ACTION

It is evident that the 1YM Lagrangian

$$\mathcal{L}_{1\mathrm{YM}} = -\frac{1}{2}F^a_{\mu\nu}(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} + g\epsilon^{abc}A^{b\mu}A^{c\nu}) + \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \tag{2.1}$$

is classically equivalent to the second-order Yang-Mills (2YM) Lagrangian

$$\mathcal{L}_{2\mathrm{YM}} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu)^2 \tag{2.2}$$

as upon substitution of the equation of motion for $F^a_{\mu\nu}$ that follows from (2.1) back into $\mathcal{L}_{1\mathrm{YM}}$, $\mathcal{L}_{2\mathrm{YM}}$ follows.

The 1YM and 2YM Lagrangians have the gauge invariance

$$\delta F^a_{\mu\nu} = g\epsilon^{abc}F^b_{\mu\nu}\theta^c, \tag{2.3a}$$

$$\delta A^a_\mu = \partial_\mu\theta^a + g\epsilon^{abc}A^b_\mu\theta^c; \tag{2.3b}$$

we are led to the path integral for $\mathcal{L}_{1\mathrm{YM}}$,

$$Z = \int \mathcal{D}A^a_\mu \mathcal{D}F^a_{\mu\nu}\Delta_{\mathrm{FP}}(A)\exp i\int d^dx(\mathcal{L}_{1\mathrm{YM}} + \mathcal{L}_{gf}), \tag{2.3c}$$

where $\Delta_{\mathrm{FP}}(A)$ is the Faddeev-Popov determinant associated with the gauge fixing Lagrangian $\mathcal{L}_{gf}$. (More than one gauge fixing may occur [6–8].) The field A^a_μ (but not $F^a_{\mu\nu}$) interacts with other "matter" fields.

If in Eq. (2.3c) we perform the shift

$$F^a_{\mu\nu} \to F^a_{\mu\nu} + (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu) \tag{2.4}$$

then we find that

$$Z = \int \mathcal{D}A^a_\mu \mathcal{D}F^a_{\mu\nu}\Delta_{\mathrm{FP}}(A)\exp i \times \int d^dx\left[\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \mathcal{L}_{2\mathrm{YM}} + \mathcal{L}_{gf}\right]. \tag{2.5}$$

The integral over $F^a_{\mu\nu}$ decouples and the usual generating functional associated with $\mathcal{L}_{2\mathrm{YM}}$ is recovered with its three-point and four-point vertices. [In its unshifted form, Eq. (2.3c) results in the three propagators $\langle AA\rangle$, $\langle FF\rangle$ and $\langle AF\rangle$ and the vertex $\langle FAA\rangle$ [2,3].]

We can also make the shift

$$F^a_{\mu\nu} \to F^a_{\mu\nu} + (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) \tag{2.6}$$

leaving us with

$$Z = \int \mathcal{D}A^a_\mu \mathcal{D}F^a_{\mu\nu}\Delta_{\mathrm{FP}}(A)\exp i \times \int d^dx\left[\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2}(F^a_{\mu\nu} + \partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(g\epsilon^{abc}A^b_\mu A^c_\nu) + \mathcal{L}_{gf}\right]. \tag{2.7}$$

When the generating functional Z is written in this form we see that there are now two propagators $\langle FF\rangle$ and $\langle AA\rangle$ as well as two three-point functions $\langle FAA\rangle$ and $\langle AAA\rangle$ (but no mixed propagators $\langle AF\rangle$ or four-point vertex $\langle AAAA\rangle$).

This possibility of altering the Feynman rules in YM theory is exploited when examining the first-order (Palatini) form of the Einstein-Hilbert action.

III. THE FIRST-ORDER EINSTEIN-HILBERT ACTION

Rather than using $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ as independent fields in the 1EH Lagrangian of Eq. (1.1), it proves convenient to use [9]

$$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} \tag{3.1a}$$

and

$$G^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{1}{2}(\delta^\lambda_\mu\Gamma^\sigma_{\nu\sigma} + \delta^\lambda_\nu\Gamma^\sigma_{\mu\sigma}) \tag{3.1b}$$

so that now we have

$$\mathcal{L}_{1\mathrm{EH}} = h^{\mu\nu}\left(G^\lambda_{\mu\nu,\lambda} + \frac{1}{d-1}G^\lambda_{\mu\lambda}G^\sigma_{\nu\sigma} - G^\lambda_{\mu\sigma}G^\sigma_{\nu\lambda}\right). \tag{3.2}$$

The canonical structure of this action has been examined in Refs. [9,10] and the resulting path integral in Ref. [11]. Here, we consider using the Faddeev-Popov path integral [12]

$$Z_{1\mathrm{EH}} = \int \mathcal{D}h^{\mu\nu}\mathcal{D}G^\lambda_{\mu\nu}\Delta_{\mathrm{FP}}(h)\exp i\int d^dx[\mathcal{L}_{1\mathrm{EH}} + \mathcal{L}_{gf}]. \tag{3.3}$$

Directly using the form of Eq. (3.2) makes it impossible to define a propagator for $h^{\mu\nu}$ and $G^\lambda_{\mu\nu}$. (This is easily seen if one were to attempt to find a propagator for fields ϕ and V^λ with the Lagrangian $\mathcal{L} = \phi V^\lambda_{,\lambda}$.) In Ref. [4], $h^{\mu\nu}$ is expanded about a flat metric $\eta^{\mu\nu} = \mathrm{diag}(+,+,+,\ldots,-)$ so that

$$h^{\mu\nu}(x) = \eta^{\mu\nu} + \phi^{\mu\nu}(x); \tag{3.4}$$

the propagators $\langle\phi\phi\rangle$, $\langle GG\rangle$, $\langle\phi G\rangle$ and the vertex $\langle\phi GG\rangle$ are given in Ref. [3]. However, it is not immediately evident how this form of $Z_{1\mathrm{EH}}$ yields results consistent with those that follow from the 2EH Lagrangian $\mathcal{L}_{2\mathrm{EH}}$.

To show this equivalence, we start by writing Eq. (3.2) as

$$\mathcal{L}_{1\mathrm{EH}} = G^{\lambda}_{\mu\nu}(-h^{\mu\nu}_{,\lambda}) + \frac{1}{2}M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(h)G^{\lambda}_{\mu\nu}G^{\sigma}_{\pi\tau}, \tag{3.5}$$

where

$$M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(h) = \frac{1}{2}\left[\frac{1}{d-1}(\delta^{\nu}_{\lambda}\delta^{\tau}_{\sigma}h^{\mu\pi} + \delta^{\mu}_{\lambda}\delta^{\tau}_{\sigma}h^{\nu\pi} + \delta^{\nu}_{\lambda}\delta^{\pi}_{\sigma}h^{\mu\tau} + \delta^{\mu}_{\lambda}\delta^{\pi}_{\sigma}h^{\nu\tau}) - (\delta^{\tau}_{\lambda}\delta^{\nu}_{\sigma}h^{\mu\pi} + \delta^{\tau}_{\lambda}\delta^{\mu}_{\sigma}h^{\nu\pi} + \delta^{\pi}_{\lambda}\delta^{\nu}_{\sigma}h^{\mu\tau} + \delta^{\pi}_{\lambda}\delta^{\mu}_{\sigma}h^{\nu\tau})\right]. \tag{3.6}$$

From Eq. (3.5) we obtain the equation of motion

$$h^{\mu\nu}_{,\lambda} = M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(h)G^{\sigma}_{\pi\tau} \tag{3.7}$$

from which we see that [upon using Eq. (3.6) and with $h_{\mu\lambda}h^{\lambda\nu} = \delta^{\nu}_{\mu}$]

$$H_{\pi\tau,\lambda} \equiv -h_{\pi\mu}h_{\tau\nu}h^{\mu\nu}_{,\lambda} + h_{\tau\mu}h_{\lambda\nu}h^{\mu\nu}_{,\pi} + h_{\lambda\mu}h_{\pi\nu}h^{\mu\nu}_{,\tau} = 2\left(\frac{1}{d-1}h_{\pi\tau}G^{\sigma}_{\lambda\sigma} - h_{\lambda\sigma}G^{\sigma}_{\pi\tau}\right). \tag{3.8}$$

Upon contracting Eq. (3.8) with $h^{\tau\lambda}$ we see that

$$G^{\sigma}_{\pi\sigma} = -\frac{d-1}{2(d-2)}h_{\mu\nu}h^{\mu\nu}_{,\pi} \tag{3.9}$$

and so by Eq. (3.8),

$$G^{\rho}_{\pi\tau} = \frac{1}{2}h^{\rho\lambda}\left(-\frac{1}{d-2}h_{\pi\tau}h_{\mu\nu}h^{\mu\nu}_{,\lambda} - H_{\pi\tau,\lambda}\right). \tag{3.10}$$

From Eq. (3.8) it is apparent that

$$(M^{-1})^{\rho\ \ \lambda}_{\pi\tau\mu\nu}(h) = \frac{-1}{2(d-2)}h^{\rho\lambda}h_{\pi\tau}h_{\mu\nu} + \frac{1}{4}h^{\rho\lambda}(h_{\pi\mu}h_{\tau\nu} + h_{\pi\nu}h_{\tau\mu}) - \frac{1}{4}(h_{\tau\mu}\delta^{\rho}_{\nu}\delta^{\lambda}_{\pi} + h_{\pi\mu}\delta^{\rho}_{\nu}\delta^{\lambda}_{\tau} + h_{\tau\nu}\delta^{\rho}_{\mu}\delta^{\lambda}_{\pi} + h_{\pi\nu}\delta^{\rho}_{\mu}\delta^{\lambda}_{\tau}). \tag{3.11}$$

(We have

$$(M^{-1})^{\rho\ \ \lambda}_{\alpha\beta\,\mu\nu}M^{\mu\nu\,\gamma\delta}_{\lambda\ \ \sigma} = \Delta^{\gamma\delta}_{\alpha\beta}\delta^{\rho}_{\sigma} \equiv \frac{1}{2}(\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} + \delta^{\delta}_{\alpha}\delta^{\gamma}_{\beta})\delta^{\rho}_{\sigma}.) \tag{3.12}$$

In the Lagrangian of Eq. (3.5) we insert Eq. (3.10) and obtain

$$\mathcal{L}_{1\mathrm{EH}} = -\frac{1}{2}h^{\mu\nu}_{,\lambda}(M^{-1})^{\lambda\ \ \sigma}_{\mu\nu\,\pi\tau}(h)h^{\pi\tau}_{,\sigma} \tag{3.13}$$

which is just the second-order EH Lagrangian $\mathcal{L}_{2\mathrm{EH}}$. This demonstrates that classically, $\mathcal{L}_{1\mathrm{EH}}$ and $\mathcal{L}_{2\mathrm{EH}}$ are equivalent.

We now make the shift

$$G^{\lambda}_{\mu\nu} \to G^{\lambda}_{\mu\nu} + (M^{-1})^{\lambda\ \ \sigma}_{\mu\nu\,\pi\tau}(h)h^{\pi\tau}_{,\sigma} \tag{3.14}$$

in the path integral of Eq. (3.3). We then find that

$$Z_{1\mathrm{EH}} = \int \mathcal{D}h^{\mu\nu}\mathcal{D}G^{\lambda}_{\mu\nu}\Delta_{\mathrm{FP}}(h)\exp i \times \int d^dx\left[\frac{1}{2}G^{\lambda}_{\mu\nu}M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(h)G^{\sigma}_{\pi\tau} + \frac{1}{2}h^{\mu\nu}_{,\lambda}(M^{-1})^{\lambda\ \ \sigma}_{\mu\nu\,\pi\tau}(h)h^{\pi\tau}_{,\sigma} + \mathcal{L}_{gf}\right]. \tag{3.15}$$

The expansion of Eq. (3.4) can now be made in Eq. (3.15). Since M is linear in $h^{\mu\nu}$, it follows that

$$M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(\eta+\phi) = M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(\eta) + M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(\phi). \tag{3.16}$$

Consequently, any Feynman diagrams contributing to Green's functions with only the field $\phi^{\mu\nu}$ on external legs and which involve the field $G^{\lambda}_{\mu\nu}$ on internal lines necessarily have the field $G^{\lambda}_{\mu\nu}$ appearing in a closed loop. But the propagator for the field $G^{\lambda}_{\mu\nu}$ is independent of momentum [see Eq. (3.11)] and hence the loop momentum integral associated with any loop coming from the field $G^{\lambda}_{\mu\nu}$ is of the form

$$\int d^dkP(k^{\mu}), \tag{3.17}$$

where $P(k^{\mu})$ is a polynomial in the loop momentum k^{μ}. If we use dimensional regularization [13,14] then such loop momentum integrals vanish.

Consequently, for Green's functions involving only the field $\phi^{\mu\nu}$ on external legs, the only contribution to Feynman diagrams comes from the last two terms in the argument of the exponential in Eq. (3.15); from Eq. (3.13) we see that this is just the generating functional associated with $-\mathcal{L}_{2\mathrm{EH}}$ and so these Green's functions can be derived by using either the first-order or the second-order form of the EH action.

Using the second-order form with the Lagrangian of Eq. (3.13) results in an infinite series of vertices involving the field $h^{\mu\nu}$ (see Ref. [4]). To obtain them, we note that when Eq. (3.16) is substituted into Eq. (3.12), we schematically obtain

$$(M^{-1})(\eta+\phi) = M^{-1}(\eta) - M^{-1}(\eta)M(\phi)M^{-1}(\eta) + M^{-1}(\eta)M(\phi)M^{-1}(\eta)M(\phi)M^{-1}(\eta) - \cdots. \tag{3.18}$$

The first term in Eq. (3.18) is associated with the propagator for the $\phi^{\mu\nu}$ field in the second-order formalism while each subsequent term is associated with a vertex. This means that direct use of the 2EH Lagrangian becomes exceedingly complicated if more than the one-loop two-point Green's function is to be computed [15,16].

We now show that the 1EH generating functional can be used to compute Green's functions with only the two propagators $\langle\phi\phi\rangle$, $\langle GG\rangle$ and the three point functions $\langle GG\phi\rangle$, $\langle G\phi\phi\rangle$ and $\langle\phi\phi\phi\rangle$. First the expansion of Eq. (3.4) is made and then the shift occurs,

$$G^{\lambda}_{\mu\nu} \to G^{\lambda}_{\mu\nu} + (M^{-1})^{\lambda\ \ \sigma}_{\mu\nu\ \pi\tau}(\eta) h^{\pi\tau}_{,\sigma}. \tag{3.19}$$

[This is the shift of Eq. (3.14) with h being replaced by η.] This leads to Eq. (3.3) becoming

$$\begin{aligned} Z_{1\mathrm{EH}} = \int \mathcal{D}h^{\mu\nu}\mathcal{D}G^{\lambda}_{\mu\nu}\Delta_{\mathrm{FP}}(h)\exp i \\ \times \int d^d x\bigg[\frac{1}{2}G^{\lambda}_{\mu\nu}M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(\eta)G^{\sigma}_{\pi\tau} - \frac{1}{2}\phi^{\mu\nu}_{,\lambda}M^{-1\lambda\ \ \sigma}_{\ \ \mu\nu\,\pi\tau}(\eta)\phi^{\pi\tau}_{,\sigma} \\ + \frac{1}{2}(G^{\lambda}_{\mu\nu} + \phi^{\alpha\beta}_{,\rho}(M^{-1})^{\rho\ \ \lambda}_{\alpha\beta\,\mu\nu}(\eta))(M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(\phi)) \\ \times (G^{\sigma}_{\pi\tau} + (M^{-1})^{\sigma\ \ \xi}_{\pi\tau\,\gamma\delta}(\eta)\phi^{\gamma\delta}_{,\xi}) + \mathcal{L}_{gf}\bigg]. \end{aligned} \tag{3.20}$$

The contributions coming from the various terms in the argument of the exponential appearing in Eq. (3.20) that lead to the Feynman rules can be immediately seen to be

$$G - G\colon \frac{1}{2}G^{\lambda}_{\mu\nu}M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(\eta)G^{\sigma}_{\pi\tau}\colon, \tag{3.21a}$$

$$\phi - \phi\colon -\frac{1}{2}\phi^{\mu\nu}_{,\lambda}M^{-1\lambda\ \ \sigma}_{\ \ \mu\nu\,\pi\tau}(\eta)\phi^{\pi\tau}_{,\sigma} - \frac{1}{2\alpha}(\phi^{\mu\nu}_{,\nu})^2\colon, \tag{3.21b}$$

$$G - G - \phi\colon \frac{1}{2}M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(\phi)G^{\lambda}_{\mu\nu}G^{\sigma}_{\pi\tau}\colon, \tag{3.21c}$$

$$G - \phi - \phi\colon G^{\lambda}_{\mu\nu}M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(\phi)M^{-1\sigma\ \ \xi}_{\ \ \pi\tau\,\gamma\delta}(\eta)\phi^{\gamma\delta}_{,\xi}\colon, \tag{3.21d}$$

$$\phi - \phi - \phi\colon \frac{1}{2}\phi^{\alpha\beta}_{,\rho}M^{-1\rho\ \ \lambda}_{\ \ \alpha\beta\,\mu\nu}(\eta)M^{\mu\nu\,\pi\tau}_{\lambda\ \ \sigma}(\phi)M^{-1\sigma\ \ \xi}_{\ \ \pi\tau\,\gamma\delta}(\eta)\phi^{\gamma\delta}_{,\xi}\colon. \tag{3.21e}$$

In Eq. (3.21b) we have used the gauge fixing Lagrangian

$$\mathcal{L} = -\frac{1}{2\alpha}(\phi^{\mu\nu}_{,\nu})^2. \tag{3.22}$$

With this gauge fixing, the contribution coming from the Faddeev-Popov determinant Δ_{FP} in Eq. (3.20) involves the Feynman rules that follow from [3,4]

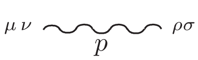
$$\begin{aligned} \mathcal{L}_{\mathrm{ghost}} = \bar{d}_\mu[\partial^2\eta^{\mu\nu} + (\phi^{\rho\sigma}_{,\rho})\partial_\sigma\eta^{\mu\nu} - (\phi^{\rho\mu}_{,\rho})\partial^\nu \\ + \phi^{\rho\sigma}\partial_\rho\partial_\sigma\eta^{\mu\nu} - (\partial_\rho\partial^\nu\phi^{\rho\mu})]d_\nu, \end{aligned} \tag{3.23}$$

where d^μ and $\bar{d}^\mu$ are Fermionic vector ghost fields. These are found to be

$$\bar{d} - d\colon \bar{d}_\mu\partial^2 d_\nu\colon, \tag{3.24a}$$

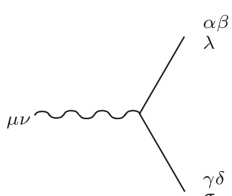
$$\begin{aligned} \bar{d} - d - \phi\colon \bar{d}_\mu[(\phi^{\rho\sigma}_{,\rho})\partial_\sigma\eta^{\mu\nu} - (\phi^{\rho\mu}_{,\rho})\partial^\nu + \phi^{\rho\sigma}\partial_\rho\partial_\sigma\eta^{\mu\nu} \\ - (\partial_\rho\partial^\nu\phi^{\rho\mu})]d_\nu\colon. \end{aligned} \tag{3.24b}$$

Let us now consider an explicit calculation of a one-loop radiative correction. From Eqs. (3.21) and (3.24) we readily find the following momentum space Feynman rules (all vertex momenta are inwards and $p + q + r = 0$):

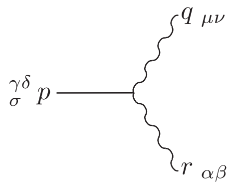


$$: \quad \frac{(1-\alpha)\,(p^\nu p^\sigma\eta^{\mu\rho} + p^\nu p^\rho\eta^{\mu\sigma} + p^\mu p^\sigma\eta^{\nu\rho} + p^\mu p^\rho\eta^{\nu\sigma} - 2p^\rho p^\sigma\eta^{\mu\nu} - 2p^\mu p^\nu\eta^{\rho\sigma})}{p^4} - \frac{\eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma} - (2-\alpha)\eta^{\mu\nu}\eta^{\rho\sigma}}{p^2} \tag{3.25a}$$

$$: \quad \frac{1}{4}\eta^{\lambda\rho}\left(\eta_{\mu\tau}\eta_{\nu\pi} + \eta_{\mu\pi}\eta_{\nu\tau} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\pi\tau}\right) - \frac{1}{4}\left(\delta^\lambda_\tau\delta^\rho_\mu\eta_{\nu\pi} + \delta^\lambda_\tau\delta^\rho_\nu\eta_{\mu\pi} + \delta^\lambda_\pi\delta^\rho_\nu\eta_{\mu\tau} + \delta^\lambda_\pi\delta^\rho_\mu\eta_{\nu\tau}\right) \equiv \mathcal{D}^{\lambda\ \ \rho}_{\mu\nu\ \pi\tau} \tag{3.25b}$$

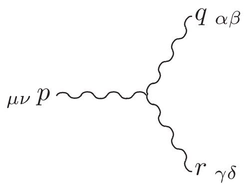


$$: \quad \frac{1}{8}\left\{\left[\left(\frac{\delta^\beta_\mu\delta^\delta_\nu\delta^\alpha_\lambda\delta^\gamma_\sigma}{d-1} - \delta^\beta_\mu\delta^\delta_\nu\delta^\alpha_\sigma\delta^\gamma_\lambda + \mu\leftrightarrow\nu\right) + \alpha\leftrightarrow\beta\right] + \gamma\leftrightarrow\delta\right\} + (\lambda,\alpha,\beta)\longleftrightarrow(\sigma,\gamma,\delta) \tag{3.25c}$$



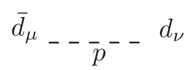
$$: \quad \frac{ir_\theta}{4}\left\{\left[\left(\frac{1}{d-1}\delta^\gamma_\mu\delta^\delta_\sigma\mathcal{D}^{\theta}{}_{\alpha\beta\nu\rho}{}^{\rho} - \delta^\gamma_\mu\mathcal{D}^{\theta}{}_{\alpha\beta\nu\sigma}{}^{\delta} + \mu\leftrightarrow\nu\right) + \alpha\leftrightarrow\beta\right] + \gamma\leftrightarrow\delta\right\} \tag{3.25d}$$

$$+ \ (q,\alpha,\beta)\longleftrightarrow(r,\mu,\nu)$$

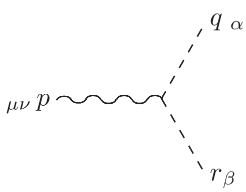


$$: \quad \frac{q_\kappa r_\theta}{8}\left\{\left[\left(\mathcal{D}^{\kappa}{}_{\alpha\beta\mu\sigma}{}^{\pi}\mathcal{D}^{\theta}{}_{\gamma\delta\nu\pi}{}^{\sigma} - \frac{1}{d-1}\mathcal{D}^{\kappa}{}_{\alpha\beta\mu\sigma}{}^{\sigma}\mathcal{D}^{\theta}{}_{\gamma\delta\nu\pi}{}^{\pi} + \mu\leftrightarrow\nu\right) + \alpha\leftrightarrow\beta\right] + \gamma\leftrightarrow\delta\right\} \tag{3.25e}$$

$$+ \ \text{six permutations of } \ (p,\mu,\nu) \ \ (q,\alpha,\beta) \ \ (r,\gamma,\delta)$$



$$: \quad -\frac{\eta^{\mu\nu}}{p^2} \tag{3.25f}$$



$$: \quad \frac{1}{2}\left[\eta_{\alpha\beta}\left(q_\mu r_\nu + q_\nu r_\mu\right) - q_\beta\left(p_\mu\eta_{\alpha\nu} + p_\nu\eta_{\alpha\mu}\right)\right] \tag{3.25g}$$

The one-loop contributions to the two-point function $\langle\phi\phi\rangle$ are given by the Feynman diagrams of Fig. 1. After loop integration, the result can only depend (by covariance) on the five tensors shown in Table I, so that each diagram in Fig. 1 can be written as

$$\Pi^I_{\mu\nu\,\alpha\beta}(k) = \sum_{i=1}^{5}\mathcal{T}^i_{\mu\nu\,\alpha\beta}(k)C^I_i(k); \quad I = \text{a, b, c and d}. \tag{3.26}$$

The coefficients C^I_i can be obtained solving the following system of five algebraic equations:

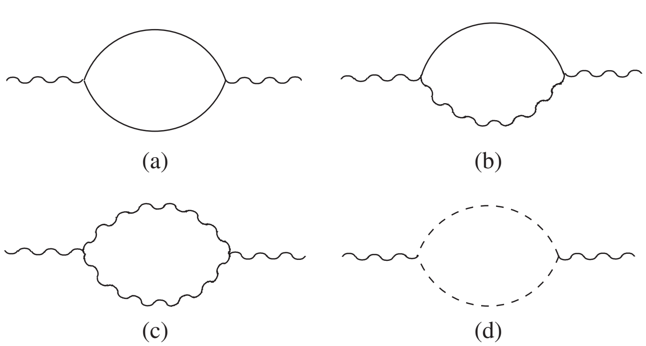


FIG. 1. One-loop contributions to $\langle\phi\phi\rangle$.

$$\sum_{i=1}^{5}\mathcal{T}^i_{\mu\nu\,\alpha\beta}(k)\mathcal{T}^{j\mu\nu\,\alpha\beta}(k)C^I_i(k)$$
$$= \Pi^I_{\mu\nu\,\alpha\beta}(k)\mathcal{T}^{j\mu\nu\,\alpha\beta}(k) \equiv J^{Ij}(k); \quad j = 1,\ldots,5. \tag{3.27}$$

Using the Feynman rules for $\Pi^I_{\mu\nu\,\alpha\beta}(k)$ the integrals on the right-hand side have the following form,

$$J^{Ij}(k) = \int\frac{d^dp}{(2\pi)^d}s^{Ij}(p,q,k), \tag{3.28}$$

where $q = p + k$; p is the loop momentum, k is the external momentum and $s^{Ij}(p,q,k)$ are scalar functions. Using the relations

$$p\cdot k = (q^2 - p^2 - k^2)/2, \tag{3.29a}$$

TABLE I. The five independent tensors built from $\eta_{\mu\nu}$ and k_μ, satisfying the symmetry conditions $\mathcal{T}^i_{\mu\nu\,\alpha\beta}(k) = \mathcal{T}^i_{\nu\mu\,\alpha\beta}(k) = \mathcal{T}^i_{\mu\nu\,\beta\alpha}(k) = \mathcal{T}^i_{\alpha\beta\,\mu\nu}(k)$.

$\mathcal{T}^1_{\mu\nu\,\alpha\beta}(u,k) = k_\mu k_\nu k_\alpha k_\beta$
$\mathcal{T}^2_{\mu\nu\,\alpha\beta}(u,k) = \eta_{\mu\nu}\eta_{\alpha\beta}$
$\mathcal{T}^3_{\mu\nu\,\alpha\beta}(u,k) = \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}$
$\mathcal{T}^4_{\mu\nu\,\alpha\beta}(u,k) = \eta_{\mu\nu}k_\alpha k_\beta + \eta_{\alpha\beta}k_\mu k_\nu$
$\mathcal{T}^5_{\mu\nu\,\alpha\beta}(u,k) = \eta_{\mu\alpha}k_\nu k_\beta + \eta_{\mu\beta}k_\nu k_\alpha + \eta_{\nu\alpha}k_\mu k_\beta + \eta_{\nu\beta}k_\mu k_\alpha$

$$q \cdot k = (q^2 + k^2 - p^2)/2, \tag{3.29b}$$

$$p \cdot q = (p^2 + q^2 - k^2)/2, \tag{3.29c}$$

the scalars $s^{Ij}(p,q,k)$ can be reduced to combinations of powers of p^2 and q^2. As a result, the integrals $J^{Ij}(k)$ can be expressed in terms of combinations of the following well-known integrals,

$$I^{ab} \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2)^a (q^2)^b} = \frac{(k^2)^{d/2-a-b}}{(4\pi)^{d/2}} \frac{\Gamma(a+b-d/2)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(d/2-a)\Gamma(d/2-b)}{\Gamma(d-a-b)} \tag{3.30}$$

(this has also been considered in [17]). The only non-vanishing (i.e. nontadpole) integrals are the ones with both $a > 0$ and $b > 0$. As we have pointed out earlier the integrals $J_i^{\mathrm{a}}(k)$ and $J_i^{\mathrm{b}}(k)$, associated respectively with the diagrams (a) and (b) of Fig. 1, are tadpolelike and do not contribute (either a or b is not positive). For a general gauge parameter, $\alpha \neq 1$, the diagram (c) in Fig. 1 involves the following three kinds of integrals:

$$I^{11} = \frac{(k^2)^{d/2-2}}{2^d \pi^{d/2}} \frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^2}{\Gamma(d-2)}, \tag{3.31a}$$

$$I^{12} = I^{21} = \frac{(3-d)I^{11}}{k^2}, \tag{3.31b}$$

$$I^{22} = \frac{(3-d)(6-d)I^{11}}{k^4}. \tag{3.31c}$$

The ghost loop diagram only involves I^{11}.

A straightforward computer algebra code can now be set up in order to implement the steps described above and to obtain the structures C_i^{c} and C_i^{d}. The results are the following:

$$C_1^{\mathrm{c}} = \frac{1}{8(d-1)} \left[\frac{1}{8}(d^3 - 2d^2 + 96d - 64) - 4(\alpha-1)(4d^2 - 21d + 24) + 4(\alpha-1)^2(d^3 - 7d^2 + 22d - 26)\right] I^{11}, \tag{3.32a}$$

$$C_2^{\mathrm{c}} = \frac{1}{8(d-1)(d-2)^2} \left[\frac{1}{8} d(-7d^2 + 4d + 52) + 4(\alpha-1)(2d^3 - 16d^2 + 41d - 30) - 2(\alpha-1)^2(d^4 - 12d^3 + 68d^2 - 179d + 162)\right] k^4 I^{11}, \tag{3.32b}$$

$$C_3^{\mathrm{c}} = \frac{1}{32(d-1)} \left[\frac{1}{2}(4d^2 + 5d - 16) - 16(\alpha-1)(d-4)(d-1) + 4(\alpha-1)^2(d^3 - 8d^2 + 30d - 43)\right] k^4 I^{11}, \tag{3.32c}$$

$$C_4^{\mathrm{c}} = \frac{1}{16(d-1)(d-2)} \left[\frac{1}{4}(d^3 - 2d^2 + 40d + 16) - 8(\alpha-1)(3d^2 - 18d + 20) + 8(\alpha-1)^2(d^3 - 7d^2 + 22d - 26)\right] k^2 I^{11}, \tag{3.32d}$$

$$C_5^{\mathrm{c}} = \frac{1}{32(d-1)} \left[\frac{1}{2}(-4d^2 - 5d + 20) + 16(\alpha-1)(d-4)(d-1) - 4(\alpha-1)^2(d^3 - 8d^2 + 31d - 44)\right] k^2 I^{11}, \tag{3.32e}$$

$$C_1^{\mathrm{d}} = -\frac{(d-2)(d^2 + 8d + 8)}{16(d^2-1)} I^{11}, \tag{3.33a}$$

$$C_2^{\mathrm{d}} = C_3^{\mathrm{d}} = -\frac{d}{16(d^2-1)} k^4 I^{11}, \tag{3.33b}$$

$$C_4^{\mathrm{d}} = -\frac{(d^2 + 2d + 2)}{16(d^2-1)} k^2 I^{11}, \tag{3.33c}$$

$$C_5^{\mathrm{d}} = -\frac{1}{16(d^2-1)} k^2 I^{11}. \tag{3.33d}$$

In the special case when $\alpha = 1$ the result is in complete agreement with Ref. [4]. The final expression for the one-loop contribution to $\langle \phi\phi \rangle$ can now be expressed as

$$\Pi_{\mu\nu\,\alpha\beta} = \sum_{i=1}^{5} (C_i^{\mathrm{c}} + C_i^{\mathrm{d}}) \mathcal{T}^i_{\mu\nu\,\alpha\beta}. \tag{3.34}$$

IV. DISCUSSION

Establishing the equivalence between the first- and second-order forms of the Yang-Mills Lagrangians at both the classical and quantum levels is straightforward; this was demonstrated in Sec. II above. It is not so easy to show at both the classical and quantum levels that the first- and second-order forms of the Einstein-Hilbert action are equivalent. In Sec. III above we have shown that this equivalence holds provided it is possible to discard tadpole diagrams (which are regulated to zero when using dimensional regularization). [One feature of this demonstration whose significance is not immediately apparent is the difference in sign between $\mathcal{L}_{1EH}$ in (3.13) and the $hM^{-1}(h)h$ term in Eq. (3.15).]

We have also shown that by rewriting the 1EH action judiciously, it is possible to have just two propagating fields and three three-point functions. This may prove to be an advantage when considering higher order diagrams in the loop expansion in (super-)gravity.

It is quite straightforward to adopt the methods of Refs. [15,16,18,19], involving the use of geodesic coordinates in conjunction with a background field for $\phi^{\mu\nu}$, to determine counterterms while working with the 1EH Lagrangian.

It would also be interesting to compute the one-loop correction to the two-point function $\langle\phi\phi\rangle$ using the transverse-traceless gauge of Ref. [6].

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