

Nonultralocality and causality in the relational framework of canonical quantum gravity

P. G. N. de Vegvar*

SWK Research 1438 Chuckanut Crest Drive, Bellingham, Washington 98229, USA

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The relational framework of canonical quantum gravity with nonultralocal constraints is explored. After demonstrating the absence of anomalies, a spatially discretized version of the relational framework is introduced. This allows the application of Lieb-Robinson bounds to on-shell monotonic gauge flow when there is a continuous external “time” parameter. An explicit Lieb-Robinson bound is derived for the differential on-shell evolution of the operator norm of the commutator of discretized Dirac observables, demonstrating how a local light conelike causal structure emerges. Ultralocal constraints do not permit such a structure to arise via Lieb-Robinson bounds. Gauge and $(3 + 1)$ -diffeomorphism invariance of the light cone is discussed along with the issues of quantum fluctuations, the nature of the nonlocalities, the spatial continuum limit, and the possible links to noncommutative geometry.

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I. INTRODUCTION AND MOTIVATION

Locality and causality have played pivotal roles in gravitational physics for nearly a century. While the concept and important consequences of local light cones are well established in general relativity, causality in canonical loop quantum gravity and in spin foam models has proven more difficult to elucidate. Almost 20 years ago Smolin [1] pointed out the importance of the emergence of classical long range correlations for nonperturbative theories of quantum gravity in some appropriate semiclassical limit. These correlations are then reflected in how one thinks about causality in quantum field theory on a fixed classical background spacetime, specifically that local observables at spacelike separation commute (microcausality). During the intervening years, there have been several studies of causality in the background independent fully quantum regime, particularly utilizing spin foam (covariant) models. However, generally the spin foam models do not support causal correlations, unless they either are assumed at the outset [2,3] or involve some kind of alteration of the vertex amplitude intended to describe a local orientation [4]. On the other hand, approaching the issue of semiclassical causality from the canonical point of view is even more conceptually challenging since that approach is a “timeless” formalism. Consequently, the consensus expectation is that microcausality will emerge from some as yet to be developed semiclassical limit of quantum gravity; however, to date such a causal limit for quantum gravity remains lacking. Thus it is mysterious that microcausality occupies such a foundational place in general relativity, quantum field theory, and the standard model of particle physics, yet is still so elusive from a background independent quantum gravity point of view. In a broader context, the semiclassical regime of quantum gravity is of

importance not only from the perspective of causal correlations, but also as a general testing ground to examine whether a theory of quantum gravity can behave in familiar classical ways in some suitable limit.

Here we take a simple first step toward understanding these questions starting from an unexpected direction. We adopt the relational framework approach to canonical quantum gravity which has been developed over several decades [5–9]. Then we explore the case where the constraints are nonultralocal and, for reasons to be discussed, limit our considerations to the on-shell physics. Next, we apply Lieb-Robinson bounds, originally introduced in the 1970s to describe solid-state spin systems [10], to a spatially discretized version of the relational framework and demonstrate how a suitably gauge invariant differential local light cone for discretized Dirac observables may be constructed. In essence, the local light-cone emerges from on-shell nonultralocality of constraints in a quite general sense within spatially discrete relational framework models with smooth monotonic gauge flow described by an external “time” parameter. If the constraints are taken to be ultralocal, then this Lieb-Robinson-based causal structure collapses. Quantum fluctuations act to disrupt the local light-cone structure, and a set of general criteria are presented which are sufficient for the Lieb-Robinson-based local light-cone structure to survive the quantum-classical tug of war.

The outline of the remainder of the paper is as follows: The relational framework is briefly recapitulated in Sec. II. Section III studies nonultralocality, its freedom from anomalies, and introduces “patchy” gauge flow. It concludes with a description of the spatially discretized model that is used later on. Section IV provides an introduction to two nonrelativistic versions of earlier Lieb-Robinson bounds from the literature: The first for Heisenberg operator evolution via time independent Hamiltonians, and another more mathematically

*Paul.deVegvar@post.harvard.edu

sophisticated approach for time-dependent Hamiltonians, both on general lattices or networks. In Sec. V a relativistic differential-time expression for the Lieb-Robinson local light cone is derived for the case of an external time parameter acting as the synchronizing conductor of the relational framework's clock variable symphony orchestra. Section VI discusses gauge invariance and other properties of the relational local light cone. Through a series of questions and answers, Sec. VII examines issues related to the continuum limit of the spatial discretization, the nature of nonlocality necessary for the Lieb-Robinson local light cone, a possible link to noncommutative geometry, and the role of quantum fluctuations. The paper concludes with a brief summary and self-criticism of the model in Sec. VIII.

II. REVIEW OF THE RELATIONAL FRAMEWORK

In this section we briefly review the necessary points of the relational framework formalism. For further details please see [7,8,11]. The essential idea behind the relational framework is to construct Dirac (gauge invariant) observables from gauge variant (partial) observables. This all starts from the classical phase space \mathcal{M} description of a reparametrization invariant system whose dynamics is described by a (canonical) Hamiltonian that consists entirely of a linear combination of constraints. In the relational framework, quantization occurs on the reduced phase space. We start by describing the classical formalism. Consider then a set of first-class constraints C_I with $I \in \mathcal{I}$, an arbitrary index set. For the case of canonical four-dimensional general relativity, the index I includes both a continuous three-coordinate index $y(I)$, labeling a point σ on the three-dimensional manifold Σ , as well as a discrete index $i(I)$. The latter index ranges from 0 through $N_c - 1$, and labels the N_c first-class constraints (gauge conditions) at each point σ . These constraints satisfy the first-class condition $\{C_I, C_J\} = f_{IJ}^K C_K$, where generally f_{IJ}^K may be a structure function, possibly depending on phase space functions. We have assumed all second-class constraints have previously been handled by taking \mathcal{M} to be the surface in phase space where they all vanish, and that the Poisson bracket used above is the Dirac bracket. Next select a set of gauge variant phase space functions T^I , $I \in \mathcal{I}$ called clock functions or clock variables that coordinatize the gauge orbit of any point in phase space within a neighborhood of the (classical) constraint surface (shell) $\overline{\mathcal{M}} \doteq \{m \in \mathcal{M} | C_I(m) = 0, \forall I \in \mathcal{I}\}$. The T^I might include matter (nongravitational) degrees of freedom. Then $A_I^J \doteq \{C_I, T^J\}$ is locally nonsingular as an (I, J) "matrix," and one can define the transformed equivalent first-class constraints

$$C'_I \doteq \sum_J [A^{-1}]_I^J C_J. \quad (1)$$

These obey $\{C'_I, T^J\} \approx \delta_I^J$, where \approx denotes weak equality, that is equality on-shell, and the Hamiltonian vector fields $X_{C'_I} \doteq X_I$ weakly commute (i.e., commute on shell). Gauge transformations for any phase space function f and set β^I of reals can be written as

$$\alpha_\beta(f) \doteq \exp(X_\beta) \cdot f = \sum_{n=0}^{\infty} \frac{1}{n!} (X_\beta)^n \cdot f, \\ X_\beta \doteq \sum_I \beta^I X_I. \quad (2)$$

If one is given a set of real-valued phase space constants (clock parameters) τ^I , a weakly gauge invariant (Dirac) observable associated with partial (gauge variant) observables f and T^I is

$$\mathcal{O}_f(\tau) \doteq \mathcal{O}[f](\tau) \doteq [\alpha_\beta(f)]_{\alpha_\beta(T^I)=\tau^I}. \quad (3)$$

The motivating idea is that $\mathcal{O}[f](\tau)$ represents the value of f when the clock variables T^I take the values τ^I ; i.e., it is a gauge slice or fixing. $\mathcal{O}[f](\tau)$ is a phase space function, and one must compute it first treating β^I as phase space constants, and only subsequently set $\beta^I = \tau^I - T^I$.

One also finds $\alpha_\beta(T^I) \approx T^I + \beta^I$ and $\mathcal{O}[T^I](\tau) \doteq \alpha_\beta(T^I)_{\alpha_\beta(T^I)=\tau^I} \approx \tau^I$ on shell. Also on the constraint surface $\mathcal{O}[f](\tau)$ may be formally expanded as

$$\mathcal{O}[f](\tau) \approx \sum_{\{k_i=0\}}^{\infty} \left(\prod_I \frac{(\tau^I - T^I)^{k_i}}{k_i!} \right) \left(\prod_I (X_I)^{k_i} \right) \cdot f. \quad (4)$$

One can also derive

$$\mathcal{O}[f](\tau) + \mathcal{O}[f'](\tau) = \mathcal{O}[f + f'](\tau), \quad (5)$$

$$\mathcal{O}[f](\tau)\mathcal{O}[f'](\tau) \approx \mathcal{O}[ff'](\tau), \quad (6)$$

$$\{\mathcal{O}[f](\tau), \mathcal{O}[f'](\tau)\} \approx \{\mathcal{O}[f](\tau), \mathcal{O}[f'](\tau)\}_D \\ \approx \mathcal{O}[\{f, f'\}_D](\tau), \quad (7)$$

where the Dirac bracket is defined as

$$\{f, f'\}_D \doteq \{f, f'\} - \{f, C_I\} [A^{-1}]_I^J \{T^J, f'\} \\ + \{f', C_I\} [A^{-1}]_I^J \{T^J, f\}. \quad (8)$$

The formalism simplifies considerably if one can choose canonical coordinates so that the clock variables T^I are themselves some canonical coordinates. Then one has a complete set of canonical pairs partitioned as (q^a, p_a) and (T^I, P_I) , where the P_I are the canonical momenta conjugate to the T^I . Hence in a local neighborhood of the constraint surface one can write the constraints as the equivalent set

$$\tilde{C}_I = P_I + h_I(q^a, p_a, T^J) \approx 0, \quad (9)$$

and setting $P_I = -h_I(q^a, p_a, T^J)$ formally solves the constraints. One can also show that the canonical Dirac observables $Q^a(\tau) \doteq \mathcal{O}[q^a](\tau)$ and $P_a(\tau) \doteq \mathcal{O}[p_a](\tau)$ satisfy equal- τ canonical Poisson bracket relations. As discussed in [7,11], the \tilde{C}_I comprise a strongly Abelian constraint algebra and obey $C'_I = \tilde{C}_I + \mathcal{O}(C^2)$; hence the Hamiltonian vector fields X_I of C'_I and $X_{\tilde{I}}$ of \tilde{C}_I weakly commute. The relations (5), (6), and (7) may be summarized by saying that \mathcal{O} induces a weak algebra homomorphism with respect to pointwise addition and multiplication as well as a weak Dirac algebra homomorphism on $\{f, f'\}_D$. When neither f nor f' depend on any P_I , their Dirac bracket reduces to the Poisson bracket, and then there is also a weak Poisson algebra homomorphism.

Next define

$$\begin{aligned} H_I(\tau) &= H_I(Q^a(\tau), P_a(\tau), \tau) \\ &\doteq \mathcal{O}[h_I](\tau) \approx h_I(Q^a(\tau), P_a(\tau), \tau). \end{aligned} \quad (10)$$

If f is any phase space function depending only on q_a, p_a , but not on T^I, P_I , one has

$$\frac{\partial}{\partial \tau^I} \mathcal{O}[f](\tau) \approx \{H_I(\tau), \mathcal{O}[f](\tau)\}. \quad (11)$$

That is, $H_I(\tau)$ generates the τ -parametrized gauge flow of f on the constraint surface. So if one specializes to a parametrization invariant dynamical system whose canonical Hamiltonian vanishes, one may refer to the $H_I(\tau)$ as the (τ dependent) physical Hamiltonians. In the following we will make extensive use of (11), and it is important to realize that both that partial differential equation (PDE) as well as its integrability condition hold only on shell [7]. So we will henceforth limit ourselves to on-shell physics.

In order to quantize the system on the reduced phase space where the classical constraints are valid, the gauge invariant canonical variables $Q^a(\tau), P_a(\tau); T^I(\tau), P_I(\tau)$ are mapped to operators $\hat{Q}^a(\tau), \hat{P}_a(\tau); \hat{T}^I(\tau), \hat{P}_I(\tau)$ which generate the quantum algebra \mathfrak{U} with the usual equal- τ canonical commutation relations. Given \mathfrak{U} , its representation (carrier) Hilbert space \mathcal{H} may be generated via the Gelfand-Naimark-Segal (GNS) construction employing any positive linear functional (state) on \mathfrak{U} . Here one is reducing phase space before quantizing; that is, all the constraints are satisfied at the classical level. We assume that for all τ , the physical Hamiltonians $H_I(Q^a(\tau), P_a(\tau), \tau)$ are represented as densely defined self-adjoint operators on \mathcal{H} .

III. NONULTRALOCALITY

We say h_I is *ultralocal* if it only depends on the canonical fields or their spatial gradients (of any finite order) at the point $y(I) \in \Sigma$. If all the h_I are ultralocal, then $\{h_I, h_J\} \propto \delta(y(I), y(J))$ (or its spatial derivatives), and therefore one has $\{H_I(\tau), H_J(\tau)\} \approx 0$ for $y(I) \neq y(J)$. In this case, $H_I(\tau)$ and $H_J(\tau)$ have no common q^a, p_a, T^K for $y(I) \neq y(J)$.

Ultralocality produces significant mathematical convenience and simplification. Moreover, all the commonly used classical gravitational constraint algebras, such as Arnowitt-Deser-Misner (ADM) and Holst, including possible scalar matter fields, possess ultralocal h_I ; see [11]. In fact, all the known interactions in the standard model of particle physics are ultralocal as well. However, a field theory is not required to be ultralocal, just that measurements so far are consistent with ultralocality. Here we keep an open mind and explore the consequences of nonultralocal h_I . One should realize from the outset that the *constraint* \tilde{C}_I algebra remains Abelian for nonultralocal h_I , even though neither h_I nor H_I will possess weakly Abelian algebras. We will see later on that the quantity $[\hat{H}_I(\tau), \hat{H}_J(\tau)]$ for $y(I) \neq y(J)$ plays a crucial role in Lieb-Robinson bounds.

It is sensible to first confirm that the quantum dynamics remains anomaly-free even for non-Abelian $\hat{H}_I(\tau)$. This means that all the classical gauge symmetries are faithfully reproduced in the quantum theory. Han [11] has proposed the condition that the H_I form an Abelian algebra to be used as a definition for freedom from anomalies. It is demonstrated below that this is overly restrictive, and we provide an alternative condition for the quantum dynamics to be anomaly-free. Following [11], one seeks to solve the Schrödinger equation

$$\frac{\partial}{\partial \tau_I} \hat{U}(\tau, \tau') = \hat{H}_I(\tau) \hat{U}(\tau, \tau') \quad (12)$$

for a unitary propagator $\hat{U}(\tau, \tau')$ between two Schrödinger states $\Psi(\tau), \Psi'(\tau')$ at initial “multifingered time” τ' and final value τ . Let \mathcal{T} denote the space for τ , and let $c: \mathbb{R} \rightarrow \mathcal{T}$ be a path from τ' to τ parametrized by a real-valued “external time” t . We will show that $U(\tau, \tau')$ is independent of the choice of path c as long as $dc^I/dt \geq 0, \forall I, t$. This is a mathematical representation of the general covariance (gauge invariance) of the quantum dynamics. We will need a little more terminology. The real-valued fields $\tau^I(t)$ specify one τ at any given t for each $y \in \Sigma$ and each gauge $i = 0 \dots N_c - 1$. We will refer to a single set of configurations (all y, i) $\{\tau^I(t)\}_i$ for all t as a “slicing.” One configuration (all y, i) at *one* given t is called a “slice” from a slicing. A gauge transformation is then a change of *slicing* from $\{\tau(t)\}_t$ to $\{\tilde{\tau}(t)\}_t$; i.e., a change of multifingered time. Independent from slicing invariance, t can be smoothly reparametrized to $t' = f(t)$ (a 1-diff).

Start by rewriting the τ -evolution PDE (12) as an integral equation,

$$\hat{U}(\tau, \tau') = \hat{1} + \sum_I \int_{\tau'}^{\tau_I} dr \hat{H}_I(\sigma) \hat{U}(\sigma, \tau'). \quad (13)$$

Here σ is a variable like τ or τ' taking a value in \mathcal{T} whose components are

$$\sigma_J = \begin{cases} r, & \text{for } J = I, \\ \tau'_J & \text{otherwise,} \end{cases}$$

where I is the summation index in (13), and r is the (real) variable of integration. Iterating this leads to a Dyson expansion,

$$\begin{aligned} \hat{U}(\tau, \tau') &= \hat{1} + (-i) \sum_I \int_{\tau'}^{\tau_I} d\sigma_I^{(1)} \hat{H}_I(\sigma^{(1)}) \\ &+ (-i)^2 \sum_{I,J} \int_{\tau'}^{\tau_I} d\sigma_I^{(1)} \int_{\tau'_J}^{\sigma_I^{(1)}} d\sigma_J^{(2)} \\ &\times \hat{H}_I(\sigma^{(1)}) \hat{H}_J(\sigma^{(2)}) \hat{U}(\sigma^{(2)}, \tau') \quad (14) \\ &= \hat{1} + \sum_{n=1}^{\infty} (-i)^n \sum_{I_1, \dots, I_n} \int_{\tau'}^{\tau_{I_1}} d\sigma_{I_1}^{(1)} \\ &\times \int_{\tau'_{I_2}}^{\sigma_{I_1}^{(1)}} d\sigma_{I_2}^{(2)} \cdots \int_{\tau'_{I_n}}^{\sigma_{I_{n-1}}^{(n-1)}} d\sigma_{I_n}^{(n)} \\ &\times \{ \hat{H}_{I_1}(\sigma^{(1)}) \hat{H}_{I_2}(\sigma^{(2)}) \cdots \hat{H}_{I_n}(\sigma^{(n)}) \}, \quad (15) \end{aligned}$$

where n is the depth of iteration, and $I_1, \dots, I_n \in \mathcal{I}$. One may concretely picture each fixed I as an “ I -channel” from τ' to τ . Each $\sigma^{(m)}$ is a τ -vector (a point in \mathcal{T}), having real components $\sigma_K^{(m)}$, $K \in \mathcal{I}$. The I -subscripts on the $\hat{H}_I(\sigma)$ correspond to the τ_I integrated over when its corresponding argument σ_I is integrated.

Suppose the $\sigma^{(n)}$ satisfy $\sigma_K^{(m-1)} \geq \sigma_K^{(m)}$ for all m and K , where here $K \in \mathcal{I}$ plays a role like the index I in τ_I . We refer to this as the *ordering hypothesis* for the path through \mathcal{T} -space from the initial τ' to the final τ . Now consider the n th term in the sum on the right-hand side (RHS) of (15). Suppose among the n channels $I_1, \dots, I_n \in \mathcal{I}$, $J_1 \in \mathcal{I}$ occurs p_1 times, $\dots, J_M \in \mathcal{I}$ occurs p_M times, $1 \leq M \leq n$, with $0 \leq p_1, \dots, p_M \leq n$, and $p_1 + \dots + p_M = n$. Under the ordering hypothesis one can recast Eq. (15) as

$$\begin{aligned} \hat{U}(\tau, \tau') &= \hat{1} + \sum_{n=1}^{\infty} (-i)^n \sum_{1 \leq M \leq n} \sum_{\substack{\{p_1 \dots p_M\}=1 \\ p_1 + \dots + p_M = n}}^M \left[\prod_{k=1}^M \left(\sum_{i(J_k)=0}^{N_c-1} \int_{\Sigma} d^3y(J_k) \mu(y(J_k)) \right) \right]_{\text{NCC}} \\ &\times \left[\prod_{k=1}^M \left(\frac{1}{p_k!} \int_{\tau'_{J_k}}^{\tau_{J_k}} d\tau_{J_k}^{(1)} \cdots \int_{\tau'_{J_k}}^{\tau_{J_k}} d\tau_{J_k}^{(p_k)} \right) \right] T_{\tau} \{ \hat{H}_{I_1}(\sigma^{(1)}) \cdots \hat{H}_{I_n}(\sigma^{(n)}) \}. \quad (16) \end{aligned}$$

Here the channels $I_j \in \mathcal{I}$, $j = 1, \dots, n$ are drawn from the $(p_1 \times J_1 + p_2 \times J_2 + \dots)$ J 's. The sum over the I_1, \dots, I_n has been decomposed into sums over the gauge index $i(J_k)$ and integrals over the 3-manifold Σ with respect to the volume element $d^3y \mu(y)$, where y coordinatizes Σ . The existence of a such a volume element is assured once Σ is taken to be an oriented manifold, and it is not generally necessary to have a metric tensor on Σ for that. However, $\hat{U}(\tau, \tau')$ on the left-hand side (LHS) of (16) must be invariant under smooth coordinate reparametrizations of Σ , and this requires the volume element to be similarly invariant for scalar \hat{H}_I . It is straightforward to see that this can be done if it is possible to induce a metric tensor $g_{ab}(y)$ on Σ , where the volume element then takes the standard form $d^3y |\det g_{ab}(y)|^{1/2}$. $g_{ab}(y)$ does not have to be any physical metric. In fact, if the smooth 3-manifold Σ smoothly embeds into any Euclidean space $E = \mathbb{R}^n$, then the Euclidean metric tensor on E induces a suitable metric tensor on Σ . The (strong) Whitney embedding theorem states that if Σ is m -dimensional Hausdorff and second countable, then Σ smoothly embeds into E for $n = 2m$. Taking Σ to have those properties, one then has a volume element on it to render $\hat{U}(\tau, \tau')$ coordinate reparametrization invariant on Σ ; moreover this is true for any such invariant volume element. Alternatively, taking Σ to be a paracompact differentiable 3-manifold assures it has a Riemannian structure, whose metric tensor can then be

used to construct an invariant 3-volume element in the standard way.

The notation NCC on the RHS of (16) stands for “noncollisional channels” and arises from the following considerations: Each I -channel has a gauge index i and position y on Σ . i and y are independent degrees of freedom for each channel; however, once the number of distinct channels has been fixed to be M , two channels with the same i values cannot occupy the same position y . That is the channels cannot “collide,” as the associated merger or splitting of channels would alter the previously fixed number of channels M , so “NCC” can also be thought of as necessary to avoid double or under counting. The $\{p_k\}$ give the number of τ -steps or integrations along each distinct channel J_k . Any single $\sigma^{(k)} \in \mathcal{T}$ has the components: *one* from the sequence $(\tau_{J_1}^{(1)}, \dots, \tau_{J_1}^{(p_1)})$ of real integration variables in channel J_1 ; \dots ; *and any one* from the sequence $(\tau_{J_M}^{(1)}, \dots, \tau_{J_M}^{(p_M)})$ of integration variables in channel J_M ; *and* (if not integrated) that component of $\sigma^{(k)}$ is set equal to the corresponding component of the initial τ' . Because of the ordering hypothesis, $\sigma^{(k-1)}$ differs from $\sigma^{(k)}$ at only one index value J , where $\sigma_J^{(k-1)} \neq \sigma_J^{(k)}$, and then $I_k = J$. The “ τ -ordering operator” T_{τ} acts to order the τ arguments in each I -channel independently so that within each channel J they increase from right to left:

$\sigma_J^{(n)} \leq \dots \leq \sigma_J^{(2)} \leq \sigma_J^{(1)}$, $\forall J$. Notice that the ordering hypothesis has allowed us to remove the path-dependent limits on the multiple τ -integrations.

Equation (16) sums (averages) over all the paths in \mathcal{T} from τ' to τ by advancing monotonically in τ_I within each I -channel independently, stepwise over the n integrations, from initial τ' to final τ , as enacted by the T_τ operation. The averaging over the paths from the initial to final τ -slices is carried out for each fixed n, M , first at fixed (NCC) channel configuration $\{(i(J_k), y(J_k))\}_k$ and fixed stepping configuration $\{p_k\}_k$ by the corresponding τ -integrations. Then the channel configuration and stepping configuration are separately averaged at fixed n, M (the result is independent of which of the latter two averages is performed first). Now suppose we have some ‘‘external time’’ parameter t so that $\tau_I = \tau_I(t)$. As long as the path $c^I(t) \doteq \tau_I(t)$ between the fixed initial and final τ obeys $dc^I(t)/dt \geq 0$ for all $I \in \mathcal{I}$ and relevant values of t , one fulfills the ordering hypothesis. This mild monotonicity condition is consistent with the physical picture of multifingered time as ‘‘flowing forwards everywhere.’’ The overall sign of the monotonicity condition may be reversed, so multifingered time then globally flows ‘‘backwards’’; however, the choice of that sign does not affect the conclusions.

By adding over all such ways of channel-wise monotonic advancement from τ' to τ , the overall RHS of (16) is insensitive to a change of slicing (gauge) $\tau_I(t) \rightarrow \tilde{\tau}_I(t)$ for fixed initial τ' and final τ , and $\hat{U}(\tau, \tau')$ depends only on those initial and final configurations. This happens regardless of the commuting or noncommuting properties of the $\hat{H}_I(\tau)$. It occurs because the monotonicity condition takes care of the required τ_I -ordering within each I -channel separately (T_τ becomes a t -ordering), and the operator orderings among different I -channels (inside T) on the RHS are averaged over (as a sum over monotonic paths or slicings between the fixed initial τ' and final τ configurations). This absence of path or slicing dependence of $\hat{U}(\tau, \tau')$ is the relational framework manifestation of general covariance (gauge invariance) for the quantum dynamics: we have obtained freedom from anomalies for τ -paths monotonic in an external time parameter. Under these conditions the propagator \hat{U} more resembles the familiar one from standard (fixed background geometry) quantum field theory. In the absence of monotonicity, the use of the components of τ as integration variables, such as in Eqs. (13), (14), and (15) becomes ill-defined: Some I -channels could then have ranges of τ_I which are traversed multiple times in both senses as t advances. This accords with the intuition that these locally negative lapses among generally positive ones are indeed somehow physically anomalous. In fact, a similar notion in a different guise was used in [4] to generate ‘‘causal’’ spin-foam vertices and amplitudes. Monotonicity may be viewed as the relational framework analog of global hyperbolicity on Lorentzian

manifolds; however, unlike the latter, monotonicity assumes no background causal structure.

If one specializes to the ultralocal case, so $\hat{H}_I(\tau)$ and $\hat{H}_J(\tau)$ have no common $\hat{Q}^a, \hat{P}_a, \hat{T}^K$ for $y(I) \neq y(J)$, and to the gauge diagonal case, for which the same thing occurs when $i(I) \neq i(J)$, then $[\hat{H}_I(\tau), \hat{H}_J(\tau)] \approx 0$ for any $I \neq J$. Consequently the T_τ -ordered product in (16) factors into separate T_{τ_I} for each I -channel, and $\hat{U} = \prod_I \hat{U}_I$, as obtained earlier by Han [11].

It is also interesting to compare the T_τ -ordering in (16) with the well-known T -ordering from standard quantum field theory. Weinberg [12] gives the following Dyson expansion for the S -matrix (operator) in Minkowskii spacetime or special relativity:

$$\hat{S} = \hat{1} + \sum_{n=1}^{\infty} \int d^4x_1 \dots d^4x_n T\{\hat{\mathcal{H}}(x_1) \dots \hat{\mathcal{H}}(x_n)\}, \quad (17)$$

where $\hat{H} = \hat{H}_0 + \hat{V}$, $\hat{V}(t) = \exp(i\hat{H}_0 t) \hat{V} \exp(-i\hat{H}_0 t)$, $\hat{V}(t) = \int d^3x \hat{\mathcal{H}}(\vec{x}, t)$ in the interaction picture, with \hat{H}_0 the free Hamiltonian and \hat{V} the interaction. This is globally Lorentz invariant except for the T -ordered operator product. The T -order of two spacetime points x_1 and x_2 (the order of their 0-coordinates) is globally Lorentz invariant unless $(x_1 - x_2)^2 > 0$ ($x_1 - x_2$ spacelike), so (17) introduces no special frame if (but not only if) the $\hat{\mathcal{H}}(x)$ commute at spacelike distances. While this is often referred to as a kind of causality condition, in this sense it arises from the invariance of \hat{S} that occurs because global Lorentz transforms $x \rightarrow x' = \Lambda x$ alter the spatial and temporal components of 4-vectors and so can then reorder the T sequence among the $\{x_n\}$. By contrast, in the relational framework representation just discussed, t and all the τ_I increase smoothly within both slicings $\tau_I(t)$ and $\tilde{\tau}_I(t)$ connecting the fixed initial and final τ 's, and the gauge transform is no longer simply related to the spatial coordinates y . Thus the relational framework has no built-in analog of special relativity's ‘‘causality from global Lorentz invariance.’’ The T_τ in Eq. (16) will not generally lead to ‘‘causality’’ by itself, regardless of whether the constraints are ultralocal or not.

The quantity $[\hat{H}_I(\tau), \hat{H}_J(\tau)]$ for $y(I) \neq y(J)$ is important in the subsequent sections. Therefore we spend some time to study it as well as its classical precursor $\{H_I(\tau), H_J(\tau)\}$.

We start by computing classically,

$$\{H_I(\tau), H_J(\tau)\} \approx \mathcal{O}[\{h_I, h_J\}_D](\tau) = \mathcal{O}[\{h_I, h_J\}](\tau), \quad (18)$$

where the first relation has used Eq. (7), and the second equality used that the h_I do not depend on any P_J inside some phase space neighborhood of the constraint surface. We also have $\tilde{C}_I = P_I + h_I$ with $\{\tilde{C}_I, \tilde{C}_J\} = 0$ (recall the \tilde{C}_I form a strongly Abelian constraint algebra), and one derives

$$0 = \{P_I + h_I, P_J + h_J\} = \{h_I, h_J\} + \{P_I, h_J\} + \{h_I, P_J\} \\ = \{h_I, h_J\} + \frac{\delta h_I}{\delta T^I} - \frac{\delta h_J}{\delta T^J}. \quad (19)$$

Thus,

$$\{H_I(\tau), H_J(\tau)\} \approx \mathcal{O}\left[\frac{\delta h_I}{\delta T^J} - \frac{\delta h_J}{\delta T^I}\right](\tau). \quad (20)$$

We use the same sign conventions as Han [11], namely

$$\{p_a, q^b\} = (+)\delta_a^b, \\ \{P_I, h_J\} = (+)X_{P_I} \cdot h_J = \frac{\delta h_J}{\delta T^I}, \\ X_f = \frac{\delta f}{\delta p_i} \frac{\delta}{\delta q^i} - \frac{\delta f}{\delta q^i} \frac{\delta}{\delta p_i}. \quad (21)$$

One also has from (18)

$$\mathcal{O}\{\{h_I, h_J\}\}(\tau) = \mathcal{O}\{\{\tilde{C}_I, h_J\}\}(\tau) - \mathcal{O}\{\{P_I, h_J\}\}(\tau) \\ = \sum_{\{k_I\}} \left(\prod_J \frac{(\tau^J - T^J)^{k_J}}{k_J!} \right) \left(\prod_J (X_J)^{k_J} \right) \tilde{X}_I \cdot h_J - \mathcal{O}\left[\frac{\delta h_J}{\delta T^I}\right](\tau) \\ \approx \sum_{\{k_I\}} \left(\prod_J \frac{(\tau^J - T^J)^{k_J}}{k_J!} \right) \tilde{X}_I \left(\prod_J (X_J)^{k_J} \right) \cdot h_J - \mathcal{O}\left[\frac{\delta h_J}{\delta T^I}\right](\tau) \\ \approx \sum_{\{k_I\}} \left(\prod_J \frac{(\tau^J - T^J)^{k_J}}{k_J!} \right) X_I \left(\prod_J (X_J)^{k_J} \right) \cdot h_J - \mathcal{O}\left[\frac{\delta h_J}{\delta T^I}\right](\tau) \\ = \frac{\partial H_J(\tau)}{\partial \tau^I} - \mathcal{O}\left[\frac{\delta h_J}{\delta T^I}\right](\tau). \quad (22)$$

Consequently,

$$\{H_I(\tau), H_J(\tau)\} \approx \frac{\partial H_J(\tau)}{\partial \tau^I} - \mathcal{O}\left[\frac{\delta h_J}{\delta T^I}\right](\tau). \quad (23)$$

Combining (20) and (23) one arrives at

$$\frac{\partial H_J(\tau)}{\partial \tau^I} \approx \mathcal{O}\left[\frac{\delta h_I}{\delta T^J}\right](\tau). \quad (24)$$

Which canonical variables enter $\mathcal{Q}_{IJ} \doteq \{H_I(\tau), H_J(\tau)\}$? Since $H_I(\tau) = \mathcal{O}[h_I](\tau)$ and near shell h_I is independent of P_K 's, so H_I has no P_K 's and neither does \mathcal{Q}_{IJ} . How about the (q^a, p_a) variables? Let f be an arbitrary phase space function solely dependent on the (q^a, p_a) 's (and not containing any T^K, P_K variables). One then has

$$\{H_I(\tau), \mathcal{O}[f](\tau)\} \approx \mathcal{O}\{\{h_I, f\}_D\}(\tau) = \mathcal{O}\{\{h_I, f\}\}(\tau) \\ \approx \mathcal{O}\{\{\tilde{C}_I, f\}\}(\tau) \approx \mathcal{O}[X_I \cdot f](\tau). \quad (25)$$

In the second equality we have used that h_I, f are independent of P_K variables, in the third that f contains no T^I variables, and in the fourth that $\tilde{C}_I = C'_I + \mathcal{O}(C^2)$ and $\{f_1, f_2\} = X_{f_1} \cdot f_2$. One would like to iterate Eq. (25) to yield

$$\{H_I(\tau), \{H_J(\tau), \mathcal{O}[f](\tau)\}\} \\ \stackrel{?}{\approx} \{H_I(\tau), \mathcal{O}[X_J \cdot f](\tau)\} \\ \approx \mathcal{O}[(X_I \cdot X_J) \cdot f](\tau) \approx \mathcal{O}[(X_J \cdot X_I) \cdot f](\tau), \quad (26)$$

where in the last equality the weak commutativity of the X_I has been used. If this were true, then by using the Jacobi identity and that f is an arbitrary phase space function of the q^a, p_a variables, one would “find” that $\{H_I(\tau), H_J(\tau)\}$ could not be a phase space function of any q^a, p_a . However, the step marked with ? is invalid because there one is using the weak Eq. (25) *inside* a Poisson bracket. Such a maneuver is *inadmissible*, as no weak equation may be used before evaluating Poisson brackets. Hence $\{H_I(\tau), H_J(\tau)\}$ may still be a phase space function of q^a, p_a as well as the clock variables.

Suppose the constraints are ultralocal, meaning h_I only contains q^a, p_a with $y(a) = y(I)$ and clocks T_K with $y(K) = y(I)$. Then $\delta h_J / \delta T^I \propto \delta(y(I), y(J))$, so by (20) $\{H_I(\tau), H_J(\tau)\} \approx 0$ for $y(I) \neq y(J)$. Hence to obtain $\{H_I(\tau), H_J(\tau)\} \neq 0$ for $y(I) \neq y(J)$ requires nonultralocal constraints.

What about spatial parity (P) and time reversal (T) symmetries? Let $\mathcal{Q}_{xy}^{jk} \doteq \{H_I(\tau), H_J(\tau)\}$ where $j = i(I)$, $k = i(J)$, $x = y(I)$, and $y = y(J)$. One has $\mathcal{Q}_{xy}^{jk} = -\mathcal{Q}_{yx}^{kj}$ by antisymmetry of the Poisson bracket. One might be concerned that when $x \neq y$, for \mathcal{Q}_{xy}^{jk} to be nonvanishing could require violation of P -symmetry (in addition to

nonultralocality); i.e., that Q_{xy}^{jk} could acquire a nonzero spatially odd piece. Such a violation is only necessary provided $Q_{xy}^{jk} = Q_{xy}^{kj}$, that is, the constraints are “gauge symmetric,” meaning they satisfy

$$\frac{\delta h_x^k}{\delta T_y^j} = \frac{\delta h_x^j}{\delta T_y^k} \quad (\text{gauge symmetry}), \quad (27)$$

which does not generally hold. So a violation of P -symmetry is not generally necessary to obtain $\{H_I(\tau), H_J(\tau)\} \neq 0$ for $y(I) \neq y(J)$ if there is nonultralocality. An examination of the formal power series for $\mathcal{O}[f](\tau)$ (4) similarly shows that one does not require a violation of T -reversal symmetry either. The absence of requiring P - and/or T -violation to obtain $\{H_I(\tau), H_J(\tau)\} \neq 0$ for $y(I) \neq y(J)$ is reassuring since the gravitational interaction is not expected to violate those symmetries.

We now present a very simple toy example of nonultralocality. Suppose the Hamiltonians have the special form

$$h^i(x) = \bar{h}^i(x) + \int_{\Sigma} d^3y K^{ij}(q^a(x), p_a(x), q^b(y), p_b(y); x, y) T^j(y), \quad (28)$$

with the nonultralocal term chosen linear in the clock variables for simplicity. We take the clock variables as 3-diff scalars and the kernel K^{ij} to be a weight-one 3-density. The first term on the RHS is ultralocal and does not contribute to $\{H_I(\tau), H_J(\tau)\}$. One finds

$$\begin{aligned} \mathcal{Q}(\tau) &\doteq \{H_I^x(\tau), H_m^z(\tau)\} \approx \mathcal{O}\left[\frac{\delta h_l^x}{\delta T_m^z} - \frac{\delta h_m^z}{\delta T_l^x}\right](\tau) \\ &\approx \mathcal{O}[K^{lm}(q^a(x), p_a(x), q^b(z), p_b(z); x, z) \\ &\quad - K^{ml}(q^b(z), p_b(z), q^a(x), p_a(x); z, x)](\tau). \end{aligned} \quad (29)$$

Next we examine the τ and external time gauge flow in more detail. This is the bridge we will need to cross to get to the Lieb-Robinson bounds. From the classical gauge flow equation (11) and again following Han’s sign conventions that $\{P_a, Q^b\} \approx \delta_a^b$, $[\hat{P}_a(\tau), \hat{Q}^a(\tau)] = (-i)\{P_a(\tau), Q^b(\tau)\} = (-i)\delta_a^b$ (\hbar is set to unity), one has the on-shell quantum gauge flow equation

$$\frac{\partial}{\partial \tau_I} \hat{\mathcal{O}}[f](\tau) \approx (i)[\hat{H}_I(\tau), \hat{\mathcal{O}}[f](\tau)], \quad (30)$$

where f may depend on q^a , p_a canonical variables but not the T^I , P_I types. Adopting the Ansatz

$$\hat{\mathcal{O}}[f](\tau) = \exp[i\hat{M}(\tau)](\hat{\mathcal{O}}[f](0)) \exp[-i\hat{M}(\tau)] \quad (31)$$

for some self-adjoint operator \hat{M} independent of f , one infers from the gauge flow equation that

$$\left[\frac{\partial \hat{M}(\tau)}{\partial \tau^I} - \hat{H}_I(\tau), \hat{\mathcal{O}}[f](\tau)\right] \approx 0. \quad (32)$$

Since f is an arbitrary phase space function of q^a , p_a , one has that on-shell $\partial \hat{M}(\tau)/\partial \tau^I - \hat{H}_I(\tau)$ may depend on the \hat{T}^K but not on the \hat{q}^a , \hat{p}_a , or \hat{P}_K . Here we will assume the simplest case, that is,

$$\frac{\partial \hat{M}(\tau)}{\partial \tau^I} \approx \hat{H}_I(\tau), \quad (33)$$

which is sufficient, but not necessary. Now decompose \hat{M} as

$$\hat{M}(\tau) = \sum_Z \hat{M}_Z(\tau), \quad (34)$$

where Z is called a “patch” and is just the support of \hat{M}_Z on Σ . The motivation behind this is as follows: Each $I \in \mathcal{I}$ contains a continuous spatial coordinate $y(I)$ as well as a discrete gauge index $i(I)$. For nonlocal h_I and H_I , besides the canonical variables at $y(I)$ (called the “central site”) there are other “nearby” canonical variables living at $y' \neq y(I)$ which also enter h_I and H_I as their “entourage.” We define the patch $Z(I)$ to consist of the central site $y(I)$ together with all those nearby y' where its entourage resides. We assume the patches to be bounded and not to take up all of Σ . Denoting the central site of Z by $y_c(Z)$ we can set

$$H_Z \doteq H_{Z(I)} = H_I|_{y(I)=y_c(Z)}. \quad (35)$$

Using (33) with fixed I ,

$$H_{Z(I)}(\tau) \approx \sum_X \partial M_X(\tau)/\partial \tau^I. \quad (36)$$

Later on we will be more interested in the external time t gauge flow of $\hat{\mathcal{O}}[f](\tau)$ than in the τ -flow, so we let

$$\begin{aligned} \hat{\mathbb{H}}(t) &\doteq \sum_I \frac{\partial \tau^I(t)}{\partial t} \hat{H}_I(\tau) \\ &\approx \sum_I \frac{\partial \tau^I(t)}{\partial t} \sum_X \frac{\partial \hat{M}_X(\tau)}{\partial \tau^I} = \frac{\partial \hat{M}(t)}{\partial t}, \end{aligned} \quad (37)$$

$$\hat{M}(t) \doteq \sum_X \hat{M}_X(\tau(t)). \quad (38)$$

It then follows from the τ gauge flow equation (30) that

$$\begin{aligned}
\frac{d}{dt} \hat{\mathcal{O}}[f](t) &= \sum_I \frac{\partial \tau^I(t)}{\partial t} \frac{\hat{\mathcal{O}}[f](\tau(t))}{\partial \tau^I} \\
&= (i) \left[\sum_I \frac{\partial \tau^I(t)}{\partial t} \hat{H}_I(\tau), \hat{\mathcal{O}}[f](\tau) \right] \\
&= (i) [\hat{\mathbb{H}}(t), \hat{\mathcal{O}}[f](\tau(t))]. \tag{39}
\end{aligned}$$

As expected, one sees that $\hat{\mathbb{H}}(t)$ generates gauge flow in external time. More explicitly, setting $\hat{\mathcal{O}}[f](t) \doteq \hat{\mathcal{O}}[f](\tau(t))$, (39) implies

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0} \hat{\mathcal{O}}[f](t + \epsilon) &= \hat{\mathcal{O}}[f](t) + i\epsilon [\hat{\mathbb{H}}(t), \hat{\mathcal{O}}[f](t)] + \mathcal{O}(\epsilon^2) \\
&= \exp(i\epsilon \hat{\mathbb{H}}(t)) \hat{\mathcal{O}}[f](t) \exp(-i\epsilon \hat{\mathbb{H}}(t)). \tag{40}
\end{aligned}$$

We would like to put $\hat{\mathbb{H}}(t) \doteq \sum_Z \hat{H}_Z(t)$ and figure out what $\hat{H}_Z(t)$ is; hence by (37)

$$\begin{aligned}
\sum_Z \hat{H}_Z(t) &\doteq \hat{\mathbb{H}} = \sum_I \frac{\partial \tau^I(t)}{\partial t} \hat{H}_I(\tau) \\
&= \sum_Z \sum_{I: Z(I)=Z} \frac{\partial \tau^I(t)}{\partial t} \hat{H}_I(\tau) \\
&= \sum_Z \left(\frac{\partial \tau^I(t)}{\partial t} \hat{H}_I(\tau) \right)_{Z(I)=Z}, \tag{41}
\end{aligned}$$

where the last equality follows from the fact that there is only one patch $Z(I)$ with central site $y(I)$ corresponding to Z : $y(I) = y_c(Z)$. Consequently,

$$\sum_Z \hat{H}_Z(t) = \sum_Z \sum_{i=0}^{N_c-1} \left(\frac{\partial \tau^i(y_c(Z), t)}{\partial t} \right) \hat{H}_i^{y_c(Z)}(\tau(t)). \tag{42}$$

And pulling it all together

$$\hat{\mathbb{H}}(t) \doteq \sum_Z \hat{H}_Z(t), \tag{43}$$

$$\hat{H}_Z(t) = \sum_{i=0}^{N_c-1} \left(\frac{\partial \tau^i(y_c(Z), t)}{\partial t} \right) \hat{H}_i^{y_c(Z)}(\tau(t)). \tag{44}$$

$\hat{\mathbb{H}}(t)$ is a ‘‘patchy’’ Hamiltonian generating relational framework gauge flow in external time, whose patches Z are based on the nonultralocality of the original h_I . This patchy representation of t gauge flow unlocks the door to applying the Lieb-Robinson bounds to be introduced in Sec. IV below.

Before immediately moving on to the Lieb-Robinson bound, we will need to know a little more about $[\hat{H}_X(t), \hat{H}_Y(t)]$. Here we discuss its classical counterpart $\mathcal{Q} \doteq \{H_X(t), H_Y(t)\}$ to gain some intuition about it before proceeding.

\mathcal{Q} is built from

$$\{H_{i_1}^{y_1}, H_{i_2}^{y_2}\} \approx \mathcal{O} \left[\frac{\partial h_{i_1}^{y_1}}{\partial T_{i_2}^{y_2}} - \frac{\partial h_{i_2}^{y_2}}{\partial T_{i_1}^{y_1}} \right](\tau). \tag{45}$$

Denote

$$f_{i_2}^{i_1}(x, y) \doteq \mathcal{O} \left[\frac{\partial h_{i_1}^x}{\partial T_{i_2}^y} \right](\tau). \tag{46}$$

We have $\mathcal{Q}(x, y, t) \doteq \{H_X(t), H_Y(t)\} = -\mathcal{Q}(y, x, t)$ where $x = y_c(X)$ and $y = y_c(Y)$, and thus

$$\mathcal{Q}(x, y, t) = \sum_{i_1, i_2=0}^{N_c-1} \left[\frac{\partial \tau_{i_1}(x)}{\partial t} \frac{\partial \tau_{i_2}(y)}{\partial t} \right] (f_{i_2}^{i_1}(x, y) - f_{i_1}^{i_2}(y, x)). \tag{47}$$

As a reminder, \mathcal{Q} may be a phase space function of the T^K, q^a, p_a variables, but not P_K variables, or it could just be a phase space constant, and so include τ 's. As a simple example, here we will try to construct f from just the τ 's. We take $f_j^i(x, y)$ to be (x, y) symmetric (P -even) and (i, j) antisymmetric, such as $f_j^i(x, y) = (\tau_i(x) - \tau_j(x)) + (\tau_i(y) - \tau_j(y))$. Then for the sum in (47) to be nonvanishing, we have to (i_1, i_2) antisymmetrize the brackets in that expression to obtain

$$\mathcal{Q}(x, y, t) = \sum_{i_1, i_2=0}^{N_c-1} \left[\frac{\partial \tau_{i_1}(x)}{\partial t} \frac{\partial \tau_{i_2}(y)}{\partial t} - \frac{\partial \tau_{i_2}(x)}{\partial t} \frac{\partial \tau_{i_1}(y)}{\partial t} \right] f_{i_2}^{i_1}(x, y), \tag{48}$$

which is overall (x, y) antisymmetric as required. So as $y \rightarrow x$, $\mathcal{Q} \rightarrow 0$, but away from $x = y$, \mathcal{Q} is nonvanishing. We expect it to decay as H_X 's and H_Y 's patches X and Y cease to overlap, but that behavior is not well captured by this toy model for \mathcal{Q} .

Lieb-Robinson bounds were originally intended to study spin systems imbedded in a solid-state lattice, so they are naturally discretized. This lattice may be extended to include a general network and is not limited to a periodic tessellation of 3-space by polyhedra. Discretization achieves significant mathematical simplifications, so we will follow that approach in this initial investigation of relational framework with nonultralocal constraints. We discuss the limitations and issues related to discretization and its continuum limit in Sec. VII. A continuum approach will be left for future research.

Here we describe the discretization of the 3-manifold Σ into a (generalized) lattice Λ . Associated with each lattice site $j \in \Lambda \subset \Sigma$ is a D -dimensional Hilbert space. Unless otherwise indicated Λ will have finite size (cardinality); alternatively, Λ may be taken to be a finite sublattice of

some countable lattice $\Gamma \subset \Sigma$. The Lieb-Robinson bound does not depend on the dimensionality D , and the Hilbert space for the entire system is taken to be the tensor product of the site-based spaces. Capital Latin letters from the end of the alphabet (previously referring to Hamiltonian patches) will now denote sets of lattice sites, and $|X|$ designates the cardinality of X . We say an operator \hat{O} is supported on a set Y of sites if \hat{O} may be expressed as $\hat{O} = \hat{1}_{\Lambda \setminus Y} \otimes \hat{P}$, where $\hat{1}_{\Lambda \setminus Y}$ is the identity operator on sites not in Y , and \hat{P} is an operator defined on Y . In the following sections we will be most interested in the complete (Dirac) observables $\mathcal{O}[f](\tau)$, where f is a phase space function containing neither T^I nor P_I canonical variables, and use the unitary Hamiltonian patchy gauge flow (39), (43), and (44). We assume we can take the discretized $\hat{H}_I(\tau) \approx \hat{H}_j^i(Q^a(\tau), P_a(\tau), \tau)$ as a self-adjoint operator on the Hilbert space which is the tensor product of Hilbert spaces over lattice sites $j(a)$ included in its arguments.

We must take a moment to carefully resolve any potential issues that might arise from discretizing operators like the $\hat{\mathcal{O}}[f](\tau)$, and to confirm that the discretized equations behave as expected, especially from a gauge-flow point of view. To this end, we have to define what precisely is meant by the spatial discretization Δ of a continuum operator constructed as a sum of products of the canonical variables. The discretization map Δ is defined to act linearly with respect to any sum of operators. Acting on a product of continuum operators, Δ annihilates (“apodizes” or cuts off) any product which contains one or more factors of canonical variables that are not on the lattice. From this definition follows $\Delta(\hat{A} \hat{B}) = \Delta(\hat{A})\Delta(\hat{B})$. The procedure we follow is to “Diracify” first by constructing the continuum Dirac operator $\hat{\mathcal{O}}[f](\tau)$ from f , and then to discretize by acting with Δ . One seeks to demonstrate that the continuum gauge-flow equation (30) holds when all operators are replaced by their discretized images under Δ , i.e., that Δ is a gauge-flow homomorphism. This is made easier after one notes that the continuum gauge-flow equation (30) is an operator equation with both sides (weakly) equal to the operator corresponding to the classical expression

$$\sum_{k_j=0}^{\infty} \left(\prod_J \frac{(\tau^J - T^J)^{k_j}}{k_j!} \right) X_I \left(\prod_J (X_J)^{k_j} \right) \cdot f; \quad (49)$$

see Eq. (2.13) in [11]. Then by restricting the free index I to have $y(I) \in \Lambda$, i.e. to be on the lattice, and applying the Δ map, it is straightforward to show that

$$\frac{\partial}{\partial \tau^I} \Delta(\hat{\mathcal{O}}[f](\tau)) \approx (i) [\Delta(\hat{H}_I(\tau)), \Delta(\hat{\mathcal{O}}[f](\tau))]. \quad (50)$$

One has to interpret the $\tau^J = \tau^J \hat{1}$ terms as $\Delta(\hat{1}) = \hat{1}_{\Lambda} = \otimes_{j \in \Lambda} \hat{1}_j$, where $\hat{1}_j$ is the identity operator on the

Hilbert space at site j . This way all the J 's appearing in the sums inside (50) are on lattice, and there is no on-shell operator flow to/from the lattice from/to nonlattice sites. It is also simple to show that $[\Delta(\hat{C}_I), \Delta(\hat{C}_J)] \approx 0$, so the discretized constraints are weakly Abelian. Henceforth we drop the Δ whenever it is clear from the context that we are discussing a discretization.

If i, j are lattice sites on Σ , the Lieb-Robinson bound requires a 3-metric $d(i, j)$. If A, B are sets of lattice sites, for future use we define

$$d(A, B) \doteq \text{dist}(A, B) \doteq \min_{i \in A, j \in B} d(i, j), \quad (51)$$

$$\text{diam}(A) \doteq \max_{i, j \in A} d(i, j). \quad (52)$$

When Lieb-Robinson bounds were first applied to solid-state spin systems, introducing the static metric $d(i, j)$ was innocuous; however, in applying Lieb-Robinson bounds to gravitational physics there are several issues of serious concern. Already at the purely classical level, d will acquire a dependence on geometric variables included in the $Q^a(\tau)$, so if one has a continuously varying external time parameter t , $d(i, j)$ will inherit a continuous t dependence as well, while d still describes a discretized 3-geometry of Σ . We will show in Sec. V how this t dependence can be accommodated within the Lieb-Robinson bound. Still at the classical level, in the continuum $d(x, y)$ for $x, y \in \Sigma$ could be taken as the proper geodesic distance between x and y . But once Σ has been discretized, the voxelated classical 3-geometric information and the replacement of the PDE describing a geodesic by a finite difference equation will introduce a classical discretization “error” into $d(i, j)$. Of course, one expects this classical error to become negligible in the limit where the (proper) lattice cell size becomes much smaller than any classical length characterizing the 3-geometry. On the quantum level, once the classical phase space functions are mapped into operators, the well-known more difficult issues of quantum fluctuations, nonvanishing expectations of variances, choices of quantum state, and so on, immediately arise. The quantum clock “operators” \hat{T}^I themselves, however, do not appear in the physical Hilbert space of the quantum theory, since they are not gauge invariant, so there are no worries from that direction: only $f(T^I = \tau^I)$ can be promoted to a physical quantum operator [13]. In the fully developed quantum regime, of course, there is not even a well-defined 3-geometry at all, so the best one might hope for is that one can find some kind of semiclassical regime or limit that supports or approximates a 3-metric like $d(x, y)$. For now, we will work at a level (classical or semiclassical) where we may safely assume we do have a sufficiently accurate t -dependent $d(i, j)$ on the lattice, and discretization error, quantum fluctuations, and semiclassical consistency will be discussed later in Sec. VII after we see what

the Lieb-Robinson bound can tell us about relational framework operator gauge flow in external time with nonultralocal constraints.

IV. INTRODUCTION TO LIEB-ROBINSON BOUNDS

Here we provide a brief and hopefully self-contained introduction to Lieb-Robinson bounds. The definitions and theorems will be presented together with some intuition, but we refer the more interested reader to Refs. [14] and [15] for the detailed derivations.

From a pedagogical point of view, it is best to start with the simplest case first: A nonrelativistic spin system on a one-dimensional lattice [14]. So consider the one-dimensional transverse Ising model for N spins with Hamiltonian

$$H = -J \sum_{i=1}^{N-1} S_i^z S_{i+1}^z + B \sum_{i=1}^N S_i^x. \quad (53)$$

This spin Hamiltonian has the form $H = \sum_Z H_Z$ with H_Z supported on Z . Lieb-Robinson bounds are most suited to cases where $\|H_Z\|$ decays rapidly with $\text{diam}(Z) > 1$. Using (53) and the metric $d(i, j) = |i - j|$, we see that the Zeeman term has diameter 0 and the Ising (exchange) interaction has diameter one. So $\|H_Z\| = 0$ for $\text{diam}(Z) > 1$, and these are examples of “finite range” interactions. There are also other forms of decaying interactions such as exponential, and so on. One could also place the spins at the vertices of a graph. Then H is again a sum of H_Z , each Z being two vertices, with H_Z nonvanishing only if an edge of the graph links them. In that case the metric $d(i, j)$ could be chosen as the shortest path metric, and it gives $\|H_Z\| \neq 0$ only if $\text{diam}(Z) = 0, 1$. (This is quite different from the spin networks usually considered in loop quantum gravity.)

When discussing these kinds of spin systems it is natural to give operators the (nonrelativistic) time dependence given by Heisenberg evolution,

$$\mathcal{O} = \exp[iHt]\mathcal{O}(0)\exp[-iHt], \quad (54)$$

where for simplicity we have taken H to be (explicitly) time independent. Then one has the following [14]:

Theorem (L-R): Suppose for all sites $i \in \Lambda$ one has the L-R condition

$$\sum_{X \ni i} \|H_X\| |X| \exp[\mu \text{diam}(X)] \leq s, \quad (55)$$

for some positive real constants s, μ . Let A_X and B_Y be (bosonic) operators supported on sets X, Y , respectively. Then if $d(X, Y) > 0$, one has

$$\begin{aligned} & \| [A_X(t), B_Y(0)] \| \\ & \leq 2 \|A_X\| \|B_Y\| \sum_{i \in X} \exp[-\mu \text{dist}(i, Y)] [\exp(2s|t|) - 1] \end{aligned} \quad (56)$$

$$\leq 2 \|A_X\| \|B_Y\| |X| \exp[-\mu \text{dist}(X, Y)] [\exp(2s|t|) - 1]. \quad (57)$$

The physical interpretation of this bound is made especially lucid by an argument due to Hastings [14], which we reproduce here because of its later importance: Given an operator A with support X as above, let $B_\ell(X)$ be the ball radius ℓ about X : $B_\ell(X) = \{i : \text{dist}(i, X) \leq \ell\}$. Construct the following operator:

$$A_X^\ell(t) = \int dU U A_X(t) U^\dagger, \quad (58)$$

where one integrates over unitaries U supported on $\Lambda \setminus B_\ell(X)$ using the Haar measure. $A_X^\ell(t)$ has support $B_\ell(X)$. Since $U A_X(t) U^\dagger = A_X(t) + U[A_X(t), U^\dagger]$, one has

$$\|A_X^\ell(t) - A_X(t)\| \leq \int dU \| [A_X, U] \|. \quad (59)$$

Using Lieb-Robinson bound (57) to bound the integrand on the RHS, we see $A_X^\ell(t)$ is exponentially operator norm close to $A_X(t)$ provided ℓ is sufficiently large compared to $2s|t|/\mu$. That is, a time-evolved operator $A_X(t)$ may be approximated to exponential accuracy by an operator $A_X^\ell(t)$ supported on $B_\ell(X)$. Therefore $B_\ell(X)$ has the interpretation of an effective t -dependent support for $A_X(t)$, and the (norm) “leakage” of $A_X(t)$ out of the “light cone” $B_\ell(X)$ is exponentially small.

Most commonly the Lieb-Robinson bound is cast into the following form: Suppose the L-R condition (55) holds; then there is a constant v_{LR} that depends on s, μ such that for $\ell = \text{dist}(X, Y)$, and $\ell \geq v_{LR}t$,

$$\| [A_X(t), B_Y(0)] \| \leq \frac{v_{LR}|t|}{\ell} g(\ell) |X| \|A_X\| \|B_Y\|, \quad (60)$$

and $g(\ell)$ decays exponentially with ℓ . From the theorem, $v_{LR} = 2s/\mu$. $A_X(t)$ can be approximated by $A_X^\ell(t)$ supported on the set of sites within distance $\ell = v_{LR}|t|$ of X by an error whose norm is bounded by $\ell^{-1} v_{LR}|t| g(\ell) |X| \|A_X\|$. Bounds on the leakage of information (von Neumann entropy) out of the light cone were studied in [16]. For H_Z of finite nonzero range, i.e., $\|H_Z\| = 0$ for $\text{diam}(Z) > R$ for some R , the bound may be further improved [14]. If $R = 1$, $\|H_Z\| \leq J$, then one finds $g(\ell)$ decays *faster* than exponentially, roughly $g(\ell) \sim \exp(-a\ell^2)$, for positive constant a . However, if H_Z has range 0, the discrete equivalent of ultralocality, then μ is undefined since $\text{diam}(X) = 0$, and there is no more Lieb-Robinson light cone.

The intuition underlying the exponential decay is the following: From the proof [14] one finds that the n th order term of the exponential comes from a chain H_{Z_1}, \dots, H_{Z_n} such that $Z_1 \cap X \neq \emptyset, Z_1 \cap Z_2 \neq \emptyset, \dots, Z_{n-1} \cap Z_n \neq \emptyset, Z_n \cap Y \neq \emptyset$, i.e., a chain of n patches $Z_k, k = 1, \dots, n$ each supporting a local patch of H . Successive H_k are mutually noncommuting as their ranges overlap, but more distant ones commute as their supports are mutually disjoint. So it is crucial for Lieb-Robinson bounds that $[H_{Z_1}, H_{Z_2}] \neq 0$ for $Z_1 \cap Z_2 \neq \emptyset$ and $[H_{Z_1}, H_{Z_2}] = 0$ for $Z_1 \cap Z_2 = \emptyset$ for all Hamiltonian patches Z_1, Z_2 . This is why the (classical) relational framework analog $\{H_I(\tau), H_J(\tau)\}$ was studied earlier in Sec. III, where the relation between the Lieb-Robinson bound patchy H_Z and the relational framework $H_I(\tau)$ is given by (44). These chains of successively overlapping Hamiltonian patches generate the effective operator support $A_X^\ell(t)$ (light cone).

While there are clear similarities with some features of relational framework nonultralocality, the relational framework Hamiltonians $H_I(\tau)$ have a nontrivial (and nonunitary) $\tau(t)$ flow; see (23). In particular, the simple Heisenberg evolution with a time independent Hamiltonian (54) does not apply to $\hat{O}[f](\tau(t))$, and one requires a Lieb-Robinson bound for t dependent Hamiltonians. A nonrelativistic Lieb-Robinson bound including this possibility was derived by Nachtergaele, Vershynina, and Zagrebnov (NVZ) in 2011 [15], which will now be sketched.

NVZ start with vertices $x \in \Gamma$, where Γ is a countable set of vertices. They assume: There exists a nonincreasing real-valued function $F: [0, \infty) \rightarrow (0, \infty)$ such that

$$\|F\| \doteq \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x, y)) < \infty, \quad \text{and} \quad (61)$$

$$C \doteq \sup_{x, y \in \Gamma} \sum_{z \in \Gamma} \frac{F(d(x, z))F(d(z, y))}{F(d(x, y))} < \infty. \quad (62)$$

For $\mu > 0$ define $F_\mu(x) \doteq \exp(-\mu x)F(x)$, so $\|F_\mu\| < \|F\|$, $C_\mu < C$. The Hilbert space of states for the subsystem at $x \in \Gamma$ is \mathcal{H}_x . For finite $\Lambda \subset \Gamma$ the Hilbert space associated with Λ is $\mathcal{H}_\Lambda \doteq \otimes_{x \in \Lambda} \mathcal{H}_x$. The algebra of observables supported on Λ is $\mathcal{A}_\Lambda \doteq \otimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x)$, where $\mathcal{B}(\mathcal{H}_x)$ is the set of bounded linear operators on \mathcal{H}_x . If $\Lambda_1 \subset \Lambda_2$, then identify \mathcal{A}_{Λ_1} with the subalgebra $\mathcal{A}_{\Lambda_1} \otimes \hat{1}_{\Lambda_2 \setminus \Lambda_1}$ of \mathcal{A}_{Λ_2} , and so $\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$. The algebra of local observables is defined as

$$\mathcal{A}_\Gamma^{\text{loc}} \doteq \bigcup_{\Lambda \subset \Gamma} \mathcal{A}_\Lambda. \quad (63)$$

The C^* -algebra of quasilocal observables \mathcal{A} is the norm completion of $\mathcal{A}_\Gamma^{\text{loc}}$. The support of $A \in \mathcal{A}_\Lambda$ is the minimal set $X \subset \Lambda$ such that $A = A' \otimes \hat{1}_{\Lambda \setminus X}$ for some $A' \in \mathcal{A}_X$. The generator of the operator dynamics is defined for each finite

volume $\Lambda \subset \Gamma$, and we confine our interest to Hamiltonian interactions (NVZ were also able to include suitable dissipative terms). This interaction is such that for each finite $X \subset \Gamma$ and for all t , $\Phi(t, X)$ is an operator in \mathcal{A}_X and $\Phi^*(t, X) = \Phi(t, X)$. The evolution map $\mathcal{L}_\Lambda(t)$, for any finite $\Lambda \subset \Gamma$ and time t , is a bounded linear map $\mathcal{A}_\Lambda \rightarrow \mathcal{A}_\Lambda$,

$$\mathcal{L}_\Lambda(t)(A) \doteq \sum_{Z \subset \Lambda} (i)[\Phi(t, Z), A] \doteq \sum_{Z \subset \Lambda} \Psi_Z(t)(A). \quad (64)$$

The $\Psi_Z(t)$ are bounded linear maps acting on \mathcal{A}_X , for any $X \subset \Lambda$ such that $X \supset Z$, which are of the form $\Psi_Z(t) \otimes \text{id}_{X \setminus Z}$. The $\Psi_Z(t)$ have norms that generally depend on X , but are uniformly bounded as $\|\Psi_Z(t)\| \leq 2\|\Phi(t, Z)\|$. Let $M_n = \mathcal{B}(\mathbb{C}^n)$ be the $n \times n$ complex matrices. We say a map $\Psi \in \mathcal{B}(\mathcal{A}_Z)$ is *completely bounded* iff, $\forall n \geq 1$ the linear maps $\Psi \otimes \text{id}_{M_n}$, defined on $\mathcal{A}_Z \otimes M_n$, are bounded with uniformly bounded norm

$$\|\Psi\|_{\text{cb}} \doteq \sup_{n \geq 1} \|\Psi \otimes \text{id}_{M_n}\| < \infty. \quad (65)$$

Specifically, $\|\Psi_Z\|_{\text{cb}}$ is a map defined on \mathcal{A}_Λ , $\forall \Lambda \subset \Gamma$ such that $Z \subset \Lambda$, which is *independent* of the choice of Λ in Γ .

To obtain a Lieb-Robinson bound, NVZ make the following two hypotheses: Given Γ, d, F as above,

- (1) For all finite $\Lambda \subset \Gamma$, \mathcal{L}_Λ is norm continuous in t , hence uniformly continuous on compact intervals.
- (2) For each Λ , there exists $\mu > 0$ such that for all $t \in \mathbb{R}$,

$$\|\Psi\|_{t, \mu} \doteq \sup_{s \in [0, t]} \sup_{x, y \in \Lambda} \sum_{Z \ni x, y} \frac{\|\Psi_Z\|_{\text{cb}}}{F_\mu(d(x, y))} < \infty. \quad (66)$$

One also finds

$$\begin{aligned} \|\mathcal{L}_\Lambda(t)\| &\leq \sum_{Z \subset \Lambda} \|\Psi_Z(t)\| \leq \sum_{x, y \in \Lambda} \sum_{Z \ni x, y} \|\Psi_Z(t)\|_{\text{cb}} \\ &\leq \|\Psi\|_{t, \mu} |\Lambda| \|F\| \doteq M_t. \end{aligned} \quad (67)$$

By definition of $\|\Psi\|_{t, \mu}$ one has $M_s \leq M_t$ for $s < t$.

Fix some large time $T > 0$, and for all $A \in \mathcal{A}_\Lambda$ let $A(t)$ for $t \in [0, T]$ be a solution of the ordinary differential equation (ODE)

$$\frac{d}{dt} A(t) = \mathcal{L}_\Lambda A(t) \quad \text{with} \quad A(0) = A. \quad (68)$$

Because $\|\mathcal{L}_\Lambda(t)\| \leq M_T < \infty$, this ODE has a unique solution defined by $\gamma_{t, s}^\Lambda(A) = A(t)$ for $0 \leq s \leq t \leq T$, where $A(t)$ is the unique solution of (68) for $t \in [s, T]$ with initial condition $A(s) = A$. We say a linear map $\gamma: \mathcal{A} \rightarrow \mathcal{B}$ for C^* -algebras \mathcal{A}, \mathcal{B} is *completely positive* if the maps $\gamma \otimes \text{id}_n: \mathcal{A} \otimes M_n \rightarrow \mathcal{B} \otimes M_n$ are positive for all $n \geq 1$. Here positive means positive algebra elements (i.e.,

of form A^*A) are mapped to positive algebra elements. NVZ showed that the map $\gamma_{t,s}$ is a unit preserving, completely positive map. For the one parameter group of automorphisms induced by the Hamiltonian generators $\Phi(t, Z)$, the NVZ version of the Lieb-Robinson bound states the following:

There are constants v, μ, c such that for $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$,

$$\|[A, B(t)]\| \leq C(A, B) \exp[-\mu(d(X, Y) - vt)], \quad (69)$$

where $C(A, B) = c\|A\|\|B\| \min(|X|, |Y|)$. More specifically, given assumptions (1) and (2) above, NVZ's theorem 2 states that

$$\begin{aligned} \|[A(s), B(t)]\| &\leq (2/C_\mu)\|A\|\|B\|\|F\| \min(|X|, |Y|) \\ &\times \exp(-\mu d(X, Y)) \\ &\times [\exp(\|\Psi\|_{t,\mu} C_\mu(t-s)) - 1], \end{aligned} \quad (70)$$

for $X, Y \subset \Lambda, X \cap Y = \emptyset$. Notice the bound is *uniform* over the chosen Λ . To extend to uniformity over Γ , the definition of $\|\Psi\|_{t,\mu}$ in assumption (2) above should have the sup over x, y altered from Λ to Γ .

In order to adapt this result to relational framework gauge flow, we need to know that the derivation introduces the quantity

$$C_B(X, t) \doteq \sup_{\mathcal{T} \in \mathcal{B}_X} \frac{\|\mathcal{T} \gamma_{t,s}^\Lambda(B)\|}{\|\mathcal{T}\|_{cb}}, \quad (71)$$

where for $X \subset \Lambda, \mathcal{B}_X$ is the subspace of $\mathcal{B}(\mathcal{A}_X)$ of completely bounded linear maps vanishing on the identity. Here NVZ use the cb-norm (in contrast to the standard norm) to make the denominator independent of $\Lambda \subset \Gamma$. NVZ's derivation gives

$$\begin{aligned} C_B(X, t) &\leq C_B(X, s) \\ &+ \sum_{Z \cap X \neq \emptyset} \int_s^t \|\mathcal{L}_Z(r)\| C_B(Z, r) \, dr, \quad \text{with} \end{aligned} \quad (72)$$

$$\|\mathcal{L}_Z(t)\| = \|\Psi_Z(t)\| \leq \|\Psi\|_{t,\mu} \sum_{x,y \in Z} F_\mu(d(x, y)). \quad (73)$$

One also has that $C_B(Z, s) = \|B\|$ if $Z \cap Y \neq \emptyset$ and otherwise vanishes. Iterating these equations produces a Dyson expansion,

$$C_B(X, t) \leq \|B\| \sum_{n=1}^{\infty} \frac{1}{n!} (t-s)^n a_n, \quad (74)$$

$$a_n = (\|\Psi\|_{t,\mu})^n (C_\mu)^{n-1} \sum_{x \in X, y \in Y} F_\mu(d(x, y)). \quad (75)$$

This implies Eqs. (69) and (70) above with the (non-relativistic) spatially uniform Lieb-Robinson velocity

$$v_{LR} = \|\Psi\|_{t,\mu} \frac{C_\mu}{\mu} \leq \|\Psi\|_{T,\mu} \frac{C_\mu}{\mu}, \quad (76)$$

for all $t \in [0, T]$ (temporally uniform bound). This bound on v_{LR} can be utilized to bound the norm leakage of operators outside the Lieb-Robinson light cone, analogously to what was performed earlier for the time-independent Hamiltonian case.

V. LIEB-ROBINSON BOUNDS FOR THE DISCRETIZED RELATIONAL FRAMEWORK

We now apply the NVZ version of a Lieb-Robinson bound [15] to the relational framework discretized as previously described on some lattice or network. It is essential to handle appropriately the fact that the 3-metric $d(x, y)$ is both slicing $\{\tau(s)\}$ and slice s (external time) dependent, and the Lieb-Robinson bound should preserve the necessary gauge invariance. Strictly speaking, $d(x, y)$ should then be denoted as $d(x, y; \{\tau(s)\}, s)$ for the slice at external time s in slicing $\{\tau(s)\}$; however, we will continue to use the abbreviated form $d(x, y)$ for convenience. The reader should bear in mind the suppressed slicing and slice dependence.

The key initial step is to replace NVZ's $\Phi(t, Z)$ by

$$\Phi(t, Z) \rightarrow H_Z(t) = \sum_{i=0}^{N_c-1} \left(\frac{\partial \tau_i^{j_c(Z)}(t)}{\partial t} \right) H_i^{j_c(Z)}(\tau(t)), \quad (77)$$

from the spatially discretized version of (44). Equation (72) may be iterated as $C_B(X, t) \leq \sum_{n=1}^{\infty} C_B^{(n)}(X, t)$, where the $n=0$ iterate vanishes since $X \cap Y = \emptyset$. For simplicity we first focus on the $n=1$ term,

$$C_B^{(1)}(X, t) \doteq \sum_{Z \cap X \neq \emptyset} \int_s^t \|\Psi_Z(r)\| C_B(Z, s) \, dr \quad (78)$$

$$\leq 2\|B\|(t-s) \sum_{j \in X} \sum_{Z \ni j, Z \cap Y \neq \emptyset} \sup_{\tilde{s} \in [s, t]} \|H_Z(\tilde{s})\|. \quad (79)$$

The factor 2 on the RHS comes from $\|\Psi_Z(r)\| \leq 2\|H_Z(r)\|$, where 2 enters from bounding $\|[H_Z(r), A]\|$ by $2\|H_Z(r)\|\|A\|$. The general idea of the rest of the derivation is to insert strategically placed uniforming bounds (sups) after introducing an appropriate factor of F_μ . The ranges of the sups are also important and have to be selected with care. We also use $\sup(AB) \leq \sup(A) \sup(B)$ for $A, B > 0$. Inserting a factor $1 = F_\mu(d(j, k))/F_\mu(d(j, k))$, one has

$$\begin{aligned} C_B^{(1)}(X, t) &\leq 2\|B\|(t-s) \sum_{j \in X, k \in Y} \sum_{Z \ni j, k} \sup_{\tilde{s} \in [s, t]} \\ &\times \left[\frac{\|H_Z(\tilde{s})\|}{F_\mu(d(j, k))} F_\mu(d(j, k)) \right], \end{aligned} \quad (80)$$

and using the RHS of (77),

$$\begin{aligned}
C_B^{(1)}(X, t) &\leq 2\|B\|(t-s) \sum_{j \in X, k \in Y} \sum_{Z \ni j, k} \left(\sup_{\tilde{s} \in [s, t]} \right) \left(\sup_{i_2 \in [0, N_c - 1]} \right) \\
&\times \left[N_c \left| \frac{\partial \tau_{i_2}^{j_c(Z)}(\tilde{s})}{\partial \tilde{s}} \right| \right] \\
&\times \left(\sup_{\tau(\tilde{s})} \right) \left(\sup_{i_1 \in [0, N_c - 1]} \right) \left(\sup_{\tilde{s} \in [0, T]} \right) \left[\frac{\|H_{i_1}^{j_c(Z)}(\tau(\tilde{s}))\|}{F_\mu(d(j, k))} \right] \\
&\times \left(\sup_{\tilde{s} \in [s, t]} \right) F_\mu(d(j, k)), \tag{81}
\end{aligned}$$

where a bounding sup over slicings $\{\tau(\tilde{s})\}$ has been inserted into the middle factor. Henceforth we will take $\partial \tau_i^j(t)/\partial t \geq 0$, for all t, i, j , which is just the monotonicity condition necessary for freedom from anomalies discussed earlier, so the absolute values in the first line may be omitted. We have also “extended” the sup over \tilde{s} in the factor containing $\|H\|$ from $\tilde{s} \in [s, t]$ to $\tilde{s} \in [0, T]$, with

$0 \leq s < t \leq T$, where recall T is some “large” external time.

Next, first bound the first factor, containing the \tilde{s} derivatives of τ , by taking an overall sup over $j_c \in \Gamma$, thereby rendering that factor independent of Z . Returning then to the middle factor with $\|H\|$, bring in the sum over Z , and expand $j \in X, k \in Y$ to $x, y \in \Lambda \supset Z$, for some chosen $\Lambda \subset \Gamma$ of finite cardinality. By expanding j, k to $x, y \in \Lambda$, more positive terms were added. Hence the sup factor containing $\|H_{i_1}^j(\tau)\|$ may be bounded by

$$\left(\sup_{\tau(\tilde{s})} \right) \left(\sup_{i \in [0, N_c - 1]} \right) \left(\sup_{\tilde{s} \in [0, T]} \right) \left(\sup_{x, y \in \Lambda} \right) \sum_{Z \ni x, y} \left[\frac{\|H_i^{j_c(Z)}(\tau(\tilde{s}))\|}{F_\mu(d(x, y))} \right], \tag{82}$$

which is now conveniently independent of j, k . Therefore the j, k sum on the far left may then be moved all the way to the right to act only on the third factor, containing only $F_\mu(d(j, k))$. Thus,

$$\begin{aligned}
C_B^{(1)}(X, t) &\leq 2\|B\|(t-s) \left(\sup_{j_c \in \Gamma} \right) \left(\sup_{i \in [0, N_c - 1]} \right) \left(\sup_{\tilde{s} \in [s, t]} \right) \left[N_c \frac{\partial \tau_{i_c}^{j_c}(\tilde{s})}{\partial \tilde{s}} \right] \\
&\times \left(\sup_{\tau(\tilde{s})} \right) \left(\sup_{i \in [0, N_c - 1]} \right) \left(\sup_{\tilde{s} \in [0, T]} \right) \left(\sup_{x, y \in \Lambda} \right) \sum_{Z \ni x, y} \frac{\|H_i^{j_c(Z)}(\tau(\tilde{s}))\|}{F_\mu(d(x, y))} \left(\sum_{j \in X, k \in Y} \left(\sup_{\tilde{s} \in [s, t]} \right) F_\mu(d(j, k)) \right). \tag{83}
\end{aligned}$$

This may be written more compactly as

$$C_B^{(1)}(X, t) \leq 2\|B\|(t-s) I_{X, Y}(F_\mu) \left[\left(\sup_{j_c \in \Gamma} \right) \left(\sup_{i \in [0, N_c - 1]} \right) \left(\sup_{\tilde{s} \in [s, t]} \right) \left(N_c \frac{\partial \tau_{i_c}^{j_c}(\tilde{s})}{\partial \tilde{s}} \right) \right] \|H\|_{T, \mu}, \tag{84}$$

where we have set

$$I_{X, Y}(F_\mu) \doteq \sum_{j \in X} \sum_{k \in Y} \left(\sup_{\tilde{s} \in [s, t]} \right) F_\mu(d(j, k))|_{\tau(\tilde{s}), \tilde{s}}. \tag{85}$$

Similar to NVZ, we assume there exists a real $\mu > 0$ such that

$$\|H\|_{T, \mu} \doteq \left(\sup_{\tau(\tilde{s})} \right) \left(\sup_{i \in [0, N_c - 1]} \right) \left(\sup_{\tilde{s} \in [0, T]} \right) \left(\sup_{x, y \in \Lambda} \right) \sum_{Z \ni x, y} \frac{\|H_i^{j_c(Z)}(\tau(\tilde{s}))\|}{F_\mu(d(x, y))} < \infty. \tag{86}$$

From (86) we see $\|H\|_{T, \mu}$ is a bound temporally uniform over $\tilde{s} \in [0, T]$ and spatially uniform over the chosen finite subset $\Lambda \subset \Gamma$. This occurs because $\Lambda \subset \Gamma$ is *any* finite lattice containing $Z \subset \Lambda$, and $\|H_k^{j_c(Z)}(\tau(\tilde{s}))\|$ is independent of the choice of Λ within Γ . One may extend the definition of $\|H\|_{T, \mu}$ to be spatially uniform over countable Γ as in NVZ, by changing the sup over x, y in (86) from Λ to Γ .

We now examine $I_{X, Y}(F_\mu)$ in some more detail. Recall $F_\mu(x) \doteq \exp(-\mu x)F(x)$ is a positive real-valued, nonincreasing function of its non-negative real argument; thus for a fixed slicing

$$\sup_{\tilde{s} \in [s, t]} F_\mu(d(x, y)) \leq F_\mu(\inf_{\tilde{s} \in [s, t]} d(x, y)), \tag{87}$$

and

$$I_{X, Y}(F_\mu) \leq \sum_{x \in X} \sum_{y \in Y} F_\mu(\inf_{\tilde{s} \in [s, t]} d(x, y)). \tag{88}$$

Alternatively, in the discretized model under study, X, Y are both finite sets, so

$$\begin{aligned}
I_{X,Y}(F_\mu) &= \sup_{\tilde{s} \in [s,t]} \sum_{x \in X} \sum_{y \in Y} F_\mu(d(x,y)) \\
&\leq \sup_{\tilde{s} \in [s,t]} \min(|X|, |Y|) \sup_{y \in m(X,Y)} \sum_{x \in M(X,Y)} F_\mu(d(x,y)),
\end{aligned} \tag{89}$$

where $m(X, Y) \doteq Y$ and $M(X, Y) \doteq X$ if $|X| > |Y|$; otherwise $m(X, Y) \doteq X$ and $M(X, Y) \doteq Y$. Since F_μ is positive, this may be bounded by ‘‘expanding’’ both $m(X, Y)$ and $M(X, Y)$ to Γ to yield

$$I_{X,Y}(F_\mu) \leq \min(|X|, |Y|) \exp[-\mu \inf_{\tilde{s} \in [s,t]} d(X, Y)] \sup_{\tilde{s} \in [0,T]} \|F\|, \tag{90}$$

where as a reminder, $\|F\| \doteq \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x,y)) < \infty$ contains an implicit \tilde{s} dependence through $d(x,y)$.

When one bounds the higher order terms $n > 1$ in the Dyson expansion (74) for $C_B(X, t)$, at order n one initially inserts $n - 1$ factors of

$$\frac{F_\mu(d(x,z))F_\mu(d(z,y))}{F_\mu(d(x,y))} \leq C \frac{\exp[-\mu(d(x,z) + d(z,y))]}{\exp[-\mu d(x,y)]} < C, \tag{91}$$

where the triangle property of the 3-metric on a fixed slice of a fixed slicing has been applied. Recall the positive real constant C is defined by (62), and $F_\mu(x) = e^{-\mu x} F(x)$. There are also n factors of H_Z . Expanding $C_B(X, t) \leq \|B\| \sum_{n=1} \hat{a}_n (t-s)^n / n!$, one bounds

$$\begin{aligned}
\hat{a}_n &\leq (2\|H\|_{T,\mu})^n [(\sup_{\tau(\tilde{s})}) (\sup_{\tilde{s} \in [0,T]}) C_\mu(d(\tau(\tilde{s}), \tilde{s}))]^{n-1} \\
&\times \left[(\sup_{j_c \in \Gamma}) (\sup_{i \in [0, N_c-1]}) (\sup_{\tilde{s} \in [s,t]}) \left(N_c \frac{\partial \tau_i^{j_c}(\tilde{s})}{\partial \tilde{s}} \right) \right]^n \\
&\times \left[(\sup_{\tilde{s} \in [s,t]}) \sum_{j \in X} \sum_{k \in Y} F_\mu(d(j,k))|_{\tau(\tilde{s}), \tilde{s}} \right].
\end{aligned} \tag{92}$$

We now assemble all these intermediate steps into the final result. Define

$$\tilde{C}_\mu \doteq (\sup_{\tau(\tilde{s})}) (\sup_{\tilde{s} \in [0,T]}) C_\mu. \tag{93}$$

Because external time reparametrization (1-diff) invariance will require a Lieb-Robinson bound restricted to infinitesimal time increments $(t-s) \rightarrow 0$, we set $\delta t \doteq (t-s) \rightarrow 0$ in the above expressions. Also define

$$\begin{aligned}
& \left(\sup_{\tilde{s} \in [t, t+\delta t]} \right) \|F(d(\tau(\tilde{s}), \tilde{s}))\| \leq (\sup_{\tau(\tilde{s})}) (\sup_{\tilde{s} \in [0,T]}) (\sup_{x \in \Gamma}) \\
& \times \sum_{y \in \Gamma} F(d(x,y; \tau(\tilde{s}), \tilde{s})) \\
& \leq \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(\inf_{\tau(\tilde{s})} \inf_{\tilde{s} \in [0,T]} d(x,y)) \doteq \|\tilde{F}\|,
\end{aligned} \tag{94}$$

and $\|\tilde{F}\|$ is slicing and slice independent. Then one has the relational framework Lieb-Robinson bound,

$$\begin{aligned}
& \| [A_X(t), B_Y(t+\delta t)] \|_{|\tau(s)} \\
& \leq \frac{2}{C_\mu} \|A\| \|B\| \|\tilde{F}\| \min(|X|, |Y|) \\
& \times [\exp(-\mu \mathcal{E}(\tau(t))) - \exp(-\mu \inf_{s \in [t, t+\delta t]} d(X, Y))],
\end{aligned} \tag{95}$$

where the $|\tau(s)$ on the LHS indicates one is referring to a single slice at some $t \in [0, T]$ within an arbitrary slicing $\tau(s)$. As a reminder, X, Y are the supports of $A(t), B(t)$, respectively, with $X \cap Y = \emptyset$. We have denoted

$$\begin{aligned}
\mathcal{E}(\tau(t)) &\doteq \left[\inf_{s \in [t, t+\delta t]} d(X, Y; \tau(s), s) \right] - \left(\frac{2\tilde{C}_\mu}{\mu} N_c \|H\|_{T,\mu} \right) (\delta t) \\
&\times \left[(\sup_{j \in \Gamma}) (\sup_{i \in [0, N_c-1]}) (\sup_{s \in [t, t+\delta t]}) \left(\frac{\partial \tau_i^j(s)}{\partial s} \right) \right].
\end{aligned} \tag{96}$$

We will refer to the quantity

$$v_{LR} \doteq \frac{2\tilde{C}_\mu}{\mu} N_c \|H\|_{T,\mu} \tag{97}$$

as the relational framework Lieb-Robinson velocity.

VI. INVARIANCE AND OTHER PROPERTIES OF THE RELATIONAL FRAMEWORK LIEB-ROBINSON BOUND

The relational framework Lieb-Robinson bound (95) has a LHS that refers to the t -differential (δt) behavior of the norm of an operator commutator between discretized observables A, B near a single slice of some arbitrary slicing. Typically we take the operators A, B to have the relational framework form $\Delta(\hat{\mathcal{O}}[f](\tau))$, so they are discretized Dirac observables. The RHS has many factors, some of which are uniformly bounded over slicings and slices, and others that are slicing and/or slice dependent. We now describe those dependencies. Recall that a change of slicing is a gauge transformation, and a choice of slice s within a slicing is a gauge fixing. By definition (86), $\mu \sim 1/\text{diam}(Z)$ is chosen so that $\|H\|_{T,\mu}$ is finite and includes sups over all slicings $\tau(\tilde{s})$ and over all slices at \tilde{s} within those slicings. Therefore, in addition to being spatially and temporally uniform, μ and $\|H\|_{T,\mu}$ are slicing and slice

uniform as well. Moreover, $\tilde{C}_\mu \doteq \sup_{\tau(s)} \sup_{s \in [0, T]} C_\mu$ is μ and F dependent, and hence slicing and slice uniform too. Thus $(2/\mu)\tilde{C}_\mu \|H\|_{T, \mu}$ and then v_{LR} are both slicing, slice, and spatiotemporally uniform. The same conclusion holds for the prefactor of the exponentials on the RHS of (95).

By comparison with the simple t independent Hamiltonian nonrelativistic Lieb-Robinson bound (57), we see the exponentially damped leakage from a local light cone is governed by the quantity \mathcal{E} in (96) above. So when

$$\delta D(t) \doteq \inf_{s \in [t, t+\delta t]} (d(X, Y; \tau(s), s)) > v_{LR} \left(\sup_{j \in \Gamma} \right) \times \left(\sup_{k \in [0, N_c - 1]} \right) \left(\sup_{s \in [t, t+\delta t]} \right) \left[\frac{\partial \tau_k^j(s)}{\partial s} \delta t \right], \quad (98)$$

there is exponentially small leakage of the operator norm from the local light cone during δt . $\delta D(t)$ is a slicing dependent and slice dependent (slice near t) quantity. It is also t reparameterization (1-diff) invariant under $t \rightarrow t' \doteq f(t)$, $\delta t \rightarrow \delta t' = f'(t)\delta t$ with $f'(t) > 0$ for $t \in [0, T]$, treating τ^l and $H_I(\tau)$ as $(3+1)$ scalars, and noting that such a relabeling of the slice from t to t' does not affect the 3-metric d on the slice. At a fixed slicing and slice, $\delta D(t)$ is also invariant under smooth coordinate reparametrizations (3-diffs) of that 3-slice because the 3-metric d is, even though neither the spatial discretization (lattice) itself nor the 3-metric tensor g_{ij} are 3-diff invariant. The factor $\sup_s (\partial \tau / \partial s) \delta t$ on the RHS of (98) has these same 3-slice properties as d . Hence one has on-shell $(3+1)$ -diff invariance of the relational framework Lieb-Robinson bound local light cone.

As shown by Dittrich [7,8], on-shell one can classically embed every slicing into a 4-manifold with a Lorentzian 4-metric. The tangent bundle of this 4-manifold may also be smoothly partitioned in a $(3+1)$ -diff invariant way using that Lorentzian metric to define local null directions and so generate a 4-metric based null cone. The relational framework Lieb-Robinson bound local light cone should coincide with or bound the 4-metric null cone, but this has not yet been explicitly established. The 4-metric null cone, however, does not address the important issue of observable commutator leakage outside the light cone, which is the crux of the relational framework Lieb-Robinson bound.

One can also derive a relational framework Lieb-Robinson bound local light-cone structure with $v'_{LR} \sup_s (\partial \tau / \partial t) \delta t$ replacing $v_{LR} \sup_s (\partial \tau / \partial t) \delta t$ in a new $\delta \tilde{D}(t)$ which is slicing independent (containing a $\sup_{\tau(s)}$) but is slice dependent (still retaining the $\sup_{s \in [t, t+\delta t]}$). That is, one fixes some external t parametrization $\tau^l(t)$, and looks at slices within δt of t over all the slicings $\tau(t)$. But this construction seems less physically natural than the one described above, where $\delta D(t)$ is both slicing and slice dependent, so we will not discuss it further.

A differential Lieb-Robinson bound local light cone can be “integrated forwards” in t from slice to slice within a single slicing to generate “support tubes” for observables. To do this, one constructs operators $A_X^\xi(t, t + \delta t)$ over the external time interval $[t, t + \delta t]$ by the Hastings method described in Sec. IV which are exponentially accurate t -dependent supports for a discretized Dirac observable gauge evolving in external time from t to $t + \delta t$. This then may be iterated for succeeding slices spaced by intervals $\delta t \rightarrow 0$. This is precisely analogous to how one “integrates” null cones on a curved Lorentzian manifold to generate causal curves from the locally Minkowskian geometry.

VII. PHYSICAL AND CONCEPTUAL QUESTIONS

We have explored nonultralocal constraints with the relational framework and derived an external time differential local light-cone structure based on Lieb-Robinson bounds using a discrete spatial lattice or network model. Several physical and conceptual issues about this spring to mind, and we discuss those here. This discussion is by necessity less mathematically rigorous and considerably speculative in some cases.

Question 1: Aside from slicing and slice dependent factors like $(\partial \tau / \partial s) \delta t$ are v_{LR} and the local light cone “the same” as the classical spacetime into which the relational framework is embedded varies? That is, basically one is taking sups over a large spacetime to construct $v_{LR} \sup_s (\partial \tau / \partial s) \delta t$. But what happens when the entire spacetime, initial conditions, and so on, are altered? Would μ stay the same? If μ changed, then according to (97) v_{LR} would also change as the spacetime under investigation was altered, a potentially fatal physical pathology if we expect v_{LR} to be (or bound) the speed of light.

One way to avoid such an early demise for the relational framework Lieb-Robinson bound would be that $1/\mu \sim \text{diam}(Z)$, the proper “typical size” of single on-shell Hamiltonian patches, is a proper length much smaller than the minimum over the classical spacetimes of any proper curvature scales $L_c(x)$ they contain. That is, one could interpret $1/\mu$ as some kind of proper finite range ξ of the on-shell constraints or Hamiltonians. Thus, if ξ is a microscopic scale relative to classical geometrical scales, $\xi \ll \min_x L_c(x)$ uniformly for all the classical spacetimes under consideration, then different classical geometries but with the *same constraints* will have the same μ , $\|H\|_{T, \mu}$, \tilde{C}_μ and thus v_{LR} . This takes the constraints to be nonlocal, homogeneous in form, and uniformly bounded; it essentially requires a large separation of physical scales, which is common throughout physics. In addition, ξ would also have to be far smaller than any particle physics lengths that have been probed so far for there not to have been any evidence yet of nonlocality. But this does not mean that ξ has to be on the order of the Planck scale $L_P \sim 10^{-35}$ m, but it would certainly require $\xi \ll 2 \times 10^{-19}$ m (1 TeV). ξ would still have to be longer than the scale needed to have a

well defined 3-metric on each slice, which at least requires that $\xi \gg L_P$.

Question 2: What kind of terms in $H_I(\tau)$ would generate the required nonultralocality? Specifically, would gradient terms (of any order) suffice to produce the nonultralocality for a Lieb-Robinson bound?

This question is closely tied to discretization. In a typical discretization one replaces $\nabla_x \psi(x) \rightarrow (\psi(j+1) - \psi(j))/b$, where b is some kind of lattice constant, and j is a site index corresponding to the continuum coordinate x . So H_I containing a pointwise gradient in the continuum, such as $\psi(x)\nabla\psi(x)$, becomes nonlocal in the discretization since it would couple sites j and $j+1$. But in the continuum H_I remains firmly ultralocal. Indeed, all the well known actions for continuum canonical gravity such as the 3+1 ADM decomposition or the Holst action are ultralocal, and for those cases $\xi = 0$, and then there is no more Lieb-Robinson local light cone.

The answer to the question is no, continuum gradients alone are insufficient for nonultralocality. The reason for the negative answer is any discretization perceives a continuum gradient as a lattice constant dependent contribution to $\xi(b) = 1/\mu(b)$. An n th order gradient is computed by discretization to give a contribution $\xi \propto nb$, which vanishes in the continuum limit $b \rightarrow 0$. Gradients have no nonzero natural scale in the continuum. Hence for a local light cone to emerge by Lieb-Robinson bounds, the nonultralocal H_I cannot be constructed from products of fields and their gradients at a single point. Both $\xi(b)$ and $\|H_I\|_{T,\mu}$ must be independent of b as $b \rightarrow 0$. Physical quantities like v_{LR} and the local light cone cannot depend on any cutoff scale like b . This is the lesson of the renormalization group for background dependent quantum field theory, and it also requires discretization independence in the continuum limit for gravitational theories. In fact, as found in [17], discretization independence implies nonlocality in 4D discrete quantum gravity.

This question and its answer lead us to ask the following:

Question 3: What is the role of the discretization in the Lieb-Robinson bound?

Contact with physical reality occurs when $b \rightarrow 0$ because any nonzero b spatial discretization by itself breaks 3-diff invariance since the lattice or network is not generally mapped into itself by a 3-diff [18]. There are, however, perfect actions for discretizations that do recover the requisite invariance as $b \rightarrow 0$ for several models [19]. Even for these special cases, the discretization will coarse grain or voxelate the metric information at least over the scale b . This classical error makes the discretized $d(j, k)$ acquire a b dependence. Again we have to assume that $\lim_{b \rightarrow 0} d(x, y; b)$ is well defined so $v_{LR} \sup_s (\partial\tau/\partial s) \delta t$ and $\delta D(t)$ are all also well defined in that continuum limit, where the classical voxelation error disappears.

Question 4: How might the required nonultralocality of $h_I, H_I(\tau)$ arise? That is, if one starts from some bare

classical constraints specified by H_I^{cl} that are ultralocal, how can one end up with effective quantum constraints generated by a nonultralocal H_I^{eff} ?

The simple answer is that the real quantitative origins of nonultralocality h_I lie beyond the scope of the present work. Indeed, this is like asking what atomic physics lies behind the exchange coupling constant J in the Ising model (53). The quantum algebra \mathcal{U} at the kinematic level is generated by $\hat{Q}_a(\tau), \hat{P}_a(\tau)$ through their canonical commutation relations. From that algebra we can find its representation carrier Hilbert space by the GNS construction from any positive linear functional (state) ρ on \mathcal{U} . The quantum constraints are $\rho[\hat{P}_I + \hat{h}_I^{\text{eff}}] = 0$; i.e., the quantum fluctuations satisfy the effective constraints in the mean (as an expectation value). A similar process occurs in background-dependent quantum field theory where quantum corrections due to loops are taken into account using a field dependent effective (“dressed”) action $\Gamma[\Phi]$ rather than the bare classical action $I[\Phi]$ [20]. Connected vacuum-vacuum quantum field theory amplitudes can be computed using tree-level (mean field) Feynman graphs with vertices using $\Gamma[\Phi]$ instead of $I[\Phi]$. A similar “dressing” by quantum fluctuations might render \hat{H}^{eff} nonultralocal. Unfortunately the quantitative details are out of present computational reach for background independent quantum gravity without the well-defined path integral technology of quantum field theory.

An alternative way to consider this problem is from the point of view of the renormalization group (RNG), asymptotic safety, and lattice gauge theories [21]. How finite-scale interactions arise in the $b \rightarrow 0$ (UV) limit is addressed by the Wilsonian RNG flow. Nontrivial scaling of the coupling constants (residing within the h_I) at a RNG fixed point can lead to a continuum theory, especially if there is a second order phase transition either within the quantum-classical crossover regime or perhaps at lower energies. The correlation lengths such as ξ (in lattice units) would diverge there and may play a role in the continuum limit of canonical quantum gravity. It is also interesting to note that asymptotic safety does not use a bare action as an input, but rather produces one as an output, thereby possibly circumventing the issue of how starting from an ultralocal bare action leads to nonultralocality.

A more speculative answer is that we may not know the “true” physical action or constraints for gravity at very short but not yet Planckian lengths, only that they look local as far as we can tell from our experience at long length scales (above 2×10^{-19} m or energies up to 1 TeV). In that case, the Lieb-Robinson local light cone would be a long length scale manifestation of nonultralocality of those otherwise inaccessible short length scale constraints, a hint that we might not be aware of some deeper physics. One possibility is that noncommutative products such as the Moyal-Weyl-Groenewold \star -product and deformed diffeomorphisms could play a role. While noncommutative field theories still have Lagrangian densities and Noether

currents, the products of objects in the Lagrangian density are nonlocal [22]. The scale of that nonlocality might be associated with ξ above. However, noncommutative geometry is not a quantization of the underlying manifold in the sense that it does not promote phase space functions to operators. So noncommutative manifolds do not describe quantum fluctuations of geometry, and they are on-shell descriptions. Instead they require that the geometric and matter field actions be invariant under deformed diffeomorphisms of the noncommutative manifold, which are nearly the standard diffeomorphisms normally used to describe manifolds like Σ that leave the usual local action invariant. At length scales larger than the nonlocal effects, the noncommutative symmetries and constraints should approach the standard ones while an on-shell Lieb-Robinson light cone emerges from a relational framework based on the noncommutative action. It is worth noting in this regard that the nonlocality induced by noncommutative geometry is not of the gradient (spurious) type.

Question 5: In the relational framework Lieb-Robinson bound which variables are quantum and which are classical?

The operators \hat{A} , \hat{B} appearing in the Lieb-Robinson bound analysis include those constructed from $\hat{Q}_a(\tau)$, $\hat{P}_a(\tau)$, which may describe either geometrical or possibly matter degrees of freedom. These are treated fully quantum mechanically from the on-shell gauge flow (11). The τ_l are real parameters and are never promoted to quantum operators, so they do not acquire fluctuations. As discussed earlier, the clock variables T^I are also not promoted to physical operators \hat{T}^I . More crucially, at the heart of the Lieb-Robinson bound lurks $d(x, y)$, the on-slice 3-metric, computed from geometrical variables. In $d(x, y)$ those variables, such as the ADM 3-metric q_{ab} , are treated classically, while in $\hat{A}(\tau)$, $\hat{B}(\tau)$ those same variables are treated quantum mechanically. So there seems to be an inconsistency. That is, inside $\hat{A}(\tau)$ and $\hat{B}(\tau)$ \hat{q}_{ab} is treated as an element of a noncommutative (quantum) C^* -algebra, but when \hat{q}_{ab} enters a computation of $d(x, y; \tau(s), s)$ it is treated as an element of a commutative (classical) C^* -algebra. However, if we use classical geodesics as a basis for computing $d(x, y)$ as an extremum over the quantum expectation $\langle \hat{q}_{ab} \rangle$, then the Lieb-Robinson local light-cone quantity $\delta D(t)$ [see (98)] is mean field (expectation value level) with respect to \hat{q}_{ab} over $[t, t + \delta t]$. Within the Hamiltonians $\hat{H}_I(\tau)$, q_{ab} is treated as a quantum operator. Thus we can say the Lieb-Robinson local light-cone construction is at least mean field with respect to fluctuations entering q_{ab} , and it must be stabilized against those fluctuations if it is to survive.

Question 6: What parameters, if any, delineate a window of survival for the mean-field Lieb-Robinson local light cone in its precarious perch among the tensions between the classical and quantum worlds?

To answer this, we introduce δd as the largest root mean square quantum fluctuation of the proper lengths $d(i, j)$, and also take the classical discretization error in proper

lengths to be roughly the same as the lattice constant b . L_c denotes the shortest classical curvature scale of any 3-geometry under consideration. We also have $\xi = 1/\mu$, the range (proper patch diameter) of the nonultralocal constraints. Since a classical metric $d(i, j)$ loses physical meaning when $d(i, j) < \delta d$, and quantum fluctuations could play a role in the origin of ξ , $\delta d \lesssim \xi$. We can then qualitatively delineate four distinct physical regimes:

(a) $\delta d \lesssim \xi < b \ll L_c$: Here the discretization is too coarse to resolve the finite range ξ of the H_I , which then appears to be ultralocal. The Lieb-Robinson local light cone does not emerge.

(b) $\delta d < b < \xi \ll L_c$: Here the Lieb-Robinson local light cone is stabilized against the quantum fluctuations δd , the discretization can resolve the nonultralocal range ξ of the H_I , but it does not resolve the quantum fluctuations. The local light cone is b (discretization) independent once $b \ll \xi$.

(c) $b < \delta d < \xi \ll L_c$: The proper range ξ is still separated and immune from quantum fluctuations, while the local light cone is still discretization (b) independent.

(d) $b < \delta d = \xi \ll L_c$: Now ξ is quantum limited, a fluctuation limited patch size. We still have discretization independence once $b < \delta d$, where the discretization error is no longer physically relevant.

In cases (b) and (c) the Lieb-Robinson local light cone will survive provided: (1) the continuum limit $b \rightarrow 0$ is well behaved, and (2) $\delta d \ll \delta D(t)$ [see (98)]. This means δd does not significantly affect the width of the local light cone at $t, t + \delta t$, and thereby limits δt from below. That is, the t -dependence of the classical 3-metric $d(x, y)$ cannot vary too quickly in external time, so one can define a mean field or classical differential local light cone. Case (b) or (c) could correspond to the noncommutative geometry scenario for nonlocality over the scale ξ in the presence of a metric solution. In the marginal case (d), where the scale of nonlocality is about that of the quantum fluctuations, the survival of the Lieb-Robinson local light cone is too close to call. Unfortunately, we really do not know which regime we live in. But if noncommutative geometry provided an action invariant under deformed diffeomorphisms and nonlocal on scale ξ , then that would be on shell, and could naturally separate ξ from quantum length fluctuations δd .

An alternative approach to relieve the classical vs quantum tension inherent in $d(i, j)$ is to use semiclassical (coherent) states Ψ [23] of 3-space. The idea is that each length on the initial slice t has a quantum expectation $\langle \hat{d} \rangle_\Psi \doteq d_\Psi$, a quantum fluctuation $\langle (\hat{d} - \langle \hat{d} \rangle_\Psi)^2 \rangle_\Psi \doteq \sigma_d^2(\Psi)$, and the classical length $d_{cl} > 0$. It is also possible to use other geometric quantities besides $d(x, y)$, such as the areas of triangles or volumes of tetrahedra. To achieve semiclassical consistency for $d(i, j)$ in a state Ψ , one would require for all sites i, j

$$\frac{|d_{cl} - d_\Psi|}{d_{cl}} \ll 1 \quad \text{and} \quad (99)$$

$$\frac{\sigma_d(\Psi)}{d_{cl}} \ll 1. \quad (100)$$

This makes the notion of classical distance insensitive to the quantum fluctuations from the state Ψ . Such a construction would also encounter difficulties in case (d) above, where the range ξ is quantum limited.

VIII. SUMMARY, SELF-CRITICISM, AND CONCLUSION

In this work we have explored the consequences of nonultralocal constraints within the context of the relational framework of canonical gravity. It was shown that this leads to an on-shell non-Abelian algebra for the physical Hamiltonians, while the constraint algebra remains Abelian. Unitary propagators stay anomaly-free for smooth monotonic gauge flow in an external time parameter t . A set of Hamiltonians that generate operator gauge flow in t with finite-ranged support patches was derived. After introducing a spatial discretization, Lieb-Robinson bounds were reviewed and applied to demonstrate an on-shell differential time local light cone. This local light cone has the properties that there is exponentially small norm leakage of discretized Dirac operator commutators outside the local light cone, it displays suitable gauge (slicing) and $(3 + 1)$ -diffeomorphism invariance, and the local light cone can be “integrated” into “support tubes” for discretized Dirac operators that resemble familiar causal curves from general relativity. This entire Lieb-Robinson bound local light-cone structure collapses for ultralocal constraints. Therefore nonultrolocality together with Lieb-Robinson bounds go an unexpectedly long way toward explaining how the standard quantum field theory version of microcausality, where local observables commute at spacelike distances, emerges from the (semi)classical relational formulation of canonical gravity at length scales greater than that characterizing the nonlocality.

Within the application of quantum field theory to fixed curved background spacetimes, one can derive the familiar causal advanced and retarded propagators as inverse wave operators (Green’s functions) for matter fields such as scalar bosons and so on. These show that the vacuum

expectation values of commutators of canonical fields vanish outside the past or future light cone. However, while this is straightforward for Minkowski spacetime, to obtain unique solutions for general curved space times one imposes the stringent requirement that the manifold be globally hyperbolic. The Lieb-Robinson approach, on the other hand, requires no such corresponding *ab initio* strong global background causal structure assumption.

The criticism of the Lieb-Robinson bound route from nonultrolocality to local light cones is abundantly clear from the responses to the questions in the previous section. While some issues, such as how two different classical spacetimes can share the same Lieb-Robinson velocity, or the role of field gradients, are quite clear, many deeper concerns remain only partially clarified, or just display our glaring ignorance. These harder nuts to crack include the following: What are the origins of nonlocality? What is the detailed microscopic meaning of the range or correlation length $\xi = 1/\mu$? What if ξ is about the size of quantum fluctuations (marginal case)? What are the specifics of the semiclassical limit or choice of quantum states necessary to ensure that quantum fluctuations do not destroy the local light cone? Is it possible to handle the continuum limit more thoroughly than simply to assume the required limit is well behaved? Might noncommutative geometry or field theory play a role in these issues? Each of these questions challenges us to probe more deeply into the “atomic” theory underlying the model of condensed matter ancestry presented here and stands as motivation for future work.

Nevertheless, it remains surprising that aspects of causality may be linked to nonultrolocality. Adopting ultralocality uncritically might be somewhat like what occurred in the 1950s with parity: A beautiful symmetry, but Nature could be a lot more interesting if She broke it once in a while.

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