Entanglement entropy renormalization for the noncommutative scalar field coupled to classical BTZ geometry

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In this work, we consider a noncommutative (NC) massless scalar field coupled to the classical nonrotational BTZ geometry. In a manner of the theories where the gravity emerges from the underlying scalar field theory, we study the effective action and the entropy derived from this noncommutative model. In particular, the entropy is calculated by making use of the two different approaches, the brick-wall method and the heat kernel method designed for spaces with conical singularity. We show that the UV divergent structures of the entropy obtained through these two different methods agree with each other. It is also shown that the same renormalization condition that removes the infinities from the effective action can also be used to renormalize the entanglement entropy for the same system. Besides, the interesting feature of the NC model considered here is that it allows an interpretation in terms of an equivalent system comprising a commutative massive scalar field but in a modified geometry: that of the rotational BTZ black hole, the result that hints at a duality between the commutative and noncommutative systems in the background of a BTZ black hole.

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I. INTRODUCTION

That the entropy can be assigned to a black hole with the magnitude proportional to the horizon area was indicated for the first time in [1,2], and later on, this idea was given strong credibility [3]. In trying to understand why the black hole has the entropy and why is it proportional to the area, two different approaches were proposed in the 1980s. One is due to 't Hooft and is referred to as the brick-wall model [4]. In this method, one considers a thermal bath of particles propagating just outside of the horizon and calculates their entropy. In the other seemingly unrelated approach, the entropy is calculated via introducing an auxiliary but important concept of reduced density matrix, which is obtained by tracing over the degrees of freedom of a quantum field that reside inside the horizon [5]. This latter approach was, in fact, a seed which triggered the whole stream of development, which over the subsequent years and in conjunction with the series of other important contributions in the field, gave rise to what may now be called the conical space or conical singularity approach to calculating the entanglement entropy [6–12].

Entropies obtained by either of these two approaches appear to be divergent quantities, which naturally raises the problem of their renormalization. In this respect, it was suggested in [13] that the leading divergence in the entropy can be removed by the standard renormalization of Newton's gravitational constant. Subsequently, the removal of the subleading divergent terms in the entropy by renormalizing the higher curvature couplings in the gravitational action was demonstrated in [14–16].

The conical singularity method is based on the simple replica trick first introduced in [8]. It has been proven particularly elegant and powerful in a number of situations of great physical interest. This method is particularly interesting when applied to black holes in trying to understand the dynamical origin of the entropy [9,14,17,18]. In that situation, the entangling surface Σ is the Killing type of black hole horizon. The characteristic feature that regular metrics with Killing type of horizon have is that their Euclidean time direction is compact with periodicity $2\pi\beta_H$, with β_H being the inverse Hawking temperature. Such property is dictated by the regularity condition. The Euclidean time τ then plays the role of the angular coordinate on the two-dimensional disk which is perpendicularly oriented with respect to the entangling surface Σ. In the small neighborhood of the surface Σ, the complete spacetime E can be represented as the direct product $E = \Sigma \times C$ of the entangling surface and the two-dimensional disk C. With τ and ρ parametrizing the disk C, and coordinates z^i, $i = 1, \ldots, d-2$ parametrizing the surface Σ, the spacetime E is then assumed to be described by the static and Euclidean metric of the general type

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = f(\rho)d\tau^2 + d\rho^2 + \gamma_{ij}(\rho, z^i)dz^i dz^j. \qquad (1)$$

Accordingly, the surface Σ is determined by the condition $\rho = 0$ so that the expansion around the black hole horizon corresponds to expanding around $\rho = 0$.

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One way to introduce a singularity into this otherwise regular spacetime manifold is to displace the black hole out of its thermal equilibrium [8,9,19] by allowing it to have a temperature T different from the Hawking temperature T_H ($T_H = 1/2\pi\beta_H$). Naturally, such displacement causes the appearance of a conical singularity which is attached to the surface Σ at the origin and has the deficit angle $\delta = 2\pi(\beta_T - \beta_H)/\beta_T$. The regularity of the metric can then be restored by relaxing the black hole back to its thermal equilibrium at the temperature $T = T_H$, resulting in the vanishing deficit angle, $\delta = 0$.

Concerning the thermal entropy S for a given field theoretical system at some generic temperature T, one may recall that it is given in terms of the partition function Z_{β_T} of the field theoretical system in question,

$$S = -\left(\beta_T \frac{\partial}{\partial \beta_T} - 1\right) \ln Z_{\beta_T}. \tag{2}$$

This relation, in turn, may be reexpressed in terms of the deficit angle δ as

$$S_{\mathrm{BH}} = \left(2\pi \frac{d}{d\delta} + 1\right) \ln Z_\delta|_{\delta=0} \tag{3}$$

to give the Bekenstein-Hawking entropy as the thermal entropy evaluated at the temperature $T = T_H$.

On the other side, one may recall the concept of the reduced density matrix which appears to be very useful in the current context. Thus, for the system described by the pure quantum state $|\psi\rangle$, the density matrix is given by $\rho_0 = |\psi\rangle\langle\psi|$. If the system under consideration is further divided by some entangling surface Σ in such a way that a portion of the degrees of freedom is located within the surface, while the rest of them are located outside the surface, then a reduced density matrix can be defined by tracing ρ_0 over the degrees of freedom residing within the entangling surface. Even more, the reduced density matrix pertaining to any of the two artificially created subsystems can be defined, and this can be done by tracing the density matrix ρ_0 over the degrees of freedom that are located within the remaining of the two subsystems. Hence, if a given system is divided into two subsystems A and B by some surface Σ, then the reduced density matrix for the subsystem B is defined by tracing over the modes of the quantum field that reside in the subsystem A,

$$\rho_B = \mathrm{Tr}_A \rho_0, \tag{4}$$

and the entanglement entropy for the subsystem B is given by von Neumann's entropy as

$$S_B = -\mathrm{Tr}\rho_B \ln \rho_B. \tag{5}$$

It is known that the reduced density matrix ρ, properly normalized as $\hat{\rho} \equiv \frac{\rho}{\mathrm{Tr}\rho}$, complies with the so-called "replica trick"

$$S = -\mathrm{Tr}\hat{\rho} \ln \hat{\rho} = \left(-\frac{d}{dn} + 1\right) \ln \mathrm{Tr}\rho^n|_{n=1}, \tag{6}$$

which provides the efficient way to calculate the entanglement entropy of the quantum (field theory) system under consideration. Trace of the nth power of the reduced density matrix appears to be a very important quantity as the following line of arguments may plainly show.

To begin with, one first has to note that calculating the expression $\mathrm{Tr}\rho^n$ corresponds to taking the path integral over field configurations that are defined on the n-sheeted covering E_n of the Euclidean spacetime E described by the metric (1). When calculating this functional integral, one has to introduce a cut in the spacetime E in order to implement the boundary conditions obeyed by the quantum field. The cut is determined by the points that lie on a half of the hyperplane $\tau = 0$. Then, in the process of taking the trace, the value of the quantum field that lies on one sheet immediately below the cut has to be identified with the value of the quantum field that lies on the next sheet immediately above the cut (see, e.g., [20] for the nice graphical presentation). In passing across the cut from one sheet to the other, fields are glued analytically, and two points which differ in angular variable by $2\pi n$ are identified. Thus, we have a manifold on which two points are identified which differ in angular variable by some integral multiple of 2π. Geometrically, this corresponds to a cone with the deficit angle $\delta = 2\pi(1 - n)$.

Reexpressing the replica formula (6) in terms of the deficit angle δ, rather than in terms of n, gives

$$S = -\mathrm{Tr}\hat{\rho} \ln \hat{\rho} = \left(2\pi \frac{d}{d\delta} + 1\right) \ln \mathrm{Tr}\rho^n|_{\delta=0}. \tag{7}$$

If we are about to correlate the two entropies, the geometric entropy (7) and the Bekenstein-Hawking entropy (3), then by comparing the relations (7) and (3), a clear physical interpretation of the quantity $\mathrm{Tr}\rho^n$ emerges; namely, it appears to be a partition function for the quantum field in some gravitational background, $Z_n = \mathrm{Tr}\rho^n$.

Thereafter, the space E_n, which is a n-fold cover of the space E, is still described by the metric (1), with the only exception that the variable τ/β_H is no longer a periodic variable with period 2π, but instead its period now is $2\pi n$. This space has a conical singularity attached to the surface Σ at the origin so that in the small neighborhood of the entangling surface Σ, the space E_n looks as a direct product $E_n = \Sigma \times C_n$ of the surface Σ and a two- dimensional cone with the deficit angle $\delta = 2\pi(1 - n)$.

Because of an Abelian isometry, which exists in the plane orthogonal to Σ and which is manifested by the periodicity with period $2\pi n$, it is possible to analytically continue the above conclusions from integer n to arbitrary noninteger α, so that in the small vicinity of Σ, it is possible to write $E_\alpha = \Sigma \times C_\alpha$. Correspondingly, the identification $Z_\alpha = \mathrm{Tr}\rho^\alpha$ may also be drawn.

At the same time, the assumption about the spacetime as being described by a smooth manifold at the energies of the order of the Planck scale was being increasingly more challenged over the past few decades. Indeed, different approaches to quantum gravity in one way or another point toward the necessity to revise such description. One of the approaches to quantum gravity takes this route, and the spacetime is assumed to be noncommutative (NC) at the microscopic level [21]. Such an assumption is not an arbitrary one, since general relativity and Heisenberg's uncertainty principle together imply that the spacetime has a noncommutative structure [22–24]. With this prospect, different types of noncommutative spacetimes and their implications to physical models have been analyzed in recent times [25,26]. Likewise, there have been various attempts to construct noncommutative theories of gravity, noncommutative black hole solutions, and noncommutative quantum cosmology [27–38]. In particular, it has been shown that the noncommutative version of the BTZ black hole is described by a κ-deformed algebra [39,40]. Similar κ-deformed algebras have been found in the noncommutative description of Kerr black holes [41] and certain noncommutative versions of cosmology [30]. It, thus, appears that there is a certain element of universality in the appearance of the κ-deformed algebras, as they occur in the noncommutative descriptions of various types of classical geometries. Therefore, it is of interest to study the properties of black holes in the framework of κ-deformed noncommutative systems.

In this paper, we set up to investigate the effects induced by noncommutativity on the coupling of matter to gravity. For that purpose, the dynamics of the scalar matter in the background of the BTZ geometry [42,43] has been sampled out as a working model for describing the matter coupled to gravity. On the other hand, as a conceptual framework for mimicking the noncommutative nature of spacetime at the Planck scale, the κ-deformed Minkowski spacetime has been envisioned as a convenient and sufficiently general one, as explained above. Likewise, it is noteworthy that a nonsmooth, grainlike nature of spacetime calls for different types of symmetries which are compatible with it, since those embraced by the ordinary Poincaré are not. Symmetries that underlie κ-deformed systems are, however, embodied within the κ-deformed Poincaré algebra [44–46], a specific type of quantum deformation of the Poincaré algebra.

The model we investigate is, therefore, based on the noncommutative scalar field coupled to the classical BTZ black hole background. It was first laid down in [47], and in [48] quasinormal modes are investigated in light of searching for the effects induced by noncommutativity. We bear on these results and expand the research along some novel lines, with the strong emphasis on two main aspects of the problem. The first one is the validity of the renormalization statement[1] for the above-described NC model in the spirit of [13] and the other is the evaluation of the entropy for the same NC model. As for the latter case, in particular, the entropy for the NC model is calculated by using two different methods, namely, the brick-wall method [4] and the heat kernel method for conical spaces and the results, especially their UV divergent structures, are confronted with each other. In trying to reach the stated milestones, we heavily rely on the heat kernel method designed for the spaces with conical singularity. The latter procedure fits neatly into the picture based on the long-lasting idea of Sakharov [49–51], in which the gravity is not a fundamental property but instead arises as a result of the quantum fluctuations that are due to the underlying quantum field theory. In the present case, the underlying quantum field theory is given by the NC scalar field in the classical BTZ background.

II. MODEL FOR NC SCALAR FIELD IN BTZ BACKGROUND AND MAPPING TO THE EQUIVALENT COMMUTATIVE MODEL

In [47], a model was put forth based on the action

$$\hat{\mathcal{S}} = \int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}(\partial_\mu\phi \star \partial_\nu\phi) = \int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}(\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} \triangleright 1) \tag{8}$$

that describes the coupling of the scalar particle to the metric of the form (in units where $8G = 1$)

$$g_{\mu\nu} = \begin{pmatrix} M - \frac{r^2}{l^2} & 0 & 0 \\ 0 & \frac{1}{\frac{r^2}{l^2}-M} & 0 \\ 0 & 0 & r^2 \end{pmatrix} \tag{9}$$

and within the noncommutative setting realized through the presence of symmetries that are described with κ-deformed Poincaré algebra. In this framework, the noncommutativity gets in by means of the insertion of the star product (A3) compatible with the quantum (deformed) symmetry. As suggested by Eq. (8), one may equally say that noncommutativity enters the formalism through the scalar field, which while describing matter, is treated as a noncommutative object (see the Appendix). The gravity, on the other hand, is treated classically. This approach, therefore, amounts to considering a noncommutative scalar field

[1]The renormalization statement refers to the assertion that for renormalizing the entropy one does not need to invent a separate procedure, since the entropy renormalization can be carried out by the same redefinition of the couplings that served to renormalize the effective gravitational action.

coupled to the classical geometrical background produced by the spinless ($J = 0$) BTZ black hole with mass M.

Salient features of the model are encoded within the radial equation of the form [47,48]

$$r\left(M-\frac{r^2}{l^2}\right)\frac{\partial^2 R}{\partial r^2}+\left(M-\frac{3r^2}{l^2}\right)\frac{\partial R}{\partial r} + \left(\frac{m^2}{r}-\omega^2\frac{r}{\frac{r^2}{l^2}-M}-a\beta\omega\frac{8r}{l^2}\frac{\frac{3r^2}{2l^2}-M}{\frac{r^2}{l^2}-M}\right)R=0, \qquad (10)$$

which is the radial component of the field equation derived from the action (8), and a is the deformation parameter $a = \frac{1}{\kappa}$ (therefrom, the phrase κ deformation). It fixes the energy scale at which NC effects are supposed to start occurring. Most frequently, it is taken to be of the order of the Planck length. l is related to the cosmological constant Λ as $l = \sqrt{-\frac{1}{\Lambda}}$. Furthermore, ω and m are, respectively, the energy and the angular momentum (magnetic quantum number) of the scalar particle. The constant β is the parameter determining the differential operator representation of the κ-Minkowski algebra. In the Appendix are given principal technical and physical arguments that lead to Eq. (10). Further details are elaborated in [47,48].

As a matter of fact, the field equation (Klein-Gordon equation) for the NC scalar field $\hat{\phi}$ can be rephrased in terms of the commutative reduction ϕ of $\hat{\phi}$, in which case, the field equation takes the general form

$$(\Box_g + \mathcal{O}(a))\phi = 0. \qquad (11)$$

By commutative reduction, we mean the result of the limiting procedure $\hat{\phi} \to \phi$, as $a \to 0$. Having said that, Eq. (10) is, in fact, the radial component of Eq. (11) that governs the dynamics of the commutative reduction ϕ of the NC scalar field $\hat{\phi}$. As it can be seen, (11) consists of two parts, the first one being the standard Klein-Gordon operator for the geometry (9) and the second one represents a novel contribution. It goes linearly with the NC scale a and above all, introduces a new physics. Note that this a-dependent term in (11) is induced by the noncommutative nature of spacetime at the Planck scale (or NC scale in general, whatever it may be). It also gives rise to the corresponding term in the radial equation (10) which scales linearly with a.

Using the substitution

$$z = 1 - \frac{Ml^2}{r^2}, \qquad (12)$$

Eq. (10) can be reexpressed as

$$z(1-z)\frac{\mathrm{d}^2R}{\mathrm{d}z^2} + (1-z)\frac{\mathrm{d}R}{\mathrm{d}z} + \left(\frac{A}{z} + B + \frac{C}{1-z}\right)R = 0, \qquad (13)$$

where the constants A, B, and C are

$$A = \frac{\omega^2 l^2}{4M} + a\beta\omega, \qquad B = -\frac{m^2}{4M}, \qquad C = 3a\beta\omega. \qquad (14)$$

Equation (13) together with the coefficients (14) describes the dynamics of massless NC scalar field with energy ω and angular momentum m probing the geometry of a BTZ black hole with mass M and vanishing angular momentum ($J = 0$). Closer inspection of its form leads to an interesting observation. Namely, the analytical form of Eq. (13) is exactly the same as that of the equation of motion that governs the massive scalar field of mass μ', energy ω, and angular momentum m, and probing the geometry of the rotational BTZ black hole with mass M' and angular moment J' [see Eq. (8) in Ref. [52]]. On purely technical grounds, the reason why this happened is due to the fact that it was possible to absorb the term with the noncommutative contribution right into those terms in the equation of motion that have already been present there in the absence of noncommutativity. As a result of this purely mathematical peculiarity, an interesting physical picture pops up. It appears that independent of the mass of the black hole whose geometry is being probed by the massless scalar particle, this scalar particle will though acquire the mass in the presence of noncommutativity and will simultaneously undergo some type of backreaction deforming the geometry through which it propagates. In particular, the latter feature pertains to both the modification of the black hole mass as well as the change in the very nature of the geometrical background that is being probed, forcing it to alter from a nonrotational into a rotational one (with the angular momentum J different from zero). In accordance with the above observations, a novel perspective may be given to spacetime noncommutativity. Not only that it may be given a role of the driving force that lies behind the mass generating mechanism, but it may also be responsible for or give rise to certain backreaction effects.

Thereby, since we have an equivalence between two equations of motion, which pertain to two completely different physical situations, a question naturally arises as to whether it is possible to find some kind of mathematical correspondence between them. The answer is positive. Namely, it appears that it is possible to find a direct mapping between the case considered here, that is, the NC massless scalar field in the nonrotational BTZ background and the physical setting where the ordinary massive scalar field probes a BTZ geometry with nonvanishing angular momentum. In what follows, the latter setting we shall refer to as the fictitious one (see Ref. [52]

for the explicit expressions that correspond to this fictitious situation).

Therefore, by comparing the constants A, B, and C appearing in (13), with the appropriate constants from Ref. [52], we get the following set of conditions:

$$A = \frac{\omega^2 l^2}{4M} + a\beta\omega = \frac{l^4}{4(r'^2_+ - r'^2_-)^2}\left(\omega r'_+ - \frac{m}{l} r'_-\right)^2 = A',$$
$$B = -\frac{m^2}{4M} = -\frac{l^4}{4(r'^2_+ - r'^2_-)^2}\left(\omega r'_- - \frac{m}{l} r'_+\right)^2 = B',$$
$$C = 3a\beta\omega = -\frac{\mu'}{4} = C', \tag{15}$$

with r'_+, r'_- being the outer, i.e., inner radius of the equivalent spinning BTZ black hole, respectively.

Note that the radii pertaining to the spinless BTZ are, respectively, given by $r_+ = l\sqrt{M}$ and $r_- = 0$. In addition, to keep the notation as simple as possible, we omit the superscript from μ' so that hereafter we have $\mu' \equiv \mu$.

Furthermore, since [42]

$$M' = \frac{r'^2_+ + r'^2_-}{l^2}, \qquad J' = \frac{2r'_+ r'_-}{l}, \tag{16}$$

we can express the parameters of the commutative fictitious situation completely in terms of the parameters defining the NC case we analyze here

$$M' = f_1(a, M), \tag{17}$$

$$J' = f_2(a, M), \tag{18}$$

$$\mu' \equiv \mu = f_3(a, M). \tag{19}$$

This mapping is similar to the one obtained in [53], where the analogy between the NC version of the Schwarzschild black hole and the commutative Reisner-Nordstrom black hole was drawn.

Turning back to the conditions (15), it is possible to solve them and get the fictitious parameters M' and J' in a closed form. This looks as

$$\frac{1}{M'}\frac{1}{\omega^2 l^2 + m^2 - 2\omega m \frac{J'}{M'}} = \left(\frac{1}{M} + \frac{4a\beta\omega}{\omega^2 l^2 - m^2}\right)^2 \frac{1}{\frac{\omega^2 l^2 + m^2}{M} + 4a\beta\omega}, \tag{20}$$

where the ratio J'/M' appearing in (20) is given by

$$\frac{J'}{M'} = \frac{\sigma\lambda\gamma^2 - \sqrt{\gamma^2\sigma^2 - \frac{\gamma^2\lambda^2}{l^2} + \frac{1}{l^2}}}{\gamma^2\sigma^2 + \frac{1}{l^2}}. \tag{21}$$

The remaining abbreviations γ, λ, and σ appearing in the last two expressions are listed as follows:

$$\gamma \equiv \frac{\frac{1}{M} + \frac{4a\beta\omega}{\omega^2 l^2 - m^2}}{\frac{\omega^2 l^2 + m^2}{M} + 4a\beta\omega}, \tag{22}$$

$$\lambda \equiv \omega^2 l^2 + m^2, \tag{23}$$

$$\sigma \equiv 2\omega m. \tag{24}$$

The mass M' and the angular momentum J' of the equivalent black hole can be expressed explicitly within the first order in the deformation,

$$M' = M\left[1 + 4a\beta\omega M\left(\frac{1}{\lambda} - \frac{2}{\omega^2 l^2 - m^2} + \frac{l^2}{\lambda^2}\frac{2\sigma^2\lambda^2 - \sigma^2 l^2 + \lambda^2}{\sigma^2 l^2 + \lambda^2}\left(\frac{1}{\omega^2 l^2 - m^2} - \frac{1}{\lambda}\right)\right)\right],$$
$$J' = 4a\beta\omega M^2 \frac{l^2}{\lambda\sigma}\frac{2\sigma^2\lambda^2 - \sigma^2 l^2 + \lambda^2}{\sigma^2 l^2 + \lambda^2}\left(\frac{1}{\omega^2 l^2 - m^2} - \frac{1}{\lambda}\right). \tag{25}$$

In order to understand the physical meaning behind the above equivalence, it should be noted that when the noncommutative parameter a (NC scale) goes to zero, the parameters of the two situations coincide with each other. In particular, the ratio J'/M' goes to 0, as expected. Moreover, it should be noted that the right-hand sides of the relations (20) and (21), besides depending on a and the "old" parameter M, also depend on the quantum state of the scalar field through their dependence on the quantum numbers m and ω. For example, the relation (25) implies that the scalar particle with zero orbital angular momentum ($m = 0$) cannot change the spin of the black hole, although it can change its mass. We may additionally portray the whole situation by saying that the more energetic the incoming scalar field is, the more intensive is its impact on the geometry it is probing. This observation speaks in favor of the above-posed assertion that the noncommutativity of the scalar field generates, possibly through some backreaction, the additional mass, and angular momentum of the system with the fictitious black hole.

To summarize this part, the picture that has emerged so far is the following. The dynamics of the NC massless scalar field in the geometry (9) is described by Eq. (11), where $\Box_g$ is the Klein-Gordon operator for the metric (9). Likewise, the dynamics of the massive commutative scalar field in the geometry

$$g_{\mu\nu} = \begin{pmatrix} M' - \frac{r^2}{l^2} & 0 & \frac{-J'}{2} \\ 0 & \frac{1}{\frac{r^2}{l^2} + \frac{J'^2}{4r^2} - M'} & 0 \\ \frac{-J'}{2} & 0 & r^2 \end{pmatrix} \tag{26}$$

is described by the equation

$$(\Box_{g'} - \mu'^2)\phi = 0, \tag{27}$$

where $\Box_{g'}$ is the Klein-Gordon operator for the metric (26). As demonstrated above, these two different physical situations are mathematically equivalent due to the fact that Eq. (11) can be rewritten and reduced to the form (27).

As already indicated, such direct mathematical correspondence enables one to give a physical interpretation to NC effects. Consequently, we may say that probing a BTZ $(M, J = 0)$ black hole with a massless NC scalar field appears to be equivalent to a fictitious commutative setting where this same scalar field (its commutative reduction, to be more precise) acquires the mass and simultaneously modifies the geometry through which it propagates, possibly through some mechanism of backreaction. We know that the gravitational background influences the particle. However, it may be that the opposite is also true, with the grainlike, noncommutative nature of spacetime providing a sufficiently suitable medium/agent for making something like this come true.

In the next three sections, the focus will be on the entropy issue for the NC model just described. In Sec. III the entropy will be calculated by using the brick-wall method. After reviewing in Sec. IV the essentials of the heat kernel method for the spaces with conical singularity, in Sec. V the entanglement entropy for the same NC model will be discussed. However, when carrying out the calculations within the latter framework, one has to work with the Euclidean metric. Therefore, before commencing the analysis of Secs. IV and V, it is necessary to make an analytic transformation of the Lorentzian metric into the Euclidean one. We do this by changing the real variables of time t and angular momentum J' into

$$\tau = it, \qquad J_E = -iJ', \tag{28}$$

leading to the metric

$$ds_E^2 = \left(\frac{r^2}{l^2} - \frac{J_E^2}{4r^2} - M'\right)d\tau^2 + \frac{dr^2}{\frac{r^2}{l^2} - \frac{J_E^2}{4r^2} - M'} + r^2\left(d\varphi - \frac{J_E}{2r^2}d\tau\right)^2. \tag{29}$$

The idea here is to carry out the calculations in the Euclidean setting and then after the final result is reached, one again switches back to the Lorentzian form by using the same transformations. One more thing that one has to keep in mind is that the conical singularity method is an off-shell method, meaning that the metric for which the entanglement entropy is calculated does not necessarily need to be a solution to any field equations. Even if it is, one inserts the specific metric into the formulas only after the final formula for the entropy is derived.

III. ENTROPY AND ITS DIVERGENT STRUCTURE FOR THE NC SCALAR FIELD IN THE CLASSICAL BTZ BACKGROUND FROM THE BRICK-WALL METHOD

The method for calculating the entropy of the black hole by using the "brick-wall model" was introduced in the seminal paper by 't Hooft [4]. For the case of the BTZ black hole, the method has been applied in [54], and in [55] it was used to study the rotational BTZ case. By following the same line of arguments as in these papers, we find from Eq. (10) that the r-dependent radial wave number has the following form [47]:

$$k^2(r,m,\omega) = -\frac{m^2}{r^2(\frac{r^2}{l^2} - M)} + \omega^2\frac{1}{(\frac{r^2}{l^2} - M)^2} + a\beta\omega\frac{8}{l^2}\frac{\frac{3r^2}{2l^2} - M}{(\frac{r^2}{l^2} - M)^2}. \tag{30}$$

In obtaining the last expression, we have used the WKB approximation, which assumes the ansatz of the form $R(r) = \mathrm{e}^{i\int k(r)\mathrm{d}r}$. According to the semiclassical quantization rule, the radial wave number is quantized as

$$\pi n = \int_{r_+ + h}^{L} k(r,m,\omega)\mathrm{d}r, \tag{31}$$

where the quantum number $n > 0$ and the angular momentum quantum number m should be fixed so that $k^2(r,m,\omega) > 0$. Note that $n \equiv n(m,\omega)$. Alongside, h and L are the ultraviolet and infrared regulators, respectively. In the subsequent calculation for the free energy and entropy, we shall take the limit $L \to \infty$ at the end of calculation and set $h \approx 0$. In Ref. [47], the leading, i.e., the most dominant term, was calculated. Here, we shall extend the calculation and isolate all UV divergent terms. Special focus will be on calculating the next-to-leading term in the entropy and free energy.

The total number ν of single particle solutions with energy not exceeding ω is given by

$$\nu \equiv \nu(\omega) = \sum_{m=-m_0}^{m_0} n(m,\omega) = \int_{-m_0}^{m_0} \mathrm{d}m\, n(m,\omega) = \frac{1}{\pi}\int_{-m_0}^{m_0}\mathrm{d}m\int_{r_+ + h}^{L} k(r,m,\omega)\mathrm{d}r, \tag{32}$$

where $m_0^2 = \frac{\omega^2 l^2}{z} + a\beta\omega \frac{8M}{z}\frac{z+1/2}{1-z}$ is fixed by the requirement $k^2(r,m,\omega) > 0$. Note that for reaching the conclusion on the value of m_0, the change of the variable (12) has been made in the above integral, so that $k(r)\mathrm{d}r = \kappa(z)\mathrm{d}z$, for an appropriate function $\kappa(z)$ [see (36) for its explicit form]. Accordingly, the bounds of integration in (32) are changed to $z_h = 1 - \frac{Ml^2}{(r_+ + h)^2}$ and $z_L = 1 - \frac{Ml^2}{L^2}$.

The free energy at the inverse temperature β_T of the black hole is

$$\mathrm{e}^{-\beta_T F} = \sum_\nu \mathrm{e}^{-\beta_T E} = \prod_\nu \frac{1}{1-\mathrm{e}^{-\beta_T E}}, \tag{33}$$

which after taking the logarithm on both sides leads to

$$\begin{aligned}\beta_T F &= \sum_\nu \ln\left(1-\mathrm{e}^{-\beta_T E}\right) = \int \mathrm{d}\nu \ln\left(1-\mathrm{e}^{-\beta_T E}\right) \\ &= -\int_0^\infty \mathrm{d}E \frac{\beta_T \nu(E)}{\mathrm{e}^{\beta_T E}-1},\end{aligned} \tag{34}$$

where the last line is obtained by the partial integration. For this, we find the free energy F as

$$F = -\frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}\omega}{\mathrm{e}^{\beta_T\omega}-1}\int_{z_h}^{z_L}\mathrm{d}z\int_{-m_0}^{m_0}\mathrm{d}m\kappa(z,m,\omega), \tag{35}$$

or more explicitly,

$$F = -\frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}\omega}{\mathrm{e}^{\beta_T\omega}-1}\int_{z_h}^{z_L}\mathrm{d}z\int_{-m_0}^{m_0}\mathrm{d}m\sqrt{\frac{1}{z(1-z)}\left[-\frac{m^2}{M}+\frac{\omega^2 l^2}{Mz}+8a\beta\omega\frac{z+\frac{1}{2}}{z(1-z)}\right]}. \tag{36}$$

The integration over m can be performed exactly and it yields

$$F = -\frac{1}{4}\int_0^\infty \frac{\mathrm{d}\omega}{\mathrm{e}^{\beta_T\omega}-1}\int_{z_h}^{z_L}\mathrm{d}z\sqrt{\frac{1}{z(1-z)}}\frac{\omega^2 l^2}{\sqrt{M}z} - \frac{1}{4}\int_0^\infty \frac{\mathrm{d}\omega}{\mathrm{e}^{\beta_T\omega}-1}\int_{z_h}^{z_L}\mathrm{d}z\sqrt{\frac{1}{z(1-z)}}8a\beta\omega\sqrt{M}\frac{z+\frac{1}{2}}{z(1-z)}.$$

After carrying out the integrations over z, one gets

$$F = -\frac{1}{4}\int_0^\infty \frac{\mathrm{d}\omega}{\mathrm{e}^{\beta_T\omega}-1}\frac{\omega^2 l^2}{\sqrt{M}}(-2)\sqrt{\frac{1-z}{z}}\bigg|_{z_h}^{z_L} - \frac{1}{4}\int_0^\infty \frac{\mathrm{d}\omega}{\mathrm{e}^{\beta_T\omega}-1}8a\beta\omega\sqrt{M}\frac{4z-1}{\sqrt{z(1-z)}}\bigg|_{z_h}^{z_L}. \tag{37}$$

Next, we extract all divergent contributions to the free energy. For that purpose, both terms in (37) are expanded in the brick-wall cutoff h. Keeping all divergent terms in the first term gives

$$(-2)\sqrt{\frac{1-z}{z}}\bigg|_{z_h}^{z_L} = \sqrt{\frac{2l\sqrt{M}}{h}}\left(1-\frac{h}{l\sqrt{M}}+O(h^2)\right),$$

while the second term leads to

$$\frac{4z-1}{\sqrt{z(1-z)}}\bigg|_{z_h}^{z_L} = \left(1-\frac{8h}{l\sqrt{M}}\right)\sqrt{\frac{l\sqrt{M}}{2h}}\left(1-\frac{2h}{l\sqrt{M}}\right)^{-1/2} = \sqrt{\frac{l\sqrt{M}}{2h}}\left(1+\frac{h}{l\sqrt{M}}+O(h^2)\right).$$

This means that only the leading term is divergent, while all other terms including the next-to-leading term are UV finite. Therefore, the total divergent part of the free energy is given as

$$F = -\frac{l^{\frac{5}{2}}}{(M)^{\frac{1}{4}}}\frac{\zeta(3)}{\beta_T^3}\frac{1}{\sqrt{2h}} - 2a\beta\frac{(M)^{\frac{3}{4}}\sqrt{l}}{\sqrt{2h}}\frac{\zeta(2)}{\beta_T^2}, \quad (38)$$

which is the exact result in the sense of the WKB method, and ζ is the Riemann zeta function.

The corresponding divergent structure of the entropy for the NC massless scalar field is calculated using the relation $S = \beta_T^2 \frac{\partial F}{\partial \beta_T}$, and it accordingly amounts to

$$\begin{aligned} S &= 3\frac{l^{\frac{5}{2}}}{(M)^{\frac{1}{4}}}\frac{\zeta(3)}{\beta_H^2}\frac{1}{\sqrt{2h}} + 4a\beta\frac{(M)^{\frac{3}{4}}\sqrt{l}}{\sqrt{2h}}\frac{\zeta(2)}{\beta_H} \\ &= S_0\left(1 + \frac{4}{3}a\beta\frac{M}{l^2}\frac{\zeta(2)}{\zeta(3)}\beta_H\right), \end{aligned} \quad (39)$$

where S_0 is the undeformed entropy for BTZ at the Hawking temperature $\beta_T = \beta_H = \frac{2\pi l^2}{r_+}$. The result for S_0 coincides with the result of [54], while the additional term that scales linearly with a is a consequence of the presumed noncommutative nature of spacetime.

A. Dimensional analysis

We work in units $\hbar = c = k_B = 8G = 1$. From the well-known relations $E = k_B T$, $E = mc^2$, $E = \hbar\omega$, $p = \frac{\hbar}{\lambda}$, $l = ct$, $\beta_T = \frac{1}{k_B T}$, we have the following dimensional relationships:

$$\begin{aligned} [E] &= [T] = [m] = [\omega] = [p], \\ &= [l]^{-1} = [E], \\ &= [t], \\ [\beta_T] &= [T]^{-1} = [E]^{-1} = ([l]^{-1})^{-1} = [l]. \end{aligned} \quad (40)$$

The more explicit relation between the brick-wall cutoff h and the invariant proper length ϵ between the horizon and the brick wall is visible [55] from $\epsilon = \int_{r_+}^{r_+ + h} \sqrt{g_{rr}} \mathrm{d}r$. Namely,

$$\epsilon = \int_{r_+}^{r_+ + h} \frac{\mathrm{d}r}{\sqrt{\frac{r^2}{l^2} - M}}$$

leads to the relation

$$r_h = r_+ + \frac{\pi\epsilon^2}{\beta_H} = r_+ + h. \quad (41)$$

From this and (40), a dimensional analysis gives $[h] = \frac{[\epsilon^2]}{[\beta_H]} = \frac{[\epsilon]^2}{[l]}$. Since h has the dimension of length, $[h] = [l]$, it follows $[l] = \frac{[\epsilon]^2}{[l]}$; that is, the geodesic invariant distance ϵ between the horizon and an imaginary brick wall also has the dimension of length, $[\epsilon] = [h] = [l]$.

Expressed in terms of the geodesic invariant distance cutoff ϵ and upon utilizing the area formula $A(\Sigma) = 2\pi r_+$, the entropy (39) takes the form

$$S = \frac{3}{8\pi^3}\zeta(3)\frac{A(\Sigma)}{\epsilon}\left(1 + \frac{4}{3}a\beta\frac{M}{l^2}\frac{\zeta(2)}{\zeta(3)}\beta_H\right) \quad (42)$$

manifesting the area law for the NC model.

IV. HEAT KERNEL METHOD FOR SPACES WITH CONICAL SINGULARITY

As we have already noted, the black hole spacetime E in the vicinity of the horizon Σ, which plays the role of the entangling surface here, may be represented by the direct product of the compact surface Σ and a two-dimensional disc C, $E = \Sigma \times C$, with the Euclidean time τ playing the role of the angular coordinate on the disc, with the period $2\pi\beta_H$. The conical singularity may then be introduced into this spacetime by virtue of displacing the black hole out of its thermal equilibrium, which is effectively achieved by allowing the horizon temperature T to depart from the Hawking temperature T_H by some small amount, thus, resulting in the Euclidean spacetime E_α with a conical singularity. The conical singularity introduced in such way is then located at the horizon, and the spacetime E_α in the neighborhood of the singular horizon surface then looks as $E_\alpha = \Sigma \times C_\alpha$, where C_α is the two-dimensional cone with the angular deficit $\delta = 2\pi(1 - \frac{T}{T_H}) \equiv 2\pi(1 - \alpha)$. This means that in the vicinity of the horizon, the metric is still described by (1), except for the fact that the Euclidean time τ is now a periodic variable with period $2\pi\beta_T$.

Moreover, in theories in which the gravity emerges from the underlying (bosonic) quantum field theory, one usually considers the quantity which is called the effective action W_{eff}. This effective action then describes the gravity theory that one is about to analyze. The crucial notion in constructing the effective action is the trace of the heat kernel of the field operator, which in the case of the bosonic scalar quantum field of mass μ coupled to the classical gravitational background described by the metric tensor $g_{\mu\nu}$ is given by the d'Alembertian operator extended with the mass term, $\Box_g - \mu^2$. On the spacetime with the regular geometry $g_{\mu\nu}$, whose metric in the near horizon region complies with the general form (1), the trace of the heat kernel of the operator $\Box_g$ is given by the well-known Schwinger-DeWitt expansion [56–62],

$$\mathrm{Tr}e^{-s\Box_g} = \frac{1}{(4\pi)^{d/2}}\sum_{n=0}^{\infty} a_n s^{n-d/2}, \quad (43)$$

where the first few coefficients a_n are given by

$$\begin{aligned} a_0 &= \int_E d^d x\sqrt{g}, \\ a_1 &= \int_E d^d x\sqrt{g}\frac{1}{6}R, \end{aligned} \quad (44)$$

$$a_2 = \int_E d^d x\sqrt{g}\left(\frac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{72}R^2 + \frac{1}{30}\Box_g R\right). \quad (45)$$

Hence, as for the first few terms in the small s expansion, there comes out in a sequence the vacuum energy term, the Einstein-Hilbert term, and the higher curvature terms, respectively.

If, on the other hand, the conical singularity is introduced, then the resulting spacetime E_α requires a modified small s expansion for the trace of the heat kernel of the field operator $\Box_g$. This is due to the fact that the Riemann curvature tensor for the space with conical singularity acquires an additional singular, delta-function-like contribution [12,62–64] when restricted to the surface Σ. At the same time, outside the surface Σ, it is completely identical to the curvature tensor of the regular smooth manifold E. Correspondingly, the small s expansion (43) for the trace of the heat kernel $K(s) = e^{-s\Box_g}$ on a space with a conical singularity appropriately modifies

$$\mathrm{Tr}_{E_\alpha} e^{-s\Box_g} = \frac{1}{(4\pi)^{d/2}}\sum_{n=0}^{\infty}(a_n^{\mathrm{reg}} + a_n^{\mathrm{sing}})s^{n-d/2}, \quad (46)$$

where the coefficients in the expansion acquire additional singular, i.e., surface integral contributions a_n^{sing}. The components a_n^{reg} of the expansion coefficients constitute their regular part. The first three a_n^{reg} are the same as a_0, a_1, and a_2 appearing in (44) and (45) above, except only for the additional factor of α multiplying the integrals in the expressions for a_0, a_1, a_2. This factor of α is due to the fact that the calculation of a_n^{reg} assumes performing the integration over E_α, instead of over E, which, in turn, amounts to carrying the integration over E followed by an additional multiplication with α. In other words, a_n^{reg} are the coefficients that would have ruled the expansion (46) as the sole coefficients if the conical singularity had not been present at all (that is, for $\alpha = 1$).

When, however, the conical singularity is switched on ($\alpha \neq 1$), the regular components a_n^{reg} get accompanied by the singular components a_n^{sing}, which for the first three look as [6,11,62,65–68]

$$a_0^{\mathrm{sing}} = 0, \qquad a_1^{\mathrm{sing}} = \frac{\pi}{3}\frac{(1-\alpha)(1+\alpha)}{\alpha}A(\Sigma), \quad (47)$$

$$a_2^{\mathrm{sing}} = \frac{\pi}{18}\frac{(1-\alpha)(1+\alpha)}{\alpha}\int_\Sigma R - \frac{\pi}{180}\frac{(1-\alpha)(1+\alpha)(1+\alpha^2)}{\alpha^3} \times \int_\Sigma\left(\sum_{k=1}^{2}R_{\mu\nu}n^{k,\mu}n^{k,\nu} - 2\sum_{k=1}^{2}\sum_{j=1}^{2}R_{\mu\nu\sigma\rho}n^{k,\mu}n^{j,\nu}n^{k,\sigma}n^{j,\rho}\right). \quad (48)$$

Here, $\alpha = \beta_H/\beta_T = T/T_H$ and $A(\Sigma)$ is the area of the horizon surface. The quantities $R, R_{\mu\nu}, R_{\mu\nu\sigma\rho}$ are, respectively, the curvature, Ricci tensor, and the curvature tensor of the regular black hole spacetime. Since the surface Σ is a codimension-two hypersurface, it has two mutually orthonormal vectors $n^{k,\mu}$, $k = 1, 2$, that are orthogonal to it. The indices μ, ν label the spacetime components of these vectors.

It has to be noted that there exists an even more general expansion for the trace of the heat kernel on both regular as well as on the conical space. The necessity for the generalization may arise for generally two reasons. One reason may be that the geometry that is being analyzed is somewhat more involved, meaning that in the vicinity of the horizon surface it may not simply be reduced to the direct product $\Sigma \times C$. This may, e.g., be the case with the geometries describing the rotational spacetimes, like, for example, that of the Kerr black hole or the rotational BTZ, where the geometry in the near horizon region is no more described by (1). In these cases, there may appear additional terms in the heat kernel expansion. In particular, the singular coefficients $a_n^{\mathrm{sing}}, n \geq 2$ might suffer a mayor revision which may consist of including the extrinsic curvature effects [62,69,70]. This means that a_n^{sing} might be affected by the additional surface integrals of the quadratic invariants like $\sum_{j=1}^{2}\kappa^{j,\mu\nu}\kappa^j_{\mu\nu}$ and $\sum_{j=1}^{2}\kappa^j\kappa^j$, which would accompany the Riemann spacetime curvature terms that already exist in the expressions for a_n^{sing}. Here, $\kappa^j = g_{\mu\nu}\kappa^{j,\mu\nu}$, and $\kappa^j_{\mu\nu} = -\gamma^\alpha_\mu\gamma^\beta_\nu\nabla_\alpha n^j_\beta$ is the extrinsic curvature of the horizon surface Σ with respect to the normal vectors $n^j, j = 1, 2$ introduced above.[2] As observed in [69,71], the presence of the extrinsic curvature terms of the above type is necessary for a_2^{sing} to manifest a general conformal invariance. Without such terms, a_2^{sing} may at best be invariant only under a highly special class of conformal transformations.

Another reason for generalizing the small s expansion (46) may appear when one considers the gravity theory that emerges from the underlying higher spin field theories. In that case, the actual field operator for the quantum field of spin σ is

$$\mathcal{O}^{(\sigma)} = \Box_g + X^{(\sigma)}, \quad (49)$$

where $\Box_g$ is the same d'Alembertian operator as before, and $X^{(\sigma)}$ is generally a matrix depending on the spin of the field. Correspondingly, the coefficients in the small s expansion of the trace of the heat kernel $K(s) = e^{-s\mathcal{O}^{(\sigma)}}$ of the operator $\mathcal{O}^{(\sigma)}$ appropriately modify.

[2]The object $\gamma_{\mu\nu} = g_{\mu\nu} - n^1_\mu n^1_\nu - n^2_\mu n^2_\nu$ is the metric of the horizon surface [62,70] induced by embedding it into a larger space with the metric $g_{\mu\nu}$.

Here we shall only stick with the scalar field operator with minimal coupling ($X^{(\sigma=0)}=0$). Moreover, for what concerns the analysis in this paper, only the coefficients $a_0^{\rm reg}, a_1^{\rm reg}, a_1^{\rm sing}$ will be important for drawing the main conclusions of this article. Namely, since the main focus here is on finding and identifying the UV divergent part of the effective action and the entanglement entropy of the classical BTZ probed by the minimally coupled massless NC scalar field, only terms $a_0^{\rm reg}, a_1^{\rm reg}$, and $a_1^{\rm sing}$ will be of interest. Higher terms govern the UV finite contributions to the effective action and the entropy in $(2+1)$ dimensions and thereby do not contain an inevitable piece of information as long as the testing of the renormalization statement and the comparison of the UV divergent structures obtained by different methods are the only things to look after.

Recall that in Sec. II, it was shown that a massless NC scalar field in the background of a classical spinless BTZ is mathematically equivalent to an ordinary (commutative) massive scalar field coupled to a classical spinning BTZ. In this respect, the calculation of the effective action and the entanglement entropy for the NC scalar field coupled to a classical nonrotating BTZ is reducible to finding the effective action and the entanglement entropy for the rotational BTZ, though with different black hole parameters $[(J=0,M)\rightarrow(J',M')]$. Hence, in our particular model, the features induced by noncommutativity can be inferred by applying the conical singularity method onto the physical system describing the massive commutative scalar field, minimally coupled to a rotating BTZ black hole.

On the other side, the term $a_1^{\rm sing}$ will be unaffected by the nonstatic nature of the rotational black hole spacetime, as shown below. The nonstatic nature of the geometry in question is not supposed to change the regular coefficients $a_0^{\rm reg}, a_1^{\rm reg}$ either. Naively, this can be expected on the grounds of the general behavior of the entanglement entropy, which for the d-dimensional curved spacetime is given by the Laurent series in the UV cutoff parameter ϵ, with the generic nth term [62,71] in the expansion scaling as $1/\epsilon^{d-2-2n}$ and with the most divergent term behaving as $1/\epsilon^{d-2}$. Therefrom, it is evident that in the $(2+1)$-dimensional case, only the first term will be UV divergent, and as being readily seen from the conical singularity method, this one draws its origin from the term in the heat kernel expansion that contains the singular coefficient $a_1^{\rm sing}$. This is why to this purpose the higher coefficients can be ignored. It still remains to give a more direct argument as to why this coefficient will stay unaffected by the intrusion of the extrinsic curvature terms that may enter the formulas for the singular coefficients, owing to the nonstatic nature of the actual geometry (in the present case the rotational BTZ).

In order to put forth the statement on insensitivity of $a_1^{\rm sing}$ to the rise of the rotational character of the black hole spacetime with the horizon Σ, one may recall [69] that for some general metric having the conical singularity located at the entangling surface Σ, the only source of modification in the singular terms in the small s expansion (46) can come from the extrinsic curvature of Σ (see the discussion above). Moreover, since the coefficient S_{d-2-2n}, which stands next to the generic term $1/\epsilon^{d-2-2n}$ (nth in a row; see [72]) in the UV divergent part of the entanglement entropy expansion, cannot depend on the direction of vectors normal to Σ, the components $\kappa^{j,\mu\nu}$ of the extrinsic curvature may appear in S_{d-2-2n} only with even powers. This is the reason [62,71] why the general coefficient S_{d-2-2n} has the form $\sum_{k+j=n}\int_\Sigma R^k\kappa^{2j}$, where R and κ represent symbolically the components of the spacetime curvature tensor, e.g., the components of the extrinsic curvature of Σ, respectively. Since the leading divergent term ($n=0$), S_{d-2}/ϵ^{d-2} in the entropy originates from the term with $a_1^{\rm sing}$ in the heat kernel expansion, it is clear that no change in the $a_1^{\rm sing}$ coefficient is possible due to the extrinsic curvature. Therefore, we may conclude that in finding the UV divergent structure of the entanglement entropy for the model considered here, we can rely on the expressions given in (44) and (47).

Once having the heat kernel expansion, the effective action on the space with conical singularity is given as

$$W(\alpha)=-\frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}{\rm Tr}_{E_\alpha}e^{-s\Box_g}. \tag{50}$$

The entanglement entropy is calculated by using the replica trick

$$S=((\alpha\partial_\alpha-1)W(\alpha))_{\alpha=1}. \tag{51}$$

If the scalar field has the mass μ, then the corresponding effective action can be written in terms of the trace of the heat kernel (46) in the following way:

$$W(\alpha)=-\frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}({\rm Tr}_{E_\alpha}e^{-s\Box_g})e^{-s\mu^2}. \tag{52}$$

V. ENTANGLEMENT ENTROPY AND ITS DIVERGENT STRUCTURE FOR THE NC SCALAR FIELD IN THE CLASSICAL BTZ BACKGROUND FROM THE CONICAL SINGULARITY METHOD

We return to the model described by Eq. (10) describing the NC scalar field in the background of the classical spinless BTZ black hole. It was shown that this system is equivalent to the massive scalar field probing the geometry described by the metric (26). As explained in the previous section, the entanglement entropy of this system may be calculated by the method of conical singularity which consists of introducing the conical defect into the spacetime (29). This conical defect is located at the horizon and has the small deficit angle $\delta = 2\pi(1-\alpha)$. The parameter $\alpha = T/T_H$ is close to 1, and it measures the departure of the black hole temperature from its equilibrium temperature T_H. With this, the Riemann curvature tensor acquires an additional δ-function-like contribution at the horizon, and the trace of the heat kernel as well as the effective action become functions of α. The entanglement entropy is then found by the replica trick (51).

Although the method assumes that the geometry to which it is applied has the near horizon limit (1), so that it can be represented as $\Sigma \times C$, and (29) is not of that kind, we can still pursue the method along the lines described in Sec. IV, as long as our primary goal is merely to extract out the UV divergent part of the effective action and entropy in the model considered. To this purpose, we only need to consider the coefficients $a_0^{\rm reg}$, $a_1^{\rm reg}$, and $a_1^{\rm sing}$.

Thus, by applying the heat kernel method to the field operator $\square_g$ defined on the background (29), one gets for the effective action

$$\begin{aligned} W(\alpha) = & -\frac{\alpha}{3}\frac{1}{(4\pi)^{3/2}}\frac{1}{\epsilon^3}\int d^3x\sqrt{g} + \frac{\alpha}{(4\pi)^{3/2}}(-1)\frac{1}{\epsilon}\int d^3x\sqrt{g}\left(\frac{1}{6}R-\mu^2\right) \\ & -\frac{\alpha}{(4\pi)^{3/2}}(\sqrt{\Lambda_{\rm IR}}-\epsilon)\int d^3x\sqrt{g}\left[\frac{1}{180}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{1}{180}R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{6}\square_g\left(\frac{1}{5}R-\mu^2\right) + \frac{1}{2}\left(\frac{1}{6}R-\mu^2\right)^2\right] \\ & + \frac{1}{(4\pi)^{3/2}}(-1)\frac{1}{\epsilon}\frac{\pi}{3}\frac{1-\alpha^2}{\alpha}A(\Sigma) - \frac{1}{(4\pi)^{3/2}}(\sqrt{\Lambda_{\rm IR}}-\epsilon)\left[\frac{\pi}{3}\frac{1-\alpha^2}{\alpha}\int_\Sigma\left(\frac{1}{6}R-\mu^2\right)\right. \\ & \left. -\frac{\pi}{180}\frac{1-\alpha^4}{\alpha^3}\int_\Sigma\left(\sum_{k=1}^{2}R_{\mu\nu}n^{k,\mu}n^{k,\nu} - 2\sum_{k=1}^{2}\sum_{j=1}^{2}R_{\mu\nu\sigma\rho}n^{k,\mu}n^{j,\nu}n^{k,\sigma}n^{j,\rho}\right) + f(\alpha)\int_\Sigma \mathcal{O}(\kappa^k\kappa_k, \kappa^{k,\mu\nu}\kappa_{k,\mu\nu})\right], \end{aligned} \quad (53)$$

where the $\Lambda_{\rm IR}$ is some conveniently chosen infrared cutoff, and $n^{k,\mu}$, $k = 1, 2$ are two mutually orthonormal vectors that are orthogonal to the horizon surface Σ of the equivalent spinning BTZ. As stated earlier, in the effective action there may also appear the additional extrinsic curvature terms of the horizon surface Σ due to the nonstatic nature of the geometry. They are indicated in the above expression within the last term, where $f(\alpha)$ is some generic function of the parameter α, with the property $f(\alpha = 1) = 0$, whose exact form is not essential for the forthcoming analysis. The gravitational action that emerges from the underlying quantum scalar field theory and which contains the divergent and finite parts is given by the regular part of the heat kernel expansion, for which purpose we have to set $\alpha = 1$ in (53),

$$\begin{aligned} W_{\rm eff} = & -\frac{1}{3}\frac{1}{(4\pi)^{3/2}}\frac{1}{\epsilon^3}\int d^3x\sqrt{g} - \frac{1}{(4\pi)^{3/2}}\frac{1}{\epsilon} \\ & \times\int d^3x\sqrt{g}\left(\frac{1}{6}R-\mu^2\right) - \frac{1}{(4\pi)^{3/2}}(\sqrt{\Lambda_{\rm IR}}-\epsilon) \\ & \times\int d^3x\sqrt{g}\left[\frac{1}{180}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - \frac{1}{180}R_{\alpha\beta}R^{\alpha\beta}\right. \\ & \left. + \frac{1}{30}\square_g R + \frac{1}{2}\left(\frac{1}{6}R-\mu^2\right)^2\right]. \end{aligned} \quad (54)$$

The BTZ solution, despite sharing many similarities with ordinary $(3+1)$-dimensional black holes, has important differences as well, rooted in the simplicity of $(2+1)$-dimensional gravity. Therefore, due to the fact that in three spacetime dimensions, the full curvature tensor is completely determined by the Ricci tensor,

$$\begin{aligned} R_{\mu\nu\rho\sigma} = & g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} \\ & -\frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R, \end{aligned}$$

the corresponding terms in (53) and (54) should be appropriately modified in accordance with that. Nevertheless, as already announced earlier, we shall only be interested in the divergent part of the effective gravitational action,

$$\begin{aligned} W_{\rm div}(\epsilon) \equiv & -\frac{1}{3}\frac{1}{(4\pi)^{3/2}}\frac{1}{\epsilon^3}\int d^3x\sqrt{g} \\ & -\frac{1}{(4\pi)^{3/2}}\frac{1}{\epsilon}\int d^3x\sqrt{g}\left(\frac{1}{6}R-\mu^2\right), \end{aligned} \quad (55)$$

as one of our main goals is to test the renormalization hypothesis within the particular NC scalar field model presented in Sec. II. This task includes the renormalization of the bare gravitational action $W_{\rm gr}(G_B, \Lambda_B)$, with G_B and Λ_B being the bare couplings, that is the bare Newton's gravitational constant and the bare cosmological constant, respectively. The standard way of treating the UV divergences in the action is to absorb them into a redefinition of the couplings. While the UV divergent part in the gravitational action has already been isolated, it still remains to set up the renormalization condition that will sweep away the divergences. To this purpose, one proceeds as follows.

Taking into account that the bare and the renormalized gravitational actions are given by

$$W_{\rm gr}(G_B,\Lambda_B)=\int d^3x\sqrt{g}\left[-\frac{1}{16\pi G_B}(R+2\Lambda_B)\right],$$
$$W_{\rm gr}(G_{\rm ren},\Lambda_{\rm ren})=\int d^3x\sqrt{g}\left[-\frac{1}{16\pi G_{\rm ren}}(R+2\Lambda_{\rm ren})\right],$$

and that the UV divergences in the effective action computed by the heat kernel method read as

$$W_{\rm div}(\epsilon)=-\frac{1}{3}\frac{1}{(4\pi)^{3/2}}\frac{1}{\epsilon^3}\int d^3x\sqrt{g}-\frac{1}{(4\pi)^{3/2}}\frac{1}{\epsilon}\int d^3x\sqrt{g}\frac{1}{6}R+\frac{\mu^2}{(4\pi)^{3/2}}\frac{1}{\epsilon}\int d^3x\sqrt{g},$$

the renormalization of the effective action can be carried out by imposing the requirement

$$W_{\rm gr}(G_B,\Lambda_B)+W_{\rm div}(\epsilon)=W_{\rm gr}(G_{\rm ren},\Lambda_{\rm ren}). \tag{56}$$

This requirement plays the role of the renormalization condition, and it leads to the redefinition of the couplings

$$\frac{\Lambda_B}{G_B}+\frac{1}{3\sqrt{\pi}}\frac{1}{\epsilon^3}-\frac{\mu^2}{\sqrt{\pi}}\frac{1}{\epsilon}=\frac{\Lambda_{\rm ren}}{G_{\rm ren}}, \tag{57}$$

$$\frac{1}{G_B}+\frac{1}{3\sqrt{\pi}}\frac{1}{\epsilon}=\frac{1}{G_{\rm ren}}. \tag{58}$$

It should be noted that the noncommutativity scale a enters the first condition through the mass μ of the scalar probe, which was given by the third relation in Eq. (15). Eventual impact of the scale a on the cosmological constant would certainly be an interesting issue to study and would perhaps deserve a special analysis, but it is beyond the scope of the present paper.

In order to test the renormalization statement, it is necessary to know what the entropy looks like for the case of the NC scalar field model of Sec. II. Hence, applying the replica trick to the effective action (53)

$$S=(\alpha\partial_\alpha-1)W(\alpha)|_{\alpha=1}, \tag{59}$$

one gets the following result for the entanglement entropy:

$$S=\frac{A(\Sigma)}{12\sqrt{\pi}}\frac{1}{\epsilon}+\frac{1}{12\sqrt{\pi}}(\sqrt{\Lambda_{\rm IR}}-\epsilon)\left[\int_\Sigma\left(\frac{1}{6}R-\mu^2\right)-\frac{1}{30}\int_\Sigma\left(\sum_{k=1}^{2}R_{\mu\nu}n^{k,\mu}n^{k,\nu}-2\sum_{k=1}^{2}\sum_{j=1}^{2}R_{\mu\nu\sigma\rho}n^{k,\mu}n^{j,\nu}n^{k,\sigma}n^{j,\rho}\right)+(\alpha\partial_\alpha f(\alpha))_{\alpha=1}\int_\Sigma\mathcal{O}(\kappa^k\kappa_k,\kappa^{k,\mu\nu}\kappa_{k,\mu\nu})\right]. \tag{60}$$

This is the entanglement entropy for the massless NC scalar field minimally coupled to the classical nonrotational BTZ geometry (9). It is calculated by applying the heat kernel method for the spaces with conical singularity onto the mathematically equivalent system that consists of the ordinary (commutative) massive scalar field probing the rotational BTZ geometry (26). As it can be seen from Eq. (60), this entropy is an UV divergent quantity. The leading (and, at the same time, the only) UV divergent term in the entanglement entropy obtained by the heat kernel method on the space with conical singularity is given by

$$S_{\rm div}(\epsilon)=\frac{A(\Sigma)}{12\sqrt{\pi}}\frac{1}{\epsilon}, \tag{61}$$

showing that it scales linearly with the UV cutoff parameter ϵ. The horizon area $A(\Sigma)$ is here determined by

$$A(\Sigma)=2\pi r'_+=2\pi\sqrt{\frac{M'l^2}{2}\left(1+\sqrt{1-\frac{J'^2}{M'^2l^2}}\right)}, \tag{62}$$

with the mass M' and the angular momentum J' calculated in (25). It is worthy to note that the same result for the UV divergent contribution to the entanglement entropy is obtained in [73] where the heat kernel on the conical BTZ geometry is constructed by solving exactly the heat equation on a maximally symmetric constant curvature space and then utilizing the Sommerfeld formula [74] to obtain the heat kernel with the required periodicity. The heat kernel method for AdS_3 spaces was also considered in [75].

The next-to-leading term in the Eq. (60) is already UV finite. We point out that this same UV divergent structure is exhibited by the entropy that was calculated for the same NC model but within the framework of the 't Hooft brick-wall method of Sec. III. The leading term in the entropy obtained there was also linearly divergent in ϵ, while the next following term was UV finite [see Eq. (42)].

Besides comparing the brick-wall result in Eq. (42) with that in (61), the potentially interesting conclusions may also be drawn by confronting the result (42) with that in [55], where the entropy of the quantized scalar field in the background of a rotating BTZ black hole has been analyzed within the brick-wall model by distinguishing explicitly between the contributions to the entropy that are coming from the superradiant and nonsuperradiant modes. What has been shown in [55] is that, although both of these contributions to the entropy have the subleading logarithmically divergent terms, these subleading terms in the superradiant and nonsuperradiant parts come with the opposite sign, implying that in the total entropy they cancel with each other. Moreover, the leading terms in the superradiant and nonsuperradiant contributions are exactly the same, so that in the total entropy they double.

Consequently, the UV divergent structure of the total entropy for the quantized scalar field in the rotating BTZ background geometry according to [55] is determined only by a linearly UV divergent term which, in addition, is proportional to the area of the event horizon. This result, besides being consistent with that in [73], where the entanglement entropy has been analyzed for the same physical situation, though within the framework of the heat kernel method on spaces with conical singularity, is also consistent with the result (42) and the main tenet of the present paper.

In order to explain the potentially interesting implications of the above-described similarities between the results obtained here and in the literature, it is useful to note that the result (42) has been calculated by applying the brick-wall method to the case of the nonrotational BTZ geometry, though in the presence of noncommutativity. Contrary to that, the entropy (61), as well as the corresponding results in [55,73], have been obtained by analyzing a basically rotational BTZ geometry, thus, clearly indicating that the results (42) and (61) refer to two different physical situations. Nevertheless, although pertaining to two physically different settings, these results anyway appear to be structurally equivalent. By equivalent, it is meant that the entropy of each one of them is proportional to the area of the event horizon, and in addition, they have the same UV divergent structure, which is characterized by the linear divergence in ϵ. This observation may be found useful when studying some specific rotational geometry and trying to simplify the analysis by reducing the problem to a more simple but equivalent one. It may be found useful, particularly in light of the observation that a rotational black hole background geometry appears to be fundamentally different from the nonrotational one, at least as far as the quantization of the matter fields is concerned [76,77]. Even more so with regard to the entropy calculation since, along the standard nonsuperradiant modes, in these new circumstances one also has to care about the superradiant modes. The presence of these superradiant modes makes the case with a rotational geometry fundamentally different and accordingly more involved when compared to the case with a nonrotational geometry. It is also likely that these superradiant modes that call for special care in the brick-wall model, in fact, mirror the extrinsic curvature effects that one encounters in studying the rotational geometries within the framework of the conical singularity method.

Nonetheless, based on the equivalence between the nonrotational geometry in the presence of noncommutativity and the rotational geometry, the feature that was explicitly exhibited through the mapping (15), it turns out that a noncommutative spacetime may provide a suitable medium in which the bridging between these two physically different settings might be possible to enforce by connecting and correlating their respective parameters. In particular, this might help to reduce one situation that is technically more involved (a rotational one) with another which is less involved (a nonrotational one), the very feature which might appear beneficial from the practical and calculational point of view.

From the present results and from the results known in the literature, it is clear that the BTZ geometry, no matter if it is rotational or nonrotational, gives rise to an entropy that is in the leading order characterized by a linearly UV divergent contribution of the Bekenstein-Hawking type. Thereby, in the leading order, the entropy is proportional to the area $A = 2\pi r_+$ of the event horizon, where $r_+ = l\sqrt{M}$ for the nonrotational case, while for the rotational case, the radius of the outer horizon gets modified by the interference of the angular momentum. Moreover, a brief look at Eq. (42) shows that the presence of noncommutative setting does not bring any change to this conclusion. As a matter of fact, a closer inspection of Eq. (42) specifying the entropy of the nonrotational BTZ in the presence of noncommutativity leads to an interpretation of noncommutativity as giving rise to a stretch/shrinkage of the event horizon. Likewise, an interesting observation may be drawn by making an explicit comparison of this result with the result (61) or with the formula (31) in [55] or the analogous one in [73] (the latter three being basically the same up to distinct numerical prefactors). Based on this comparison, a common picture emerges in which the noncommutative contribution in (42) appears as if it has been soaked up into an effective angular momentum, which, in turn, gives rise to a stretch of the event horizon.

We now turn the attention toward the problem of validation of the renormalization statement within the framework of the NC scalar field model discussed in this article. In this respect, note that the divergency in the entanglement entropy identified in (61) can be removed by the standard renormalization procedure, during which the divergency becomes reabsorbed within the redefined coupling constants. To this end, the bare and the renormalized entropy can, respectively, be written as

$$S(G_B) = \frac{A(\Sigma)}{4G_B}, \qquad S(G_{\rm ren}) = \frac{A(\Sigma)}{4G_{\rm ren}}. \tag{63}$$

The renormalization condition,

$$S(G_B) + S_{\rm div}(\epsilon) = S(G_{\rm ren}), \tag{64}$$

together with the relation (61), then leads to the renormalization of Newton's gravitational constant,

$$\frac{1}{G_B} + \frac{1}{3\sqrt{\pi}}\frac{1}{\epsilon} = \frac{1}{G_{\rm ren}}, \tag{65}$$

in the manner as first proposed in [13]. Herefrom, it is readily seen that the same renormalization condition which removes divergences in the effective action also renormalizes the entanglement entropy. Hence, the renormalization statement for the particular model considered in this article describing the NC scalar field coupled to the classical BTZ geometry has been validated explicitly.

Upon utilizing the renormalization condition to remove the divergences in the entanglement entropy, a relation $\frac{1}{G_{\rm ind}} \sim \frac{1}{3\sqrt{\pi\epsilon}}$ naturally arises, making for a precise balance between the induced gravitational constant $G_{\rm ind}$ and the entanglement entropy so that the entanglement entropy appears to be precisely equal to the Bekenstein-Hawking entropy expressed in terms of the induced gravitational constant. As already observed, this result is congruent with [55,73] when restrained to the leading order.

Interesting enough, a study of a two-dimensional conformal field theory (CFT) within the framework of the holography and AdS/CFT correspondence may lead to the same conclusion [78,79]. That is, the information on the holographic entanglement entropy in the bulk of AdS_3 spacetime may be obtained by studying the dual CFT on the two-dimensional boundary which has a topology of a cylinder. The corresponding entanglement entropy [80,81] for the thermal two-dimensional CFT on the cylinder for a spacelike slice of length $2\pi l$ is

$$S_{\rm CFT} = \frac{l}{4G_{\rm ind}} \ln\left[\frac{l^2}{\pi^2(r'_+ + r'_-)(r'_+ - r'_-)} \times \sinh\frac{\pi(r'_+ + r'_-)}{l} \sinh\frac{\pi(r'_+ - r'_-)}{l}\right].$$

In the above expression, the renormalization was already undertaken by subtracting the vacuum contribution coming from the left and right movers describing the rotational BTZ. The macroscopic, that is, the large temperature limit $r'_+ \gg l$ and $r'_+ \gg r'_-$ then gives $\frac{\pi r'_+}{2G_{\rm ind}}$ for the leading contribution, that is, the Bekenstein entropy (61).

The calculation of the heat trace in this section was made possible due to the fact that we were able to recognize the actual operator as a Laplace operator on the definite curved background (26). It is, however, noteworthy that even in the case that we were not able to make such reduction, we would have been able to calculate the appropriate heat trace still by using the heat trace expansion developed for elliptical operators (see Refs. [60,82] and the results in Refs. [83–85] in the commutative limit), though in a somewhat more general form due to the presence of conical singularity.

In the case that we followed a-exact approach and did not truncate the star product, then bearing on the experience with the Moyal star product, certain subtle points could potentially appear. Namely, the nonanalyticity in the noncommutativity parameter may pop up at the one-loop order on a fixed background through nonanalyticity of the corresponding heat kernel expansion. On the Moyal plane, this effect may appear or not, depending on the type of the differential operator in question. For this, see Refs. [83,85] where the heat trace has been studied for the operator that, respectively, contains either only left (or right) Moyal multiplication or contains both right and left Moyal multiplication. In the case that the operator contains a Moyal star multiplication from only one side, the corresponding heat kernel expansions were constructed in [83] on the torus and in [84] on the plane.

VI. FINAL REMARKS

In the present paper, we have considered the NC scalar field model coupled to the classical nonrotational BTZ geometry. For this particular model, we have calculated the entropy within the two different frameworks, one being that of the ’t Hooft brick-wall model and the other one being that of the heat kernel method developed for the spaces with conical singularity and then we have compared the results of these two approaches. When using the heat kernel method in particular, we have relied on the small s expansion for the trace of the heat kernel of the actual field operator $\Box_g$, rather than on the exact solution of the corresponding heat equation. A comparison of the results for the entropy obtained from these two different approaches shows that they both predict the identical UV divergent structure for the entropy, with the leading term being linearly divergent in the UV cutoff parameter ϵ and the next-to-leading-order term being UV finite, as well as the rest of the expansion for the entropy. The second goal of the paper was to test the renormalization statement for the NC model considered. Here we have found that the same renormalization condition that removes divergences in the effective action is also responsible for the removal of the divergences in the entanglement entropy. Hence, the renormalization statement for the particular model describing the NC scalar field in the background of the classical BTZ geometry has been validated.

In carrying out the analysis, we have utilized the exact mathematical equivalence between the noncommutative model considered and the equivalent model consisting of the massive commutative scalar field coupled to a spinning BTZ geometry. This mathematical equivalence itself has an interesting and novel physical interpretation. Namely, it gives rise to a novel view on noncommutativity, which emerges from our analysis, by assigning it a role of a mass generating agent, as well as a driving force that lies behind the black hole spin.

Furthermore, there is an interesting observation that can be made regarding the effect of noncommutativity on the divergent character of the entropy. To wit, we have seen that the mere effect of the noncommutative nature of spacetime was to shrink or stretch the event horizon of a black hole, seemingly without any impact whatsoever on the UV structure of the entropy, neither changing it for the better nor worsening it further. However, there may be an indirect impact, as may be seen from the following line of reasoning.

As it is well known, when the UV cutoff parameter ϵ approaches 0, the entropy blows up. However, the presence of noncommutativity implicitly presumes the existence of the minimal distance scale a beyond which it is not possible to go. This, in turn, means that the limit $\epsilon \to 0$ cannot be applied to the full extent, since, otherwise, ϵ would at some stage cross the barrier set out by the noncommutative scale a. By way of, ϵ at best can reach a but cannot go beyond; i.e., it cannot reduce further. Because of the existence of this natural length barrier, one may argue that the noncommutative nature of spacetime provides a setting which avoids the problem with the divergences in the entropy because no matter how small a may be, the entropy, though very large, will still remain finite.

As a matter of fact, the UV cutoff ϵ can be fixed by relating it with the NC scale parameter a as[3]

$$\epsilon = \frac{3\zeta(3)}{2\pi^3} G \left[1 + a \left(\frac{8\pi}{3} \beta \frac{\sqrt{8GM}}{l} \frac{\zeta(2)}{\zeta(3)} - \frac{1}{2} \right) \right], \tag{66}$$

with the latter bound stemming from the comparison of (42) with (61) and by utilizing (65). Therefrom, it is readily seen that a pushes the brick-wall cutoff $h = \frac{\sqrt{8GM}}{2l} \epsilon^2$ slightly below or above the classical 't Hooft bound, depending on the value of the black hole mass M and the sign of the parameter β.

An important matter that was so far left untouched is that of UV/IR mixing, a feature characteristic for the noncommutative field theories. It was first observed [86] in the theories with the Moyal-Weyl star product, where the appearance of a nontrivial phase factor in the loop calculation regulated the ultraviolet divergence in the one-loop two point function, but at the same time introduced an infrared divergent term. A similar feature was noticed in the noncommutative ϕ^4 theory built from the κ-deformed star product [87]. It is, therefore, important to discuss why this feature does not show up in the analysis presented here. Closely related to this issue is the question of the validity of the small $a = 1/\kappa$ expansion used in this paper, so we need to address that also. First, we stress that what we have initially considered was not an interacting theory, thereby rendering even, in principle, UV/IR mixing effects unlikely to occur.[4] Instead, it was the free noncommutative scalar field propagating in a classical curved background that was considered. After carrying out the κ-expansion, the initial model turns into an effective theory described by the standard scalar field action extended with a correction that is induced by the noncommutativity. In this picture, the gravity is classical, and noncommutativity enters the formalism in a form no other than a mere coupling to gravity. Effectively, this amounts to considering a nonminimal coupling of the commutative scalar field to the metric tensor, with the noncommutativity parameter $a = 1/\kappa$ playing the role of the coupling constant. Introducing noncommutativity in such way seems to be consistent with a viewpoint where the noncommutativity is seen as a low-energy remnant of the quantum gravity, with the entire contribution due to noncommutativity being reduced to nothing more than the few novel terms in the Lagrangian. Moreover, as the NC effects are expected to occur at the Planck scale and the deformation parameter would be suppressed in powers of the Planck mass, it seems reasonable to consider the NC effects only to the lowest order.

It is also worthy to note that a phenomenon like UV/IR mixing is generally a consequence of a nonlocal character of the noncommutative field theory. In this way, the noncommutative theories that are studied via the θ-expansion method are rendered local, and, consequently, the effect like UV/IR mixing does not occur there. Since in our approach we use κ expansion and accordingly truncate the star product expansion at the first order, the nonlocality is lost right from the beginning, and, thus, nonperturbative features like UV/IR mixing are not likely to occur either. However, although the method that uses the expansion in terms of the deformation parameter loses some of the crucial nonperturbative information (due to the cutoff at the finite order of θ/a), it has, nevertheless, shown some good points and has even demonstrated some advantages over the fully nonperturbative treatment [88–94]. The θ-expansion method has also been pursued within the context of noncommutative grand unified theories [95]. Therefore, an approach based on θ expansion has been demonstrated to work finely in the model building and for this reason could legitimately be considered to lead to a minimal noncommutative extension of the corresponding commutative model. Since this approach appeared to be relatively fruitful in a variety of cases, it seems plausible to expect that it might also borrow some of its good features when carried over to our scheme.

Finally, it should be noted that in the first place, when the brick-wall method was used, it was applied to a static black hole, though in the presence of noncommutativity. On the contrary, when the conical space approach was used later on, the full equivalence with the model of rotational BTZ geometry was utilized, and the method was applied directly to this rotational case. Besides that, both results are consistent with each other, and they also turn out to be consistent with the results established in the earlier literature. Here we specifically have in mind the leading

[3]So far, we were carrying the study in the units where $8G = 1$. For the purpose of the remaining analysis, we switch to the standard unit system, which, in turn, formally corresponds to putting $8GM$ everywhere in place of M.

[4]The UV/IR effect, e.g., occurs in a ϕ^4 NC theory as a result of an extra phase factor appearing in the one-loop nonplanar diagrams.

contribution to the entropy that was obtained for the rotational BTZ geometry within the brick-wall method [55] and also the leading contribution to the entropy that was obtained for the same geometry, yet analyzed within the framework of the heat kernel method on spaces with conical singularity [73]. Moreover, a direct comparison of the two results (42) and (61) attributes to noncommutativity the role of a medium that mediates in the process of stretching a horizon through the appearance of an effective black hole angular momentum, which seemingly takes over the whole noncommutativity onto itself.

As the model of the NC scalar field in the background of the classical spinless BTZ was shown to be equivalent to the rotational BTZ geometry probed by a massive scalar field, our confirmation of the renormalization statement for the NC model may then be seen as a mere consequence of the similar statement as applied to the rotational BTZ. In this respect, the conclusions presented here are, in fact, an indirect consequence of the results that are so far acquired with regard to the entropy of the rotational BTZ [55,73]. To put it differently, they have just been rediscovered in the new context, that of the noncommutative setup mingled with the $(2+1)$-dimensional gravity.

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APPENDIX: DERIVATION OF THE RADIAL EQUATION

The starting point in the derivation of Eq. (10) is the following action

$$\hat{\mathcal{S}} = \int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}(\partial_\mu\phi \star \partial_\nu\phi) = \int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}(\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} \triangleright 1) \tag{A1}$$

describing the dynamics of the NC scalar field in the background with the classical geometry $g_{\mu\nu}$. The noncommutativity is introduced by replacing the usual pointwise multiplication between the fields in the action functional with the NC star product, i.e., $\phi(x)\phi(x) \to \phi(x) \star \phi(x)$.

The compatibility with the κ-deformed Poincaré symmetry is secured through the implementation of the star product

$$(f \star g)(x) = \lim_{\substack{y\to x\\ z\to x}}\mu_0(e^{x^\alpha(\triangle-\triangle_0)\partial_\alpha}f(y)\otimes g(z)), \tag{A2}$$

where μ_0 is the multiplication map $\mu_0(f\otimes g) = f\cdot g$ and $\triangle(\partial_\mu)$ is the coproduct for translation generators $p_\mu = i\partial_\mu$, and $\triangle_0(\partial) = \partial\otimes 1 + 1\otimes\partial$ is the primitive coproduct. This structure map belongs to a coalgebra sector of the κ-Poincaré Hopf algebra. It obviously provides a passage where the quantum symmetry pours into the description. We point out that the formula (A2) is a general one [96–98] being valid for the star products corresponding to any Lie-algebra-type of deformation, of which κ deformation is one particular example (but interestingly, the θ deformation is not). For elucidating the origin of formula (A2) and other issues related to the κ deformation, particularly, those related to the "method of realizations" and the correspondence between the star product, differential operator realization, coproduct, and the operator ordering prescription one may consult [96–98].

When expanded up to first order in a, the star product looks as

$$\begin{aligned} f(x)\star g(x) = f(x)g(x) &+ i\beta'\left(\eta^{\mu\nu}x_\mu\frac{\partial f}{\partial x^\nu}\right)\left(\eta^{\lambda\sigma}a_\lambda\frac{\partial g}{\partial x^\sigma}\right) \\ &+ i\beta(\eta^{\mu\nu}a_\mu x_\nu)\left(\eta^{\lambda\sigma}\frac{\partial f}{\partial x^\lambda}\frac{\partial g}{\partial x^\sigma}\right) \\ &+ i\bar{\beta}\left(\eta^{\mu\nu}a_\mu\frac{\partial f}{\partial x^\nu}\right)\left(\eta^{\lambda\sigma}x_\lambda\frac{\partial g}{\partial x^\sigma}\right). \end{aligned} \tag{A3}$$

Here, $\beta', \beta, \bar{\beta}$ are the parameters determining the differential operator representation of the κ-Minkowski algebra. On the other side, each choice of the operator representation corresponds [96] to a different choice of the coproduct (and different basis of κ Poincaré), which, in turn, corresponds to the vacuum of the theory, and this should be fixed by experiment, in principle. Alongside, a_μ is a 4-vector of deformation. In the subsequent analysis, we choose one particular orientation $a_\mu = (a,0,0,0)$ so that the symbol a from now on and in the main text is reserved for the time component of the deformation 4-vector. This choice of the orientation leads to the original κ-Minkowski algebra [44–46].

There exists an isomorphism between the NC algebra $\hat{\mathcal{A}}$ generated by the noncommutative coordinates $\hat{x}_\mu$ and the commutative algebra $\mathcal{A}^\star$ generated by the commutative coordinates x_μ but with $\star$ as the algebra multiplication. The star product between any two elements $f(x)$ and $g(x)$ in $\mathcal{A}^\star$ is defined as

$$f(x) \star g(x) = \hat{f}(\hat{x})\hat{g}(\hat{x}) \triangleright 1, \tag{A4}$$

where $\hat{f}(\hat{x})$ and $\hat{g}(\hat{x})$ are the elements in $\hat{\mathcal{A}}$ that are uniquely assigned to the elements $f(x)$ and $g(x)$, respectively, through the following correspondences, $\hat{f}(\hat{x}) \triangleright 1 = f(x)$, $\hat{g}(\hat{x}) \triangleright 1 = g(x)$. The element 1 is the unit element in the algebra $\mathcal{A}$, and the action $\triangleright: \mathcal{H} \mapsto \mathcal{A}$ is defined by

$$x_\mu \triangleright f(x) = x_\mu f(x), \qquad p_\mu \triangleright f(x) = i\frac{\partial f}{\partial x^\mu}. \tag{A5}$$

Here, x_μ and p_μ are the generators of the Heisenberg algebra $\mathcal{H}$ satisfying the relations,

$$[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \qquad [p_\mu, x_\nu] = i\eta_{\mu\nu}, \tag{A6}$$

where $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. Furthermore, the coordinates $\hat{x}_\mu$ define κ-Minkowski algebra [44–46], and they admit a differential operator representation within the enveloping algebra of $\mathcal{H}$ in terms of the formal power series in x_μ and p_ν. The correspondence just described between the elements of $\hat{\mathcal{A}}$ and $\mathcal{A}^\star$ provides a ground for establishing the isomorphism between these two structures.

As a next step, two approximations are in order that are motivated on the physical grounds. The first approximation is related with the observation that we are looking for, the NC correction to the lowest order in the deformation parameter. The NC effects are expected to arise at the Planck scale, and the deformation parameter would be suppressed in powers of the Planck mass. It is, therefore, logical to consider the NC effects only to the lowest order. In addition to that, we also look at the long wavelength or low frequency limit for the solutions to the wave equation describing the matter propagation in the background of the black hole. The reason for this is that these long wavelength solutions are associated with the leading contributions of the gravitational perturbations, which are inherently very weak (see the article [99] for a review). There is also a considerable effort from the experimental side to detect the low frequency signals (see [100]). It is, therefore, both logical and important to consider the long wavelength limit.

Following this line of argument, after setting in (A3) $f = g = \partial\phi$, we expand the action up to first order in the deformation parameter a_μ as

$$\hat{\mathcal{S}} = \mathcal{S}_0 + \int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}\bigg[i\beta' x^\sigma \frac{\partial^2\phi}{\partial x^\sigma \partial x^\mu}a^\lambda + i\beta(\eta^{\sigma\rho}a_\sigma x_\rho)\frac{\partial^2\phi}{\partial x_\lambda \partial x^\mu} + i\bar{\beta}\frac{\partial^2\phi}{\partial x_\alpha \partial x^\mu}a_\alpha x^\lambda\bigg](\partial_\lambda\partial_\nu\phi), \tag{A7}$$

where S_0 is the standard action functional describing the commutative scalar field coupled to $g_{\mu\nu}$. Since the action in Eq. (A7) contains the terms involving higher derivatives in the scalar field, that is, the Lagrangian is of the general form $\mathcal{L} = \mathcal{L}(\phi, \partial\phi, \partial^2\phi, x)$, the actual Euler-Lagrange equations accordingly modify, as in the case of higher derivative theories. They read as

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} - \partial_\mu\partial_\nu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\partial_\nu\phi)} = \frac{\delta\mathcal{L}}{\delta\phi}. \tag{A8}$$

The equation of motion following from (A8) is further subject to the long wavelength approximation, where we keep terms in the equations of motion that are of the lowest order in derivatives. In this approximation, the terms dependent on β' and $\bar{\beta}$ do not contribute since they are all proportional to higher derivatives of the scalar field. Consequently, in the lowest order of the long wavelength approximation, only terms depending on β will contribute to that part of the equation of motion that is induced by the noncommutativity. With the above two approximations, the equation of motion reduces to the form given in (10).

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[72] S_n is the generic coefficient that stands in front of the power $1/\epsilon^n$ in the expansion of the entanglement entropy, which has the following general form

$$S = S_{d-2}/\epsilon^{d-2} + S_{d-4}/\epsilon^{d-4} + \cdots + S_{d-2-2n}/\epsilon^{d-2-2n} + s_0 \ln\epsilon + \text{finite}.$$

This expansion is the series in terms of the UV cutoff ϵ. The coefficients S_n are close to the coefficients a_n that appear in the expansion (46) but differ from them by some multiplicative factor which occurs due to an additional integration over the small adiabatic parameter s, which is one that is supposed to be carried out in (49).
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