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Study of geodesic motion in a (2 + 1)-dimensional charged BTZ black hole

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The purpose of this study is to derive the equation of motion for geodesics in the vicinity of the spacetime of a (2 + 1)-dimensional charged BTZ black hole. In this paper, we solve geodesics for both massive and massless particles in terms of Weierstrass elliptic and Kleinian sigma hyperelliptic functions. Then we determine different trajectories of motion for particles in terms of conserved energy and angular momentum and also with effective potential.

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I. INTRODUCTION

The black hole is one of the most interesting predictions of the general theory of relativity, which has been attractive to theoretical physicists for a long time, and it still has unknown parts to study.

A black hole is a region of spacetime with a strong gravitational field that even light cannot escape from. It has an event horizon whose total area never decreases in any physical process [1]. In 1992 Banados, Teitelboim, and Zanelli (BTZ) demonstrated that there is a black hole solution to (2 + 1)-dimensional general relativity with a negative cosmological constant [2], in which it was proved that this type of black hole arises from collapsing matter [3]. In their solution of the gravitational field equation, a constant curvature in local spacetime is required [4], which is a strange result as a solution of general relativity. In a certain subset of anti–de Sitter (AdS) spacetime, they found a solution that contains all the properties of a black hole by making a special identification [4,5]. Also, the charged BTZ black hole is the analogous solution of AdS-Maxwell gravity in (2 + 1) dimensions [6–8].

The BTZ black hole is interesting because of its connections with string theory [9,10] and its role in microscopic entropy derivations [11,12]. The BTZ black hole can also be used in some ways to study black holes in quantum scales [7,13]. Unlike the Schwarzschild and Kerr black holes, the BTZ black hole is asymptotically anti–de Sitter rather than flat and has no curvature singularity at the origin [7].

Black holes have various aspects to study. One of them that we are more interested to investigate is the gravitational effects on test particles and light that reach the spacetime of a black hole. It is important because the motion of matter and light can be used to classify an arbitrary spacetime, in order to discover its structure. For this purpose, we need to solve geodesic equations that describe the motion of particles and light. The analytical solutions for many famous spacetimes [such as Schwarzschild [14], four-dimensional Schwarzschild–de Sitter [15], higher-dimensional Schwarzschild, Schwarzschild–(anti–)de Sitter, Reissner-Nordstrom and Reissner–Nordstrom–(anti–)de Sitter [16], Kerr [17], Kerr–de Sitter [18], and a black hole in f(R) gravity [19]] have been found previously. The solutions are given in terms of Weierstrass $\wp$-functions and derivatives of Kleinian sigma functions.

The interesting classical and quantum properties of the black hole have made it appropriate to use a lower-dimensional analogue that could represent the main features without unessential complications [2]. Moreover, (2 + 1)-dimensional black holes are interesting as simplified models for analyzing conceptual issues such as black hole thermodynamics [20]. In addition, the study of black holes in lower dimensions is useful to better understand the physical features (like entropy, radiated flux) in a black hole geometry [21]. Also, studying the gravitational field of (2 + 1)-dimensional black holes and motion around these black holes can be useful.

The purpose in this paper is to determine the types of a particle's motion around a (2 + 1)-dimensional charged BTZ black hole by studying its spacetime. The outline of our paper is as follows. In Sec. II we introduce the metric and obtain geodesic equations. Section III includes analytical solutions for massless and massive particles and also the resulting orbits are classified in terms of the energy and the angular momentum of test particles, and we conclude in Sec. IV.

II. METRIC AND GEODESIC EQUATIONS

The charged BTZ black hole is the solution of the (2 + 1)-dimensional Einstein-Maxwell gravity with a negative cosmological constant $\Lambda = -\frac{1}{l^2}$ [6]. In the case of a special matter source, which is a nonlinear electrodynamic term in the form of $(F_{\mu\nu}F^{\mu\nu})^s$, called Einstein-power Maxwell invariant gravity [22–24], the form of the coupled (2 + 1)-dimensional action in the presence of cosmological constant is written as follows [25]:

$$I(g_{\mu\nu}, A_\mu) = \frac{1}{16\pi}\int_{\partial M} d^3x\sqrt{-g}[R - 2\Lambda + (kF)^s]. \quad (1)$$

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Here R denotes the scalar curvature, F is the Maxwell invariant which is equal to $F_{\mu\nu}F^{\mu\nu}$ ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field and A_μ is the gauge potential), and s is an arbitrary positive nonlinearity parameter ($s \neq \frac{1}{2}$). Varying the action (1) with respect to $g_{\mu\nu}$ (the metric tensor) and A_μ (the electromagnetic field), one can obtain the equations of gravitational and electromagnetic fields as

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = T_{\mu\nu}, \tag{2}$$

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}(kF)^{s-1}) = 0, \tag{3}$$

and the energy-momentum tensor is

$$T_{\mu\nu} = 2[skF_{\mu\rho}F^{\rho}_{\nu}(kF)^{s-1} - \frac{1}{4}g_{\mu\nu}(kF)^s], \tag{4}$$

where k is a constant. When s and k go to -1, Eqs. (1)–(4) reduce to the field equations of a black hole in Einstein-Maxwell gravity. It is convenient to restrict the nonlinearity parameter to $s > \frac{1}{2}$ in order to have an asymptotically well-defined electric field. The metric of a nonrotating charged BTZ black hole can be written as follows [26]:

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\phi^2, \tag{5}$$

in which the metric function $g(r)$ using the components of Eq. (2) is obtained as [26]

$$g(r) = \frac{r^2}{l^2} - m + \begin{cases} 2q^2 ln(\frac{r}{l}) & s = 1, \\ \frac{(2s-1)^2\left(\frac{sq^2(s-1)^2}{(2s-1)^2}\right)^s}{2(s-1)} r^{\left(\frac{2(s-1)}{2s-1}\right)} & \text{otherwise.} \end{cases} \tag{6}$$

This spacetime is characterized by m (an integration constant related to the mass), q (the electric charge of the black hole), and a cosmological constant Λ. In the case of $s = \frac{3}{4}$, one can obtain a well-known metric which is called a conformally invariant Maxwell solution [26], such as

$$g(r) = \frac{r^2}{l^2} - m - \frac{(2q^2)^{\frac{3}{4}}}{2r}. \tag{7}$$

Taking $(2q^2)^{\frac{3}{4}} = K$ we have

$$ds^2 = -\left(\frac{r^2}{l^2} - m - \frac{K}{2r}\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - m - \frac{K}{2r}} + r^2 d\phi^2. \tag{8}$$

The metric (8) is stationary and axially symmetric. To describe geodesic motion in such a spacetime we need a geodesic equation, which is written as

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0, \tag{9}$$

in which $d\lambda^2 = g_{\mu\nu}dx^\mu dx^\nu$ is the proper time and $\Gamma^\mu_{\rho\sigma}$ denotes the Christoffel connections given by

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}(\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma}). \tag{10}$$

We can obtain geodesic equations using the Lagrangian equation

$$\begin{aligned} L &= \frac{1}{2}\sum_{\mu,\nu=0}^{3} g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \frac{1}{2}\epsilon \\ &= \frac{1}{2}\left[-\left(\frac{r^2}{l^2} - m - \frac{K}{2r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{(\frac{r^2}{l^2} - m - \frac{K}{2r})}\left(\frac{dr}{d\lambda}\right)^2 \right. \\ &\quad \left. + r^2\left(\frac{d\phi}{d\lambda}\right)^2\right], \end{aligned} \tag{11}$$

where ϵ for massive and massless particles has the values of 1 and 0, respectively, and λ is an affine parameter.

Using the Euler-Lagrange equation we can obtain constants of motion

$$\begin{aligned} P_t &= \frac{\partial L}{\partial \dot{t}} = -\left(\frac{r^2}{l^2} - m - \frac{K}{2r}\right)\dot{t} = -E, \\ P_\phi &= \frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi} = \mathcal{L}, \end{aligned} \tag{12}$$

where E is energy and $\mathcal{L}$ is angular momentum. Now, using Eqs. (11) and (12), we can obtain geodesic equations as follows:

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 + m\epsilon - \frac{\mathcal{L}^2}{l^2} - \frac{\epsilon r^2}{l^2} + \frac{K\epsilon}{2r} + \frac{m\mathcal{L}^2}{r^2} + \frac{K\mathcal{L}^2}{2r^3}, \tag{13}$$

$$\begin{aligned} \left(\frac{dr}{d\phi}\right)^2 &= \left(-\frac{\epsilon}{l^2\mathcal{L}^2}\right)r^6 + \left(\frac{E^2}{\mathcal{L}^2} + \frac{m\epsilon}{\mathcal{L}^2} - \frac{1}{l^2}\right)r^4 + \left(\frac{K\epsilon}{2\mathcal{L}^2}\right)r^3 \\ &\quad + mr^2 + \frac{Kr}{2} = R(r), \end{aligned} \tag{14}$$

$$\begin{aligned} \left(\frac{dr}{dt}\right)^2 &= \left(\frac{r^2}{l^2} - m - \frac{K}{2r}\right)^2 - \frac{\epsilon(\frac{r^2}{l^2} - m - \frac{K}{2r})^3}{E^2} \\ &\quad - \frac{\mathcal{L}^2(\frac{r^2}{l^2} - m - \frac{K}{2r})^3}{E^2 r^2}. \end{aligned} \tag{15}$$

These equations give a complete description of dynamics. Using Eq. (13) we can find effective potential

$$V_{\text{eff}} = \frac{\epsilon r^2}{l^2} - \frac{K\epsilon}{2r} - \frac{m\mathcal{L}^2}{r^2} - \frac{K\mathcal{L}^2}{2r^3} - m\epsilon + \frac{\mathcal{L}^2}{l^2}. \tag{16}$$

Here for convenience we define a series of dimensionless parameters as

$$\tilde{r} = \frac{r}{m}, \qquad \tilde{l} = \frac{l}{m}, \qquad \tilde{K} = \frac{K}{m}, \qquad \tilde{\mathcal{L}} = \frac{m^2}{\mathcal{L}^2}, \quad (17)$$

and then rewrite Eq. (14) as

$$\left(\frac{d\tilde{r}}{d\phi}\right)^2 = \tilde{r}^6\left(\frac{-\epsilon\tilde{\mathcal{L}}}{\tilde{l}^2}\right) + \tilde{r}^4\left(E^2\tilde{\mathcal{L}} + \epsilon\tilde{\mathcal{L}}m - \frac{1}{\tilde{l}^2}\right) + \tilde{r}^3\left(\frac{\epsilon\tilde{\mathcal{L}}\tilde{K}}{2}\right) + m\tilde{r}^2 + \frac{\tilde{K}\tilde{r}}{2} = R(\tilde{r}). \quad (18)$$

A. Comparison to other cases of parameter s

For $s = \frac{3}{2}$, the metric function is equal to $g(r) = \frac{r^2}{l^2} - m + Ar^{\frac{1}{2}}$, in which $A = 4(\frac{3q^2}{32})^{\frac{3}{2}}$, so we have

$$\left(\frac{dr}{d\phi}\right)^2 = r^6\left(-\frac{\epsilon}{\mathcal{L}^2 l^2}\right) + r^4\left(\frac{E^2}{\mathcal{L}^2} + \frac{m\epsilon}{l^2\mathcal{L}^2} - \frac{1}{l^2}\right) + r^2\left(-\frac{A\epsilon}{\mathcal{L}^2} + m\right) - Ar. \quad (19)$$

The solution of this equation is similar to Eq. (14) (i.e., for $s = \frac{3}{4}$, so that it is investigated completely in this paper). In the case of $s = 1$, the metric function $g(r)$ is

$$g(r) = \frac{r^2}{l^2} - m + 2q^2 \ln\left(\frac{r}{l}\right), \quad (20)$$

and so we have

$$\left(\frac{dr}{d\phi}\right)^2 = r^6\left(-\frac{\epsilon}{\mathcal{L}^2 l^2}\right) + r^4\left(\frac{E^2}{\mathcal{L}^2} + \frac{m\epsilon}{\mathcal{L}^2} - \frac{2\epsilon q^2}{\mathcal{L}^2}\ln\left(\frac{r}{l}\right) - \frac{1}{l^2}\right) + r^2\left(m - 2q^2\ln\left(\frac{r}{l}\right)\right), \quad (21)$$

and for $s = 3$, the metric function is $g(r) = \frac{r^2}{l^2} - m + Qr^{\frac{4}{5}}$, where $Q = \frac{25(\frac{12q^2}{25})^3}{4}$, so we have

$$\left(\frac{dr}{d\phi}\right)^2 = r^6\left(-\frac{\epsilon}{\mathcal{L}^2 l^2}\right) + r^4\left(\frac{E^2}{\mathcal{L}^2} + \frac{m\epsilon}{\mathcal{L}^2} - \frac{1}{l^2}\right) + mr^2 - \frac{Q\epsilon}{\mathcal{L}^2}r^{\frac{24}{5}} - Qr^{\frac{14}{5}}. \quad (22)$$

Equation (21) includes some logarithmic terms; Eq. (22) and other equations related to other cases of s have some terms with fractional powers of r, that, to our knowledge, cannot be solved analytically. However, they may be solved numerically similar to applied methods in Ref. [27]. Therefore, in the following we consider the conformally invariant Maxwell solution ($s = \frac{3}{4}$).

B. Possible regions for geodesic motion

Equation (18) implies that a necessary condition for the existence of a geodesic is $R(\tilde{r}) \geq 0$, and therefore, the real positive zeros of $R(\tilde{r})$ are extremal values of the geodesic motion and determine the type of geodesic. Since $\tilde{r} = 0$ is a zero of this polynomial for all values of the parameters, we can neglect it. So Eq. (18) changes to a polynomial of degree 5 as below:

$$R^*(\tilde{r}) = \tilde{r}^5\left(\frac{-\epsilon\tilde{\mathcal{L}}}{\tilde{l}^2}\right) + \tilde{r}^3\left(E^2\tilde{\mathcal{L}} + \epsilon\tilde{\mathcal{L}}m - \frac{1}{\tilde{l}^2}\right) + \tilde{r}^2\left(\frac{\epsilon\tilde{\mathcal{L}}\tilde{K}}{2}\right) + m\tilde{r} + \frac{\tilde{K}}{2}. \quad (23)$$

Using analytical solutions, one can analyze possible orbits that depend on the parameters of the test particle or light ray ϵ, E^2, l, K, and $\mathcal{L}$. In the next sections it will be shown exactly.

For a given set of parameters ϵ, l, E^2, K, and $\mathcal{L}$, the polynomial $R^*(r)$ has a certain number of positive and real zeros. If E^2 and $\mathcal{L}$ are varied, the number of zeros can change only if two zeros of $R^*(r)$ merge to one. Solving $R^*(\tilde{r}) = 0$ and $\frac{dR^*(\tilde{r})}{d\tilde{r}} = 0$ gives us E^2 and $\tilde{\mathcal{L}}$. For massive particles ($\epsilon = 1$) we have

$$\tilde{\mathcal{L}} = -\frac{\tilde{l}^2(4m\tilde{r} + 3\tilde{K})}{\tilde{r}^2(\tilde{K}\tilde{l}^2 + 4\tilde{r}^3)},$$
$$E^2 = -\frac{4\tilde{l}^4m^2\tilde{r}^2 + 4\tilde{K}\tilde{l}^4m\tilde{r} - 8\tilde{l}^2m\tilde{r}^4 + \tilde{K}^2\tilde{l}^4 - 4\tilde{K}\tilde{l}^2\tilde{r}^3 + 4\tilde{r}^6}{(4m\tilde{r} + 3\tilde{K})\tilde{l}^4\tilde{r}}, \quad (24)$$

and for massless particles ($\epsilon = 0$)

$$\tilde{\mathcal{L}} = \left(-\frac{64m^3}{27\tilde{K}^2} + \frac{1}{\tilde{l}^2}\right)\frac{1}{E^2}. \quad (25)$$

The results of this analysis are shown in Figs. 1 and 2, in which regions of different types of geodesic motion are classified.

The shape of an orbit is related to the energy and angular momentum of the test particle. Since $\tilde{r}$ must be real and positive, the acceptable physical regions can be found with the condition $E^2 \geq V_{\text{eff}}$. So the number of positive and real zeros of $R(\tilde{r})$ will characterize the shape of different orbits. Here according to the obtained results in this section, we can identify three regions for the geodesic motion of test particles:

(1) In region I, $R^*(\tilde{r})$ has two real and positive zeros ($r_1 < r_2$) such that for $R^*(\tilde{r}) \geq 0$ we have $0 < \tilde{r} < r_1$ and $\tilde{r} \geq r_2$. There are two kinds of orbits, terminating bound orbits (TBOs) and flyby orbits (FOs).

(2) In region II, $R^*(\tilde{r})$ has four real positive zeros ($r_i < r_{i+1}$) such that for $R^*(\tilde{r}) \geq 0$ they are $0 < \tilde{r} < r_1$, $r_2 < \tilde{r} < r_3$ and $r_4 \leq \tilde{r}$. Three possible orbits are

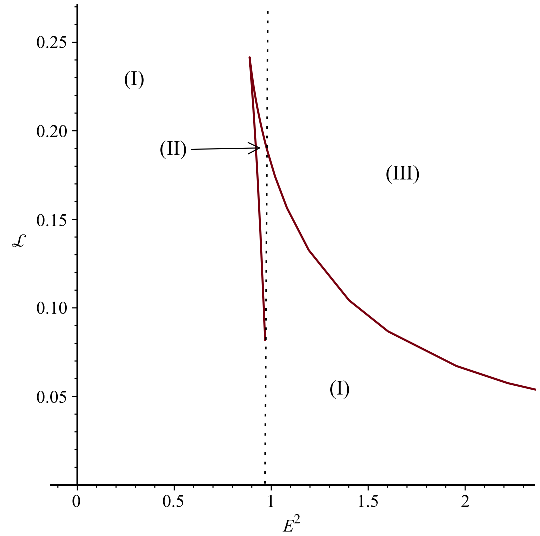

FIG. 1. Regions of different types of geodesic motion for test particles ($\epsilon = 1$). The numbers of positive real zeros in these regions are $\mathrm{I} = 2$, $\mathrm{II} = 4$, $\mathrm{III} = 0$.

terminating bound orbits, bound orbits (BOs), and flyby orbits, respectively.

(3) In region III, there is no real and positive zero for $R^*(\tilde{r})$ and $R^*(\tilde{r}) \geq 0$ for positive $\tilde{r}$; therefore there are just terminating escape orbits (TEOs).

For timelike geodesics these three regions will appear, but for null geodesics only regions I and III exist. In Fig. 3 different potentials for each of these regions are illustrated.

III. ANALYTICAL SOLUTION OF GEODESIC EQUATION

In this section we study analytical solutions of equations of motion. Using a new parameter $u = \frac{1}{\tilde{r}}$ we simplify Eq. (18) to

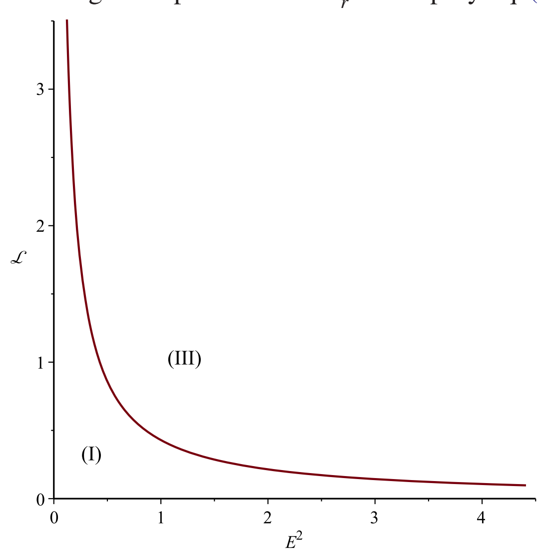

FIG. 2. Regions of different types of geodesic motion for light ($\epsilon = 0$). The numbers of positive real zeros in these regions are $\mathrm{I} = 2$, $\mathrm{III} = 0$.

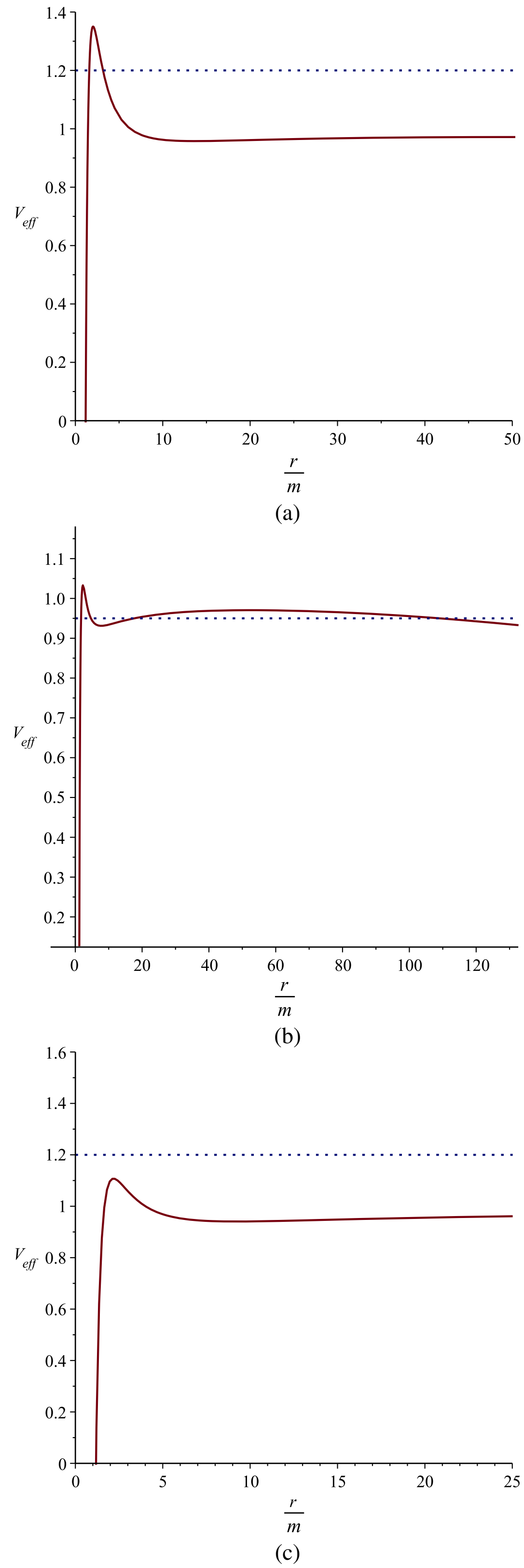

FIG. 3. Effective potentials for different regions of geodesic motion for test particles: (a) region I, (b) region II, and (c) region III. The horizontal line denotes the squared energy parameter E^2.

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{K}u^3}{2} + mu^2 + \left(\frac{\epsilon\tilde{\mathcal{L}}\tilde{K}}{2}\right)u + \left(E^2\tilde{\mathcal{L}} + \epsilon\tilde{\mathcal{L}}m - \frac{1}{\tilde{l}^2}\right) + \left(\frac{-\epsilon\tilde{\mathcal{L}}}{\tilde{l}^2}\right)\frac{1}{u^2}. \tag{26}$$

We will consider it for both particles and light rays as follows.

A. Null geodesics

For $\epsilon = 0$ Eq. (26) changes to

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\tilde{K}u^3}{2} + mu^2 + \left(E^2\tilde{\mathcal{L}} - \frac{1}{\tilde{l}^2}\right) = P_3(u) = \sum_{i=0}^{3} a_i u^i, \tag{27}$$

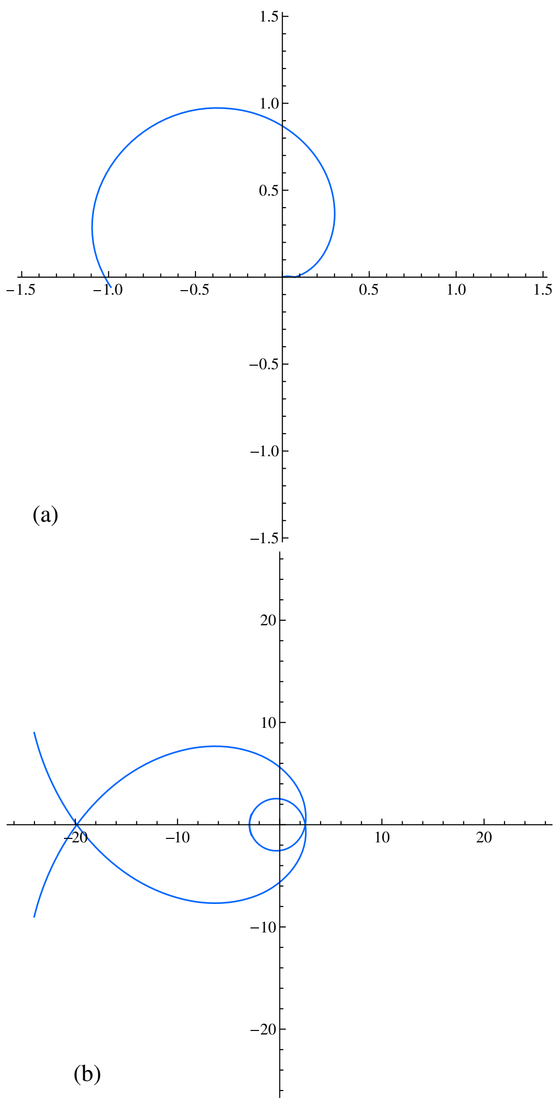

FIG. 4. Null geodesic, region I: (a) corresponding terminating bound orbit with $E^2 = 0.3$, $\mathcal{L} = 0.1$; (b) corresponding flyby orbit with $E^2 = 0.9$, $\mathcal{L} = 0.1$.

which is of the elliptic type. Another substitution $u = \frac{1}{a_3}(4y - \frac{a_2}{3}) = \frac{2}{\tilde{K}}(4y - \frac{m}{3})$ transforms Eq. (27) into Weierstrass form as below,

$$\left(\frac{dy}{d\phi}\right)^2 = 4y^3 - \alpha y - \gamma = P_3(y), \tag{28}$$

in which

$$\alpha = \frac{a_2^2}{12} - \frac{a_1a_3}{4} = \frac{m^2}{12},$$
$$\gamma = \frac{a_1a_2a_3}{48} - \frac{a_0a_3^2}{16} - \frac{a_2^3}{216} = -\frac{(E^2\tilde{\mathcal{L}}\tilde{l}^2 - 1)\tilde{K}^2}{64\tilde{l}^2} - \frac{m^3}{216} \tag{29}$$

are Weierstrass constants. Equation (28) is of the elliptic type and is solved by the Weierstrass function [15,19]

$$y(\phi) = \wp(\phi - \phi_{in}; \alpha, \gamma), \tag{30}$$

which here we have $\phi_{in} = \phi_0 + \int_{y_0}^{\infty}\frac{dy}{\sqrt{4y^3 - \alpha y - \gamma}}$ and $y_0 = \frac{1}{4}(\frac{a_3}{\tilde{r}_0} + \frac{a_2}{3}) = \frac{\tilde{K}}{8\tilde{r}_0} + \frac{m}{12}$ depends only on the initial values ϕ_0 and $\tilde{r}_0$. As a result, the analytical solution of Eq. (18) is

$$\tilde{r}(\phi) = \frac{a_3}{4\wp(\phi - \phi_{in}; \alpha; \gamma) - \frac{a_2}{3}} = \frac{\tilde{K}}{2[4\wp(\phi - \phi_{in}; \alpha; \gamma) - \frac{m}{3}]}. \tag{31}$$

Using this solution we could create the examples of null geodesics for each region of different types of orbits which are plotted in Figs. 4 and 5.

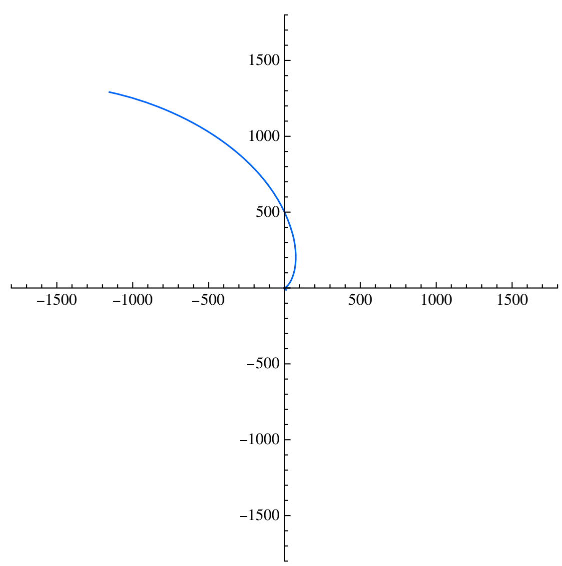

FIG. 5. Null geodesic, region III: corresponding terminating escape orbit with $E^2 = 3$, $\mathcal{L} = 0.3$.

B. Timelike geodesics

For $\epsilon = 1$ Eq. (26) changes to

$$\left(u\frac{du}{d\phi}\right)^2 = \frac{\tilde{K}u^5}{2} + mu^4 + \left(\frac{\tilde{\mathcal{L}}\,\tilde{K}}{2}\right)u^3 + \left(E^2\tilde{\mathcal{L}} + \tilde{\mathcal{L}}m - \frac{1}{\tilde{l}^2}\right)u^2 - \frac{\tilde{\mathcal{L}}}{\tilde{l}^2} = P_5(u) = \sum_{i=1}^{5} a_i u^i, \tag{32}$$

which is a polynomial of degree 5 with an analytical solution as below [15,19,28],

$$u(\phi) = -\frac{\sigma_1}{\sigma_2}(\phi_\sigma), \tag{33}$$

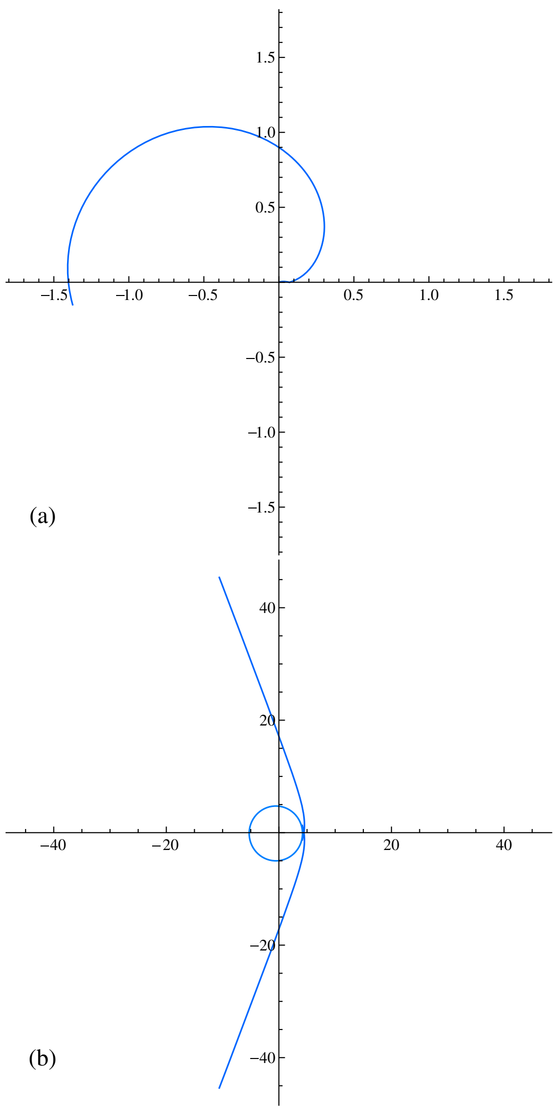

FIG. 6. Timelike geodesic, region I: (a) corresponding terminating bound orbit with $E^2 = 1.2$, $\mathcal{L} = 0.11$; (b) corresponding flyby orbit with $E^2 = 1.6$, $\mathcal{L} = 0.05$.

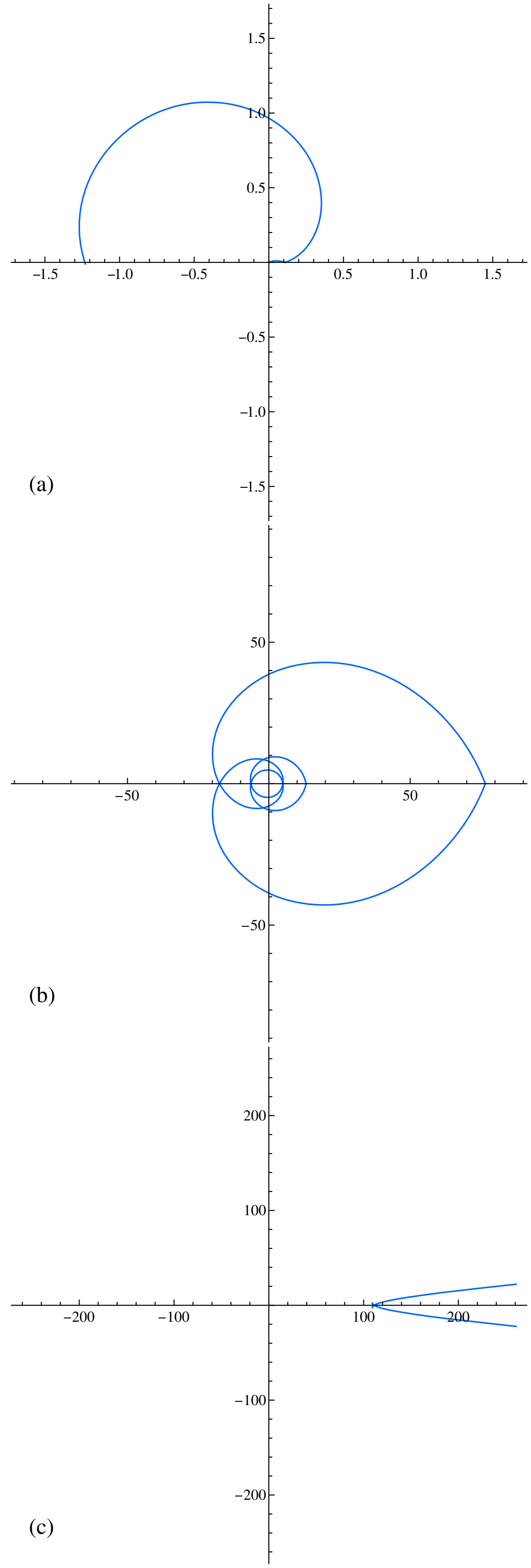

FIG. 7. Timelike geodesic, region II: (a) corresponding terminating bound orbit with $E^2 = 0.965$, $\mathcal{L} = 0.11$; (b) corresponding bound orbit with $E^2 = 0.95$, $\mathcal{L} = 0.17$; (c) corresponding flyby orbit with $E^2 = 0.95$, $\mathcal{L} = 0.17$.

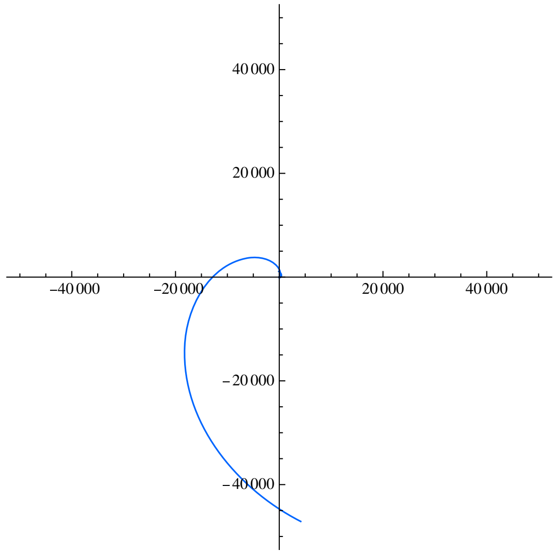

FIG. 8. Timelike geodesic, region III: terminating escape orbit with $E^2 = 2.2$, $\mathcal{L} = 0.15$.

where σ_i is the ith derivative of the Kleinian sigma function in two variables:

$$\sigma(z) = Ce^{-\frac{1}{2}z^t\eta\omega^{-1}z}\theta[g,h]((2\omega)^{-1}z;\tau). \tag{34}$$

We have some parameters here: the symmetric Riemann matrix $\tau = \omega^{-1}\acute{\omega}$; the Riemann theta function $\theta[g,h]$, which is written as

$$\theta[g;h](z;\tau) = \sum_{m\in\mathbb{Z}^g} e^{i\pi(m+g)^t(\tau(m+g)+2z+2h)}; \tag{35}$$

the period-matrix $(2\omega, 2\acute{\omega})$; the period-matrix of the second type $(2\eta, 2\acute{\eta})$; and C is a constant that can be given explicitly. Note that z is a zero of the Kleinian sigma function if and only if $(2\omega)^{-1}z$ is a zero of the theta function $\theta[g,h]$.

With Eq. (33) the solution for $\tilde{r}$ is

$$\tilde{r} = -\frac{\sigma_2}{\sigma_1}(\phi_\sigma). \tag{36}$$

This solution of $\tilde{r}$ is the analytical solution of the equation of motion for massive particles. Different types of orbits for each region of this solution are illustrated in Figs. 6–8.

C. Orbits

In region I, as we expressed before, there are two kinds of orbits [for TBO r starts in $(0, r_a]$ for $0 < r_a < \infty$ and falls into the singularity at $r = 0$; and for FO r starts from ∞, then approaches a periapsis $r = r_p$, and then goes back to ∞)] with $E^2 = 1.2$ and $\mathcal{L} = 0.11$. In region II, we have three orbits (for TBO, FO, and BO, r oscillates between two boundary values $r_p \leq r \leq r_a$ with $0 < r_p < r_a < \infty$) with $E^2 = 0.95$ and $\mathcal{L} = 0.17$. Region III has just one kind of orbit (TEO, in which r comes from ∞ and falls into the singularity at $r = 0$) with $E^2 = 1.2$ and $\mathcal{L} = 0.15$. With the help of analytical solutions, parameter diagrams Figs. 1 and 2, and effective potentials (Fig. 3), various orbits for these three regions considering $\Lambda = -\frac{1}{l^2} = \frac{1}{3}(10^{-5})$ and $q = 1.25$ are presented in Figs. 4–8.

IV. CONCLUSION

In this paper considering a three-dimensional charged BTZ black hole, we studied the motion of particles (massive) and light rays (massless). For this purpose, at first we found equations of motion (geodesic equations). Then using the effective potential and solving geodesic equations in terms of the Weierstrass elliptic function and Kleinian sigma hyperelliptic function, we classified the complete set of orbit types. We also demonstrated that for both timelike and null geodesics, there are different regions where test particles can move. These regions and possible kinds of motion are illustrated in Figs. 1–8. For timelike geodesics TBO, BO, FO, and TEO are possible and for null geodesics TBO, FO, and TEO are possible.

These results and obtained figures can be used to have an intuition about the properties of the orbits such as light deflection, periastron shift, and so on. The higher-dimensional and rotating version of this spacetime could be studied in the future.

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