Entropy emission properties of near-extremal Reissner-Nordström black holes

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Bekenstein and Mayo have revealed an interesting property of evaporating $(3 + 1)$-dimensional Schwarzschild black holes: their entropy emission rates $\dot{S}_{\text{Sch}}$ are related to their energy emission rates P by the simple relation $\dot{S}_{\text{Sch}} = C_{\text{Sch}} \times (P/\hbar)^{1/2}$, where C_{Sch} is a numerically computed dimensionless coefficient. Remembering that $(1 + 1)$-dimensional perfect black-body emitters are characterized by the same functional relation, $\dot{S}^{1+1} = C^{1+1} \times (P/\hbar)^{1/2}$ [with $C^{1+1} = (\pi/3)^{1/2}$], Bekenstein and Mayo have concluded that, in their entropy emission properties, $(3 + 1)$-dimensional Schwarzschild black holes behave effectively as $(1 + 1)$-dimensional entropy emitters. Later studies have shown that this intriguing property is actually a generic feature of all radiating $(D + 1)$-dimensional Schwarzschild black holes. One naturally wonders whether all black holes behave as simple $(1 + 1)$-dimensional entropy emitters? In order to address this interesting question, we shall study in this paper the entropy emission properties of Reissner-Nordström black holes. We shall show, in particular, that the physical properties which characterize the neutral sector of the Hawking emission spectra of these black holes can be studied *analytically* in the near-extremal $T_{\text{BH}} \to 0$ regime (here T_{BH} is the Bekenstein-Hawking temperature of the black hole). We find that the Hawking radiation spectra of massless neutral scalar fields and coupled electromagnetic-gravitational fields are characterized by the nontrivial entropy-energy relations $\dot{S}_{\text{RN}}^{\text{Scalar}} = -C_{\text{RN}}^{\text{Scalar}} \times (AP^3/\hbar^3)^{1/4} \ln(AP/\hbar)$ and $\dot{S}_{\text{RN}}^{\text{Elec-Grav}} = -C_{\text{RN}}^{\text{Elec-Grav}} \times (A^4 P^9/\hbar^9)^{1/10} \ln(AP/\hbar)$ in the near-extremal $T_{\text{BH}} \to 0$ limit (here $\{C_{\text{RN}}^{\text{Scalar}}, C_{\text{RN}}^{\text{Elec-Grav}}\}$ are analytically calculated dimensionless coefficients and A is the surface area of the Reissner-Nordström black hole). Our analytical results therefore indicate that *not* all black holes behave as simple $(1 + 1)$-dimensional entropy emitters.

I. INTRODUCTION

The entropy emission rate $\dot{S}$ and the energy emission rate (power) P of a perfect black-body (BB) emitter in a flat $(3 + 1)$-dimensional spacetime are related by the well-known Stefan-Boltzmann radiation law [1] (we use gravitational units in which $G = c = k_B = 1$)

$$\dot{S}_{\text{BB}}^{3+1} = C^{3+1} \times \left(\frac{AP^3}{\hbar^3}\right)^{1/4}, \quad (1)$$

where A is the surface area of the $(3 + 1)$-dimensional radiating black body and $C^{3+1} = (32\pi^2/1215)^{1/4}$ is a dimensionless proportionality coefficient.

However, in a very interesting work, Bekenstein and Mayo [2] have revealed the remarkable fact that the Hawking radiation spectra [3] of $(3 + 1)$-dimensional Schwarzschild black holes are characterized by the qualitatively different (and mathematically much simpler) entropy-energy relation [2]

$$\dot{S}_{\text{Sch}}^{3+1} = C_{\text{Sch}}^{3+1} \times \left(\frac{P}{\hbar}\right)^{1/2}, \quad (2)$$

where C_{Sch}^{3+1} is a numerically computed coefficient [2] which depends on the characteristic greybody factors [4] of the $(3 + 1)$-dimensional Schwarzschild black-hole spacetime.

Bekenstein and Mayo [2] have emphasized the interesting fact that the entropy-energy relation (2), which characterizes the Hawking emission spectra of $(3 + 1)$-dimensional Schwarzschild black holes, has the same functional form as the entropy-energy relation [2]

$$\dot{S}_{\text{BB}}^{1+1} = C_{\text{BB}}^{1+1} \times \left(\frac{P}{\hbar}\right)^{1/2} \quad (3)$$

which characterizes the emission spectra of $(1 + 1)$-dimensional perfect black-body emitters [one finds $C_{\text{BB}}^{1+1} = (\pi/3)^{1/2}$ for a $(1 + 1)$-dimensional perfect black-body emitter [2]]. Hence, Bekenstein and Mayo [2] have reached the intriguing conclusion that, in their entropy emission properties, $(3 + 1)$-dimensional Schwarzschild black holes behave effectively as $(1 + 1)$-dimensional [and not as $(3 + 1)$-dimensional] thermal entropy emitters [see Eqs. (2) and (3)]. It is worth noting that it was later proved [5,6] that this intriguing property of the $(3 + 1)$-dimensional Schwarzschild black holes is actually a generic characteristic

of all radiating $(D + 1)$-dimensional Schwarzschild black holes.

One naturally wonders whether this intriguing physical property of the Schwarzschild black holes is shared by all black holes? In particular, we raise here the following question: do all radiating black holes behave as simple $(1 + 1)$-dimensional entropy emitters? In order to address this interesting question, we shall analyze in this paper the entropy emission properties of Reissner-Nordström black holes. As we shall show below, the physical properties which characterize the neutral sector of the Hawking radiation spectra of these black holes can be studied *analytically* in the near-extremal $T_{BH} \to 0$ regime [here T_{BH} is the Bekenstein-Hawking temperature of the Reissner-Nordström black hole; see Eq. (6) below]. Our analytical results (to be presented below) indicate that *not* all radiating black holes behave as simple $(1 + 1)$-dimensional entropy emitters.

II. THE HAWKING RADIATION SPECTRA OF NEAR-EXTREMAL REISSNER-NORDSTRÖM BLACK HOLES

In the present section we shall study the Hawking emission of massless neutral fields from near-extremal Reissner-Nordström black holes. The semiclassical Hawking radiation power P_{RN} and the semiclassical entropy emission rate $\dot{S}_{RN}$ for one bosonic degree of freedom are given respectively by the integral relations [4,7,8]

$$P_{RN} = \frac{\hbar}{2\pi} \sum_{l,m} \int_0^\infty \frac{\Gamma\omega}{e^{\hbar\omega/T_{BH}} - 1} d\omega, \qquad (4)$$

and

$$\dot{S}_{RN} = \frac{1}{2\pi} \sum_{l,m} \int_0^\infty \left[\frac{\Gamma}{e^{\hbar\omega/T_{BH}} - 1} \ln\left(\frac{e^{\hbar\omega/T_{BH}} - 1}{\Gamma} + 1 \right) \right. $$
$$\left. + \ln\left(1 + \frac{\Gamma}{e^{\hbar\omega/T_{BH}} - 1} \right) \right] d\omega, \qquad (5)$$

where $\{l, m\}$ are the angular harmonic indices of the emitted field mode, $\Gamma = \Gamma_{lm}(\omega)$ are the black-hole-field greybody factors [4], and

$$T_{BH} = \frac{\hbar(r_+ - r_-)}{4\pi r_+^2} \qquad (6)$$

with $r_\pm = M + (M^2 - Q^2)^{1/2}$ is the semiclassical Bekenstein-Hawking temperature of the Reissner-Nordström black hole (here $\{r_+, r_-\}$ are the horizon radii of the Reissner-Nordström black hole, and $\{M, Q\}$ are the black-hole mass and charge, respectively).

The characteristic thermal factor $\omega/(e^{\hbar\omega/T_{BH}} - 1)$ that appears in the expression (4) for the semiclassical black-hole radiation power implies that the Hawking emission spectra peak at the characteristic frequency

$$\frac{\hbar\omega^{peak}}{T_{BH}} = O(1). \qquad (7)$$

Furthermore, taking cognizance of the fact that the Bekenstein-Hawking temperature (6) of a *near-extremal* [9] Reissner-Nordström black hole is characterized by the strong inequality

$$\frac{MT_{BH}}{\hbar} \ll 1, \qquad (8)$$

one finds the closely related strong inequality

$$M\omega^{peak} \ll 1 \qquad (9)$$

for the characteristic emission frequencies that constitute the Hawking black-hole radiation spectra in the near-extremal regime (8).

It is well known [4,10] that the dimensionless greybody factors $\Gamma_{lm}(\omega)$, which quantify the interaction of the emitted field modes with the effective curvature barrier in the exterior region of the black-hole spacetime, can be calculated analytically in the low-frequency regime (9) [it is worth emphasizing again that the low-frequency regime (9) dominates the neutral sector of the Hawking radiation spectra in the low-temperature (near-extremal) regime (8)]. We shall now use this fact in order to study *analytically* the physical properties which characterize the neutral sector of the Hawking black-hole radiation spectra in the near-extremal (low-temperature) limit (8).

A. The Hawking emission of massless scalar quanta

Following the analysis presented in [4], one finds the leading-order behavior

$$\Gamma_{lm} = \left[\frac{(l!)^2}{(2l)!(2l+1)!!} \right]^2 \prod_{n=1}^l \left[1 + \left(\frac{\hbar\omega}{2\pi T_{BH} \cdot n} \right)^2 \right]$$
$$\times \left(\frac{AT_{BH}\omega}{\hbar} \right)^{2l} \frac{A\omega^2}{\pi} \cdot [1 + O(AT_{BH}\omega/\hbar)^{2l+1}] \quad (10)$$

for the greybody factors which characterize the emission of scalar quanta in the low-frequency regime (9). Here

$$A = 4\pi r_+^2 \qquad (11)$$

is the surface area of the Reissner-Nordström black hole. From (10) one finds that, in the low-frequency regime (9), the scalar Hawking black-hole radiation spectrum is dominated by the fundamental $l = m = 0$ mode. [It is worth emphasizing the fact that the Hawking emission of scalar modes with $l > 0$ is suppressed as compared to the Hawking emission of the fundamental $l = m = 0$ scalar mode. This characteristic property of the Hawking black-hole emission spectra stems from the fact that the greybody

factors of the higher scalar modes (that is, scalar modes which are characterized by $l > 0$) contain higher powers (as compared to the fundamental $l = m = 0$ scalar mode) of the small quantity $\omega(r_+ - r_-) \ll 1$ [see Eq. (10)]]. In particular, one finds [see Eq. (10)]

$$\Gamma_{00} = \frac{A\omega^2}{\pi} \cdot [1 + O(AT_{\mathrm{BH}}\omega/\hbar)] \tag{12}$$

in the small frequency regime (9) which characterizes the neutral sector of the Hawking emission spectra in the low-temperature (near-extremal) regime (8).

Substituting (12) into Eqs. (4) and (5), one finds after some algebra [11]

$$P_{\mathrm{RN}} = \frac{\pi^2}{30} \frac{AT_{\mathrm{BH}}^4}{\hbar^3} \tag{13}$$

and

$$\dot{S}_{\mathrm{RN}} = -\frac{\zeta(3)}{\pi^2} \frac{AT_{\mathrm{BH}}^3}{\hbar^3} \cdot \left[\ln\left(\frac{AT_{\mathrm{BH}}^2}{\hbar^2}\right) + O(1) \right] \tag{14}$$

for the scalar Hawking radiation power and the scalar Hawking entropy emission rate of the near-extremal Reissner-Nordström black holes.

Finally, substituting (13) into (14), one can express the black-hole entropy emission rate in terms of the Hawking radiation power:

$$\dot{S}_{\mathrm{RN}}^{\mathrm{Scalar}} = -C_{\mathrm{RN}}^{\mathrm{Scalar}} \times \left(\frac{AP_{\mathrm{RN}}^3}{\hbar^3}\right)^{1/4} \ln\left(\frac{AP_{\mathrm{RN}}}{\hbar}\right), \tag{15}$$

where the analytically calculated coefficient $C_{\mathrm{RN}}^{\mathrm{Scalar}}$ is given by

$$C_{\mathrm{RN}}^{\mathrm{Scalar}} = \frac{30^{3/4}\zeta(3)}{2\pi^{7/2}}. \tag{16}$$

It is worth emphasizing the fact that the entropy emission rate (15), which characterizes the scalar Hawking emission spectra of the near-extremal Reissner-Nordström black holes, does *not* have the simple $(1 + 1)$-dimensional entropy-energy functional relation (2) which characterizes the Hawking emission spectra of the Schwarzschild black holes. It is important to emphasize that Mirza, Oboudiat, and Zare [6] have reached similar conclusions for 3-dimensional rotating Bañados, Teitelboim, and Zanelli black holes and for D-dimensional Lovelock black holes in odd and even dimensions (Refs. [5] and [6] have also considered the case of D-dimensional general relativistic black holes).

B. The Hawking emission of coupled electromagnetic-gravitational quanta

For the case of coupled electromagnetic-gravitational quanta, one finds the leading-order behavior [10]

$$\Gamma_{11} = \Gamma_{2m} = \frac{4}{9}\left(\frac{A\omega^2}{4\pi}\right)^4 \tag{17}$$

for the greybody factors in the low-frequency regime (9) which characterizes the neutral sector of the Hawking black-hole emission spectra in the near-extremal (low-temperature) regime (8). [It is worth emphasizing the fact that the Hawking emission of coupled electromagnetic-gravitational modes with $l > 2$ is suppressed as compared to the Hawking emission of coupled electromagnetic-gravitational modes with $l = 1$ and $l = 2$. This characteristic property of the Hawking black-hole emission spectra stems from the fact that the greybody factors of coupled electromagnetic-gravitational modes with higher-l values (that is, electromagnetic-gravitational modes which are characterized by $l > 2$) contain higher powers (as compared to the $l = 1$ and $l = 2$ modes) of the small quantity $\omega r_+ \ll 1$ [10]].

Substituting (17) into Eqs. (4) and (5), one finds after some algebra [12]

$$P_{\mathrm{RN}} = \frac{4\pi^5}{297} \frac{A^4 T_{\mathrm{BH}}^{10}}{\hbar^9} \tag{18}$$

and

$$\dot{S}_{\mathrm{RN}} = -\frac{560\zeta(9)}{\pi^5} \frac{A^4 T_{\mathrm{BH}}^9}{\hbar^9} \cdot \left[\ln\left(\frac{AT_{\mathrm{BH}}^2}{\hbar^2}\right) + O(1) \right] \tag{19}$$

for the electromagnetic-gravitational Hawking radiation power and the electromagnetic-gravitational Hawking entropy emission rate of the near-extremal Reissner-Nordström black holes.

Finally, substituting (18) into (19), one can express the black-hole entropy emission rate in terms of the Hawking radiation power:

$$\dot{S}_{\mathrm{RN}}^{\mathrm{Elec\text{-}Grav}} = -C_{\mathrm{RN}}^{\mathrm{Elec\text{-}Grav}} \times \left(\frac{A^4 P_{\mathrm{RN}}^9}{\hbar^9}\right)^{1/10} \ln\left(\frac{AP_{\mathrm{RN}}}{\hbar}\right), \tag{20}$$

where the analytically calculated coefficient $C_{\mathrm{RN}}^{\mathrm{Elec\text{-}Grav}}$ is given by

$$C_{\mathrm{RN}}^{\mathrm{Elec\text{-}Grav}} = \frac{112\zeta(9)}{\pi^{19/2}} \left(\frac{297}{4}\right)^{9/10}. \tag{21}$$

It is worth emphasizing again that the entropy emission rate (20), which characterizes the electromagnetic-gravitational

Hawking emission spectra of the near-extremal Reissner-Nordström black holes, does *not* have the simple $(1+1)$-dimensional entropy-energy functional relation (2) which characterizes the Hawking radiation spectra of the Schwarzschild black holes.

III. SUMMARY

In a very interesting paper [2], Bekenstein and Mayo have revealed that, in their entropy emission properties, $(3+1)$-dimensional Schwarzschild black holes behave effectively as $(1+1)$-dimensional [and not as $(3+1)$-dimensional] thermal entropy emitters [see Eqs. (2) and (3)]. Later studies [5,6] have extended the analysis of [2] to higher dimensional black holes and proved that all radiating $(D+1)$-dimensional Schwarzschild black holes are characterized by this intriguing physical property.

One naturally wonders whether this interesting property of the Schwarzschild black holes is shared by all black holes? In particular, motivated by the results of [2,5,6], we have raised here the following question: do all radiating black holes behave as simple $(1+1)$-dimensional entropy emitters? In order to address this intriguing question, we have explored in this paper the entropy emission properties of Reissner-Nordström black holes. In particular, we have shown that the physical properties which characterize the neutral sector of the Hawking emission spectra of these black holes can be studied *analytically* in the low-temperature (near-extremal) $T_{\text{BH}} \to 0$ regime.

We have explicitly shown that the analytically derived expressions for the Hawking entropy emission rates of massless scalar fields and coupled electromagnetic-gravitational fields by near-extremal Reissner-Nordström black holes [see Eqs. (15) and (20)] do not have the simple $(1+1)$-dimensional entropy-energy functional relation (2) which characterizes the Hawking emission spectra of the Schwarzschild black holes. Our analytical results therefore indicate that *not* all black holes behave as simple $(1+1)$-dimensional entropy emitters.

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[12] Here we have used the relations $\frac{e^{\hbar\omega/T_{\text{BH}}}-1}{\frac{4}{9}(A\omega^2/4\pi)^4} \gg 1$ and $\ln\left[1 + \frac{\frac{4}{9}(A\omega^2/4\pi)^4}{e^{\hbar\omega/T_{\text{BH}}}-1}\right] \simeq \frac{\frac{4}{9}(A\omega^2/4\pi)^4}{e^{\hbar\omega/T_{\text{BH}}}-1} \ll 1$ in the low-frequency regime $M\omega \ll 1$ [see Eq. (9)] which characterizes the neutral sector of the Hawking black-hole emission spectra in the low-temperature (near-extremal) regime (8).