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Remarks on a SUSY exact action in 3D supergravity

Norihiro Iizuka[1,*] and Akinori Tanaka[2,†]

[1]*Department of Physics, Osaka University, Toyonaka, Osaka 560-0043, Japan*

[2]*Interdisciplinary Theoretical Science Research Group, RIKEN, Wako 351-0198, Japan*

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We consider $2+1$-dimensional off-shell $\mathcal{N}=1$ pure supergravity that is constructed from graviton, gravitino and auxiliary field. We show that the R^2 supersymmetric invariant and $R^2_{\mu\nu}$ supersymmetric invariant are expressed as local supersymmetric exact terms up to mass terms for the gravitino. In both cases, the mass parameter is proportional to the off-shell supersymmetric cosmological constant.

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I. INTRODUCTION

Calculating the quantum gravity partition function in a reasonable way is one of the most important and fundamental questions in theoretical physics. Even in the conventional quantum field theory with spin 0, 1/2, 1 fields, the exact computation is extremely difficult in many cases, and we are often tempted to use perturbative analysis. However, if there are some supersymmetries, one can utilize these symmetries to reduce the path integral to the finite-dimensional matrix models [1,2]. In this procedure, the existence of the supersymmetric "exact" Lagrangian is extremely important because adding such a term into the path integral weight does not change the final result, and the WKB computation turns out to be exact by taking its coupling constant to be infinite (or zero).

By applying this technique to the gravity path integral, we would like to make the gravity path integral well defined. In [3], the authors considered such a possibility in terms of the supersymmetric Chern-Simons formulation of three-dimensional gravity. In this paper, we discuss another possibility: localization computation with local supersymmetry based on supergravity [4]. We will focus on $2+1$-dimensional $\mathcal{N}=1$ supergravity [5,6], and we start with reviewing some known facts on the theory.

II. THREE-DIMENSIONAL $\mathcal{N}=1$ OFF-SHELL SUPERGRAVITY

We focus on the component expression with Lorentz signature $\eta^{ab}=\mathrm{diag}(-1,+1,+1)$, where the alphabet runs for local Lorentz indices a, $b=0$, 1, 2. The fundamental degrees of freedom are as follows: graviton $e_\mu{}^a$, gravitino ψ_μ, and real auxiliary field S. Local supersymmetry is defined by an arbitrary Majorana spinor parameter ϵ which depends on the coordinates as follows [5],

$$\delta e_\mu{}^a=\frac{1}{2}(\bar{\epsilon}\gamma^a\psi_\mu),\qquad \delta\psi_\mu=D_\mu(\hat{\omega})\epsilon+\frac{1}{2}S\gamma_\mu\epsilon, \tag{1}$$

$$\delta S=\frac{1}{4}(\bar{\epsilon}\gamma^{\mu\nu}\psi_{\mu\nu}(\hat{\omega})-\frac{1}{4}(\bar{\epsilon}\gamma^\mu\psi_\mu)S, \tag{2}$$

where the covariant derivative is defined by $D_\mu(\omega)=\partial_\mu+\frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$, the hatted spin connection contains the contribution from torsion, and $\psi_{\mu\nu}(\omega)=\frac{1}{2}(D_\mu(\omega)\psi_\nu-D_\nu(\omega)\psi_\mu)$. See [5] for more details.

Under these transformations, the following Lagrangians are invariant up to a total derivative term:

$$L_{EH}=e(R-\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu(\hat{\omega})\psi_\rho-2S^2), \tag{3}$$

$$L_C=e\left(S+\frac{1}{8}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\right). \tag{4}$$

The first one is the usual Einstein-Hilbert term. The second one corresponds to the cosmological constant term. Just by integrating out the auxiliary field, it generates the usual negative cosmological constant term, and the resultant Lagrangian turns out to be the so-called $\mathcal{N}=(1,0)$ AdS supergravity. In addition to that, one can find a supersymmetric gravitational CS term, but we omit it for simplicity. Other supersymmetric terms can be found in [5] as follows,

$$\begin{aligned}L_{R^2_{\mu\nu}}=&-\frac{1}{4}eR^{\mu\nu ab}(\Omega^+)R_{\mu\nu ab}(\Omega^+)\\&-2e\bar{\psi}_{ab}(\Omega^-)\gamma^\mu D_\mu\psi^{ab}(\Omega^-)\\&+\frac{1}{2}eR_{\mu\nu ab}(\Omega^+)\bar{\psi}_\rho\gamma^{\mu\nu}\gamma^\rho\psi^{ab}(\Omega^-)\\&+eS\bar{\psi}_{ab}(\Omega^-)\psi^{ab}(\Omega^-)-\frac{1}{2}e\bar{\psi}^{ab}(\Omega^-)\psi_{ab}(\Omega^-)\bar{\psi}_\mu\psi^\mu\\&+\frac{1}{8}e\bar{\psi}^{ab}(\Omega^-)\psi_{ab}(\Omega^-)\psi_\mu\gamma^{\mu\nu}\psi_\nu,\end{aligned} \tag{5}$$

where $\Omega_\mu^{\pm ab}=\hat{\omega}_\mu{}^{ab}\pm S\varepsilon_\mu{}^{ab}$ and $(\Omega^\pm)$ means that the corresponding object is defined by the covariant derivative with respect to $\Omega^\pm$. This is the $R^2_{\mu\nu}$-type supersymmetric

*iizuka@phys.sci.osaka-u.ac.jp
†akinori.tanaka@riken.jp

Lagrangian. A nice property is that $L_{R^2_{\mu\nu}}$ can be represented as the supersymmetric Yang-Mills action by considering the pair of indices ab as the gauge index and regarding the gauge field $A^I_\mu = \Omega^{+ab}_\mu$ and the gaugino $\chi^I = \psi^{ab}(\Omega^-)$,

$$\begin{aligned} L_{\rm SYM} &= -\frac{1}{4}eF^{\mu\nu I}F_{\mu\nu}{}^I - 2e\bar{\chi}^I\gamma^\mu(D_\mu\chi)^I + \frac{1}{2}eF_{\mu\nu}{}^I\bar{\psi}_\rho\gamma^{\mu\nu}\gamma^\rho\chi^I \\ &\quad + eS\bar{\chi}^I\chi^I - \frac{1}{2}e\bar{\chi}^I\chi^I\bar{\psi}_\mu\psi^\mu + \frac{1}{8}e\bar{\chi}^I\chi^I\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu. \\ &= L_{R^2_{\mu\nu}}, \end{aligned} \tag{6}$$

where we use the identifications $A^I_\mu = \Omega^{+ab}_\mu$ and $\chi^I = \psi^{ab}(\Omega^-)$ in the final equality.

In addition, one can also find the following R^2-type supersymmetric term,

$$\begin{aligned} L_{R^2} &= \frac{1}{16}e\hat{R}^2(\Omega^+) + \frac{1}{4}e\bar{\psi}_{\mu\nu}(\Omega^-)\gamma^{\mu\nu}\not{D}\psi_{\rho\sigma}(\Omega^-) - e\partial^\mu S\partial_\mu S \\ &\quad - \frac{1}{8}eS\bar{\psi}_{\mu\nu}(\Omega^-)\gamma^{\mu\nu}\gamma^{\rho\sigma}\psi_{\rho\sigma}(\Omega^-) \\ &\quad + \frac{1}{2}e\bar{\psi}_\mu\gamma^\nu\gamma^\mu\partial_\nu S\gamma^{\rho\sigma}\psi_{\rho\sigma}(\Omega^-) \\ &\quad - \frac{1}{32}e\bar{\psi}_{\mu\nu}(\Omega^-)\gamma^{\mu\nu}\gamma^{\rho\sigma}\psi_{\rho\sigma}(\Omega^-)\bar{\psi}_\lambda\psi^\lambda \\ &\quad + \frac{1}{64}e\bar{\psi}_{\mu\nu}(\Omega^-)\gamma^{\mu\nu}\gamma^{\rho\sigma}\psi_{\rho\sigma}(\Omega^-)\bar{\psi}_\lambda\gamma^{\lambda\tau}\psi_\tau, \end{aligned} \tag{7}$$

where the hatted curvature is defined by

$$\hat{R}(\Omega^+) = R(\hat{\omega}) + 6S^2 + 2\bar{\psi}_\mu\gamma_\nu\psi^{\mu\nu}(\Omega^-) + \frac{1}{2}S\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu. \tag{8}$$

Similarly, one can regard this L_{R^2} matter Lagrangian as follows,

$$\begin{aligned} L_{\rm matter} &= -e\partial^\mu\phi\partial_\mu\phi - \frac{1}{4}e\bar{\lambda}\gamma^\mu D_\mu\lambda + \frac{1}{16}ef^2 + \frac{1}{8}eS\bar{\lambda}\lambda \\ &\quad + \frac{1}{2}e\bar{\psi}_\mu\gamma^\nu\gamma^\mu\partial_\nu\phi\lambda + \frac{1}{32}e\bar{\lambda}\lambda\bar{\psi}_\mu\psi^\mu - \frac{1}{64}e\bar{\lambda}\lambda\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu. \\ &= L_{R^2}, \end{aligned} \tag{9}$$

where we use the identifications for the scalar $\phi = S$, for the spinor $\lambda = \gamma^{\mu\nu}\psi_{\mu\nu}(\Omega^-)$, and for the auxiliary scalar $f = \hat{R}(\Omega^\pm)$ in the final equality.

III. SUSY EXACT TERMS

For the localization calculation, the most important feature is the following point: To obtain the partition function $Z = \lim_{t\to 0}Z(t)$, we define $Z(t)$ as

$$Z(t) = \int \mathcal{D}e_\mu{}^a\mathcal{D}\psi_\mu\mathcal{D}Se^{iS+it\delta V}, \tag{10}$$

and furthermore this $Z(t)$ does not depend on the parameter t. Then we can take the $t\to\infty$ limit to conduct the computation, and in this limit, all the contributions of the path integral are localized on the field configurations which satisfy $\delta V = 0$ [7]. In quantum field theory, this technique achieved great success [1,2] and uncovered structures of the interacting supersymmetric field theories in various dimensions. The necessary ingredients for this t independence are (1) off-shell supersymmetry δ, (2) supersymmetric invariant action $S = \int L$, and (3) supersymmetric exact action δV where V is a certain functional of the fields, which satisfy $\delta^2 V = 0$. Naively, we expect that its analog to the supergravity provides us an unknown structure of quantum gravity. In this paper, we try to uncover it.

In order to apply the above localization argument, the missing piece is the supersymmetric exact action δV, and we find that the following actions are candidates for the appropriate actions δV,

$$\begin{aligned} L_{R^2_{\mu\nu}+\rm cosm} &= -\frac{1}{8}L_{R^2_{\mu\nu}} - \frac{1}{4}L_C\bar{\psi}^{ab}(\Omega^-)\psi_{ab}(\Omega^-) \\ &\quad \left(= -\frac{1}{8}L_{\rm SYM} - \frac{1}{4}L_C\bar{\chi}\chi\right), \end{aligned} \tag{11}$$

$$\begin{aligned} L_{R^2+\rm cosm} &= L_{R^2} + \frac{1}{4}L_C\bar{\psi}_{ab}(\Omega^-)\gamma^{ab}\gamma^{cd}\psi_{cd}(\Omega^-) \\ &\quad \left(= L_{\rm matter} - \frac{1}{4}L_C\bar{\lambda}\lambda\right), \end{aligned} \tag{12}$$

where L_C is the supersymmetric cosmological constant given in (4). In fact, one can verify the following relations:

$$\delta(e[\bar{\chi}\delta\chi]) = (\bar{\epsilon}\epsilon)L_{R^2_{\mu\nu}+\rm cosm} \tag{13}$$

$$\delta(e[\bar{\lambda}\delta\lambda]) = (\bar{\epsilon}\epsilon)L_{R^2+\rm cosm}. \tag{14}$$

These relations show that above $L_{R^2_{\mu\nu}+\rm cosm}$ and $L_{R^2+\rm cosm}$ are SUSY exact terms. Of course, the Lagrangians $L_{R^2_{\mu\nu}}$ and L_{R^2} preserve supersymmetry. However, in each case (11) or (12), one has a *mass term* for the fermion, and it is a typical *supersymmetry breaking term*, where the supersymmetry breaking is given by the supersymmetric cosmological constant term L_C.

One might wonder why these SUSY exact actions are not SUSY invariant. The reason is as follows. In the rigid limit, we have $\delta^2 = 0$ in the field theoretical sense, and one can show SUSY invariance just by adding additional δ to (13) or (14). However, if we do not take a rigid limit ($\psi_\mu = 0$), then we have $\delta^2 \neq 0$. As a result, (13) and (14) are *not* SUSY invariant, even though they are SUSY exact.

IV. NAIVE ATTEMPT TOWARD GRAVITY LOCALIZATION

Let us discuss the localization argument on supergravity based on the results in the previous section. As explained above, the only embarrassing term is the mass term for the graviton ψ_μ, or equivalently χ or λ in (11) or (12), which breaks the SUSY invariance of SUSY exact term (13) and (14). To overcome the problem, here we try to eliminate it just by inserting the delta function $\delta(L_c)$ to the path integral in (10)

$$Z(t) = \int \mathcal{D}e_\mu{}^a \mathcal{D}\psi_\mu \mathcal{D}S \delta(L_C) e^{i\int d^3x L_{EH} + i\int d^3x L_C + it\delta V}, \quad (15)$$

where we take $S = \int(L_{EH} + L_C)$, and δV is the one in (11) or (12). It might look strange, but since the delta function can be written by introducing the auxiliary field φ as an integral formula,

$$\delta(L_C) = \int \mathcal{D}\varphi e^{i\int d^3x L_C \cdot \varphi}, \quad (16)$$

we can rewrite (15) as

$$Z(t) = \int \mathcal{D}e_\mu{}^a \mathcal{D}\psi_\mu \mathcal{D}S \mathcal{D}\varphi e^{i\int d^3x L_{EH} + i\int d^3x L_C(1+\varphi) + it\delta V}. \quad (17)$$

If the supersymmetric invariance for the deformed cosmological constant term and for the path integral measure are achieved, then this $Z(t)$ becomes t independent, and we can utilize the localization technique by taking the $t \to \infty$ limit. For that purpose, we require

$$\delta(\varphi L_c) = 0. \quad (18)$$

If the SUSY variation of the cosmological constant term is total divergence, say, $\delta L_C = \nabla_\mu J^\mu$, then (18) implies that $\delta\varphi$ should be defined linear with respect to φ, such as

$$\delta\varphi = -\frac{\nabla_\mu J^\mu}{L_C}\varphi. \quad (19)$$

However, this induces a quantum anomaly, *i.e.*, the Jacobian for the supersymmetry variation of φ is not one. In order to apply the conventional supersymmetric localization technique, the Jacobian for the supersymmetry variation should vanish; therefore, this naive method does not work, unfortunately.

V. CONCLUSION, DISCUSSION, AND FUTURE WORK

In this paper, we discussed a possibility for the application of localization technique to the quantum gravity path integral. We tried to conduct direct gravity path integral by constructing SUSY exact terms in 3D supergravity. Although SUSY exact terms are constructed, naive procedure for localization calculation fails. Our main discovery is that the $R_{\mu\nu}^2$ supersymmetric invariant, $L_{R_{\mu\nu}^2}$, and R^2 supersymmetric invariant, L_{R^2}, can be represented as SUSY *exact* terms *up to gravitino mass terms*, which break supersymmetric invariance and its breaking is given by the supersymmetric cosmological constant term L_C. This prevents us from applying a naive localization technique to supergravity within these setups [8]. We would like to make some comments about our (rather negative) results.

First, let us comment on the difficulty of the gravity sector localization computation. In our case, as one can find the algebraic structure of local SUSY δ on three-dimensional supergravity in [9], squared SUSY δ^2 is not zero and contains SUSY δ, too. This structure is coming from the existence of the gravitino, and it is absent in the rigid SUSY limit δ_{rigid} [10] which guarantees the localization computation because of the nilpotent nature $\delta_{\mathrm{rigid}}^2 = 0$ in many cases. However, the possibility for localization in supergravity is not excluded even for three-dimensional $\mathcal{N} = 1$ because what we find is just the relationship (11)–(14). Therefore, if one can find certain better SUSY exact terms and succeed in canceling the obstructing mass term, then it should work.

Second, the mass terms in our SUSY exact Lagrangians, (11) and (12), seem to be "universal" mass terms because they are always proportional to the supersymmetric cosmological constant L_C in (4). We have no *a priori* reason to get such a supersymmetric coefficient as the mass parameter, but there might exist certain deep reasons that could be related to the algebraic structure on supergravity.

Third, it may be good to consider the same problem with extended local supersymmetries, $\mathcal{N} \geq 2$. For example, we can find off-shell formulation of $\mathcal{N} = 2$ supergravity in [11,12]. In three dimensions, the conventional field theoretical localization computation is available only for $\mathcal{N} \geq 2$; therefore, the situation there could be better.

It will also be interesting to consider the analog of our argument with *Euclidean* supergravity. (For relevant works on three-dimensional Euclidean *pure* gravity with negative cosmological constant, see for example, [3,13,14].) The crucial difference is that, in the Euclidean signature, modular invariance is strong enough to determine (some of) the nonperturbative effects. It would be great if we could derive the summation over the modular group discussed in [14–17] in a direct supergravity localization calculation without relying on the power of modular invariance. This should be done along the lines of [3], where the sum over the modular group appears naturally as the sum over all of the localization locus, $\mathcal{F}_{\mu\nu} = 0$, which are solutions of all the complex Einstein equations.

Before we end, let us discuss the physical meaning of the conducting gravity path integral, $Z = \int[\mathcal{D}g_{\mu\nu}]e^{iS[g_{\mu\nu}]}$. Even if we succeed in conducting the metric path integral $[\mathcal{D}g_{\mu\nu}]$ exactly, whether or not it gives an *exact* partition function for quantum gravity depends on whether the metric $g_{\mu\nu}$ is a fundamental degree of freedom in quantum gravity. We have learned from holography that bulk gravity is an effective theory, which is valid and emerging typically in the large-N limit of QCD-like $SU(N)$ gauge theory as a dual effective description. Furthermore, a metric, which is dual to the gauge-singlet stress tensor, is a dominating degree of freedom only in the low-temperature phase [18,19]. In fact, in the high-temperature phase, rather than the metric, the black hole microstates are the dominating degrees of freedom [20]. Given these, how meaningful is the bulk metric path integral calculation?

To answer this, the analogy to QCD helps. Gravity in the low-temperature phase is like the chiral Lagrangian in QCD, where the dynamical degrees of freedom are pion field π's, instead of quarks and gluons. Therefore, conducting the gravity path integral $\int[\mathcal{D}g_{\mu\nu}]e^{iS[g_{\mu\nu}]}$ corresponds to conducting the pion field path integral $\int[\mathcal{D}\pi]e^{iS_{\mathrm{chiral}}}$ in the chiral Lagrangian. Of course, we know the fundamental theory behind the chiral Lagrangian is QCD, and the *exact* answer for the partition function for QCD can be obtained only by conducting the path integral for the quark-gluon fields, rather than the pion fields. The pion field path integral of the chiral Lagrangian never gives the right answer for QCD, due to its lack of quark and gluon degrees of freedom which are dominating in the high-temperature phase [21]. As one cannot describe the quark-gluon plasma by multipion fields, we expect that black hole microstates are not describable by multigravitons (see [22] for a nice overview) [23]. In this way, we expect that the naive bulk metric path integral, $Z=\int[\mathcal{D}g_{\mu\nu}]e^{iS[g_{\mu\nu}]}$, is not a nonperturbatively defined quantity, at least in the bulk where we have spacetime dimensions larger than three. (Note, however, that in three dimensions, modular invariance of the partition function is powerful enough to determine the contributions of the BTZ black hole microstates; see [3,13,14,24].) To obtain an exact partition function for full quantum gravity, we have to rely on the dual nonperturbatively defined boundary theory path integral [25].

However what we try to calculate in this paper is *not* this quantity (partition function), but rather supersymmetric index due to the fermion boundary condition. Then the situation is totally different: Index calculations in field theory quite often works to count the supersymmetric black hole microstates. This is because of supersymmetry, significant reduction of degrees of freedom occurs. Therefore, the SUSY index calculation from the bulk metric by conducting $\int[\mathcal{D}g_{\mu\nu}]$ is still meaningful even in bulk.

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APPENDIX A: SPINOR NOTATIONS AND FORMULAS

The clifford algebra is generated by the following two-by-two matrices:

$$\gamma^0=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \gamma^2=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{A1}$$

The charge conjugation matrix is

$$C=i\gamma^0. \tag{A2}$$

We always use Majorana fermions throughout this paper. Thus,

$$\bar\psi=\psi^\dagger i\gamma^0=\psi^T C \tag{A3}$$

is equivalent to $\psi^*=\psi$, and it means a real fermion. For Majorana fermions ψ,χ,ϵ, we have the following formulas:

$$\bar\psi\chi=\bar\chi\psi, \tag{A4}$$

$$\overline{(\gamma^\mu\psi)}\chi=-\bar\psi\gamma^\mu\chi, \tag{A5}$$

$$\bar\chi\gamma^\mu\psi=-\bar\psi\gamma^\mu\chi, \tag{A6}$$

$$\bar\chi\gamma^\mu\gamma^\nu\psi=\bar\psi\gamma^\nu\gamma^\mu\chi, \tag{A7}$$

$$\bar\chi\gamma^\mu\gamma^\nu\gamma^\rho\psi=-\bar\psi\gamma^\rho\gamma^\nu\gamma^\mu\chi, \tag{A8}$$

$$\epsilon(\bar\chi\psi)+2(\bar\chi\epsilon)\psi+\gamma^\mu\epsilon(\bar\chi\gamma_\mu\psi)=0 \tag{A9}$$

$$(\bar\psi\epsilon)(\bar\epsilon\chi)=-\frac{1}{2}(\bar\epsilon\epsilon)(\bar\psi\chi). \tag{A10}$$

The formula in (A10) is useful in the calculation of (13) and (14).

APPENDIX B: PROOF OF EQ. (13)

As noted in the main part of this paper, if we define $A^I_\mu=\Omega^{+ab}_\mu$, $\chi^I=\psi^{ab}(\Omega^-)$, these redefined multiplet satisfy

$$\delta A_\mu=-(\bar\epsilon\gamma_\mu\chi),\qquad \delta\chi=\frac{1}{8}\gamma^{\mu\nu}(F_{\mu\nu}+2\bar\psi_{[\mu}\gamma_{\nu]}\chi)\epsilon, \tag{B1}$$

where we omit the index I from now on. Just using these SUSY transform, we calculate

$$\delta[e(\bar\chi\delta\chi)]=\delta e\cdot(\bar\chi\delta\chi)+e(\overline{\delta\chi}\delta\chi)+e(\bar\chi\delta^2\chi) \tag{B2}$$

as follows. We use (A10) many times:

$$\delta e\cdot(\bar\chi\delta\chi)=(\bar\epsilon\epsilon)e\bigg\{-\frac{1}{32}(\bar\chi\gamma^{\mu\nu}\gamma^\rho\psi_\rho)F_{\mu\nu}-\frac{1}{16}(\bar\chi\chi)(\bar\psi_\mu\gamma^{\mu\nu}\psi_\nu) -\frac{1}{16}(\bar\chi\chi)(\bar\psi_\mu\psi^\mu)\bigg\} \tag{B3}$$

$$e(\overline{\delta\chi}\delta\chi) = (\bar{\epsilon}\epsilon)e\left\{\frac{1}{32}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8}(\bar{\psi}^\mu\gamma^\nu\chi)F_{\mu\nu} + \frac{1}{16}(\bar{\chi}\chi)(\bar{\psi}_\mu\psi^\mu) + \frac{1}{32}(\bar{\chi}\chi)(\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu)\right\} \quad \text{(B4)}$$

$$e(\bar{\chi}\delta^2\chi) = (\bar{\epsilon}\epsilon)e\left\{\left[\frac{1}{16}(\bar{\chi}\gamma^{\rho\nu}\gamma^\mu\psi_\rho) + \frac{1}{32}(\bar{\psi}_\rho\gamma^{\mu\nu}\gamma^\rho\chi) + \frac{1}{16}(\bar{\psi}^\mu\gamma^\nu\chi)\right]F_{\mu\nu} + \frac{1}{16}(\bar{\chi}\chi)(\bar{\psi}_\mu\psi^\mu - \frac{1}{64}(\bar{\chi}\chi)(\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu) + \frac{1}{4}(\bar{\chi}\gamma^\mu D_\mu\chi) - \frac{6}{16}S(\bar{\chi}\chi)\right\}. \quad \text{(B5)}$$

Summing up (B3), (B4), and (B5), we get the result in (13).

APPENDIX C: PROOF OF EQ. (14)

If we define $\phi = S$, $\lambda = \gamma^{\mu\nu}\psi_{\mu\nu}(\Omega^-)$, $f = \hat{R}(\Omega^\pm)$, then these fields satisfy

$$\delta\phi = \frac{1}{4}\bar{\epsilon}\lambda, \qquad \delta\lambda = \gamma^\nu\epsilon\left[\partial_\nu\phi - \frac{1}{4}\bar{\psi}_\nu\lambda\right] - \frac{1}{4}\epsilon f$$

$$\delta f = -\bar{\epsilon}\gamma^\mu\left[D_\mu(\hat{\omega})\lambda - \gamma^\nu\psi_\mu\left(\partial_\nu\phi - \frac{1}{4}\bar{\psi}_\nu\lambda\right) + \frac{1}{4}f\psi_\mu\right] + \frac{1}{2}S(\bar{\epsilon}\lambda). \quad \text{(C1)}$$

By using these SUSY transformations, we calculate

$$\delta[e(\bar{\lambda}\delta\lambda)] = \delta e\cdot(\bar{\lambda}\delta\lambda) + e(\overline{\delta\lambda}\delta\lambda) + e(\bar{\lambda}\delta^2\lambda), \quad \text{(C2)}$$

and each term is given as follows:

$$\delta e\cdot(\bar{\lambda}\delta\lambda) = (\bar{\epsilon}\epsilon)e\left\{-\frac{1}{4}(\bar{\psi}_\mu\gamma^\mu\gamma^\nu\lambda)\partial_\nu\phi - \frac{1}{32}(\bar{\lambda}\lambda)(\bar{\psi}_\mu\psi^\mu) - \frac{1}{32}(\bar{\lambda}\lambda)(\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu) + \frac{1}{16}f(\bar{\lambda}\gamma^\mu\psi_\mu)\right\}, \quad \text{(C3)}$$

$$e(\overline{\delta\lambda}\delta\lambda) = (\bar{\epsilon}\epsilon)e\left\{-\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}(\bar{\psi}^\mu\lambda)\partial_\mu\phi + \frac{1}{32}(\bar{\lambda}\lambda)(\bar{\psi}_\mu\psi^\mu) + \frac{1}{16}f^2\right\}, \quad \text{(C4)}$$

$$e(\bar{\lambda}\delta^2\lambda) = (\bar{\epsilon}\epsilon)e\left\{\frac{1}{4}(\bar{\psi}_\mu\gamma^\nu\gamma^\mu\lambda)\partial_\nu\phi + \frac{1}{32}(\bar{\lambda}\lambda)(\bar{\psi}_\mu\psi^\mu) - \frac{1}{32}(\bar{\lambda}\lambda)(\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu) - \frac{1}{8}S(\bar{\lambda}\lambda) - \frac{1}{4}(\bar{\lambda}\gamma^\mu D_\mu\lambda) + \frac{1}{64}(\bar{\lambda}\lambda)(\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu) - \frac{1}{16}f(\bar{\lambda}\gamma^\mu\psi_\mu)\right\}. \quad \text{(C5)}$$

Combining (C3), (C4), and (C5), we arrive at (14).

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[7] Here, we consider the partition function with Lorentz signature. So it is plausible to take the weight as e^{iS}. In the $t\to\infty$ limit, the configuration which satisfies $\delta V = 0$ is dominant thanks to the Riemann-Lebesgue lemma.

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[20] These can easily be seen from comparison of the N dependence of the entropy between thermal gas and black holes in bulk and the confinement and deconfinement phase in the boundary [18,19].

[21] If we pay attention to physics only in the confinement phase, namely, only the low-temperature phase, and neglect all nonperturbative effects, then the results of the pion path integral of the chiral Lagrangian is still meaningful as an effective theory.

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[25] It gives, at most, an approximately valid and meaningful quantity only in the low-temperature phase, i.e., the bulk phase without black holes.