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Highly nonlinear wave solutions in a dual to the chiral model

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We consider a two-dimensional scalar field theory with a nilpotent current algebra, which is dual to the Principal Chiral Model. The quantum theory is renormalizable and not asymptotically free; the theory is strongly coupled at short distances (encountering a Landau pole). We suggest it can serve as a toy model for $\lambda\phi^4$ theory in four dimensions, just as the principal chiral model is a useful toy model for Yang-Mills theory. We find some classical wave solutions that survive the strong coupling limit and quantize them by the collective variable method. They describe excitations with an unusual dispersion relation $\omega \propto |k|^{\frac{2}{3}}$. Perhaps they are the "preons" at strong coupling, the bound states of which form massless particles over long distances.

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I. INTRODUCTION

We study the field theory [1–3] with equations of motion

$$\ddot{\phi} = \lambda[\dot{\phi}, \phi'] + \phi'', \tag{1.1}$$

where ϕ is valued in a Lie algebra, $\phi: \mathbb{R}^{1,1} \to \mathfrak{su}(2)$. This follows from the action

$$S_1 \equiv \int \mathcal{L}_1 dxdt$$
$$= \int \mathrm{Tr}\left\{\frac{1}{2\lambda}\dot{\phi}^2 - \frac{1}{2\lambda}\phi'^2 + \frac{1}{3}\phi[\dot{\phi}, \phi']\right\}dxdt. \tag{1.2}$$

In the $\lambda \to 0$ limit, these equations admit linear wave solutions. But in the high-coupling regime, the theory is dominated by nonlinear effects.

S_1 is closely tied to other models and subjects, which we elaborate on in Sec. II. These include the study of slow light, the Wess-Zumino-Witten (WZW) model, and the mathematical theory of hypoelliptic operators. Of particular interest in this paper, the model described by S_1 is also classically dual to the well-studied principal chiral model, described by the action

$$S_2 = \int \mathcal{L}_2 dxdt$$
$$= \frac{1}{2f}\int \mathrm{Tr}\{(g^{-1}\dot{g})^2 - c^2(g^{-1}g')^2\}dxdt, \tag{1.3}$$

where $g: \mathbb{R}^{1,1} \to SU(2)$. This is a special case of the nonlinear sigma model, with target space $SU(2)$.

Despite their classical equivalence, S_1 and S_2 lead to entirely different quantum theories. S_2 gives an asymptotically free theory: at short distances $f \to 0$, giving us free massless excitations. But the true particles that survive to long distances are bound states of nonzero mass [4,5]. For this reason, the principal chiral model is often used as a toy model for four-dimensional Yang-Mills theory, notorious for its mathematical complexity. Not only do the two theories share similar short-distance behavior, but the existence of a mass gap in the principal chiral model has served as a proof of concept for the conjectured mass gap in Yang-Mills (though neither can yet be proven with full mathematical rigor).

S_1, on the other hand, leads to a renormalizable but not asymptotically free quantum theory. At short distances the coupling constant $\lambda \to \infty$, while at long distances we have weakly nonlinear massless excitations. It makes sense to use S_1 as a 2D toy model for strongly coupled theories, in particular four-dimensional $\lambda\phi^4$ theory.[1] The behavior of quantum field theories at high coupling is notoriously intractable, and the physical meaning of such theories is still up for debate. For this reason, it is still necessary to search for simple examples of such theories and try to gleam what meaning, if any, they have in the short-distance limit.

In addition to sharing short-distance behavior, both the S_1 model and $\lambda\phi^4$ theory can be described by hypoelliptic Hamiltonian operators with a step-3 nilpotent bracket algebra, suggesting some algebraic structure in common (Sec. II B and Appendix A). The S_1 model's relative simplicity makes it a good candidate for attempting to probe the high coupling regime of field theories in general, but the connection to $\lambda\phi^4$ theory seems the closest. Additionally, its classical duality to the principal chiral

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[1]That is, pure $\lambda\phi^4$ theory, describing a Higgs-like particle with no coupling to fermions.

model motivates a juxtaposition of the two theories in the classical and quantum formulations.

To glimpse what becomes of our theory in the high coupling limit, we take the modest approach of finding nonlinear wave-type solutions to the classical model which survive the $\lambda \to \infty$ limit (Sec. III). This set of solutions defines a mechanical system or "reduced system" in each of the dual models. While they physically appear very different, the resulting classical solutions can be mapped from one system to another. We quantize these collective variables to determine their dispersion relation (Sec. IV) in the short-distance limit for each theory. We have in mind the sine-Gordon theory, the solitons of which turn out to be the fundamental constituents that bind to form scalar particles [6,7].

These reduced quantum theories yield two different results. In particular, the reduced model of S_1 has an exotic dispersion relation in the short-distance limit. We postulate that its spectrum may hint at the fundamental constituents of the highly coupled theory, which need not behave like traditional particles at all. In Sec. V we offer concluding remarks and a side-by-side comparison of our work with S_1 and S_2.

A. Notation

We regard $\phi = \frac{1}{2i}[\phi_1\sigma_1 + \phi_2\sigma_2 + \phi_3\sigma_3] = \frac{1}{2i}\boldsymbol{\phi}\cdot\boldsymbol{\sigma}$ as a traceless anti-Hermitian matrix. Recall then that the commutator and cross product are related by

$$[X, Y] = \frac{1}{2i}(\mathbf{X}\times\mathbf{Y})\cdot\boldsymbol{\sigma}. \tag{1.4}$$

Also, we define $\mathrm{Tr}X \equiv -2\mathrm{tr}X$ so that

$$\mathrm{Tr}\phi^2 = \phi_1^2 + \phi_2^2 + \phi_3^2. \tag{1.5}$$

In relativistically invariant notation, Eqs. (1.1) and (1.2) can be written as

$$\partial^\mu\partial_\mu\phi_a - \frac{\lambda}{2}\epsilon_{abc}\epsilon^{\mu\nu}\partial_\mu\phi^b\partial_\nu\phi^c = 0, \tag{1.6}$$

$$S_1 = \frac{1}{2\lambda}\int \partial_\mu\phi^a\partial_\nu\phi^a\eta^{\mu\nu}d^2x + \frac{1}{6}\int \epsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\epsilon^{\mu\nu}d^2x, \tag{1.7}$$

where $\mu,\nu = 0, 1$ and $a, b, c = 1, 2, 3$; also, $\epsilon^{\mu\nu}$ and ϵ_{abc} are the Levi-Civitá tensors. This is a particular case of the general sigma model studied in Ref. [8] as the background of string theory, with a flat metric on the target space and a constant 3-form field ϵ_{abc}.

II. RELATION TO OTHER MODELS

A. $c \to 0$ limit and slow light

Consider the equations of motion (1.1) where the speed of linear propagation at low coupling is taken to be c rather than 1:

$$\ddot{\phi} = \lambda[\dot{\phi}, \phi'] + c^2\phi''. \tag{2.1}$$

If we rescale $\phi \to \lambda^a\phi$, $t \to \lambda^b t$, this becomes

$$\lambda^{a-2b}\ddot{\phi} = \lambda^{1+2a-b}\dot{\phi}\times\phi' + c^2\lambda^a\phi''. \tag{2.2}$$

Set $a = 2b$ and $1 + 2a = b$ to get

$$\ddot{\phi} = \dot{\phi}\times\phi' + c^2\lambda^{-\frac{2}{3}}\phi''. \tag{2.3}$$

Thus, the strong coupling limit $\lambda \to \infty$ at fixed c is equivalent to the limit $c \to 0$ with $\lambda = 1$:

$$\ddot{\phi} = \dot{\phi}\times\phi'. \tag{2.4}$$

The strongly coupled limit can be thought of as the limit in which the waves move very slowly. It has been noted in that literature [9] that when the speed of light in a medium is small, nonlinear effects are magnified. Although the specific equations appearing there are different, it is possible that the solutions of the sort we study are of interest in that context as well.

From a field theoretic context, the equivalence of these limits seems troubling. At short distances, the highly coupled theory will not be relativistic. It is a sort of "postrelativistic" regime, where $c \to 0$. This is much the opposite of the case in the theory of S_2; there the short-distance excitations are massless but form massive bounds states which survive to long distances and can be nonrelativistic in the traditional $c \to \infty$ sense. Perhaps some exotic excitations at high coupling are in fact the fundamental constituents in the S_1-model, forming as bound states the ordinary massless particles which appear in the long-distance limit. As we know from the quark model, the short-distance excitations do not need to be particles in the usual sense; they could be confined. In any case, it is important to know what solutions might survive the high coupling limit, whether they be unphysical or simply unintuitive.

We will see an example of wave solutions which classically survive the $c \to 0$ limit, continuing to propagate through nonlinearity alone. Since the energy density is constant, these solutions do not violate causality; they are analogous to the continuous wave solutions in a medium where the phase velocity is greater than c.

B. Sub-Riemannian geometry and the strong coupling limit

Many physical problems (Yang-Mills, fluid mechanics) become intractable in the strong coupling limit where the nonlinearities dominate. It would be nice to have a unified geometric approach to understanding these systems. We have such an approach in the weak coupling limit: small perturbations around a stable equilibrium are equivalent to a harmonic oscillator.

A larger picture emerges if we think in terms of Riemannian and sub-Riemannian geometry. The orbits of many mechanical systems of physical interest (again, Yang-Mills or incompressible fluids) can be thought of as geodesics in some appropriate Riemannian manifold. In the simplest case, the harmonic oscillator describes geodesics in the Heisenberg group. The anharmonic oscillator (and many nonlinear field theories with quartic coupling) can also be thought of as geodesic motion on a nilpotent Lie group, by introducing an additional generator (see Appendix A for more detail).

In the limit of strong coupling, the metric degenerates and becomes sub-Riemannian [10]. That is, the contravariant metric tensor has some zero eigenvalues so that it can be written as $\sum_j X_j \otimes X_j$ for some vector fields X_j which may have a linear span smaller than the tangent space. Moreover, in the cases of interest, these vector fields satisfy the celebrated Hörmander condition: X_j along with their repeated commutators span the tangent spaces at every point. In such a case, there are still geodesics connecting every pair of sufficiently close points (Chow-Rashevskii theorem [10]). Thus, we can define a distance between pairs of points as the shortest length of geodesics.

These ideas came to the notice of many physicists following a model for the self-propulsion of an amoeba [11], though they have roots in the Carnot-Caratheodory geometric formalism of thermodynamics and in control theory. Hörmander [12] discovered independently that the same criterion is sufficient for the sub-Riemannian Laplace operator $\Delta = \sum_j X_j^2$ to be *hypoelliptic*, meaning the solution f to the inhomogenous equation $\Delta f = u$ is smooth whenever the source u is smooth. This can be thought of the quantum version of the above condition on subgeodesic connectivity.

This kind of sub-Riemannian geometry may present a powerful geometric framework for strongly coupled field theories. The example we work out in this paper is arguably the simplest interesting case of a strongly coupled field theory, and the solutions we study correspond to sub-Riemannian geodesics in the limit $\lambda \to \infty$. We hope to apply such geometric ideas to other cases in the future, using this as a prototype.

C. Relation to the WZW model

We can also regard our equations as a limiting case of the Wess-Zumino-Witten model[2] [13]

$$S_{\rm WZW} = \frac{1}{4\lambda_1^2}\int {\rm tr}\partial^\mu g \partial_\mu g^{-1} d^2x + \frac{n}{24\pi}\int_{M_3} {\rm tr}(g^{-1}dg)^3 \quad (2.5)$$

as $n \to \infty$ and $\lambda_1 \to 0$, keeping $\lambda = \lambda_1^2 (n/2\pi)^{\frac{2}{3}}$ fixed.[3] To see this, let $g(x) = e^{bi\sigma_a \phi^a(x)}$, and expand in powers of b:

$$S_{\rm WZW} = \frac{b^2}{2\lambda_1^2}\int \partial^\mu \phi^a \partial_\mu \phi^a + \frac{n}{24\pi} b^3 \int_{M_3} 2\epsilon_{abc} d\phi^a d\phi^b d\phi^c + \cdots \quad (2.6)$$

To this order the WZW term is an exact differential, so we can write it as an integral over space-time,

$$S_{\rm WZW} = \frac{1}{2}\frac{b^2}{\lambda_1^2}\int \partial^\mu \phi^a \partial_\mu \phi^a + \frac{n}{12\pi} b^3 \int \epsilon_{abc} \phi^a d\phi^b d\phi^c + \cdots \quad (2.7)$$

$S_{\rm WZW}$ reduces to S_1 if we identify $b^3 = 2\pi/n$ and λ as above. By taking this limit, we can easily get the renormalization of our model. Recall that [13] the one loop renormalization group equation of the $O(N)$ WZW model is

$$\frac{d\lambda_1^2}{d\log\Lambda} = -\frac{\lambda_1^4(N-2)}{2\pi}\left[1 - \left(\frac{\lambda_1^2 n}{4\pi}\right)^2\right]. \quad (2.8)$$

We need the particular case of $N = 4$ corresponding to the target space being $S^3 \approx SU(2)$. Thus, in our limit $n \to \infty$, $\lambda_1 \to 0$ keeping λ fixed,

$$\frac{d\lambda}{d\log\Lambda} = \frac{\lambda^4}{4\pi}. \quad (2.9)$$

It is useful to take this limit rather than calculating loop corrections from scratch, as the renormalization group evolution of the WZW has been studied to high order [14,15]. Including these higher order terms does not alter the short-distance divergence of λ.

D. Duality with the principal chiral model

We have now seen that the S_1 model is strongly coupled in the short-distance limit. Yet, as a classical field theory, it can be viewed [1,2] as a dual to the asymptotically free principal chiral model with equation of motion

$$\partial^\mu[g^{-1}\partial_\mu g] = 0, \qquad g\colon \mathbb{R}^{1,1} \to SU(2). \quad (2.10)$$

To see this, we define the currents

[2] Witten's Tr is our tr. His λ is our λ_1.

[3] Here, M_3 is a 3-manifold of which the two-dimensional space-time is the boundary. We do not require λ_1^2 to take the conformally invariant value $\frac{4\pi}{n}$.

$$I = \frac{1}{\lambda}\dot{\phi}, \qquad J = \phi' \tag{2.11}$$

so the equations of motion become

$$\dot{J} = \lambda I', \qquad \dot{I} = \lambda[I, J] + \frac{1}{\lambda}J'. \tag{2.12}$$

We can solve the second equation with the relations

$$I = \frac{1}{\lambda^2}g^{-1}g', \qquad J = \frac{1}{\lambda}g^{-1}\dot{g}. \tag{2.13}$$

Then the first equation becomes

$$\partial_0[g^{-1}\dot{g}] = \partial_1[g^{-1}g'], \tag{2.14}$$

which is the nonlinear sigma model. Thus, the same classical equations of motion follow from the action

$$S_2 = \frac{1}{2f}\int \mathrm{Tr}\{(g^{-1}\dot{g})^2 - c^2(g^{-1}g')^2\}dxdt \tag{2.15}$$

if we identify $f = \lambda^2$. A summary of relevant correspondences in the dual models can be found in Table I at the end of Sec. V.

We also briefly note that our theory is closely related to the sigma model on the Heisenberg group (see Ref. [16]).

III. REDUCTION TO A MECHANICAL SYSTEM

We will look at propagating waves of the form

$$\phi(t, x) = e^{Kx}R(t)e^{-Kx} + mKx, \qquad K = \frac{i}{2}k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.1}$$

for constants k, m. These solutions are equivariant under translations: the "potential" ϕ changes by an internal rotation and a constant shift under translation, while the currents only change only by the internal rotation. Thus, the energy density is constant. They are to be contrasted with soliton solutions, which have energy density concentrated at the location of the soliton. They are more analogous to the plane wave solutions of the wave equation, or a continuous wave laser beam. Moreover, the currents

$$I = \frac{1}{\lambda}e^{Kx}\dot{R}e^{-Kx}, \qquad J = e^{Kx}\{[K, R] + mK\}e^{-Kx} \tag{3.2}$$

are periodic in space with wavelength $\frac{2\pi}{k}$. Defining

$$L \equiv [K, R] + mK, \qquad S \equiv \dot{R} + \frac{1}{\lambda}K, \tag{3.3}$$

we can write the equations of motion and identity (2.12) in a symmetric form,

$$\dot{L} = [K, S], \qquad \dot{S} = \lambda[S, L]. \tag{3.4}$$

This new choice of variables will allow us to connect to the dual theory, identify the conserved quantities, and pass to the quantum theory more easily.

A. Reduced system Lagrangian

Three conserved quantities follow immediately:

$$\begin{aligned} s^2k^2 &\equiv \mathrm{Tr}S^2 \\ C_1k^2 &\equiv \mathrm{Tr}SL \\ C_2k^2 &\equiv \mathrm{Tr}\left[\frac{1}{2}L^2 - \frac{1}{\lambda}KS\right]. \end{aligned} \tag{3.5}$$

The quantity s will be of importance in the dual picture, while the other constants have less obvious roles there. Moreover, we have the identity

$$\mathrm{Tr}KL = mk^2. \tag{3.6}$$

Of the six independent variables in S and L, only two remain after taking into account these constants of motion. The dynamics are described by the effective Lagrangian density (dropping a total time derivative and an overall factor of volume of space divided by λ)

$$\begin{aligned} \mathcal{L}_1 &= \mathrm{Tr}\left\{\frac{1}{2}\dot{R}^2 + \frac{\lambda}{3}R[\dot{R}, [K, R] + mK] - \frac{1}{2}([K, R] + mK)^2\right\} \\ &= \mathrm{Tr}\left\{\frac{1}{2}\left(S - \frac{1}{\lambda}K\right)^2 + \frac{\lambda}{3}R\left[S - \frac{1}{\lambda}K, L\right] - \frac{1}{2}L^2\right\} \end{aligned} \tag{3.7}$$

and Hamiltonian density

$$H_1 = \mathrm{Tr}\left[\frac{1}{2}\left(S - \frac{1}{\lambda}K\right)^2 + \frac{1}{2}L^2\right], \tag{3.8}$$

B. Reduction to one degree of freedom

It is useful to work with the first two components of R as a single complex variable. Defining $Z = R_1 + iR_2$, we can write explicitly

$$L = \frac{k}{2}\begin{pmatrix} im & \bar{Z} \\ -Z & -im \end{pmatrix}. \tag{3.9}$$

To describe the third component, we define

$$u \equiv \frac{1}{k}\dot{R}_3 - \frac{1}{\lambda}, \tag{3.10}$$

allowing us to write a similarly compact expression for S,

$$S = \frac{1}{2i}\begin{pmatrix} uk & \dot{\bar{Z}} \\ \dot{Z} & -uk \end{pmatrix}. \tag{3.11}$$

The three conserved quantities (3.5) can now be written in terms of Z and u as

$$\begin{aligned} s^2k^2 &= u^2k^2 + |\dot{Z}|^2 \\ C_1k^2 &= \frac{ik}{2}[\bar{Z}\dot{Z} - \dot{\bar{Z}}Z] - mk^2u \\ C_2k^2 &= \frac{k^2}{2}\left(m^2 + \frac{2u}{\lambda} + |Z|^2\right). \end{aligned} \tag{3.12}$$

Using the identity

$$\left(\frac{d}{dt}|Z|^2\right)^2 = 4|Z|^2|\dot{Z}|^2 + (\dot{\bar{Z}}Z - \bar{Z}\dot{Z})^2, \tag{3.13}$$

we can combine these three equations to eliminate Z and yield an ordinary differential equation (ODE) for $u(t)$,

$$\dot{u}^2 = k^2\lambda^2\left\{\left[2C_2 - m^2 - \frac{2}{\lambda}u\right](s^2 - u^2) - [mu + C_1]^2\right\}. \tag{3.14}$$

C. Solution in terms of elliptic functions

The ODE for $u(t)$ describes an elliptic curve. Setting $u = av + b$, we can pick the constants

$$a = \frac{2}{k^2\lambda}, \qquad b = \frac{C_2\lambda}{3} \tag{3.15}$$

to bring our ODE to Weierstrass normal form in terms of v:

$$\dot{v}^2 = 4v^3 - g_2v - g_3. \tag{3.16}$$

The somewhat unsightly expressions for g_2 and g_3 can be obtained by symbolic computation:

$$\begin{aligned} g_2 &= \frac{1}{3}k^4\lambda^2(3C_1\lambda m + C_2^2\lambda^2 + 3s^2) \\ g_3 &= \frac{1}{108}k^6\lambda^4(27C_1^2 + 18C_1C_2\lambda m + 4C_2^3\lambda^2 \\ &\quad - 36C_2s + 27m^2s^2). \end{aligned} \tag{3.17}$$

The solution to the Weierstrass differential equation (3.16) is then

$$v(t) = \wp(t+\alpha) \Rightarrow u(t) = \frac{2}{k^2\lambda}\wp(t+\alpha) + \frac{C_2\lambda}{3}, \tag{3.18}$$

where $\wp$ is the Weierstrass P-function and α is a complex constant determined by the initial conditions. We can most immediately solve for $R_3(t)$. Recalling (3.10), we have

$$\dot{R}_3 = \frac{2}{k\lambda}\wp(t+\alpha) + k\left(\frac{C_2\lambda}{3} + \frac{1}{\lambda}\right). \tag{3.19}$$

In order to obtain a sensible solution, $\wp(t+\alpha)$ must be real and bounded. This requires $\mathrm{Im}(\alpha) = |\omega_2|$, where ω_2 is the imaginary half-period of the Weierstrass P-function (which depends on the elliptic invariants g_2, g_3). The real part of α merely shifts our solution in time, so we can take $\alpha = \omega_2$ for simplicity. Using the relationship

$$\int \wp(u)du = -\zeta(u), \tag{3.20}$$

where ζ is the Weierstrass ζ-function, and taking $R_3(0) = 0$ gives the solution

$$R_3(t) = \frac{2}{k\lambda}[\zeta(\omega_2) - \zeta(t+\omega_2)] + \left(\frac{C_2\lambda}{3} + \frac{1}{\lambda}\right)kt. \tag{3.21}$$

The solution for the other two components is found by making the substitution $Z = re^{i\theta}$ in (3.12). Writing $|Z^2| = r^2$ quickly yields

$$r^2(t) = \frac{4}{3}C_2 - m^2 - \frac{4}{k^2\lambda^2}\wp(t+\omega_2). \tag{3.22}$$

Note that the choice of $\mathrm{Re}(\alpha) = 0$ we made earlier implies that $t = 0$ is a turning point of the radial variable, as $\wp'(\omega_2)$ is necessarily 0. It is useful to write

$$r^2(t) = \frac{4}{k^2\lambda^2}[\wp(\Omega) - \wp(t+\omega_2)], \tag{3.23}$$

where

$$\wp(\Omega) = k^2\lambda^2\left(\frac{C_2}{3} - \frac{m^2}{4}\right). \tag{3.24}$$

Then we can use the identity

$$\wp(z) - \wp(\Omega) = -\frac{\sigma(z+\Omega)\sigma(z-\Omega)}{\sigma^2(z)\sigma^2(\Omega)}, \tag{3.25}$$

where σ is the Weierstrass σ-function, in order to simplify a later result. We obtain the solution

$$r(t) = \frac{2}{\lambda k\sigma(\Omega)}\frac{\sqrt{\sigma(t+\omega_2+\Omega)\sigma(t+\omega_2-\Omega)}}{\sigma(t+\omega_2)}. \tag{3.26}$$

To find $\theta(t)$ from (3.12), we substitute $(\dot{\bar{Z}}Z - \bar{Z}\dot{Z}) = -2ir^2\dot{\theta}$, obtaining

$$\dot{\theta} = \frac{C_3}{r^2} + \frac{km\lambda}{2} = \frac{k^2\lambda^2 C_3}{4[\wp(\Omega) - \wp(t+\omega_2)]} + \frac{km\lambda}{2}, \tag{3.27}$$

where

$$C_3 \equiv k\left[\frac{m^3\lambda}{2} - C_1 - m\lambda C_2\right]. \tag{3.28}$$

Using the identity

$$\int \frac{dz}{\wp(z)-\wp(\Omega)} = \frac{1}{\wp'(\Omega)}\left[2z\zeta(\Omega) + \log\frac{\sigma(z-\Omega)}{\sigma(z+\Omega)}\right] \tag{3.29}$$

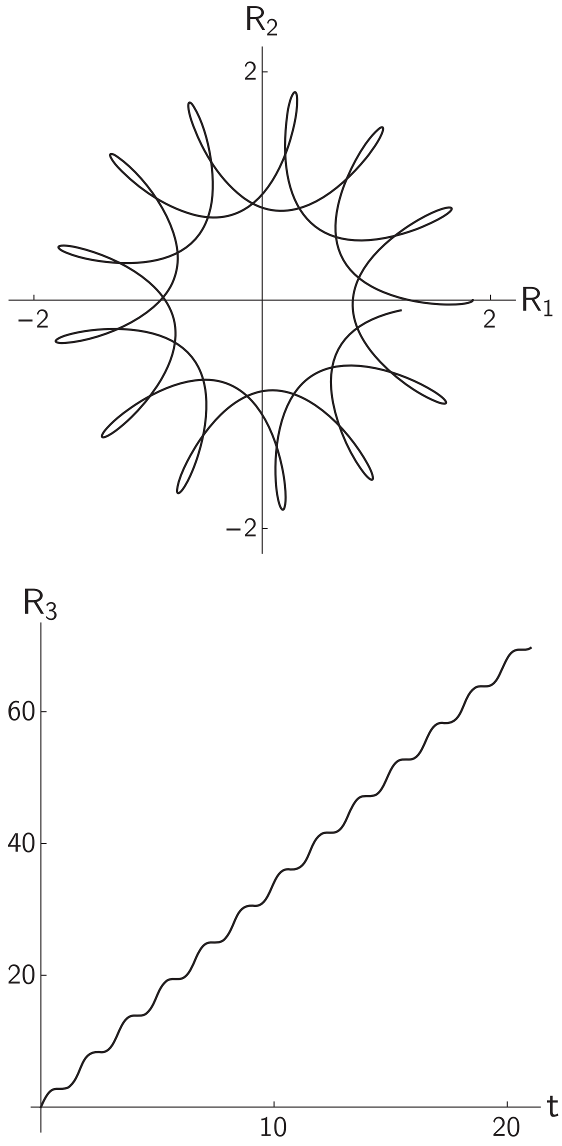

FIG. 1. The orbit in the R_1–R_2 plane (above) and the evolution of R_3 with time (below). The sample solution is plotted for $0 < t < 21$ and uses parameters $k = 1$, $\lambda = 2.4$, $C_1 = 0.5$, $C_2 = 1$, $s = 2$, $m = 0.5$.

and taking $\theta(0) = 0$, we have

$$\theta(t) = \frac{k^2\lambda^2 C_3}{4\wp'(\Omega)}\left[2t\zeta(\Omega) + \log\frac{\sigma(t+\omega_2-\Omega)\sigma(\omega_2+\Omega)}{\sigma(t+\omega_2+\Omega)\sigma(\omega_2-\Omega)}\right] + \frac{km\lambda}{2}t. \tag{3.30}$$

We can use the Weierstrass differential equation (3.16) directly to obtain $\wp'(\Omega) = (i/2)k^2\lambda^2 C_3$, leading to a seemingly remarkable cancellation. We then have

$$e^{i\theta(t)} = \sqrt{\frac{\sigma(t+\omega_2+\Omega)\sigma(\omega_2-\Omega)}{\sigma(t+\omega_2-\Omega)\sigma(\omega_2+\Omega)}} \cdot \exp\left[-\left(\zeta(\Omega) + \frac{ikm\lambda}{2}\right)t\right]. \tag{3.31}$$

Finally, a few terms cancel in the overall expression for Z, yielding

$$Z(t) = \left[\frac{2}{\lambda k\sigma(\Omega)}\sqrt{\frac{\sigma(\omega_2-\Omega)}{\sigma(\omega_2+\Omega)}}\right]\frac{\sigma(t+\omega_2+\Omega)}{\sigma(t+\omega_2)} \cdot \exp\left[-\left(\zeta(\Omega) + \frac{ikm\lambda}{2}\right)t\right]. \tag{3.32}$$

A sample solution is plotted in Fig. 1. We can see that, in the R_1–R_2 plane, the solution traces an oscillating curve in between some inner and outer radius. Meanwhile, the solution propagates in the R_3 direction with nonuniform speed. This behavior is typical over all parameter values we tested.

IV. MECHANICAL INTERPRETATION AND QUANTIZATION OF THE REDUCED SYSTEMS

The equations of motion following from the ansatz (3.4), defining the reduced system for S_1, can be written as

$$\ddot{R} = \lambda[\dot{R}, [K, R] + mK] + [K, [K, R]]. \tag{4.1}$$

These are the equations of motion of a particle in a static electromagnetic field, given by (working in cylindrical polar coordinates where $R_1 = r\cos\theta$, $R_2 = r\sin\theta$, $z = R_3$)

$$\vec{B} = kr\hat{\theta} + mk\hat{z}, \qquad \vec{E} = k^2 r\hat{r}, \tag{4.2}$$

which follow from the vector and scalar potentials

$$\vec{A} = \frac{\lambda k}{2}(mr\hat{\theta} + r^2\hat{z}), \qquad V = \frac{k^2}{2}r^2. \tag{4.3}$$

The classical Hamiltonian is then

$$H_1 = \frac{1}{2}p_r^2 + \frac{1}{2}\frac{[p_\theta - A_\theta]^2}{r^2} + \frac{1}{2}[p_z - A_z]^2 + V(r). \quad (4.4)$$

It is clear that p_θ and p_z are conserved. This formulation lends some physical intuition to the solutions found in Sec. III. We can pass to the quantum theory as usual by finding the covariant Laplacian in cylindrical coordinates,

$$\hat{H}_1\psi = -\frac{\hbar^2}{2}\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\psi}{\partial r}\right] + \frac{1}{2r^2}[-i\hbar\partial_\theta - A_\theta]^2\psi + \frac{1}{2}[-i\hbar\partial_z - A_z]^2\psi + V(r)\psi. \quad (4.5)$$

The conservation of p_θ, p_z leads us to seek a solution to the Schrodinger equation of the separable form

$$\psi(r,\theta,z) = \frac{1}{\sqrt{r}}\rho(r)e^{il\theta}e^{i\frac{p_z z}{\hbar}} \quad (4.6)$$

for integer $l = \frac{p_\theta}{\hbar}$. The system is then reduced to a one-dimensional Schrodinger equation,

$$-\frac{\hbar^2\rho''(r)}{2} + U(r)\rho(r) = E\rho(r), \quad (4.7)$$

with effective potential

$$\begin{aligned} U(r) &= -\frac{\hbar^2}{8r^2} + \frac{1}{2}\frac{[\hbar l - A_\theta]^2}{r^2} + \frac{1}{2}[p_z - A_z]^2 + V(r) \\ &= \frac{1}{2}\left[\frac{\hbar^2[l^2 - \frac{1}{4}]}{r^2} - \frac{\hbar k\lambda m l}{r} + \left(\frac{k^2\lambda^2 m^2}{4} + p_z^2\right) + (k - \lambda p_z)kr^2 + \frac{\lambda^2 k^2 r^4}{4}\right]. \end{aligned} \quad (4.8)$$

At low coupling ($\lambda \to 0$), we have

$$-\hbar^2\rho'' + \left\{\frac{\hbar^2[l^2 - \frac{1}{4}]}{r^2} + p_z^2 + \frac{k^2}{2}r^2\right\}\rho = E\rho, \quad (4.9)$$

and we see by dimensional analysis

$$[\hbar k] = 1/L^2 \Rightarrow E \sim |\hbar k|. \quad (4.10)$$

These are weakly coupled massless excitations. But in the high coupling limit ($\lambda \to \infty$), we have

$$-\hbar^2\rho'' + \left\{\frac{k^2\lambda^2}{4}(m^2 + r^4)\right\}\rho = E\rho, \quad (4.11)$$

which yields a much more peculiar spectrum,

$$[\hbar^2 k\lambda]^2 = 1/L^6 \Rightarrow E \sim |\hbar^2\lambda k|^{2/3}. \quad (4.12)$$

If this dispersion relation describes some fundamental constituents of the theory, then they are certainly not particles in the traditional sense. We propose that this may be a glimpse of some postrelativistic constituents as mentioned in Sec. II A.

A. Quantization of the dual reduced system

In the dual picture (nonlinear sigma model), our ansatz picks out a class of solutions that correspond to a different mechanical system. Though the equations of motion in each picture can be mapped to one another via the duality, the correspondence is not immediately obvious, and the systems will appear very different upon quantization.

After the ansatz, the duality relations (2.13) read

$$g^{-1}g' = \lambda e^{Kx}\left(S + \frac{1}{\lambda}K\right)e^{-Kx}, \quad g^{-1}\dot{g} = \lambda e^{Kx}Le^{-Kx}. \quad (4.13)$$

Writing $g = h(t,x)e^{-Kx}$ yields

$$h^{-1}h' = \lambda S h^{-1}\dot{h} = \lambda L. \quad (4.14)$$

We further suppose that h is separable as $h(t,x) = F(x)Q(t)$. Then the equation for S can be separated as

$$F^{-1}(x)F'(x) = \lambda Q(t)S(t)Q^{-1}(t). \quad (4.15)$$

Both sides are equal to some constant traceless matrix C. Since $Q(t)$ is only unique up to multiplication on the left by a constant matrix in $\mathbb{SU}(2)$, we can use this to choose C to be diagonal and thus proportional to K. We then have

$$Q(t)S(t)Q^{-1}(t) = sK, \quad (4.16)$$

implying that $\mathrm{Tr}S^2 = s^2k^2$. (4.15) is satisfied if

$$F(x) = e^{\lambda sKx}. \quad (4.17)$$

Thus, the full corresponding ansatz for the field variable in the dual theory is

$$g(t,x) = e^{\lambda sKx}Q(t)e^{-Kx}, \quad (4.18)$$

where Q is related to the previous variables by

$$S = sQ^{-1}(t)KQ(t), \qquad L = \frac{1}{\lambda}Q^{-1}\dot{Q}. \quad (4.19)$$

The dual Lagrangian can now be written as

$$\mathcal{L}_2 = \frac{1}{2f^2}\mathrm{Tr}[(Q^{-1}\dot{Q})^2 - (\lambda sQ^{-1}KQ - K)^2]. \quad (4.20)$$

It is useful to parametrize Q in terms of the Euler angles:

$$Q = e^{\frac{i}{2}\sigma_3\gamma}e^{\frac{i}{2}\sigma_1\beta}e^{\frac{i}{2}\sigma_3\alpha}. \quad (4.21)$$

The traces in $\mathcal{L}_2$ can then be computed directly, yielding

$$\mathcal{L}_2 = \frac{1}{\lambda^2}\left\{\frac{\dot\alpha^2+\dot\beta^2+\dot\gamma^2}{2} + \cos\beta\dot\alpha\dot\gamma - V(\beta)\right\}, \tag{4.22}$$

where (dropping a constant shift)

$$V(\beta) = -2k^2\lambda s\cos\beta. \tag{4.23}$$

As a mechanical system, this is the well-known spinning top (isotropic Lagrange top). It is instructive to write

$$\mathcal{L}_2 = \frac{1}{\lambda^2}\left[\frac{1}{2}g_{ij}\dot\alpha^i\dot\alpha^j - V\right], \tag{4.24}$$

where

$$g_{ij} = \begin{pmatrix} 1 & 0 & \cos\beta \\ 0 & 1 & 0 \\ \cos\beta & 0 & 1 \end{pmatrix} \tag{4.25}$$

is the metric of the rotation group and V is the gravitational potential of the top. The overall constant $\frac{1}{\lambda^2}$ in the action leads to a rescaling of $\hbar\mapsto\hbar\lambda^2$ upon quantization.

To pass to the quantum theory, we find the Laplacian operator with respect to the metric g of Eulerian coordinates,

$$\nabla^2\psi = \frac{1}{\sqrt{g}}\partial_i[\sqrt{g}g^{ij}\partial_j\psi]. \tag{4.26}$$

The Hamiltonian is then

$$\hat H_2 = -\frac{\hbar^2\lambda^4}{2}\left[\frac{\partial_\alpha^2+\partial_\gamma^2-2\cos\beta\partial_\alpha\partial_\gamma}{\sin^2\beta} + \partial_\beta^2 + \cot\beta\partial_\beta\psi\right] + V\psi. \tag{4.27}$$

We can again reduce the Schrodinger equation $\hat H_2\psi = E\psi$ to a one-dimensional Schrodinger equation with the ansatz

$$\psi(\alpha,\gamma,\beta) = e^{im_\alpha\alpha}e^{im_\gamma\gamma}\frac{B(\beta)}{\sqrt{\sin\beta}}, \tag{4.28}$$

yielding

$$-\hbar^2\lambda^4\frac{B''(\beta)}{2} + U(\beta)B(\beta) = EB(\beta), \tag{4.29}$$

where

$$U(\beta) = -\frac{\hbar^2\lambda^4}{8} + \frac{\hbar^2\lambda^4}{2\sin^2\beta}\left[{m_\alpha}^2 + m_\gamma^2 - 2m_\alpha m_\gamma\cos\beta - \frac{1}{4}\right] - 2k^2\lambda s\cos\beta. \tag{4.30}$$

This can be studied by standard techniques for periodic potentials (Floquet theory, Bloch waves, etc.) We content ourselves with a quick look at low energy excitations: small oscillations around the classical equilibrium $q = 0$ and setting $m_\alpha = 0 = m_\gamma$. Changing variables $\beta = \hbar\lambda^2 q$ and expanding around the classical minimum $q = 0$ gives

$$-\frac{1}{2}\frac{d^2B}{dq^2} + \left\{q^2(\hbar^2k^2s\lambda^5) - \frac{1}{8q^2} - 2k^2s\right\}B \approx EB. \tag{4.31}$$

The solutions involve Laguerre polynomials, and the spectrum is, in this approximation, $E_n \approx \sqrt{2}(2n+1)\hbar k\sqrt{s}\lambda^{\frac{5}{2}}$. If we remove the zero-point energy ($n = 0$), we have the energy of n free particles each of energy $e_1 = \hbar k\sqrt{8s}\lambda^{\frac{5}{2}}$. This is the dispersion relation of massless particles, except for a rescaling of the speed.

V. CONCLUSIONS AND OUTLOOK

Because they only exist in the short-distance limit, it is difficult to say whether objects like "preons" we discuss could correspond to directly observable objects in an experiment. Quarks were not considered at first to be directly observable things either, as they could not be created as isolated particles. In the S_1-model's strong coupling limit, the Minkowski geometry of space-time appears to be lost, and wave propagation is sustained entirely by the nonlinearity. However, these waves do not appear to transmit information, and perhaps any postrelativistic effects are hidden by some sort of confinement when they form bound states.

It is at least intriguing to question whether highly coupled theories have fundamental constituents with such an exotic nature that they have been overlooked. Drawing parallels with $\lambda\phi^4$ theory, it is tempting to speculate that the Higgs particle of the standard model is such a composite of some strongly bound preons existing only at short distances. Were this the case, one could sensibly describe a "pure Higgs" at short distances.

For a more complete understanding, we must quantize the whole theory rather than just its mechanical reduction. Since the equations have a Lax pair, it should be possible to perform a canonical transformation to angle variables and then pass to the quantum theory. Such a quantization was achieved for sine-Gordon theory [7], proving that the solitons are fermions which bind to form the scalar waves. A similar analysis of our model is a lengthy endeavor, and we hope to return to this later after laying the groundwork and motivation here.

TABLE I. A comparison of results in the dual models.

Nilpotent field theory (S_1)		Principal chiral model (S_2)
$\mathcal{L}_1 = \mathrm{Tr}\{\frac{1}{2\lambda}\dot{\phi}^2 - \frac{1}{2\lambda}\phi'^2 + \frac{1}{3}\phi[\dot{\phi}, \phi']\}$	Lagrangian density	$\mathcal{L}_2 = \frac{1}{2\lambda^2}\mathrm{Tr}\{(g^{-1}\dot{g})^2 - (g^{-1}g')^2\}$
$I = \frac{1}{\lambda}\dot{\phi},\ J = \phi'$	Currents	$I = g^{-1}g',\ J = g^{-1}\dot{g}$
$\lambda I' = \dot{J}$	Current identity	$\dot{I} - \frac{1}{\lambda}J' + \lambda[J, I] = 0$
$\dot{I} - \frac{1}{\lambda}J' + \lambda[J, I] = 0$	Equation of motion	$\lambda I' = \dot{J}$
Reduced system of S_1		Reduced system of S_2
$\phi(t,x) = e^{Kx}R(t)e^{-Kx} + mKx$	Wave ansatz	$g(t,x) = e^{\lambda sKx}Q(t)e^{-Kx}$
$I = e^{Kx}\dot{R}e^{-Kx}$	Wave currents	$I = e^{Kx}\{\lambda sQ^{-1}KQ - K\}e^{-Kx}$
$J = e^{Kx}\{\lambda[K,R] + mK\}e^{-Kx}$		$J = e^{Kx}\{Q^{-1}\dot{Q}\}e^{-Kx}$
$S = \dot{R} + \frac{1}{\lambda}K$	Common variables	$S = sQ^{-1}KQ$
$L = [K,R] + mK$		$L = \frac{1}{\lambda}Q^{-1}\dot{Q}$
$\dot{L} = [K, S]$	Current identity	$\dot{S} = \lambda[S, L]$
$\dot{S} = \lambda[S, L]$	Equation of motion	$\dot{L} = [K, S]$
$H_1 = \mathrm{Tr}\{\frac{1}{2}(S - \frac{1}{\lambda}K)^2 + \frac{1}{2}L^2\}$	Hamiltonian $H_1 = H_2$	$H_2 = \mathrm{Tr}\{\frac{1}{2}(S - \frac{1}{\lambda}K)^2 + \frac{1}{2}L^2\}$
$\mathcal{L}_1 = \mathrm{Tr}\{\frac{1}{2}(S - \frac{1}{\lambda}K)^2 + \frac{1}{3}R[S-K, L] - \frac{1}{2}L^2\}$	Lagrangian $\mathcal{L}_1 \neq \mathcal{L}_2$	$\mathcal{L}_2 = \frac{1}{2}\mathrm{Tr}\{L^2 - (S-K)^2\}$
$E \sim \lvert k\rvert^{2/3}\ \ (\lambda \to \infty)$	Short-range dispersion	$E \sim \lvert k\rvert\ \ (\lambda \to 0)$

We present a side-by-side comparison of comparison of our work with the two models in Table I below.

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APPENDIX: QUADRATIC HAMILTONIANS WITH NILPOTENT BRACKET ALGEBRAS

Many classical systems have a quadratic Hamiltonian together with Poisson brackets (or commutators in the quantum case) which generate a Lie algebra,

$$H = \frac{1}{2}h^{ab}v_a v_b, \qquad \{v_a, v_b\} = c^c_{ab}v_c, \tag{A1}$$

where c^c_{ab} are the structure constants of the bracket algebra and v_a the dynamical variables. The quadratic nature of the Hamiltonian immediately affords a geometric interpretation: h_{ab} defines a left-invariant metric, $h \in \mathfrak{g}\vee\mathfrak{g}$ on the Lie group G (with generating algebra $\mathfrak{g}$). The equations of motion then describe geodesics on the Lie group under this metric.

A nilpotent Lie algebra of step n is a Lie algebra in which all repeated brackets of order n vanish. The combination of a quadratic Hamiltonian *and* a nilpotent bracket algebra can allow one to solve for the spectrum of a quantum system algebraically, using only the representation structure of the associated Lie group. This is done by using a representation to generate raising and lowering operators, as is familiar in the case of the harmonic oscillator. While we did not take this (somewhat ambitious) approach here, it is carried out in Ref. [17] for a magnetic system very similar to the one we discuss in Sec. IV. It is worth at least mentioning this point of view, as it connects the model studied here to other, more well-known models.

1. Nilpotent mechanical systems

The simplest system of this type is a harmonic oscillator,

$$H = \frac{1}{2}(p^2 + \omega^2 q^2), \qquad \{p, q\} = 1. \tag{A2}$$

Here, the canonical variables p and q form a step-2 nilpotent Lie algebra, where all double commutators vanish. Of course, any Hamiltonian in terms of the canonical variables will have this bracket algebra, but what happens in the nonquadratic case? Consider the anharmonic oscillator,

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + \lambda q^4. \tag{A3}$$

We can recast this as a quadratic Hamiltonian with step-3 nilpotent bracket algebra by defining $q_2 = q^2$ and then treating this as a distinct element of the algebra. We then have

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + \lambda q_2^2 \tag{A4}$$

$$\{p, q_2\} = 2q, \qquad \{p, q\} = 1, \qquad \{q_2, q\} = 0, \tag{A5}$$

where one can see that all triple commutators vanish. Thus, the classical anharmonic oscillator describes geodesics in the corresponding nilpotent Lie group. It is then possible to solve such quantum theories using the methods of Ref. [17].

The mechanical reduction of our field theory gives another example. The equations of motion (3.4) follow from the Hamiltonian (3.8) with Poisson brackets

$$\begin{aligned} \{S_a, S_b\} &= \lambda\epsilon_{abc}L_c, \\ \{L_a, L_b\} &= 0, \\ \{S_a, L_b\} &= \epsilon_{abc}K_c. \end{aligned} \tag{A6}$$

This a step-3 nilpotent Lie algebra. Its representation on the space of functions of R is what we used to quantize the theory. In the dual picture, the Poisson brackets would not be nilpotent; the Lie algebra it is the semidirect product of $SU(2)$ with an Abelian algebra.

2. Nilpotent field theories

Interestingly, we find that some of the most important field theories of particle physics can be described in this way. The bosonic part of the standard model consists of Yang-Mills theory coupled to a Higgs sector described as a $\lambda\phi^4$ scalar field theory.

In field theory, the obvious analog of the anharmonic oscillator is pure $\lambda\phi^4$ theory. Identifying $\phi_2(x) = \phi^2(x)$, this theory can be described by

$$H = \frac{1}{2}\int [\pi^2(x) + m^2\phi^2(x) + \lambda\phi_2^2]dx \tag{A7}$$

$$\begin{aligned} \{\pi(x), \phi_2(y)\} &= 2\phi(x)\delta(x-y), \\ \{\pi(x), \phi(y)\} &= \delta(x-y). \end{aligned} \tag{A8}$$

Pure $\lambda\phi^4$ theory (without coupling to fermions) in four dimensions remains intractable in the short-distance limit; it is not an asymptotically free theory. We see here that it follows from a Hamiltonian with one degree of extra nilpotency in the bracket algebra. This suggests that perhaps the theory is more easily tamed with an algebraic approach.

Another famously puzzling theory, Yang-Mills theory, can be cast in the same language. Here, the Poisson brackets and Hamiltonian are best expressed in terms of the electric field

$$E[a] = \int E^{bi}a_{bi}dx \tag{A9}$$

(where a is a smooth test function) and the magnetic field $B = dA + A\wedge A$:

$$\begin{aligned} \{E[a], B\} &= da + [A, a], \\ \{E[a], A\} &= a, \\ \{A_{aj}(x), A_{bj}(y)\} &= 0, \end{aligned} \tag{A10}$$

$$H = \frac{1}{2}\int (E^2 + B^2)dx. \tag{A11}$$

Yang-Mills theory, however, is an asymptotically free theory. The fact that it can be brought to the same form as pure $\lambda\phi^4$ theory suggests some commonality in the structure of the two theories, though they might appear glaringly different due to their short-distance behavior.

We pause to note that not all systems are nilpotent. The simplest example would be the rigid rotor, which has the angular momentum momentum bracket algebra, in which repeated commutators do not vanish. Such a Lie algebra is perhaps misleadingly labeled as *simple* in the mathematics literature. Also, the Euler equations of an ideal fluid can be formulated with a quadratic Hamiltonian on the Lie algebra of vector fields. Nilpotent Lie algebras could be useful as approximations here.

3. Current algebra of S_1

The equations of motion (2.12) follow from the Hamiltonian

$$H_1 = \frac{1}{2}\int \left[\lambda I_a I_a + \frac{1}{\lambda}J^a J^a\right]dx \tag{A12}$$

and the Poisson brackets from S_1,

$$\begin{aligned} \{J^a(x), J^b(y)\}_1 &= 0 \\ \{I_a(x), J^b(y)\}_1 &= -\delta_a^b\delta'(x-y) \\ \{I_a(x), I_b(y)\}_1 &= \epsilon_{abc}J^c\delta(x-y). \end{aligned} \tag{A13}$$

So this theory can also be cast as a quadratic Hamiltonian with step-3 nilpotent algebra. This further motivates the analogy between our model and $\lambda\phi^4$ theories.

It is natural, in nilpotent Lie algebras, to take the singular limit of the metric where the coefficient of the higher-step generators shrinks to zero. (This geometry has been well studied in the simplest case of the Heisenberg group [10]). This is precisely the strong coupling limit $\lambda \to \infty$ of our theory: the second term in the Hamiltonian $\frac{1}{2}\int[\lambda I_a I_a + \frac{1}{\lambda}J^a J^a]dx$ tends to zero. In this limit the cometric is not invertible.

The resulting sub-Riemannian geometry still has geodesics connecting nearby points; the Hormander condition is satisfied because the commutator of the surviving generators $I_a(x)$ generate the remaining ones J_a. The

Chow-Rashevsky theorem does not directly apply here as we are dealing with an infinite-dimensional manifold. But it does suggest that there are propagating solutions even in the limit $\lambda \to \infty$. We found some examples numerically first and then found analytic solutions including these examples. So at least in this case, the intuition provided by the sub-Riemannian geometry was useful in understanding the strong coupling limit.

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