Spacetime dynamics of spinning particles: Exact electromagnetic analogies

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We compare the rigorous equations describing the motion of spinning test particles in gravitational and electromagnetic fields, and show that if the Mathisson-Pirani spin condition holds then exact gravito-electromagnetic analogies emerge. These analogies provide a familiar formalism to treat gravitational problems, as well as a means for comparing the two interactions. Fundamental differences are manifest in the symmetries and time projections of the electromagnetic and gravitational tidal tensors. The physical consequences of the symmetries of the tidal tensors are explored comparing the following analogous setups: magnetic dipoles in the field of nonspinning/spinning charges, and gyroscopes in the Schwarzschild, Kerr, and Kerr–de Sitter spacetimes. The implications of the time projections of the tidal tensors are illustrated by the work done on the particle in various frames; in particular, a reciprocity is found to exist: in a frame comoving with the particle, the electromagnetic (but not the gravitational) field does work on it, causing a variation of its proper mass; conversely, for "static observers," a stationary gravitomagnetic (but not a magnetic) field does work on the particle, and the associated potential energy is seen to embody the Hawking-Wald spin-spin interaction energy. The issue of hidden momentum, and its counterintuitive dynamical implications, is also analyzed. Finally, a number of issues regarding the electromagnetic interaction and the physical meaning of Dixon's equations are clarified.

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I. INTRODUCTION

Analogies between the equations of motion for gyroscopes in a gravitational field and magnetic dipoles in an electromagnetic field have been known for a long time, and were presented in many different forms throughout the years. This is the case for both the force and the spin evolution equations for these test particles in external fields. The former was first found by Wald [1] in the framework of linearized theory: he showed that the gravitational force exerted on a spinning pole-dipole test particle (hereafter a gyroscope), whose center of mass is *at rest* in a *stationary field*, takes the form $\vec{F}_G = K\nabla(\vec{H}\cdot\vec{S})$, where $\vec{H}$ is the so-called "gravitomagnetic field," K is some constant (depending on the precise definition of $\vec{H}$, e.g. [2–5]), and $\vec{S}$ is the particle's angular momentum. This formula is similar to the formula for the electromagnetic force on a magnetic dipole, $\vec{F}_{\rm EM} = \nabla(\vec{B}\cdot\vec{\mu})$, where $\vec{B}$ is the magnetic field and $\vec{\mu}$ is the dipole's magnetic moment. The analogy was later cast in an *exact* form by one of the authors in [6], using the *exact* "gravitoelectromagnetic" (GEM) inertial fields from the so-called $1+3$ "quasi-Maxwell" formalism. The force was seen therein to consist of an electromagneticlike term of the form above plus a term interpreted as the "weight of the energy" of the gravitomagnetic dipole, and the limit of validity of the analogy was extended to arbitrarily strong *stationary* fields *and* when the gyroscope's worldline *is tangent to any timelike Killing vector field* (which comprehends e.g. circular trajectories with arbitrary speed in axisymmetric spacetimes). In a different framework, it was later shown that there is actually an exact, *covariant*, and *fully general* analogy relating the two forces; such analogy is made explicit *not* in the framework of the GEM inertial fields, but by using instead the tidal tensors of both theories, introduced in [7].

The analogy between the so-called "precession" of a gyroscope in a gravitational field and the precession of a magnetic dipole under the action of a magnetic field was noticed long ago, in the framework of linearized theory, by a number of authors, e.g. [3–5,8–11], who pointed out that the spin vector of a gyroscope at rest in a stationary field evolves as $d\vec{S}/dt = K\vec{S}\times\vec{H}$. This formula is similar to the formula for the precession of a magnetic dipole in a magnetic field, $d\vec{S}/dt = \vec{\mu}\times\vec{B}$. The analogy was later cast in *exact* forms in e.g. [4,6,12,13]; these are not covariant, holding only in specific frames, but (in the more general

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formulations in [12–14] and herein) the test particle can be moving with arbitrary velocity in an *arbitrary field*.

These analogies provide a familiar formalism to treat otherwise complicated gravitational effects, as well as a means to compare the two interactions. In this paper we explore them, exemplifying their usefulness in some applications, and the insight they provide into fundamental, yet not well known aspects of both interactions.

We will also make use of a third exact gravito-electromagnetic analogy (see e.g. [15–17]), this one a *purely formal* one (see [14]), relating the quadratic scalar invariants of the Maxwell and Weyl tensors [15,16,18], which proves useful in some applications.

A. The equations of motion

In this paper we start, in Sec. II, by writing the general relativistic equations describing the motion of spinning test particles with gravitational and electromagnetic pole-dipole moments, subject to gravitational and electromagnetic external fields, in terms of quantities with a clear physical meaning. This turns out not to be a straightforward task, as the covariant equations for this problem are still not generally well understood, with different methods and derivations leading to different versions of the equations, the relationship between them not being clear. Perhaps more surprising is the fact that it is the electromagnetic sector that has been posing more difficulties, with a number of misconceptions arising in the physical interpretation of the quantities involved. These issues are clarified in Appendix A, where the relation between the different versions of the equations and their physical interpretation are discussed in detail.

In order to form a determined system, the equations of motion need to be supplemented by a spin condition; the latter is even today still regarded as an open question, with a long history of debates concerning which one is the "best" condition (see [19] for a review and list of references). In Sec. II A we briefly discuss its meaning and the problem of the relativistic definition of center of mass. This is of relevance here because the two physical analogies mentioned above (for the force and for the spin precession) rely on a specific choice—the Mathisson-Pirani spin condition.

Also related with the spin supplementary condition is an issue central to the understanding of the dynamics of a spinning particle: the decoupling of the 4-velocity U^α from the 4-momentum P^α, discussed in Sec. II D. In general, U^α is not parallel to P^α; the particle is said to possess "hidden momentum," for which another exact analogy is seen to emerge. The hidden momentum is known to lead to counterintuitive behaviors of the spinning particles; examples are the bobbings studied in [20], and the Mathisson helical motions themselves, where a particle accelerates without the action of any force [21]. Herein (Sec. III A) we present another, perhaps even more surprising consequence: a magnetic dipole with radial initial velocity in the field of a point charge accelerates in approximately the *opposite* direction to the force.

B. The main realizations

Most of our applications, Secs. III–V of this paper, will deal with the tidal tensor formalism introduced in [7], and the exact analogy it unveils: both the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope are given by a contraction of a rank 2 magnetic type tidal tensor ($B_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$), with the dipole/spin 4-vector. Here $B_{\alpha\beta}$ gives the tidal effects of the magnetic field and $\mathbb{H}_{\alpha\beta}$ is the magnetic part of the Riemann tensor, both measured in the particle's rest frame. This makes this formalism specially suited to comparing the two forces—it amounts to simply comparing the two tidal tensors. Such comparison is done through Einstein's and Maxwell's equations, as they also can be written in terms of tidal tensors. Apart from the nonlinearity of $\mathbb{H}_{\alpha\beta}$, the tensorial structure differs when the fields vary along the test particle's worldline, since this endows $B_{\alpha\beta}$ with an antisymmetric part, and a nonvanishing time projection along that worldline, whereas its gravitational counterpart is spatial relative to that worldline and, in vacuum, symmetric. Both these aspects are related with the laws of electromagnetic induction (and the absence of a counterpart in the gravitational tidal effects); we discuss them in two separate sections, as described below.

In Sec. III we explore the physical consequences of the different symmetries of the gravitational and electromagnetic tidal tensors. They are seen to imply e.g. that particles moving in a nonhomogeneous electromagnetic field always measure a nonvanishing $B_{\alpha\beta}$ (thus feel a force), which is not necessarily the case in gravity. The following analogous setups are compared: magnetic dipoles in the field of nonspinning/spinning charges, and gyroscopes in the Schwarzschild, Kerr, and Kerr-dS spacetimes. It is seen that in the cases where $B_{\alpha\beta}$ reduces to $B_{[\alpha\beta]}$, we have $\mathbb{H}_{\alpha\beta} = 0$ (thus no force) in the gravitational analogue. Geodesic motions for spinning particles are even found to exist in the Schwarzschild (radial geodesics) and in the Kerr-dS (circular equatorial geodesics) spacetimes.

In Sec. IV we explore the physical content of the time projections of the forces in different frames, which are related with the rate of work done on the test particle by the external fields. In order to obtain the relationship, we start by deriving the general equation yielding the variation of energy of a particle with multipole structure with respect to an arbitrary congruence of observers. We then show that the electromagnetic force has a nonvanishing time projection along U^α, which is the power transferred to the dipole by Faraday's induction, reflected in a variation of its proper mass m. The projection of the gravitational force along U^α, by contrast, vanishes (as $\mathbb{H}_{\alpha\beta}$ is spatial relative to U^α), leading to the conservation of the gyroscope's mass.

Also of particular interest in this context are the time projections as measured by "static observers," analyzed in Sec. IV B. For these observers, the time projection of the electromagnetic force vanishes, meaning that the total work done on the magnetic dipole is zero. This reflects the well-known fact that the work done by the stationary magnetic field is zero; in this framework, it is seen to arise from an exchange of energy between three forms, translational kinetic energy, proper mass m, and "hidden energy," occurring in a way such that their variations cancel out, keeping the total energy constant. In the gravitational case, since m is constant, such cancellation does not occur and (by contrast with its electromagnetic counterpart) a stationary field *does work* on mass currents, so that there exists an associated potential spin-curvature potential energy, of which the Wald-Hawking spin-spin interaction energy [1,22] is seen to be a special case.

In Sec. V we study the weak field and slow motion regime, and show that the above-mentioned differences between the two interactions appear at leading order (thus are *not* negligible) therein, which is commonly overlooked in the literature concerning this regime.

C. Beyond the pole-dipole

In Sec. VI we go beyond the pole-dipole approximation, including the moments of quadrupole order, to clarify the mechanism by which the proper mass of a spinning particle in an electromagnetic field varies, and solve an apparent contradiction of the former approximation: on the one hand, as stated above, the mass m of a particle with magnetic moment varies due (from the point of view of the particle's frame) to the work done on it by the induced electric field (which, by having a curl, should torque the body). On the other hand, the associated torque is not manifest in the dipole order equations. In Sec. VI A we show that such torque is indeed exerted on the particle (governed by the time projection of the magnetic tidal tensor, $B^{\alpha}{}_{\beta}U^{\beta}$), but it involves quadrupole order moments of the charge, which is why it does not show up to dipole order. The subtlety here is that *the work it does*, and the associated variation of mass/kinetic energy of rotation, *is of dipole order* (yielding indeed the time projection of the dipole force along its worldline, as obtained in Sec. IV A). Then in Sec. VI B we study the analogous gravitational problem, showing that, as expected (as $\mathbb{H}_{\alpha\beta}$ is spatial with respect to U^{α}), no analogous torque exists.

D. Notation and conventions

(1) *Signature and signs.*—We use the signature $-+++$; $\epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{-g}[\alpha\beta\gamma\delta]$ denotes the Levi-Civita tensor, and we follow the orientation $[1230] = 1$ (i.e., in flat spacetime, $\epsilon_{1230} = 1$). $\epsilon_{ijk} \equiv \epsilon_{ijk0}$ is the 3D alternating tensor. Our convention for the Riemann tensor is $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \cdots$.

(2) We use bold fonts to denote tensors $\mathbf{T}$ (including 4-vectors $\mathbf{P}$), and arrows for the spatial components $\vec{P}$ of a 4-vector $\mathbf{P}$. Greek letters $\alpha, \beta, \gamma, \ldots$ denote 4D spacetime indices, running 0–3; Roman letters $i, j, k, \ldots$ denote spatial indices, running 1–3. Following the usual practice, we sometimes use component notation $T^{\alpha\beta}$ to refer to a tensor $\mathbf{T}$.

(3) By u^{α} we denote a generic unit timelike vector, which can be interpreted as the instantaneous 4-velocity of some observer. $U^{\alpha} \equiv dz^{\alpha}/d\tau$ is the tangent vector to the test body's representative worldline $z^{\alpha}(\tau)$, taken to be a suitably defined center of mass (CM). U^{α} is thus the CM 4-velocity.

(4) *Time and space projectors.*—$(\top^{u})^{\alpha}{}_{\beta} \equiv -u^{\alpha}u_{\beta}$ and $(h^{u})^{\alpha}{}_{\beta} \equiv \delta^{\alpha}{}_{\beta} + u^{\alpha}u_{\beta}$ denote, respectively, the projectors parallel and orthogonal to a unit timelike vector u^{α}; they may be interpreted as the time and space projectors in the local rest frame of an observer with 4-velocity u^{α}.

(5) *Tensors resulting from a measurement process.*—$(A^{u})^{\alpha_1\ldots\alpha_n}$ denotes the tensor $\mathbf{A}$ as measured by an observer $\mathcal{O}(u)$ of 4-velocity u^{α}. For example, $(E^{u})^{\alpha} \equiv F^{\alpha}{}_{\beta}u^{\beta}$, $(E^{u})_{\alpha\beta} \equiv F_{\alpha\gamma;\beta}u^{\gamma}$ and $(\mathbb{E}^{u})_{\alpha\beta} \equiv R_{\alpha\mu\beta\nu}u^{\mu}u^{\nu}$ denote, respectively, the electric field, electric tidal tensor, and gravitoelectric tidal tensor as measured by $\mathcal{O}(u)$. For the space components of a vector in a given frame we use the notation $\vec{A}(u)$; for example, $\vec{E}(u)$ denotes the space components of $(E^{u})^{\alpha}$. When $u^{\alpha} = U^{\alpha}$ (i.e., the particle's CM 4-velocity) we drop the superscript [e.g. $(E^{U})^{\alpha} \equiv E^{\alpha}$], or the argument of the vector: $\vec{E}(U) \equiv \vec{E}$.

(6) *Electromagnetic field.*—The Maxwell tensor $F^{\alpha\beta}$ and its dual $\star F^{\alpha\beta}$ decompose in terms of the electric $(E^{u})^{\alpha} \equiv F^{\alpha}{}_{\beta}u^{\beta}$ and magnetic $(B^{u})^{\alpha} \equiv \star F^{\alpha}{}_{\beta}u^{\beta}$ fields measured by an observer of 4-velocity u^{α} as

$$F_{\alpha\beta} = 2u_{[\alpha}(E^{u})_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}u^{\delta}(B^{u})^{\gamma}; \tag{1}$$

$$\star F_{\alpha\beta} = 2u_{[\alpha}(B^{u})_{\beta]} - \epsilon_{\alpha\beta\gamma\delta}u^{\delta}(E^{u})^{\gamma}. \tag{2}$$

(7) *Static observers.*—In stationary, asymptotically flat spacetimes, we dub "static observers" the *rigid* congruence of observers whose worldlines are tangent to the temporal Killing vector field $\xi = \partial/\partial t$; they may be interpreted as the set of points rigidly fixed to the "distant stars" (the asymptotic inertial rest frame of the source). In the Kerr spacetime, these correspond to the observers of zero 3-velocity in Boyer-Lindquist coordinates. This agrees with the convention in e.g. [23,24]. (The denomination "static observers" has, however, a different meaning in some literature, e.g. [25], where it designates rigid, *vorticity-free* congruences tangent to a timelike Killing vector field, existing only in *static*

spacetimes.) In the case of the electromagnetic systems in flat spacetime, by static observers we mean the globally inertial rest frame of the sources.

(8) *GEM*.—This is the acronym for gravitoelectromagnetism. By "inertial GEM fields," we mean the fields of inertial forces that arise from the $1+3$ splitting of spacetime: the gravitoelectric field $\vec{G}$, which plays in this framework a role analogous to the electric field of electromagnetism, and the gravitomagnetic field $\vec{H}$, analogous to the magnetic field. We discuss these fields in detail in [14].

II. EQUATIONS OF MOTION FOR SPINNING POLE-DIPOLE PARTICLES

In most of this paper we will be dealing with the dynamics of the so-called pole-dipole spinning test particles. We consider systems composed of a test body plus background gravitational and electromagnetic fields. Let $(T_{\rm tot})^{\alpha\beta} = \Theta^{\alpha\beta} + (T_{\rm matter})^{\alpha\beta}$ denote the total energy-momentum tensor, which splits into the electromagnetic stress-energy tensor $\Theta^{\alpha\beta}$ and the energy-momentum tensor of the matter $(T_{\rm matter})^{\alpha\beta}$. Moreover, let $T^{\alpha\beta}$ and j^α denote, respectively, the energy-momentum tensor and the current density 4-vector *of the test body*. We also consider that the only matter and currents present are the ones arising from the test body: $(T_{\rm matter})^{\alpha\beta} = T^{\alpha\beta}$, $j^\alpha_{\rm tot} = j^\alpha$. In this case (see [26] for details) the conservation of total energy-momentum tensor yields (cf. e.g. [20,27,28])

$$(T_{\rm tot})^{\alpha\beta}{}_{;\beta} = 0 \Rightarrow T^{\alpha\beta}{}_{;\beta} = -\Theta^{\alpha\beta}{}_{;\beta} \Leftrightarrow T^{\alpha\beta}{}_{;\beta} = F^{\alpha\beta} j_\beta, \tag{3}$$

where $F^{\alpha\beta}$ is the Maxwell tensor of the *external* (background) electromagnetic field.

In a multipole expansion the body is represented by the moments of j^α (its "electromagnetic skeleton") and a set of moments of $T^{\alpha\beta}$, called "inertial" or "gravitational" moments (forming the so-called [29] "gravitational skeleton"). Truncating the expansion at dipole order, the equations of motion for such a particle involve only two moments of $T^{\alpha\beta}$,

$$P^{\hat\alpha} \equiv \int_{\Sigma(\tau,U)} T^{\hat\alpha\hat\beta} d\Sigma_{\hat\beta}, \tag{4}$$

$$S^{\hat\alpha\hat\beta} \equiv 2\int_{\Sigma(\tau,U)} x^{[\hat\alpha} T^{\hat\beta]\hat\gamma} d\Sigma_{\hat\gamma}, \tag{5}$$

and the electromagnetic moments [30]:

$$q \equiv \int_\Sigma j^\alpha d\Sigma_\alpha, \tag{6}$$

$$d^{\hat\alpha} \equiv \int_{\Sigma(\tau,U)} x^{\hat\alpha} j^{\hat\beta} d\Sigma_{\hat\beta}, \tag{7}$$

$$\mu^{\hat\alpha} \equiv \frac{1}{2}\epsilon^{\hat\alpha}{}_{\hat\beta\hat\gamma\hat\delta} U^{\hat\delta} \int_{\Sigma(\tau,U)} x^{\hat\beta} j^{\hat\gamma} d\Sigma. \tag{8}$$

These are taken with respect to a reference worldline $z^\alpha(\tau)$, of proper time τ and (unit) tangent vector $U^\alpha \equiv dz^\alpha/d\tau$, and to a hypersurface of integration $\Sigma(\tau, u)$, which is the spacelike hypersurface generated by all geodesics orthogonal to some timelike vector u^α at the point $z^\alpha(\tau)$; following [30] we take $u^\alpha = U^\alpha$. Also,

$$d\Sigma_\gamma \equiv -n_\gamma d\Sigma \quad (\text{at } z^\alpha : n^\alpha = U^\alpha), \tag{9}$$

where n^α is the (future-pointing) unit vector normal do $\Sigma(\tau, U)$, and $d\Sigma$ is the 3-volume element of this hypersurface. The integrations are performed in a system of Riemann *normal coordinates* $\{x^{\hat\alpha}\}$ (e.g. [23,31]) centered at the point z^α of the reference worldline (i.e., $z^{\hat\alpha} = 0$). The resulting expressions, however, are *tensors* (see below), which can be expressed in any frame. $P^\alpha(\tau)$ is the 4-momentum of the test particle, q its total charge (an invariant, independent of Σ), and $S^{\alpha\beta}(\tau)$, $d^\alpha(\tau)$ and $\mu^\alpha(\tau)$ are, respectively, the angular momentum, and the *intrinsic* electric and magnetic dipole moments about the point $z^\alpha(\tau)$ of the reference worldline. It is useful to introduce also the magnetic dipole 2-form $\mu_{\alpha\beta}$ by

$$\mu_{\alpha\beta} \equiv \epsilon_{\alpha\beta\gamma\delta}\mu^\gamma U^\delta; \qquad \mu^\alpha = \frac{1}{2}\epsilon^\alpha{}_{\beta\gamma\delta} U^\beta \mu^{\gamma\delta}. \tag{10}$$

In some applications we will assume $\mu^{\alpha\beta}$ to be proportional to the spin tensor: $\mu^{\alpha\beta} = \sigma S^{\alpha\beta}$, where σ is the gyromagnetic ratio. The moments d^α and μ^α are dubbed "intrinsic" because they are evaluated in a frame comoving with the particle's representative point $z^\alpha(\tau)$ (where $U^i = 0$). If this frame is inertial, they take the forms $d^\alpha = (0, \vec{d})$ and $\mu^\alpha = (0, \vec{\mu})$, where $\vec{d}$ and $\vec{\mu}$ are given by the usual textbook definitions (e.g. [32]): $\vec{d} = \int \rho_c \vec{x} d^3x$, $\vec{\mu} = \int \vec{x} \times \vec{j} d^3x/2$.

Expressions (4), (5), (7) and (8) are integrals of tensors over Σ (i.e., they add tensor components at different points in a curved spacetime), which requires a justification. By using Riemann normal coordinates, one is implicitly using the exponential map to pull back the integrands from the spacetime manifold to the tangent space at z^α, and integrating therein, which is a well-defined tensor operation, see [31,33]. (Note also that, by being associated to the exponential map, such coordinates are naturally adapted to integrations over geodesic hypersurfaces Σ). Other schemes to perform such integrations were proposed in the literature, based on bitensors in [20,27,34–36], or less sophisticated ones in e.g. [37]. In the pole-dipole approximation [where $T^{\alpha\beta}$ and j^α are nonvanishing only in a very small region around $z^\alpha(\tau)$, so that only terms linear in $x^{\hat\alpha}$ are kept] they are *all equivalent* (see Appendix A 1 and [33,36]).

The motion of the test particle is described by the reference worldline $z^\alpha(\tau)$; its choice will be discussed below. The equations of motion that follow from (3) are [30,34,38] (see Appendix A for a discussion)

$$\frac{DP^\alpha}{d\tau} = qF^\alpha{}_\beta U^\beta + \frac{1}{2}F^{\mu\nu;\alpha}\mu_{\mu\nu} - \frac{1}{2}R^\alpha{}_{\beta\mu\nu}S^{\mu\nu}U^\beta + F^\alpha{}_{\gamma;\beta}U^\gamma d^\beta + F^\alpha{}_\beta \frac{Dd^\beta}{d\tau}, \quad (11)$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} + 2\mu^{\theta[\beta}F^{\alpha]}{}_\theta + 2d^{[\alpha}F^{\beta]}{}_\gamma U^\gamma, \quad (12)$$

where $F^{\alpha\beta}$ is the background Maxwell tensor.

The first term in (11) is the Lorentz force; the second term, $\frac{1}{2}F^{\mu\nu;\alpha}\mu_{\mu\nu} \equiv F^\alpha_{\rm EM}$, is the force due to the tidal coupling of the electromagnetic field to the magnetic dipole moment; and the third, $-\frac{1}{2}R^\alpha{}_{\beta\mu\nu}S^{\mu\nu}U^\beta \equiv F^\alpha_{\rm G}$, is the Mathisson-Papapetrou spin-curvature force. The last two terms are the force exerted on the electric dipole, consisting of a tidal term $F^\alpha{}_{\gamma;\beta}U^\gamma d^\beta$ and of a *nontidal* term $F^\alpha{}_\beta Dd^\beta/d\tau$. Note that the terms involving μ^α and d^α are substantially different; this can be traced back to the intrinsic difference between the two types of dipole—μ^α being the dipole moment of the spatial current density $(h^U)^\alpha{}_\beta j^\beta$, and d^α the dipole moment of the charge density $\rho_c \equiv -j^\alpha U_\alpha$, cf. Eqs. (7) and (8). The former can be modeled by a current loop, the latter by a pair of oppositely charged monopoles, and these two types of objects behave differently as test particles; in this paper we shall discuss some dynamical implications.

Up until now, the reference worldline $z^\alpha(\tau)$, relative to which the moments in Eqs. (11) and (12) are taken, is still undefined. Had we made an exact expansion keeping all the infinite multipole moments as in [35,36], such worldline would be *arbitrary*. Herein, however, it must be assumed that it passes through the body (or close enough), so that the pole-dipole approximation is valid; it will be chosen as being prescribed by a suitably defined center of mass of the test particle. As discussed in the next section, that is done through a supplementary condition $S^{\alpha\beta}u_\beta = 0$, for some timelike unit vector field u^α. If $F^{\alpha\beta} = 0$ there are 13 unknowns in Eqs. (11) and (12) (P^α, three independent components of U^α, and six independent components of $S^{\alpha\beta}$) for only ten equations. The condition $S^{\alpha\beta}u_\beta = 0$, for a *definite* u^α, closes the system as it kills off three components of $S^{\alpha\beta}$. In the general case where $F^{\alpha\beta} \neq 0$ one also needs to give the laws of evolution for $\mu^{\alpha\beta}$ and d^α in order for the system to be determined, cf. [28].

A. Center of mass (CM) and spin supplementary condition

In relativistic physics, the center of mass of a spinning body is observer dependent. This is illustrated in Fig. 1 of [21]. Thus one needs to specify the frame where it is to be evaluated. That amounts to supplementing Eqs. (11) and (12) (which, as discussed above, would otherwise be undetermined) by the spin supplementary condition $S^{\alpha\beta}u_\beta = 0$, as we will show next. The vector $(d_G^u)^\alpha \equiv -S^{\alpha\beta}u_\beta$ yields the "mass dipole moment" (i.e. the mass times the displacement of the reference worldline relative to the center of mass) as measured by the observer $\mathcal{O}$ of 4-velocity u^α. In order to see this consider, at the point z^α of the reference worldline, a system of Riemann normal coordinates $\{x^{\hat\alpha}\}$ momentarily comoving with $\mathcal{O}$ (i.e., $\partial_{\hat 0} = \mathbf{u}$ *at* z^α). In this frame, $u^{\hat i} = 0$ and $S^{\hat i\hat\beta}u_{\hat\beta} = S^{\hat i\hat 0}u_{\hat 0} = -S^{\hat i\hat 0}$. From Eq. (5) we have

$$S^{\hat i\hat 0} = \int_{\Sigma(\tau,u)} x^{\hat i}T^{\hat 0\hat\gamma}d\Sigma_{\hat\gamma} \equiv m(u)x^{\hat i}_{\rm CM}(u), \quad (13)$$

where $m(u) = -P^\alpha u_\alpha$ denotes the mass as measured by $\mathcal{O}$, and we used the fact that $\Sigma(\tau,u)$ coincides with the spatial hypersurface $x^{\hat 0} = 0$. We see that $S^{\hat i\hat 0}$ is by definition the mass dipole in the frame $\{x^{\hat\alpha}\}$: $S^{\hat i\hat 0} = m(u)x^{\hat i}_{\rm CM}(u) \equiv (d_G^u)^{\hat i}$, and $x^{\hat i}_{\rm CM}(u) = S^{\hat i\hat 0}/m(u)$ is the position of the center of mass. Thus the condition

$$S^{\alpha\beta}u_\beta = 0, \quad (14)$$

implying in this frame $S^{\hat i\hat 0} = 0 \Rightarrow x^{\hat i}_{\rm CM}(u) = 0$, is precisely the condition that the reference worldline is the center of mass (or "centroid") as measured in this frame (or, equivalently, that the mass dipole vanishes for such an observer). For details on how the center of mass position changes in a change of observer, we refer the reader to [21,33]. The set of all the possible positions of the center of mass, as measured by every possible observer, forms a worldtube (the "minimal worldtube" [39]), which is typically very narrow, and always contained within the convex hull of the body's worldtube (see [33,40,41]).

Usually one prefers equations of motion that do not depend on a CM measured by some "external" observer, but instead the field u^α to be defined in terms of the timelike vectors (P^α or U^α) already present in Eqs. (11) and (12). This is the case of the two most common [19] conditions in the literature: the Frenkel-Mathisson-Pirani [29,42–44] condition $S^{\alpha\beta}U_\beta = 0$ (hereafter the Mathisson-Pirani condition, as it is best known) and the Tulczyjew-Dixon [34,45] condition $S^{\alpha\beta}P_\beta = 0$. The former seems the most natural choice, as it amounts to computing the center of mass in its proper frame, i.e., *in the frame where it has zero 3-velocity*. It also arises in a natural fashion in some derivations [46,47] (see also [48]), and has been argued [49–51] to be the only one that can be applied in the case of massless particles. It turns out, however, that it does not determine the worldline uniquely. For instance, in the case of a free particle in flat spacetime, it is known to lead, in addition to the expected straightline motion, to an infinite set of helical motions, notably found by Mathisson [43], and which have been poorly understood and subject of some misconceptions in the literature. These were clarified in [21], where it was shown that the different worldlines compatible with this condition are just equivalent descriptions of the same physical motion.

The Tulczyjew-Dixon condition $S^{\alpha\beta}P_\beta = 0$ amounts to computing the CM *in the frame where it has zero 3-momentum*. This condition determines uniquely the CM worldline [27,52,53]; there is no ambiguity in this case, since P^α is given in advance by (4), and for this reason it is preferred by a number of authors. For a free particle in flat spacetime, the worldline specified by $S^{\alpha\beta}P_\beta = 0$ corresponds to Mathisson's nonhelical solution; but in the presence of gravitational/electromagnetic field, P^α cannot in general be parallel to U^α under these spin conditions [cf. Eqs. (29) and (34), below], so the solutions do not coincide.

The fundamental point to be emphasized here is that these two conditions, as well as other reasonable conditions in the literature (such as the Corinaldesi-Papapetrou condition [54], the "parallel" condition in [39], or the Newton-Wigner condition [55,56], used in Hamiltonian and effective field theory approaches [57–63]), are, as shown explicitly in [33], *equivalent* descriptions of the motion of the test particle, the choice between them being a matter of convenience.

In most of this paper we will adopt the Mathisson-Pirani condition, since it is the one that leads to the exact gravitoelectromagnetic analogies we use. If the Tulczyjew-Dixon condition is chosen instead, one still recovers the same analogies to a good approximation. The spin conditions, their effective differences and their suitability for the applications in this paper, as well as their impact on the gravitoelectromagnetic analogies, are discussed in detail in Appendix C. Therein we show that exact analogies turn out to exist for an arbitrary spin condition, only the corresponding equations are slightly more complicated.

With the Mathisson-Pirani condition, we have $S^{\mu\nu} = \epsilon^{\mu\nu\tau\lambda}S_\tau U_\lambda$, where S^α is the spin 4-vector, which has components $(0, \vec{S})$ in an orthonormal frame comoving with the CM. Substituting into Eqs. (11) and (12) (and also performing the contractions with U^α) we obtain

$$\frac{DP^\alpha}{d\tau} = qE^\alpha + E^{\alpha\beta}d_\beta + B^{\beta\alpha}\mu_\beta - \mathbb{H}^{\beta\alpha}S_\beta + F^\alpha{}_\beta \frac{Dd^\beta}{d\tau}; \quad (15)$$

$$\frac{D_F S^\mu}{d\tau} = \epsilon^\mu{}_{\alpha\beta\nu}U^\nu(d^\alpha E^\beta + \mu^\alpha B^\beta), \quad (16)$$

where $E^\alpha \equiv F^{\alpha\beta}U_\beta$ and $B^\alpha \equiv \star F^{\alpha\beta}U_\beta$ are the electric and magnetic fields *as measured by the test particle*, and $E_{\alpha\beta} \equiv F_{\alpha\mu;\beta}U^\mu$, $B_{\alpha\beta} \equiv \star F_{\alpha\mu;\beta}U^\mu$ and $\mathbb{H}_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\sigma}U^\mu U^\sigma$ are, respectively, the electric, magnetic and gravitomagnetic tidal tensors as defined in [7,14], *as measured by the test particle*. The operator $D_F/d\tau$ denotes the Fermi-Walker derivative along U^α (e.g. [23,64]), which, for some vector V^μ, reads

$$\frac{D_F V^\mu}{d\tau} = \frac{DV^\mu}{d\tau} - 2U^{[\mu}a^{\nu]}V_\nu, \quad (17)$$

where $a^\alpha \equiv DU^\alpha/d\tau$ is the CM acceleration.

B. Force on gyroscope vs force on magnetic dipole—exact analogy based on tidal tensors

Herein we are interested in *purely magnetic* dipoles, i.e., dipoles whose electric moment vanishes in the CM frame; this is ensured by the condition $d^\alpha = 0$. In this case, Eq. (15) simplifies to

$$\frac{DP^\alpha}{d\tau} = qF^{\alpha\beta}U_\beta + B^{\beta\alpha}\mu_\beta - \mathbb{H}^{\beta\alpha}S_\beta. \quad (18)$$

These equations manifest the physical analogy $B_{\alpha\beta} \leftrightarrow \mathbb{H}_{\alpha\beta}$, summarized in Table I: (i) both the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope are determined by a contraction of the spin/magnetic dipole 4-vector with a magnetic type tidal tensor. $B_{\alpha\beta}$ may be cast as the derivative of the magnetic field $B^\alpha = \star F^\alpha{}_\beta U^\beta$ as measured in the *inertial* frame *momentarily* comoving with the test particle: $B_{\alpha\beta} = B_{\alpha;\beta}|_{U=\mathrm{const}}$. For this reason it is dubbed the *magnetic tidal tensor*, and its gravitational counterpart $\mathbb{H}_{\alpha\beta}$ the *gravitomagnetic tidal tensor* [7]. (ii) It turns out that $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ obey the formally similar equations (I.2) and (I.3) in Table I, which in one case are Maxwell's equations, and in the other are

TABLE I. Analogy between the electromagnetic force on a magnetic dipole and the gravitational force on a gyroscope.

Electromagnetic force on a magnetic dipole		Gravitational force on a spinning particle	
$F^\beta_{\mathrm{EM}} = B_\alpha{}^\beta\mu^\alpha$; $B^\alpha{}_\beta \equiv \star F^\alpha{}_{\mu;\beta}U^\mu$	(I.1a)	$F^\beta_{\mathrm{G}} = -\mathbb{H}_\alpha{}^\beta S^\alpha$; $\mathbb{H}^\alpha{}_\beta \equiv \star R^\alpha{}_{\mu\beta\nu}U^\mu U^\nu$	(I.1b)
Equations for the magnetic tidal tensor		Equations for the gravitomagnetic tidal tensor	
$B^\alpha{}_\alpha = 0$	(I.2a)	$\mathbb{H}^\alpha{}_\alpha = 0$	(I.2b)
$B_{[\alpha\beta]} = \frac{1}{2}\star F_{\alpha\beta;\gamma}U^\gamma - 2\pi\epsilon_{\alpha\beta\sigma\gamma}j^\sigma U^\gamma$	(I.3a)	$\mathbb{H}_{[\alpha\beta]} = -4\pi\epsilon_{\alpha\beta\sigma\gamma}J^\sigma U^\gamma$	(I.3b)
$B_{\alpha\beta}U^\alpha = 0$; $B_{\alpha\beta}U^\beta = \epsilon^{\beta\gamma}{}_{\alpha\delta}E_{[\beta\gamma]}U^\delta$	(I.4a)	$\mathbb{H}_{\alpha\beta}U^\alpha = \mathbb{H}_{\alpha\beta}U^\beta = 0$	(I.4b)

exact Einstein's equations. That is, the traces (I.2) are, respectively, the time projection (with respect to U^α) of the electromagnetic Bianchi identity $\star F^{\alpha\beta}{}_{;\beta}=0$ and the time-time projection of the algebraic Bianchi identities $\star R^{\gamma\alpha}{}_{\gamma\beta}=0$; the antisymmetric parts (I.3a) are, respectively, the space projection of Maxwell's equations $F^{\alpha\beta}{}_{;\beta}=4\pi j^\alpha$ and the time-space projection of Einstein's equations $R_{\mu\nu}=8\pi(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T^\alpha{}_\alpha)$. The electromagnetic equations take a familiar form in an inertial frame: Eq. (I.2a) becomes $\nabla\cdot\vec{B}=0$; the space part of (I.3a) is the Maxwell-Ampère law $\nabla\times\vec{B}=\partial\vec{E}/\partial t+4\pi\vec{j}$. The latter means that the space part of $B_{[\alpha\beta]}$ encodes the curl of B^α, which is actually a more general statement, holding in arbitrarily accelerated frames: denote by U^α the 4-velocity of the rest observers in such frames; if the frame is nonrotating and nonshearing, $U_{\alpha;\beta}=-a_\alpha U_\beta$, cf. Eq. (70) below, and we have

$$\epsilon^{\beta\gamma}{}_{\alpha\delta}B_{\gamma\beta}U^\delta=\epsilon^{\beta\gamma}{}_{\alpha\delta}B_{\gamma;\beta}U^\delta\Rightarrow\epsilon^{ikj}B_{jk}=(\nabla\times\vec{B})^i. \quad (19)$$

Expressing also the second member of (I.3a) in terms of the electric and magnetic fields E^α and B^α measured in this frame, we obtain, in 3-vector notation,

$$\nabla\times\vec{B}=\frac{D\vec{E}}{d\tau}-\vec{a}\times\vec{E}+4\pi\vec{j} \quad (20)$$

which is the generalization of Maxwell-Ampère law for accelerated frames [cf. Eq. (19) of [17]]. This equation, as well as Eq. (23) below, is of use in the particle's CM frame (as it in general accelerates).

Note this important aspect of Eq. (I.3a), considering for simplicity the vacuum case $j^\alpha=0$: it tells us that when the field $F_{\alpha\beta}$ varies along the particle's worldline (of 4-velocity U^α), that endows $B_{\alpha\beta}$ with an antisymmetric part, implying that $B_{\alpha\beta}$ itself is nonvanishing. Hence, whenever the particle moves in a nonhomogeneous field, a force will be exerted on it (except for special orientations of $\vec{\mu}$). From Eqs. (19) and (20), this can be interpreted, taking the perspective of the inertial frame momentarily comoving with the particle, as the time-varying electric field inducing a curl in the magnetic field $\vec{B}$ (and thus a nonvanishing magnetic tidal tensor). The fact that its gravitomagnetic counterpart $\mathbb{H}_{\alpha\beta}$ is symmetric in vacuum tells us that no analogous induction phenomenon occurs in gravity. The physical consequences shall be explored in Sec. III below.

There is an electric counterpart to this analogy, relating the electric tidal tensor $E_{\alpha\beta}$ with the electric part of the Riemann tensor:

$$E_{\alpha\beta}\equiv F_{\alpha\mu;\beta}U^\mu\leftrightarrow\mathbb{E}_{\alpha\beta}\equiv R_{\alpha\mu\beta\nu}U^\mu U^\nu,$$

which is manifest in the worldline deviations of both theories, see [7], and together they form the gravitoelectromagnetic analogy based on tidal tensors [7,14]. These tensors obey the following equations, which will be useful in this work:

$$E_{[\alpha\beta]}=\frac{1}{2}F_{\alpha\beta;\gamma}U^\gamma;\quad\text{(a)}\quad\mathbb{E}_{[\alpha\beta]}=0.\quad\text{(b)} \quad (21)$$

Equation (21a) results from the space projection (relative to U^α) of the identity $\star F^{\alpha\beta}{}_{;\beta}=0$, and Eq. (21b) from the time-space projection of the identity $\star R^{\gamma\alpha}{}_{\gamma\beta}=0$. Contracting (21a) with the spatial 3-form $\epsilon_{\alpha\beta\gamma\delta}U^\delta$ yields Eq. (I.4a) of Table I. Again, for a (nonrotating and nonshearing) arbitrarily accelerated frame we have

$$\epsilon^{\beta\gamma}{}_{\alpha\delta}E_{\gamma\beta}U^\delta=\epsilon^{\beta\gamma}{}_{\alpha\delta}E_{\gamma;\beta}U^\delta\Rightarrow\epsilon^{ikj}E_{jk}=(\nabla\times\vec{E})^i. \quad (22)$$

Expressing also the second member of (21a) in terms of the fields E^α and B^α measured in this frame, we obtain, in 3-vector notation:

$$\nabla\times\vec{E}=-\frac{D\vec{B}}{d\tau}-\vec{a}\times\vec{E}, \quad (23)$$

which is a generalization of *Maxwell-Faraday equation* $\nabla\times\vec{E}=-\partial\vec{B}/\partial t$ for accelerated frames, cf. Eq. (20) of [17].

The fact that the gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$ is symmetric again means that there is no analogous gravitational induction effect, and this is a key observation for the applications in Secs. IV A and VI.

Equations (I.4) are the time projections of the tidal tensors with respect to *the observer U^α measuring them* (if U^α is the particle's CM 4-velocity, they are the time projection in its rest frame); they are zero in the gravitational case, as $\mathbb{H}_{\alpha\beta}$ is spatial relative to U^α, and nonzero in the electromagnetic case, which again is related to electromagnetic induction, as the right Eq. (I.4a) corresponds to the spatially projected Eq. (21a). This means that $F^\alpha_{\rm G}$ is spatial with respect to U^α, whereas $F^\alpha_{\rm EM}$ is not, which has important implications on the work done by the fields on the particle, as will be discussed in Sec. IV.

Finally, note that $F^\alpha_{\rm EM}=B^{\beta\alpha}\mu_\beta$ is the covariant generalization of the familiar textbook 3D expression $\vec{F}_{\rm EM}=\nabla(\vec{\mu}\cdot\vec{B})$, the latter being valid only in the *inertial* frame *momentarily* comoving with the particle.

C. Spin precession—exact analogy based on inertial GEM fields from the $1+3$ formalism

For *purely magnetic* dipoles ($d^\alpha=0$), Eq. (16) for the spin evolution under the Mathisson-Pirani condition simplifies to

$$\frac{D_F S^\mu}{d\tau}=\epsilon^\mu{}_{\alpha\beta\nu}U^\nu\mu^\alpha B^\beta, \quad (24)$$

or equivalently [cf. Eq. (17)]

$$\frac{DS^\mu}{d\tau} = S_\nu a^\nu U^\mu + \epsilon^\mu{}_{\alpha\beta\nu} U^\nu \mu^\alpha B^\beta, \tag{25}$$

where B^β is the magnetic field *as measured by the test particle*. The first term in (25) embodies the Thomas precession. The second term is a covariant form for the familiar torque $\tau = \vec{\mu} \times \vec{B}$ causing the Larmor precession of a magnetic dipole under a magnetic field.

Consider now an orthonormal frame $\mathbf{e}_{\hat{\alpha}}$ carried by an observer of 4-velocity U^α, such that $\mathbf{U} = \mathbf{e}_{\hat{0}}$, comoving with the test particle. In such frame, $S^{\hat{0}} = 0$ and $U^{\hat{\alpha}} = \delta^{\hat{\alpha}}_{\hat{0}}$, and Eq. (25) reduces to (see [14])

$$\frac{DS^{\hat{i}}}{d\tau} = (\vec{\mu} \times \vec{B})^{\hat{i}} \Leftrightarrow \frac{dS^{\hat{i}}}{d\tau} = (\vec{S} \times \vec{\Omega} + \vec{\mu} \times \vec{B})^{\hat{i}}, \tag{26}$$

where $\vec{\Omega}$ is angular velocity of rotation of the spatial axes $\mathbf{e}_{\hat{i}}$ relative to a tetrad Fermi-Walker transported along the center of mass worldline. If $B^\alpha = 0$, Eqs. (24)–(26) tell us that S^α undergoes Fermi-Walker transport, i.e., it follows the local "compass of inertia" [3,65] (the so-called gyroscope "precession", of frequency $-\vec{\Omega}$, is thus in fact just minus the rotation of the $\mathbf{e}_{\hat{i}}$ relative to a locally nonrotating frame, and therefore, *locally*, an artifact of the reference frame, manifest only in the ordinary derivative $d\vec{S}/d\tau$). Up until now $\vec{\Omega}$ is arbitrary; of special interest is, in asymptotically flat spacetimes, the case where the triads $\mathbf{e}_{\hat{i}}$ are locked to the so-called "frame of the distant stars." If the spacetime is stationary, such a frame is set up by choosing the congruence of static observers (cf. point 7 of Sec. I D), and demanding $\vec{\Omega}$ to match their vorticity $\vec{\omega}$ [defined in Eq. (70) below], $\vec{\Omega} = \vec{\omega}$. That is, demanding the triads $\mathbf{e}_{\hat{i}}$ to *corotate* [12,66] with the observers, relative to Fermi-Walker transport; we dub such frame *congruence adapted*. This ensures that the axes $\mathbf{e}_{\hat{i}}$ point to fixed neighboring observers, cf. Eq. (41) of [14]. Since the observer congruence is rigid and, *at infinity*, inertial, the axes $\mathbf{e}_{\hat{i}}$ locked to it are locked to the inertial frame at infinity (the rest frame of the "distant stars"), and Eq. (26) yields the precession of spinning particle with respect to the distant stars. For more details we refer to Secs. 3.1 and 3.3 of the companion paper [14].

Note the analogy between the two terms of the second Eq. (26); when the frame is congruence adapted, then $\vec{\Omega} = \vec{H}/2$, where $\vec{H}$ is the "gravitomagnetic" or Coriolis field felt in such frame, which plays in the *exact* geodesic equations [e.g. Eq. (58) of [14]] the same role as the magnetic field $\vec{B}$ in the electromagnetic Lorentz force. Moreover, the field equations for $\vec{H}$ exhibit striking similarities with the Maxwell equations for $\vec{B}$ in an accelerated, rotating frame, see Table 2 of [14]. In the linear regime, for stationary fields, they become similar to Maxwell's equations in a Lorentz frame, as is well known [1,3,11,20,67,68]. That tells us that analogous setups generate fields alike. A well-known realization is the similarity between the gravitomagnetic field produced by a spinning mass (as measured by the congruence of static observers), and the magnetic field produced by a spinning charge, e.g. Eqs. (6.1.9) and (6.1.25) of [3].

The analogy in Eq. (26) is valid for arbitrary fields, unlike the case of most gravitoelectromagnetic analogies[1] based on GEM *inertial fields* (not tidal tensors), which do not hold (in the sense of a one to one correspondence) when one considers time-dependent fields [14,69] (another exception is the hidden momentum analogy, presented in the next section).

Finally, note that, if we assume $\vec{\mu} = \sigma\vec{S}$, then the quantity $S^2 = S^\alpha S_\alpha = S^{\alpha\beta}S_{\alpha\beta}/2$ is a constant of the motion, which is immediately seen contracting (25) with S^μ.

D. Momentum of the spinning particle—"hidden momentum" and exact analogy based on inertial GEM fields from the 1 + 3 formalism

The momentum (4) of a spinning particle is not in general parallel to its center of mass 4-velocity U^α. In order to see that, let us re-write the spin evolution equation (12) as

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} + \tau^{\alpha\beta}, \tag{27}$$

where we denoted

$$\tau^{\alpha\beta} \equiv 2\mu^{\theta[\beta}F^{\alpha]}{}_\theta + 2d^{[\alpha}F^{\beta]}{}_\gamma U^\gamma, \tag{28}$$

which is sometimes called the dipole "torque" tensor [although only its spatial part contributes to the actual torque, cf. Eq. (106)]. Consider a generic spin condition $S^{\alpha\beta}u_\beta = 0$, where u^α denotes the 4-velocity of an arbitrary observer $\mathcal{O}(u)$ [as discussed in Sec. II A, this condition means that we take as reference worldline the center of mass as measured by $\mathcal{O}(u)$]. An expression for P^α can be obtained contracting (27) with u_β, leading to

$$P^\alpha = \frac{1}{\gamma(u,U)}\left(m(u)U^\alpha + S^{\alpha\beta}\frac{Du_\beta}{d\tau} + \tau^{\alpha\beta}u_\beta\right), \tag{29}$$

where $\gamma(U,u) \equiv -U^\alpha u_\alpha$, $m(u) \equiv -P^\alpha u_\alpha$, and in the second term we used $S^{\alpha\beta}u_\beta = 0$. We split P^α in its projections parallel and orthogonal to the CM 4-velocity U^α:

[1] In the framework of the GEM inertial fields, the force on the gyroscope [6,14] and the equation for the geodesics of (nonspinning) test particles (e.g. [6,12,14]) can be exactly described by equations analogous to the ones from electromagnetism, but *only* if the fields are stationary *and* the gyroscope it at rest with respect to a stationary observer (i.e., its *worldline* is tangent to a timelike Killing vector), or, in the case of the geodesic equation, if one considers a frame adapted to a rigid congruence of stationary observers. See Secs. 3.2 and 3.6 of [14].

$$P^\alpha = P^\alpha_{\rm kin} + P^\alpha_{\rm hid}; \qquad P^\alpha_{\rm kin} \equiv mU^\alpha, \qquad P^\alpha_{\rm hid} \equiv (h^U)^\alpha{}_\beta P^\beta. \tag{30}$$

We dub the parallel projection $P^\alpha_{\rm kin} = mU^\alpha$ "kinetic momentum" associated with the motion of the center of mass. This is the most familiar part of P^α, formally similar to the momentum of a monopole particle. The component $P^\alpha_{\rm hid}$ orthogonal to U^α is the so-called hidden momentum [20]. The reason for the latter denomination is seen taking the perspective of the particle's center of mass frame (i.e., the frame where $\vec{U} = 0$): the 3-momentum is in general not zero therein, $\vec{P} = \vec{P}_{\rm hid} \neq 0$; however, by definition, the particle's CM is at rest in that frame, and so this momentum must be somehow hidden in the spinning particle. The hidden momentum $P^\alpha_{\rm hid}$ consists of two parts of distinct origin: $P^\alpha_{\rm hid} = P^\alpha_{\rm hidI} + P^\alpha_{\rm hidEM}$, where

$$P^\alpha_{\rm hidI} \equiv \frac{1}{\gamma(u,U)} (h^U)^\alpha{}_\sigma S^{\sigma\beta} \frac{Du_\beta}{d\tau}; \tag{31}$$

$$P^\alpha_{\rm hidEM} \equiv \frac{1}{\gamma(u,U)} (h^U)^\alpha{}_\sigma \tau^{\sigma\beta} u_\beta. \tag{32}$$

The term $P^\alpha_{\rm hidI}$, which we dub "inertial" hidden momentum (the reason for such denomination will be clear below), is a gauge term that depends only on the spin supplementary condition, i.e., on the choice of the vector field u^α [the 4-velocity of the observers $\mathcal{O}(u)$ relative to which the CM is being computed]. This type of hidden momentum was first discussed in [20] (dubbed "kinematical" therein). It is in general not zero when $Du^\alpha/d\tau \neq 0$; this comes as a natural consequence of what we discussed in Sec. II A: the position of the CM of a spinning body depends on the vector u^α relative to which it is computed; if that vector varies along the reference worldline, it is clear that this is reflected in the velocity U^α of the CM (which in general will accelerate even without the action of any forces; see Figs. 1 and 2 of [33]). Since the momentum P^α remains the same, U^α will in general not be parallel to P^α, and so the centroid is not at rest in the frame where $P^i = 0$; conversely, the momentum is not zero in the CM frame (hidden momentum). If we take a field u^α such that $Du^\alpha/d\tau = 0$ (which was proposed in [39] as one of the possible spin supplementary conditions), i.e., if we take as reference worldline the center of mass as measured with respect to a field u^α that *is parallel transported along it*, then $P^\alpha_{\rm hidI}$ (as well as the motion effects induced by it, such as the bobbings studied in [20], or the helical motions discussed in [21]) is made to vanish.

The term $P^\alpha_{\rm hidEM}$ is what we dub "electromagnetic" hidden momentum; it is a still not well-known feature of relativistic electrodynamics (despite its discovery [70] dating back from the 1960's, and having since been discussed in number of papers, e.g. [20,70–77]). It is associated with the electromagnetic torque tensor $\tau^{\alpha\beta}$, and consists of a part which is gauge and arises, again, from the choice of centroid (vanishing for suitable choices, see [33] for details), plus a part that is not gauge, whose motion effects (such as the bobbings in electromagnetic systems studied in [20]) cannot in general be made to vanish by any choice of center of mass.

With the Mathisson-Pirani condition $S^{\alpha\beta}U_\beta = 0$, the hidden momentum in Eqs. (30)–(32) takes the suggestive form

$$P^\alpha_{\rm hidI} \equiv -\epsilon^\alpha{}_{\beta\gamma\delta} S^\beta a^\gamma U^\delta; \quad P^\alpha_{\rm hidEM} \equiv \epsilon^\alpha{}_{\beta\gamma\delta} \mu^\beta E^\gamma U^\delta, \tag{33}$$

and so the particle's total momentum, Eq (30), reads

$$P^\alpha = mU^\alpha - \epsilon^\alpha{}_{\beta\gamma\delta} S^\beta a^\gamma U^\delta + \epsilon^\alpha{}_{\beta\gamma\delta} \mu^\beta E^\gamma U^\delta, \tag{34}$$

where $E^\alpha = F^\alpha{}_\beta U^\beta$ is the electric field *as measured in the particle's CM frame* (of 4-velocity U^α), and a^α its acceleration. In the particle's CM frame (where $U^i = 0$), and in vector notation, the space part reads ($P^0_{\rm hid} = 0$)

$$\vec{P}_{\rm hid} = \vec{P} = -\vec{S} \times \vec{a} + \vec{\mu} \times \vec{E} = \vec{S} \times \vec{G} + \vec{\mu} \times \vec{E}. \tag{35}$$

The term $\vec{P}_{\rm hidEM} = \vec{\mu} \times \vec{E}$ is the most usual form for the electromagnetic hidden momentum in the literature, e.g. [32,71–74,78]. It equals *minus* the electromagnetic field momentum $\vec{P}_\times$ generated by a magnetic dipole when placed in an external electromagnetic field, which, in the particle's frame, reads (see [26])

$$\vec{P}_\times = \int \vec{E} \times \vec{B}_{\rm dipole} = -\vec{\mu} \times \vec{E} = -\vec{P}_{\rm hidEM}.$$

It should be noted however that $\vec{P}_{\rm hidEM}$ (unlike $\vec{P}_\times$) is *purely mechanical in nature* (*not* field momentum, even though it is ultimately originated by the action of the electromagnetic field), as explained in [71,72,78] using simple models. This hidden momentum implies that, in the presence of an electromagnetic field, the spatial momentum of a dipole whose center of mass is *at rest* is in general not zero. As explained in detail in [26], this actually plays a crucial role in the conservation laws: consider a magnetic dipole at rest in a stationary field; it is $\vec{P}_{\rm hidEM}$ which allows for the total spatial momentum $\vec{P}_{\rm tot} \equiv \vec{P}_{\rm matter} + \vec{P}_{\rm EM}$ to vanish, as required by the conservation equations $(T_{\rm tot})^{\alpha\beta}{}_{;\beta} = 0$ for a stationary configuration.

Equations (33)–(35) manifest an *exact* analogy: $G^\alpha = -a^\alpha$ is the gravitoelectric field (as defined in [6,12,14]) associated to the CM frame, which is a field of "inertial forces," and so $P^\alpha_{\rm hidI}$ is the inertial analogue of $P^\alpha_{\rm hidEM}$, with S^α and G^α in the roles of μ^α and E^α. The analogy above is useful to understand the famous helical solutions allowed by the condition $S^{\alpha\beta}U_\beta = 0$: we show in [21,75] that they are a phenomena which can be cast as analogous to the bobbings of a magnetic dipole in an external electric field (studied in Sec. III. B. 1 of [20]), in both cases the effect being driven not by a force but solely by an interchange between kinetic and hidden momentum.

E. Mass of the spinning particle

We take the scalar $m = -P^\alpha U_\alpha$ as "the proper mass"[2] [41] of the spinning particle. It is simply the time projection of P^α in the particle's CM frame, i.e., the particle's energy as measured in its center of mass rest frame. Whereas for a monopole particle m is a constant of the motion, for a spinning particle with dipole moments that is not the case in general. It follows from the definition of m that

$$\frac{dm}{d\tau} = -\frac{DP^\alpha}{d\tau}U_\alpha - P^\alpha a_\alpha = -\frac{D_F P^\alpha}{d\tau}U_\alpha; \qquad (36)$$

i.e., $dm/d\tau$ is the time projection, in the CM frame, of the Fermi-Walker derivative of the momentum. Noting that $P^\alpha a_\alpha = P^\alpha_{\mathrm{hidEM}} a_\alpha$, and using the orthogonality $P^\alpha_{\mathrm{hidEM}} U_\alpha = 0$, we can rewrite this equation as

$$\frac{dm}{d\tau} = -\left(\frac{DP^\alpha}{d\tau} - \frac{DP^\alpha_{\mathrm{hidEM}}}{d\tau}\right)U_\alpha. \qquad (37)$$

Thus $dm/d\tau$ equals also the time projection, in the CM frame, of the force $DP^\alpha/d\tau$ *subtracted* by the derivative of the electromagnetic hidden momentum $DP^\alpha_{\mathrm{hidEM}}/d\tau$. Let us see the meaning of the first term. Contracting (15) with U^α, and noting that $B^{\beta\alpha}U_\alpha = U_\gamma D\star F^{\beta\gamma}/d\tau$, we obtain

$$-\frac{DP^\alpha}{d\tau}U_\alpha = -\frac{D\star F^{\beta\gamma}}{d\tau}U_\gamma \mu_\beta + E_\beta \frac{Dd^\beta}{d\tau}, \qquad (38)$$

showing that the force has a time projection if the Maxwell tensor and/or the electric dipole vector vary along the CM worldline. Now, noting from Eqs. (34) and (2) that $P^\alpha a_\alpha = \star F^{\beta\gamma} a_\gamma \mu_\beta$, and putting Eqs. (36) and (38) together, we see that

$$\frac{dm}{d\tau} = -\mu_\gamma \frac{DB^\gamma}{d\tau} + E_\gamma \frac{Dd^\gamma}{d\tau}. \qquad (39)$$

Hence the mass of a particle possessing electric and magnetic dipole moments is not constant in general. The two contributions are substantially different: the mass variation due to the coupling of the field to the magnetic dipole occurs when the magnetic field varies along the particle's worldline; it may be interpreted as *essentially* the rate of work done on the magnetic dipole through Faraday's law of induction (Fig. 3 below), as we shall see in detail in Sec. IV A. The second term corresponds to the work done on the electric dipole by the electric field when the dipole vector varies, e.g., when it rotates; this term has nothing to do with induction, and is nonzero even for constant, uniform electric fields. The case of electric dipoles is discussed in detail in Appendix B 2.

We are interested mostly in *purely magnetic* dipoles, $d^\alpha = 0$; in this case, if we take $\mu^\alpha = \sigma S^\alpha$, with σ a constant, and, since from Eq. (25), $B^\mu DS_\mu/d\tau = 0$, we have [28,79–81]

$$\frac{dm}{d\tau} = -\sigma \frac{d}{d\tau}(S_\mu B^\mu) \qquad (40)$$

$$\Rightarrow m = m_0 - \sigma S_\mu B^\mu = m_0 - \sigma \vec{S}\cdot\vec{B}, \qquad (41)$$

where m_0 is a constant. Thus, if $\vec{\mu} = \sigma\vec{S}$, the mass m is the sum of a constant plus a variable part $-\vec{\mu}\cdot\vec{B}$, about which we would like to make some remarks. The expression $-\vec{\mu}\cdot\vec{B}$ is commonly dubbed in elementary textbooks "magnetic potential energy"; for this reason some authors [27,38,81] have interpreted this term as meaning that the potential energy contributes to the particle's mass. We argue (in agreement with the analysis in [82–85]), that the term $-\vec{\mu}\cdot\vec{B}$ is actually *internal* (not potential) energy of the test particle; in fact, we shall see (Sec. VI A 3) that, for a quasirigid body, it is essentially rotational kinetic energy, associated with the rotation of the body around its center of mass. What it actually does is to ensure that *the net work done by the magnetic field on a magnetic dipole is zero* (hence no potential energy can be assigned to it). Potential energy comes into play instead in the case of a monopole charged particle or of an electric dipole in an electric field; but in neither case does it contribute to the mass [m is a constant for a monopole particle, as well as for an electric dipole if d^α is parallel transported, cf. Eq. (39)]. These issues are discussed in detail in Sec. IV B 1 and Appendix B 4.

It is also important to understand that the varying mass m (and its variable part $-\vec{\mu}\cdot\vec{B}$) is real and physically measurable, not just a matter of definition [i.e. not an issue that goes away by redefining m_0 in Eq. (41) as the particle's mass], for m is the *inertial* mass of the particle. In order to see that, take for simplicity the case when $P^\alpha_{\mathrm{hid}} = 0$; we have

$$\frac{DP^\alpha}{d\tau} = ma^\alpha + \frac{dm}{d\tau}U^\alpha,$$

i.e., the projection of the force in the orthogonal space to U^α is ma^α (thus, in the CM frame, $D\vec{P}/d\tau = m\vec{a}$). This inertial

[2] This is the most natural definition of the body's mass if one uses the Mathisson-Pirani spin condition, since it is the quantity which is conserved when $F^{\alpha\beta} = 0$, cf. Eq. (39). If one uses the Tulczyjew-Dixon condition $S^{\alpha\beta}P_\beta = 0$ instead, then the conserved quantity is $M \equiv \sqrt{-P^\alpha P_\alpha}$ (not m), i.e., the particle's energy as measured in the zero 3-momentum frame (see e.g. [19]).

mass is measurable, for instance in collisions. The angular velocity of rotation of a spinning body (since, as mentioned above, in the case of a quasirigid body, $-\vec{\mu}\cdot\vec{B}$ is kinetic energy of rotation) is measurable as well.

In the purely gravitational case, by contrast, the proper mass is a constant ($m = m_0$); the implications for the work done by the fields on the particle are discussed in Secs. IV B and VI C.

F. Center of mass motion

Equations (I.1) of Table I yield the *force* on the spinning particle in the electromagnetic and gravitational case; *not* the acceleration $a^\alpha \equiv DU^\alpha/d\tau$, as $P^\alpha \neq m_0 U^\alpha$ in general. Setting $m \equiv m_0 + m'$ in Eq. (34), and noting, from decomposition (2), that $\epsilon^\alpha{}_{\beta\gamma\sigma}\mu^\beta E^\gamma U^\sigma = \star F^{\beta\alpha}\mu_\beta + \mu^\beta B_\beta U^\alpha$, we can write

$$P^\alpha = m_0 U^\alpha - \epsilon^\alpha{}_{\beta\gamma\delta} S^\beta a^\gamma U^\delta + (m' + \mu^\alpha B_\alpha) U^\alpha + \star F_\beta{}^\alpha \mu^\beta.$$

This is simplified if we consider purely magnetic dipoles ($d^\alpha = 0$), and assume $\mu^\alpha = \sigma S^\alpha$; in that case, cf. Eq. (41), $m' = -\mu^\alpha B_\alpha$, and the third term vanishes. Differentiating, using (18), and noting that, *if* $j^\alpha = 0$, $\star F_{\alpha\beta;\tau} U^\tau = 2B_{[\alpha\beta]}$, cf. Eq. (I.3a) of Table I, we have, in a region where the charge current density j^α is zero (most of the applications in this paper deal with vacuum),

$$m_0 a^\alpha = q F^{\alpha\beta} U_\beta + B^{\alpha\beta}\mu_\beta - \mathbb{H}^{\beta\alpha} S_\beta - \star F_\beta{}^\alpha \frac{D\mu^\beta}{d\tau} + \epsilon^\alpha{}_{\beta\gamma\delta} U^\delta \frac{D}{d\tau}(S^\beta a^\gamma). \quad (42)$$

Note the reversed indices in the second term as compared to the expression for the force (I.1a). This leads to a counterintuitive dynamical behavior, as we shall exemplify in Sec. III A.

III. DYNAMICAL MANIFESTATIONS OF THE SYMMETRIES OF THE MAGNETIC TIDAL TENSORS

According to Table I, both in the case of the electromagnetic force on a magnetic dipole and in the case of the gravitational force on a gyroscope, it is the magnetic/gravitomagnetic tidal tensor, *as seen by the test particle* of 4-velocity U^α, that determines the force exerted upon it. The explicit analogy in Table I is thus ideally suited to compare the two forces, because in this framework it amounts to comparing $B_{\alpha\beta}$ to $\mathbb{H}_{\alpha\beta}$. The most important differences between them are: (i) $B_{\alpha\beta}$ is linear in the electromagnetic potentials and vector fields, whereas $\mathbb{H}_{\alpha\beta}$ is not linear in the metric tensor, nor in the GEM "vector" fields (for a detailed discussion of this aspect, we refer to Secs. 3.5 and 6 of [14]); (ii) *in vacuum*, $\mathbb{H}_{[\alpha\beta]} = 0$ (symmetric tensor), whereas $B_{\alpha\beta}$ is generically nonsymmetric, $B_{[\alpha\beta]} \neq 0$, even in vacuum; (iii) time components: $\mathbb{H}_{\alpha\beta}$ is spatial with respect to U^α, whereas $B_{\alpha\beta}$ is not. The two latter differences, which are clear from Eqs. (I.3) and (I.4), are the ones in which we are most interested in the present work. In this section we start with the physical consequences of the symmetries, and in the next section we discuss the time projections.

Equation (I.3a) of Table I reads *in vacuum* ($j^\alpha = 0$)

$$B_{[\alpha\beta]} = \frac{1}{2}\star F_{\alpha\beta;\gamma} U^\gamma; \quad (43)$$

this tells us that when the field $F_{\alpha\beta}$ varies along the worldline of the observer U^α, that endows $B_{\alpha\beta}$ with an antisymmetric part, implying that $B_{\alpha\beta}$ itself is nonvanishing. Now, since, in the force (I.1a), $B_{\alpha\beta}$ is the magnetic tidal tensor *as measured by the particle* (i.e., U^α is the test particle's 4-velocity), this means that whenever the particle moves in a nonhomogeneous field, a force will be exerted on it (except possibly for special orientations of $\vec{\mu}$). In the inertial frame momentarily comoving with the particle, this can be interpreted as being due to the time varying (in this frame) electric field, which induces, via the law $\nabla\times\vec{B} = \partial\vec{E}/\partial t$, *a curl* in the magnetic field $\vec{B}$, and implies that the particle sees a nonvanishing magnetic tidal tensor, cf. Eqs. (19) and (20).

The gravitomagnetic counterpart $\mathbb{H}_{\alpha\beta}$, by contrast, is symmetric in vacuum, which means that no analogous induction phenomenon occurs in gravity. Indeed, even in nonhomogeneous fields, there can be velocity fields for which $\mathbb{H}_{\alpha\beta} = 0$, i.e., for which *gyroscopes feel no force* (regardless of the direction of their spin $\vec{S}$). We know that from the curvature invariants, which we now briefly discuss.

In vacuum the Riemann tensor becomes the Weyl tensor (ten independent components), which can be irreducibly decomposed (see e.g. [17]) with respect to a unit timelike 4-vector u^α into two spatial tensors, the gravitoelectric $(\mathbb{E}^u)_{\alpha\beta} \equiv R_{\alpha\gamma\beta\delta} u^\gamma u^\delta$ and gravitomagnetic $(\mathbb{H}^u)_{\alpha\beta} \equiv \star R_{\alpha\gamma\beta\delta} u^\gamma u^\delta$ tidal tensors measured by u^α:

$$R_{\alpha\beta}{}^{\gamma\delta} = 4\{2u_{[\alpha}u^{[\gamma} + g_{[\alpha}{}^{[\gamma}\}(\mathbb{E}^u)_{\beta]}{}^{\delta]} + 2\{\epsilon_{\alpha\beta\mu\nu}(\mathbb{H}^u)^{\mu[\delta}u^{\gamma]}u^\nu + \epsilon^{\gamma\delta\mu\nu}(\mathbb{H}^u)_{\mu[\beta}u_{\alpha]}u_\nu\}. \quad (44)$$

The tensors $\mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ are both symmetric and traceless (in vacuum), possessing five independent components each, thus encoding the ten independent components of $R_{\alpha\beta\gamma\delta}$. Again in vacuum, one can construct the two quadratic scalar invariants (e.g. [15,16,86]),

$$\mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{1}{8}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \equiv \frac{1}{8}\mathbf{R}\cdot\mathbf{R}, \quad (45)$$

$$\mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{1}{16}R_{\alpha\beta\gamma\delta}\star R^{\alpha\beta\gamma\delta} \equiv \frac{1}{16}\star\mathbf{R}\cdot\mathbf{R}. \quad (46)$$

Note that, in spite of the dependence of $(\mathbb{E}^u)_{\alpha\beta}$ and $(\mathbb{H}^u)_{\alpha\beta}$ on the observer 4-velocity u^α, the combinations (45) and (46) are independent of u^α (for this reason we dropped the u superscript therein).

There is an analogy (a purely *formal* one, cf. [14]) with the decomposition of the Maxwell tensor in electric and magnetic parts [15–17], and the invariants they form, which is illuminating for the problem at hand. With respect to a unit timelike 4-vector u^α, the Maxwell tensor (six independent components) splits irreducibly into the two spatial vectors (three independent components each) $(E^u)^\alpha \equiv F^{\alpha\beta}u_\beta$ and $(B^u)^\alpha \equiv \star F^{\alpha\beta}u_\beta$, as can be seen from the explicit decomposition (1), analogous to (44). The fields $(E^u)^\alpha$ and $(B^u)^\alpha$ are covariant definitions for, respectively, the electric and magnetic fields as measured by an observer of 4-velocity u^α. In spite of their u^α dependence, combining them one can construct the two quadratic scalar invariants (e.g. [15,16,64]),

$$E^\alpha E_\alpha - B^\alpha B_\alpha = -\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} \equiv -\frac{1}{2}\mathbf{F}\cdot\mathbf{F}, \qquad (47)$$

$$E^\alpha B_\alpha = -\frac{1}{4}F_{\alpha\beta}\star F^{\alpha\beta} \equiv -\frac{1}{4}\star\mathbf{F}\cdot\mathbf{F}, \qquad (48)$$

[where again we dropped the u superscripts in $(E^u)^\alpha$ and $(B^u)^\alpha$] formally similar to the quadratic invariants (45) and (46). These are actually the only two[3] independent scalar invariants one can construct from $F_{\alpha\beta}$. They have the following interpretation [18,64,87]: (i) if $E^\alpha B_\alpha \neq 0$ then the electric E^α and magnetic B^α fields are both nonvanishing for all observers; (ii) if $E^\alpha E_\alpha - B^\alpha B_\alpha > 0$ (< 0) and $E^\alpha B_\alpha = 0$, then there are observers for which B^α (E^α) is zero.

In the gravitational case, it turns out (cf. [18,86,88]) that, for Petrov type D spacetimes (case of the examples studied below), and in vacuum, one obtains formally equivalent statements to (i) and (ii) above, replacing $\mathbf{F}$ by $\mathbf{R}$. That is: (i) $\star\mathbf{R}\cdot\mathbf{R} \neq 0 \Rightarrow \mathbb{E}_{\alpha\gamma}$ and $\mathbb{H}_{\alpha\gamma}$ are both nonvanishing for all observers; (ii) $\star\mathbf{R}\cdot\mathbf{R} = 0$, $\mathbf{R}\cdot\mathbf{R} > 0(< 0) \Rightarrow$ there are observers for which $\mathbb{H}_{\alpha\gamma}$ $(\mathbb{E}_{\alpha\gamma})$ vanishes. When, *at a given point*, observers exist for which $\mathbb{H}_{\alpha\gamma} = 0$ $(\mathbb{E}_{\alpha\gamma} = 0)$, the curvature tensor is dubbed "purely electric" ("purely magnetic") *at that point*, see e.g. [86,88–90]. Further details and comments on this classification (for general spacetimes), will be given in [18]. The velocity fields for which $\mathbb{H}_{\alpha\gamma} = 0$ will be exemplified below in gravitational fields—Schwarzschild and Kerr spacetimes—with a clear electromagnetic analogue—a static point charge and a spinning charge, respectively—and we shall see that indeed $\mathbb{H}_{\alpha\gamma}$, and therefore $F^\alpha_{\rm G}$, may vanish for moving spinning particles, which contrasts with the electromagnetic analogue.

[3] This contrasts with the gravitational case, where (45) and (46) are not the only algebraically independent invariants one can construct from $R_{\alpha\beta\gamma\delta}$. In vacuum (the simplest case), they reduce to four, two cubic invariants existing in addition to the quadratic invariants (45) and (46), see e.g. [16,86].

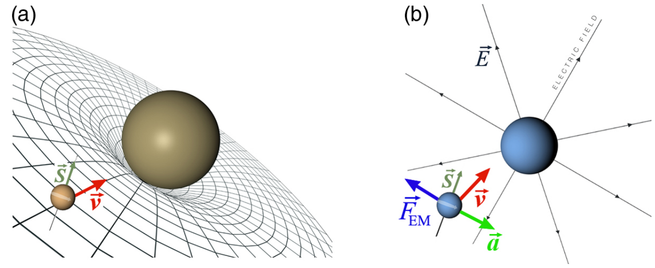

FIG. 1. An illustration of the physical consequences of the different symmetries of the tidal tensors. A gyroscope dropped from rest in Schwarzschild spacetime will move radially along a geodesic towards the source, with no force exerted on it. A magnetic dipole in (initially) radial motion in a Coulomb field, by contrast, feels a force. Due to the hidden momentum, the force is approximately *opposite* to the acceleration.

A. Radial motion in Schwarzschild spacetime

The Schwarzschild spacetime is a Petrov type D solution whose quadratic curvature invariants read

$$\mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \frac{6M^2}{r^6}, \qquad \mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = 0. \qquad (49)$$

In accordance with the classification above, this means that this is a *purely electric* spacetime, i.e., everywhere there are observers for which $\mathbb{H}_{\alpha\beta} = 0$. Let us find such observers. The nonzero components of the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\nu}U^\mu U^\nu$ seen by an observer of arbitrary 4-velocity $U^\alpha = (U^t, U^r, U^\theta, U^\phi)$, are, in Schwarzschild coordinates, givenby $(\alpha \equiv 3M\sin\theta/r)$

$$\mathbb{H}_{r\theta} = \alpha U^\phi U^t; \qquad \mathbb{H}_{r\phi} = \alpha U^t U^\theta;$$
$$\mathbb{H}_{\theta t} = -\alpha U^\phi U^r; \qquad \mathbb{H}_{\phi t} = \alpha U^r U^\theta. \qquad (50)$$

The condition $\mathbb{H}_{\alpha\beta} = 0$ implies $U^\phi = U^\theta = 0$, whilst leaving U^r arbitrary. Thus, observers at rest, or in radial motion, measure a vanishing $\mathbb{H}_{\alpha\beta}$. Since, according to Eqs. (I.1) of Table I, it is the gravitomagnetic tidal tensor, *as seen by the test particle*, that determines the force on it, this means that no force is exerted on a gyroscope at rest or in radial motion:

$$F^\alpha_{\rm G} = -\mathbb{H}^{\beta\alpha}S_\beta = 0,$$

i.e., it moves along a geodesic (it is the trivial solution of the equations of motion with the Mathisson-Pirani condition, see Appendix C 1), regardless of its spin $\vec{S}$. For instance, a gyroscope dropped from rest will fall towards the singularity along a radial geodesic just like a monopole particle, see Fig. 1(a).

This is not possible in the electromagnetic analogue, due to the symmetries of $B_{\alpha\beta}$. Consider a magnetic dipole, of 4-velocity U^α, in the field of a static point charge Q; the force exerted on it is $F^\alpha_{\rm EM} = B^{\beta\alpha}\mu_\beta$, cf. Table I, where $B_{\alpha\beta} = \star F_{\alpha\mu;\beta}U^\mu$ is the magnetic tidal tensor as seen by the particle. The components of $B_{\alpha\beta}$, for a generic U^α, are ($\alpha \equiv 3Q\sin\theta/r$)

$$B_{r\theta} = \alpha U^\phi; \qquad B_{\theta r} = \alpha 2U^\phi; \qquad B_{r\phi} = -\alpha U^\theta;$$
$$B_{\phi r} = -2\alpha U^\theta; \qquad B_{\theta\phi} = \alpha U^r; \qquad B_{\phi\theta} = -\alpha U^r. \tag{51}$$

The static observers $U^i = 0$ are the only ones measuring $B_{\alpha\beta} = 0$, as expected from Eq. (43), since the field is inhomogeneous and therefore not covariantly constant for a moving observer (i.e., $\star F_{\alpha\mu;\beta}U^\mu \neq 0$ if $U^i \neq 0$). For a radial velocity $U^\alpha = (U^t, U^r, 0, 0)$, the magnetic tidal tensor reduces to its antisymmetric part, $B_{\alpha\beta} = B_{[\alpha\beta]}$, with non-vanishing components $B_{\theta\phi} = -B_{\phi\theta} = \alpha U^r$. This means that (except for the special case $\vec{v}\|\vec{\mu}$) a force[4] will be exerted on a magnetic dipole in (initially) radial motion:

$$F^0_{\rm EM} = 0; \qquad F^i_{\rm EM} = B^{[\alpha i]}\mu_\alpha = \frac{\gamma Q}{r^3}(\vec{v}\times\vec{\mu})^i, \tag{52}$$

where $\vec{v} = \vec{U}/\gamma$ and γ is the Lorentz factor. This force comes *entirely* from the antisymmetric part of $B_{\alpha\beta}$; it is then natural, given the symmetry of $\mathbb{H}_{\alpha\beta}$ in vacuum, that it has no gravitational counterpart.

It is however important to note that, due the hidden momentum that the spinning particle possesses, the relation between this force and the particle's center of mass acceleration is not straightforward. This is manifest in Eq. (42); for flat spacetime, and a particle whose only electromagnetic moment is μ^α ($q = 0$), it reads

$$m_0 a^\alpha = B^{\alpha\beta}\mu_\beta + \epsilon^\alpha{}_{\beta\gamma\delta}\frac{D}{d\tau}U^\delta(S^\beta a^\gamma) - \star F_\beta{}^\alpha\frac{D\mu^\beta}{d\tau}.$$

The last term vanishes if one assumes $\mu^\alpha = \sigma S^\alpha$, since: $B^\alpha(U) = 0$ for radial motion; thus, from Eq. (25), $D\mu_\mu/d\tau = \sigma S_\nu a^\nu U_\mu$ and $\star F_\beta{}^\alpha D\mu^\beta/d\tau = -\sigma B(U)^\alpha S_\nu a^\nu = 0$. The second term can also be taken to a good approximation as being zero [which, as explained in Sec. V, to an accuracy of order $\mathcal{O}(S^2)$, amounts to saying that we pick the "nonhelical" solution allowed by the Mathisson-Pirani condition]. Therefore, since, in this application, $B_{(\alpha\beta)} = 0$, we are led to the conclusion that $m_0 a^\alpha \approx B^{\alpha\beta}\mu_\beta = -B^{\beta\alpha}\mu_\beta = -F^\alpha_{\rm EM}$ [see Fig. 1(b)]! This clearly shows how careful one must be with the notion of force [understood as $F^\alpha = DP^\alpha/d\tau$, with P^α defined in the usual way by Eq. (4)], because it can significantly differ from ma^α when the particle has hidden momentum.

Finally, it is worth mentioning that the vanishing of $\mathbb{H}_{\alpha\beta}$ for certain velocity fields in the Schwarzschild spacetime is analogous instead to the vanishing of the magnetic field B^α (not the tidal tensor $B_{\alpha\beta}$) in a Coulomb field. The quadratic invariants of $F^{\alpha\beta}$ have a structure formally analogous to the curvature invariants (49): $E^2 - B^2 = Q^2/r^4$, $E^\alpha B_\alpha = 0$, telling us that there are everywhere observers for which $B^\alpha = 0$. For an arbitrary U^α, the nonvanishing components of B^α are

$$B^\theta = -\frac{QU^\phi}{r^2}\sin\theta, \qquad B^\phi = \frac{QU^\theta}{r^2}\csc\theta;$$

therefore, observers at rest or in purely radial motion measure $B^\alpha = 0$, just like with the case of $\mathbb{H}_{\alpha\beta}$ in the Schwarzschild spacetime. One should however bear in mind that this one is a *purely formal* analogy, as the parallelism drawn is between gravitational tidal tensors and electromagnetic fields. The physical effects are very different: the vanishing of $\mathbb{H}_{\alpha\beta}$ for radial velocities means that a gyroscope feels no force, whereas the vanishing of B^α does *not* mean that dipoles moving radially feel no force (which they do, as discussed above), but instead that they do not undergo Larmor precession [$D\vec{S}/d\tau = 0$ in the comoving frame, cf. Eq. (25)].

B. Equatorial motion in Kerr and Kerr-dS spacetimes

In this section we compare the motions of gyroscopes in the Kerr and Kerr–de Sitter spacetimes to magnetic dipoles in the field of a spinning charge. It is shown that in the equatorial plane there are observers for which the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$ vanishes (i.e., gyroscopes moving with such velocities feel no force), and that consequently circular geodesics for gyroscopes even exist in Kerr-dS (independently of the particle's spin). This contrasts with the electromagnetic system, where observers for which $B_{\alpha\beta} = 0$ do not exist at all (consequence of the symmetries of $B_{\alpha\beta}$, i.e., the laws of electromagnetic induction, as explained above), and therefore (except for special orientations of $\vec{\mu}$) a force is always exerted on a magnetic dipole, regardless of its motion.

The vanishing of $\mathbb{H}_{\alpha\beta}$ is instead analogous to the vanishing of the magnetic *field* B^α, which likewise occurs in the equatorial plane, for asymptotically similar velocity fields. That gives useful insight into the gravitational

[4]The force (52) may seem at first sight to contradict what one might naively expect from the textbook expression $F^i_{\rm EM} = -\nabla^i(\vec{B}\cdot\vec{\mu}) \equiv B^{j;i}\mu_j$, which holds in the particle's momentarily comoving inertial frame, because the radially moving dipole indeed sees a vanishing magnetic field $\vec{B}$. However *its curl is nonzero* [implying $B_{i;j} = B_{ij} \neq 0$, cf. Eq. (22)], which, taking the perspective of such frame, is induced by the time-varying electric field, by virtue of vacuum equation $\nabla\times\vec{B} = \partial\vec{E}/\partial t$.

problem; for this reason we shall start by the simpler electromagnetic case.

1. A magnetic dipole in the field of a spinning charge

Velocity field for which $B^\alpha = 0$.—We start by the electromagnetic system, which will serve as a guide for the gravitational case. The electromagnetic field produced by a spinning charge (magnetic moment $\vec{\mu}_s$) is described by the 4-potential $A^\alpha = (\phi, \vec{A})$:

$$\phi = \frac{Q}{r}, \qquad \vec{A} = \frac{\vec{\mu}_s \times \vec{r}}{r^3} = \frac{\mu_s}{r^3}\vec{e}_\phi. \tag{53}$$

The invariant structure for this electromagnetic field is

$$\begin{cases} \vec{E}^2 - \vec{B}^2 = \frac{Q^2}{r^4} - \frac{\mu_s^2(5+3\cos 2\theta)}{2r^6} > 0, \\ \vec{E}\cdot\vec{B} = \frac{2\mu_s Q\cos\theta}{r^5} (= 0 \,\text{in the equatorial plane}), \end{cases} \tag{54}$$

the first inequality always holding assuming the classical gyromagnetic ratio $\mu_s/J = Q/2M$ (corresponding to a source in which the charge and mass are identically distributed). Expressions (54) tell us that in the equatorial plane $\theta = \pi/2$ there are observers that measure B^α to be zero (since $\vec{E}\cdot\vec{B} = 0$ and $\vec{E}^2 - \vec{B}^2 > 0$ therein). It is straightforward to obtain the 4-velocity of such observers. The magnetic field $B^\alpha = \star F^{\alpha\beta}U_\beta$ seen by an arbitrary observer of 4-velocity $U^\alpha = (U^t, U^r, U^\theta, U^\phi)$ is given by

$$B^r = \frac{2\mu_s\cos\theta}{r^3}U^t, \qquad B^\theta = \left(\frac{\mu_s U^t}{r^4} - \frac{U^\phi Q}{r^2}\right)\sin\theta,$$
$$B^\phi = \frac{QU^\theta}{r^2\sin\theta}, \qquad B^t = \frac{\mu_s}{r^3}(2U^r\cos\theta + rU^\theta\sin\theta).$$

Thus, the condition $B^r = 0$ implies $\theta = \pi/2$ (i.e., equatorial plane, as expected); in the equatorial plane, $B^t = 0$ implies $U^\theta = 0$, and $B^\theta = 0$ implies

$$\frac{d\phi}{dt} = \frac{U^\phi}{U^t} = \frac{\mu_s}{Qr^2} = \frac{J}{2Mr^2} \equiv \omega_{(\mathbf{B}=0)}, \tag{55}$$

where *in the third* equality again we assumed $\mu_s/J = Q/2M$. Therefore, observers with angular velocity (55) measure a vanishing magnetic field in the equatorial plane. No restriction is imposed on the radial component of the velocity, apart from the normalization condition $U^\alpha U_\alpha = -1$. The velocity field corresponding to the case $U^r = 0$ is plotted in Fig. 2(a). The vanishing of B^α for these observers comes from an exact cancellation between the magnetic field generated by the relative translation of the source and the field produced by its rotation. It means that a magnetic dipole possessing a velocity of the form (55) does

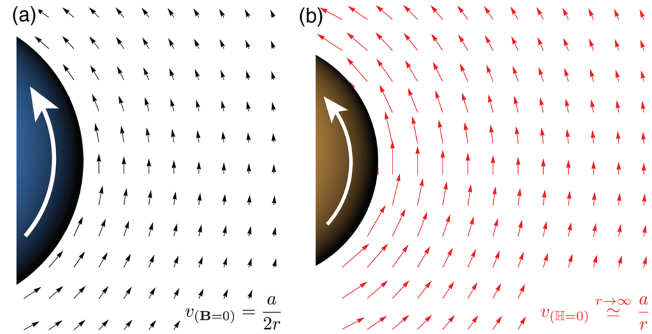

FIG. 2. (a) Velocity field $\vec{v}_{(\mathbf{B}=0)}$, which makes the magnetic *field* B^α vanish in the equatorial plane of a spinning charge; magnetic dipoles with such velocities do not undergo Larmor *precession*. (b) Velocity field $\vec{v}_{(\mathbb{H}=0)}$ for which the gravitomagnetic *tidal tensor* $\mathbb{H}_{\alpha\beta}$ vanishes in the equatorial plane of Kerr spacetime; gyroscopes moving with such velocities feel *no force*, $F^\alpha_G = 0$. If $\Lambda > 0$ (Kerr-dS spacetime), circular geodesics for gyroscopes even exist (Sec. III B 3). Asymptotically, $\vec{v}_{(\mathbb{H}=0)}$ and $\vec{v}_{(\mathbf{B}=0)}$ match up to a factor of 2. The velocity $\vec{v}_{(\mathbb{H}=0)}$ however has no *physical* electromagnetic analogue: due to the laws of electromagnetic induction, for a moving dipole $B_{[\alpha\beta]} \neq 0 \Rightarrow B_{\alpha\beta} \neq 0$ *always*, generically implying $F^\alpha_{EM} \neq 0$.

not undergo Larmor precession, since the second term of Eq. (25) vanishes.

In [26] we investigate the corresponding gravitational problem, i.e., if there are boost velocities for which gyroscopes in the Kerr spacetime do not precess.

$B_{\alpha\beta}$ never vanishes.—The force (I.1a) exerted on the dipole, however, does *not* vanish, as it is only the magnetic field B^α, not the tidal tensor $B_{\alpha\beta}$, that vanishes for the velocity fields of the type (55). As measured by a generic observer U^α, $B_{\alpha\beta}$ has the following components in the equatorial plane:

$$B_{r\theta} = \alpha(r^2QU^\phi - 3\mu_s U^t); \quad B_{\theta r} = \alpha(2r^2QU^\phi - 3\mu_s U^t);$$
$$B_{r\phi} = -\alpha Qr^2U^\theta; \qquad B_{\phi r} = -2\alpha Qr^2U^\theta;$$
$$B_{r\phi} = -\alpha Qr^2U^\theta; \qquad B_{\phi r} = -2\alpha Qr^2U^\theta;$$
$$B_{tr} = 3\alpha\mu_s U^\theta; \qquad B_{t\theta} = 3\alpha\mu_s U^r, \tag{56}$$

with $\alpha \equiv 1/r^3$. Thus we see that in order to make $B_{(\alpha\beta)}$ vanish, we must have $U^\theta = U^r = 0$ and

$$\frac{d\phi}{dt} = \frac{U^\phi}{U^t} = 2\frac{\mu_s}{Qr^2} = \frac{J}{Mr^2} \equiv \omega_{(B_{(\alpha\beta)}=0)} \tag{57}$$

[differs from a factor of 2 from the angular velocity (55) which makes B^α vanish; the second equality again assumes $\mu_s/J = Q/2M$]. However, $B_{[\alpha\beta]}$ only vanishes if $\vec{v} = 0$; hence it is not possible to find any observer for which $B_{\alpha\beta} = B_{(\alpha\beta)} + B_{[\alpha\beta]} = 0$. Again, the fact that $B_{[\alpha\beta]}$ cannot vanish for a moving observer is a direct consequence of

Maxwell's equations, or the laws of electromagnetic induction: a dipole moving relative to the spinning charge always sees a varying electromagnetic field; that endows $B_{\alpha\beta}$ with an antisymmetric part, by virtue (from the point of view of a momentarily comoving inertial frame) of the vacuum equation $\nabla\times\vec{B}=\partial E/\partial t$, or, covariantly, by Eq. (I.3a). Note that this is true even if one considers a dipole in a circular equatorial trajectory around the central source: $D\star F_{\alpha\beta}/d\tau=2B_{[\alpha\beta]}\neq 0$ along such worldline, which is due to the variation of the electric field along the curve (it is constant in magnitude, but varying in direction).

2. A gyroscope in Kerr spacetime

Velocity for which $\mathbb{H}_{\alpha\beta}=0$.—From what we learned in the electromagnetic problem, we expect the existence of observers for which $\mathbb{H}_{\alpha\beta}$ vanishes, based on two observations. First, we have seen that in the equatorial plane of the electromagnetic system there are velocities (57) for which the magnetic tidal tensor reduces to its antisymmetric part $B_{\alpha\beta}=B_{[\alpha\beta]}$; since the gravitomagnetic tidal tensor is symmetric in vacuum: $\mathbb{H}_{\alpha\beta}=\mathbb{H}_{(\alpha\beta)}$, it is natural to expect, in the spirit of the analogy, that $\mathbb{H}_{\alpha\beta}=0$ in the corresponding gravitational setup. Second, there is a close analogy between the invariants of the two systems. The Kerr spacetime is of Petrov type D, hence a classification for the curvature tensor based on quadratic invariants formally analogous to the one for $F_{\alpha\beta}$ applies, as discussed in Sec. III. The invariants (45) and (46) read (e.g. [91])

$$\begin{aligned}\mathbf{R}\cdot\mathbf{R}&=\frac{48M^2}{\Sigma^6}(a^4\cos^4\theta-14a^2r^2\cos^2\theta+r^4)\\&\quad\cdot(r^2-a^2\cos^2\theta)\\ \star\mathbf{R}\cdot\mathbf{R}&=\frac{96M^2ra}{\Sigma^6}(a^2\cos^2\theta-3r^2)(3a^2\cos^2\theta-r^2)\cos\theta,\end{aligned}\tag{58}$$

where $\Sigma\equiv r^2+a^2\cos^2\theta$. For large r we have the structure:

$$\begin{cases}\mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma}-\mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma}\overset{r\to\infty}{\simeq}\frac{6M^2}{r^6}>0,\\ \mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma}\overset{r\to\infty}{\simeq}\frac{18JM\cos\theta}{r^7}(=0\,\text{in the equatorial plane}),\end{cases}$$

formally analogous to its electromagnetic counterpart (54). Note in particular that the result $\mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma}=0$ for the equatorial plane ($\theta=\pi/2$) is *exact*, cf. Eq. (58). Since $\mathbf{R}\cdot\mathbf{R}>0$ therein, this means that *in the equatorial plane there are observers for which $\mathbb{H}_{\alpha\beta}$ vanishes,* in analogy with the vanishing of B^α in the equatorial plane of the field of a spinning charge. It is straightforward to determine the 4-velocity of such observers. In the equatorial plane, the nonzero components of the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}\equiv\star R_{\alpha\mu\beta\nu}U^\mu U^\nu$ seen by an arbitrary observer of 4-velocity $U^\alpha=(U^t,U^r,U^\theta,U^\phi)$, are given (exactly) by

$$\begin{aligned}\mathbb{H}_{r\theta}&=\alpha[(2a^2+r^2)U^\phi U^t-a(a^2+r^2)(U^\phi)^2-a(U^t)^2],\\ \mathbb{H}_{r\phi}&=\alpha(a^2+r^2)(aU^\phi-U^t)U^\theta,\\ \mathbb{H}_{rt}&=\alpha a(aU^\phi-U^t)U^\theta,\\ \mathbb{H}_{\theta\phi}&=\alpha a[(a^2+r^2)U^\phi-aU^t]U^r,\\ \mathbb{H}_{\theta t}&=-\alpha[(a^2+r^2)U^\phi-aU^t]U^r,\\ \mathbb{H}_{\phi\phi}&=-2\alpha aU^rU^\theta=\mathbb{H}_{tt},\\ \mathbb{H}_{\phi t}&=\alpha(2a^2+r^2)U^rU^\theta,\end{aligned}\tag{59}$$

where $\alpha\equiv 3M/r^3$. It is easily seen that in order to make all components vanish we must have $U^\theta=0$ (i.e. the observer must move in the equatorial plane, as expected and in analogy with the electromagnetic case above) and

$$\frac{d\phi}{dt}=\frac{U^\phi}{U^t}=\frac{a}{a^2+r^2}\equiv\omega_{(\mathbb{H}=0)}.\tag{60}$$

Thus, observers with angular velocity $\omega=\omega_{(\mathbb{H}=0)}$ measure a vanishing gravitomagnetic tidal tensor in the equatorial plane. Again, no restriction is imposed on U^r, apart from the normalization condition $U^\alpha U_\alpha=-1$. The velocity field corresponding to the case $U^r=0$ is plotted in Fig. 2(b). It is interesting to note that, asymptotically, $\omega_{(\mathbb{H}=0)}$ matches the angular velocity (57) for which the *symmetric part* of the magnetic tidal tensor $B_{\alpha\beta}$ vanishes in the electromagnetic analogue [and, up to a factor of 2, the angular velocity (55) for which B^α vanishes].

As discussed above, $\omega_{(\mathbb{H}=0)}$ has no electromagnetic counterpart; the magnetic tidal tensor $B_{\alpha\beta}$ can never vanish for a moving observer, due to Eq. (43), i.e., the laws of electromagnetic induction. We have thus here another illustration of the physical consequences of the different symmetries of $\mathbb{H}_{\alpha\beta}$ as compared to $B_{\alpha\beta}$, signaling the absence of electromagneticlike induction effects in the physical gravitational forces. Note that these differences are manifest even in the weak field and slow motion regime, since taking the field to be weak (either by going far away from the source, or by taking a to be small) only amounts to making the velocity for which $F^\alpha_{\rm G}$ vanishes smaller, since $|v|\approx a/r$. That illustrates how misleading the usual treatments in the literature on gravitoelectromagnetism in the framework of the linearized theory (e.g. [2,3]) can be, naively casting the force on a gyroscope as an expression of the type $\vec{F}_{\rm G}=K\nabla(\vec{S}\cdot\vec{H})$ (similar to the electromagnetic force on a magnetic dipole). This regime is studied in detail in Sec. V.

Finally, it is interesting to note that the angular velocity (60) appeared before in apparently unrelated contexts; it coincides with the angular velocity of the "Carter canonical observers" (e.g. [92]), which are observers that measure the photons of the principal null congruences (see page 902 of [23]) to be in purely radial motion. It also appeared in a recent paper [93], Eq. (30) therein, where it is shown that the Kerr metric can be obtained by a rescaling of an orthonormal tetrad field in Minkowski space, constructed from spheroidal coordinates in differential rotation, each spheroidal shell $r = \text{constant}$ rotating rigidly with an angular velocity that is precisely $\omega_{(\mathbb{H}=0)}$.

No circular geodesics for spinning material particles in Kerr spacetime.—the vanishing of $F^\alpha_{\rm G}$ for gyroscopes moving with angular velocity (60) makes one wonder if a spinning particle can move along circular geodesics around a Kerr black hole, which we shall now check. Equation (60) corresponds to *prograde motion*; the angular velocity of prograde circular geodesics reads (e.g. [94])

$$\omega_{\rm geo} \equiv \frac{U^\phi_{\rm geo}}{U^t_{\rm geo}} = \frac{1}{a + \sqrt{\frac{r^3}{M}}}. \tag{61}$$

Equating this expression to (60), we obtain $r = a^2/M$; this solution, however, lies inside the horizon: since $r_+ = M + \sqrt{M^2 - a^2}$, the condition $r \geq r_+$ implies

$$\frac{a^2}{M} \geq M + M\sqrt{1 - a^2/M^2} \Leftrightarrow 1 - A^2 \leq -\sqrt{1 - A^2},$$

where we defined the dimensionless parameter $A \equiv a/M$. Note that $A = 1$ is the extreme Kerr case, and $A > 1$ corresponds to a naked singularity; therefore (excluding the naked singularity scenario) the circular orbit would exist only in the extreme case, it would be precisely at the horizon, and thus it would be a null geodesic. Otherwise, no circular geodesics exist with angular velocity (60), and so $\mathbb{H}_{\alpha\beta} \neq 0$ along any timelike circular geodesic.

The only possibility of having $F^\alpha_{\rm G} = \mathbb{H}^{\beta\alpha} S_\beta = 0$ would then be if S^α was an eigenvector of $\mathbb{H}_\beta{}^\alpha$ corresponding to a zero eigenvalue; that does not lead to circular geodesics however, because S^α cannot remain an eigenvector. For $U^\alpha = (U^t, 0, 0, U^\phi)$, the only eigenvectors of $\mathbb{H}_\beta{}^\alpha$ with zero eigenvalue are U^α and $\mathbf{e}_\phi$; S^α (orthogonal to U^α) cannot remain in the eigenspace spanned by U^α and $\mathbf{e}_\phi$, by virtue of the transport law (25), which can be seen as follows. Consider a frame rigidly rotating with an angular velocity $\omega_{\rm geo}$ corresponding to a geodesic at some value of r (the associated coordinates are obtained from the Boyer-Lindquist coordinates by the transformation $t' = t$, $r' = r$, $\theta' = \theta$, $\phi' = \phi - \omega_{\rm geo} t$), and the orthonormal basis $\mathbf{e}_{\hat{\alpha}'}$ tied to it, such that $\mathbf{e}_{\hat{\gamma}'} = \mathbf{U}$, and $\mathbf{e}_{\hat{r}'}$, $\mathbf{e}_{\hat{\theta}'}$ $\mathbf{e}_{\hat{\phi}'}$ follow from normalizing $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $(\mathbf{h}^U)\cdot\mathbf{e}_\phi$, respectively. [Here $(\mathbf{h}^U)$ is the projector orthogonal to U^α, cf. Sec. I D.] In such a frame the gyroscope's CM is at rest, therefore Eq. (25) applies, $dS^{\hat{i}}/d\tau = (\vec{S} \times \vec{\Omega})^{\hat{i}}$; moreover the gravitomagnetic field $\vec{H} = 2\vec{\Omega}$ takes the very simple form $\vec{H} = -2\sqrt{M/r^3}\vec{e}_{\hat{\theta}'}$, cf. Eq. (41) of [94]. Hence, for an initial $\vec{S} = S^{\hat{\phi}'}\vec{e}_{\hat{\phi}'}$, we have $d\vec{S}/d\tau = S\Omega\vec{e}_{\hat{r}}$ and therefore $\vec{S}$ cannot remain parallel to $\mathbf{e}_{\hat{\phi}'}$ (thus S^α does not remain in the eigenspace of U^α and $\mathbf{e}_\phi$). We then conclude that no circular geodesics for spinning classical particles are possible in the Kerr spacetime.

3. Circular geodesics in Kerr-dS spacetimes

The failure to obtain circular geodesics for spinning material particles in the previous section was due to the fact that the angular velocity of circular geodesics in the Kerr spacetime dies off as $r^{-2/3}$, whereas the angular velocity for which $\mathbb{H}_{\alpha\beta} = 0$ dies off as r^{-2}; in other words, geodesics are "too fast." But they should be possible in other spacetimes; in this spirit, Kerr-de Sitter comes as natural candidate, since a repulsive Λ should "slow down" the circular geodesics. This is indeed the case. In Boyer-Lindquist coordinates, the metric takes the form (e.g. [95])

$$ds^2 = -\frac{\Delta_r}{\chi^2\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\chi^2\Sigma}[adt - (a^2 + r^2)d\phi]^2, \tag{62}$$

where

$$\Delta_r \equiv r^2 - 2Mr + a^2 - \frac{\Lambda}{3}r^2(r^2 + a^2);$$
$$\chi \equiv 1 + \frac{\Lambda}{3}a^2;$$
$$\Delta_\theta = 1 + \frac{\Lambda}{3}a^2\cos^2\theta;$$
$$\Sigma \equiv r^2 + a^2\cos^2\theta.$$

Since $\Lambda \neq 0 \Rightarrow R_{\mu\nu} = \Lambda g_{\mu\nu}$, the vacuum classification based on scalar invariants used in the previous section does not apply herein to the Riemann tensor. However, a similar classification holds for the Weyl tensor $C_{\alpha\beta\gamma\delta}$ (again, since it is of Petrov type D), see e.g. [86]. The relationship between $\mathbb{H}_{\alpha\beta}$ and the magnetic part of the Weyl tensor, $\mathcal{H}_{\alpha\beta} \equiv \star C_{\alpha\mu\beta\nu}U^\mu U^\nu$, can be obtained from the expression of $R_{\alpha\beta\gamma\delta}$ in terms of $C_{\alpha\beta\gamma\delta}$, e.g. Eq. (2) of [91]; it reads

$$\mathbb{H}_{\alpha\beta} = \mathcal{H}_{\alpha\beta} + \frac{1}{2}\epsilon_{\alpha\beta\sigma\gamma}U^\gamma R^{\sigma\lambda}U_\lambda.$$

This tells us that, for this spacetime, $\mathbb{H}_{\alpha\beta} = \mathcal{H}_{\alpha\beta}$. Therefore, solving for $\mathbb{H}_{\alpha\beta} = 0$ amounts to solving for $\mathcal{H}_{\alpha\beta} = 0$, which reduces to the same procedure of the previous section, but this time using the invariants of the Weyl tensor. The invariants have a similar structure, similarly leading to the conclusion that in the equatorial plane there are observers for which $\mathcal{H}_{\alpha\beta} = \mathbb{H}_{\alpha\beta} = 0$. Actually, the gravitomagnetic tidal tensor for the metric (62) is obtained by simply multiplying expressions (59) by $9/(3 + a^2\Lambda)^2$:

$$(\mathbb{H}_{\text{Kerr-dS}})_{\alpha\beta} = \frac{9}{(3 + a^2\Lambda)^2}(\mathbb{H}_{\text{Kerr}})_{\alpha\beta}.$$

Thus, the angular velocity of the observers for which $\mathbb{H}_{\alpha\beta} = 0$ is given by the same Eq. (60). Now we need to check if this velocity field can correspond to circular geodesics. We can easily derive the geodesic equations from the Euler-Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial U^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \tag{63}$$

with Lagrangian $\mathcal{L} = g_{\mu\nu}U^\mu U^\nu/2$. To compute the circular geodesics we only need the r-equation, $d(g_{rr}U^r)/d\tau = g_{\mu\nu,r}U^\mu U^\nu/2$, which for circular equatorial orbits yields

$$(\omega_{\text{geo}})_\pm = \frac{-Ma + \frac{\Lambda}{3}ar^3 \pm \sqrt{Mr^3 - \frac{\Lambda}{3}r^6}}{r^3 - a^2M + \frac{\Lambda}{3}a^2r^3}, \tag{64}$$

which reduces to the Kerr case, Eq. (61), when $\Lambda = 0$.

There are two things we need to check: first, that the geodesics lie outside the black hole event horizon (and inside the cosmological horizon), and second, that the geodesics are timelike. The horizons are located at the real roots of $\Delta_r = 0$, which gives the equation

$$r^2 - 2Mr + a^2 - \frac{\Lambda}{3}r^2(r^2 + a^2) = 0. \tag{65}$$

To find spinning particles that follow circular geodesics, we have to equate the *prograde* solutions of Eq. (64) to (60),

$$\frac{a}{a^2 + r^2} = \frac{-Ma + \frac{\Lambda}{3}ar^3 + \sqrt{Mr^3 - \frac{\Lambda}{3}r^6}}{r^3 - a^2M + \frac{\Lambda}{3}a^2r^3}. \tag{66}$$

We cannot analytically solve this equation for r in general, but for our purposes it suffices to numerically show that such an r exists for some particular cases of a and Λ. Consider for example the case $a/M = 0.8$, $\Lambda M^2 = 0.001$. Solving Eq. (66) for r, we find, as the only acceptable solution, $r \simeq 14.2025M$ (the other roots are either complex or fall within the horizon). This geodesic is timelike and lies outside the event horizon, as one can see from Eq. (65). Obviously, several other solutions of (66) for different values of a and Λ are possible. We generically find that, for fixed a/M, decreasing values of ΛM^2 correspond to solutions of (66) with increasing values of r.

Section III in brief.—The physical consequences of the different symmetries of $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$:

(1) In electromagnetism, due to vacuum equation $B_{[\alpha\beta]} = \star F_{\alpha\beta;\gamma}U^\gamma/2$, a force $F^\alpha_{\text{EM}} = B^{\beta\alpha}\mu_\beta$ is exerted on the dipole *whenever it moves* in an inhomogeneous field (except for very special orientations of $\vec{\mu}$).

(2) In gravity, $\mathbb{H}_{[\alpha\beta]} = 0$, and there are velocity fields for which $\mathbb{H}_{\alpha\beta} = 0$, i.e., for which *gyroscopes feel no force*;
   (a) in the examples studied, they correspond to the situations where, in the electromagnetic analogue, $B_{\alpha\beta} = B_{[\alpha\beta]}$;
   (b) there are even geodesic motions for spinning particles: radial geodesics in Schwarzschild, circular geodesics in Kerr-dS spacetimes.

(3) Formal analogy between the quadratic scalar invariants of $R_{\alpha\beta\gamma\delta}$ and $F_{\alpha\beta}$ is useful to obtain velocities for which $\mathbb{H}_{\alpha\beta} = 0$.

IV. MANIFESTATIONS OF THE TIME PROJECTIONS OF THE TIDAL TENSORS—THE WORK DONE ON THE TEST PARTICLE

A fundamental difference between the gravitational and electromagnetic interactions concerns the time projections of the forces F^α_{G} and F^α_{EM} in the different frames, which we shall explore in this section.[5] We start by explaining the meaning of the time projection of a force in a given frame, and its relation with the work done by it and the particle's energy.

Consider a congruence of observers $\mathcal{O}(u)$ with 4-velocity u^α, and let U^α denote the 4-velocity of a test particle. The following relation generically holds [12]:

$$U^\alpha = \gamma(u^\alpha + v^\alpha); \quad \gamma \equiv -u^\alpha U_\alpha = \frac{1}{\sqrt{1 - v^\alpha v_\alpha}}, \tag{67}$$

where $v^\alpha = U^\alpha/\gamma - u^\alpha$ is the velocity of the test particle relative[6] to $\mathcal{O}(u)$. The energy of the particle relative to $\mathcal{O}(u)$ is $E \equiv -P^\alpha u_\alpha$, and its rate of change per unit proper time (the "power equation") is

$$\frac{dE}{d\tau} = -F^\alpha u_\alpha - P^\alpha u_{\alpha;\beta}U^\beta, \tag{68}$$

where $F^\alpha \equiv DP^\alpha/d\tau$ denotes the 4-force. Thus we see that the variation of the particle's energy relative to $\mathcal{O}(u)$ consists of two terms: the time projection of F^α along

[5] A (very) preliminary version of some of the results herein was presented in [96].

[6] Let (t, x^i) be the coordinate system of a locally *inertial* frame momentarily comoving with the observer; in such a frame $u^i = 0$ and $v^i = dx^i/dt$ is the ordinary 3-velocity of the test particle.

u^α, plus a term depending on the variation of u^α along the particle's worldline. The first term is interpreted as the rate of work, *as measured by* $\mathcal{O}(u)$, done by the force on the test particle (per unit proper time τ). In order to better understand it, it is useful to split F^α into its components parallel and orthogonal to the particle's CM worldline,

$$F^\alpha = F^\alpha_\parallel + F^\alpha_\perp;$$
$$F^\alpha_\parallel \equiv -F^\beta U_\beta U^\alpha;$$
$$F^\alpha_\perp \equiv (h^U)^\alpha{}_\beta F^\beta;$$

the first term of (68) then reads, using (67),

$$-F^\alpha u_\alpha = -\gamma F^\beta U_\beta + F^\alpha_\perp v_\alpha. \tag{69}$$

Forces orthogonal to U^α ($F^\alpha = F^\alpha_\perp$) are the more familiar ones; it is the case of the forces on point particles with no internal structure (monopole particles). Let us start by this simplest case. Such particles have a momentum parallel to the 4-velocity, $P^\alpha = mU^\alpha$, and constant mass $m = m_0$; the force is thus parallel to the acceleration, $F^\alpha \equiv DP^\alpha/d\tau = m_0 a^\alpha$, which implies $F^\alpha_\parallel = -F^\beta U_\beta U^\alpha = 0$ (due to the condition $U^\beta U_\beta = -1$). That leads to $-F^\alpha u_\alpha = F^\alpha v_\alpha$, telling us that the time projection [in the frame $\mathcal{O}(u)$] of F^α is the familiar power $\vec{F}\cdot\vec{v}$ (see e.g. [29,41]). If we take an inertial frame, so that the second term of (68) vanishes, then $-F^\alpha u_\alpha = dE/d\tau = m_0 d\gamma/d\tau$, i.e., $F^\alpha v_\alpha = m_0 d\gamma/d\tau$ is the rate of variation of the particle's kinetic energy of translation. It is clear in particular that $dE/d\tau = F^\alpha v_\alpha = 0$ in a frame comoving with the particle. An example of such a force is the Lorentz force, $F^\alpha = qF^{\alpha\beta}U_\beta = qE^\alpha$, for which $F^\alpha U_\alpha = 0$, and whose projection along u^α reads $-u_\alpha F^\alpha = qv_\alpha E^\alpha$, yielding the power transferred by the electric force to the moving particle [relative to $\mathcal{O}(u)$].

However, if the particle has internal structure then its internal degrees of freedom may store energy (e.g., kinetic energy of rotation about the center of mass), and so the particle's proper mass $m = -P^\alpha U_\alpha$ no longer has to be a constant (cf. Sec. II E). Moreover, the momentum will not be parallel to U^α, as the particle in general possesses hidden momentum, cf. Sec. II D. These, together (as we shall see next), endow F^α with a nonvanishing component $F^\alpha_\parallel$ along U^α, which is the new ingredient. $F^\alpha_\parallel$ is the rate of work done by the force as measured in the frame comoving with the particle.

Let us turn our attention now to the second term of Eq. (68). Decomposing (e.g. [12,14,17])

$$u_{\alpha;\beta} = -a(u)_\alpha u_\beta + \omega_{\alpha\beta} + \theta_{\alpha\beta}, \tag{70}$$

where $a(u)^\alpha = u^\alpha{}_{;\beta}u^\beta$ is the observers' acceleration (*not* the particle's), $\omega_{\alpha\beta} \equiv (h^u)^\lambda_\alpha (h^u)^\nu_\beta u_{[\lambda;\nu]}$ is the vorticity, and $\theta_{\alpha\beta} \equiv (h^u)^\lambda_\alpha (h^u)^\nu_\beta u_{(\lambda;\nu)}$ is the shear/expansion tensor of the observer congruence [$\theta_{\alpha\beta} \equiv \sigma_{\alpha\beta} + \theta(h^u)_{\alpha\beta}/3$, where $\sigma_{\alpha\beta}$ is the traceless shear and θ the expansion scalar]. Let us denote by $G(u)^\alpha = -a(u)^\alpha$ the "gravitoelectric field" [12,14] measured by the observers. Decomposing $P^\alpha = mU^\alpha + P^\alpha_{\rm hid}$, cf. Eq. (30), and using (67) and (70), the second term of Eq. (68) becomes

$$\begin{aligned}-P^\alpha u_{\alpha;\beta}U^\beta = m\gamma^2[G(u)_\alpha - \theta_{\alpha\beta}v^\beta]v^\alpha \\ + \gamma P^\alpha_{\rm hid}[G(u)_\alpha - (\omega_{\alpha\beta} + \theta_{\alpha\beta})v^\beta].\end{aligned} \tag{71}$$

This part of $dE/d\tau$ depends only on the kinematical quantities of the observer congruence, not on the physical force F^α. In other words, it is an artifact of the reference frame, which vanishes if it is locally inertial. Its importance (in a nonlocal sense) cannot however be overlooked. To understand this, consider a simple example, a monopole particle in Kerr spacetime, from the point of view of the congruence of *static observers* (cf. Sec. I D, point 7). Since the congruence is rigid, $\theta_{\alpha\beta} = 0$; also, for a monopole particle, $P^\alpha_{\rm hid} = 0$, and, in a gravitational field, $F^\alpha = 0$ (the particle moves along a geodesic). Therefore, the energy variation reduces to $dE/d\tau = -P^\alpha u_{\alpha;\beta}U^\beta = m\gamma^2 G(u)_\alpha v^\alpha$, which is the rate of "work" done by the gravitoelectric "force" [6,12,14] $m\gamma^2 G(u)^\alpha$. (In the Newtonian limit, it reduces to the work of the Newtonian force $m\vec{G}$.) Hence we see that (71) is the part of (68) that encodes the change in translational kinetic energy of a particle (relative to the static observers) which occurs due to the gravitational field without the action of any (physical, covariant) force, and that is nonzero for particles in geodesic motion.

Substituting Eqs. (69) and (71) into (68), we obtain a generalization, for the case of test particles with varying m and hidden momentum, of the "power equation" (6.12) of [12] (the latter applying to monopole particles only).

A. Time components in test particle's frame

A fundamental difference between the tensorial structure of $\mathbb{H}_{\alpha\beta}$ and $B_{\alpha\beta}$ is that whereas the former is spatial, in *both* indices, with respect to the observer U^α measuring it, $(\mathbb{H}^U)_{\alpha\beta}U^\beta = (\mathbb{H}^U)_{\alpha\beta}U^\alpha = 0$ (this follows from the symmetries of the Riemann tensor), the latter is not: $(B^U)_{\alpha\beta}U^\alpha = 0$, but $(B^U)_{\alpha\beta}U^\beta = \star F_{\alpha\gamma;\beta}U^\gamma U^\beta \neq 0$ in general. This means that whereas $F^\alpha_{\rm G}$ is orthogonal to the particle's worldline, $F^\alpha_{\rm EM}$ has a nonvanishing projection along it (i.e., a time projection in the particle's CM frame), $F^\alpha_{\rm EM}U_\alpha \neq 0$. Let us see its physical meaning. First note, from Eq. (I.4a), that

$$F^\alpha_{\rm EM}U_\alpha = B^{\beta\alpha}U_\alpha\mu_\beta = \epsilon_{\beta\delta\mu\nu}U^\delta E^{[\mu\nu]}\mu^\beta, \tag{72}$$

showing that it consists of a coupling between μ^α and the space projection of the antisymmetric part of the electric

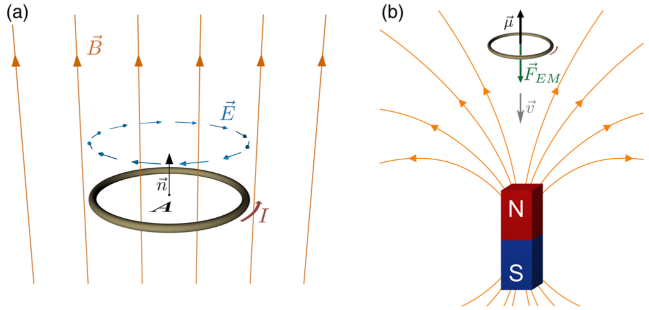

FIG. 3. A magnetic dipole (depicted as a current loop) falling in the inhomogeneous magnetic field of a strong magnet, from the point of view of two different frames: (a) the particle's rest frame; (b) the rest frame of the strong magnet (static observers). Here $\vec{\mu} = IA\vec{n}$; $A \equiv$ area of the loop; $I \equiv$ current through the loop; $\vec{n} \equiv$ unit vector normal to the loop; $\vec{E} \equiv$ induced electric field. In the dipole's frame nonvanishing work is done on it by $\vec{E}$, at a rate $\mathcal{P}_{\rm ind} = -F^{\alpha}_{\rm EM}U_{\alpha}$, which is reflected in a variation of proper mass m. From the point of view of static observers u^{α}, the work is zero ($-F^{\alpha}_{\rm EM}u_{\alpha} = \mathcal{P}_{\rm ind} + \mathcal{P}_{\rm trans} = 0$), manifesting that a stationary magnetic field does no work. That may be regarded as an exact cancellation between $\mathcal{P}_{\rm ind}$ and the rate of variation of the particle's translational kinetic energy, $\mathcal{P}_{\rm trans}$.

tidal tensor $E_{\alpha\beta}$ measured in the particle's CM frame, which, as discussed in Sec. II B, encodes Faraday's law of induction. Indeed, if one chooses the CM frame to be locally nonshearing and nonrotating (as one can always do), we may replace $E_{\alpha\beta}$ by the covariant derivative of the electric field $E_{\alpha;\beta}$, cf. Eq. (22), and Eq. (72) becomes therein, in vector notation, $F^{\alpha}_{\rm EM}U_{\alpha} = -(\nabla\times\vec{E})\cdot\vec{\mu}$. Its significance becomes clear if one thinks about the magnetic dipole as a small current loop of area A and magnetic moment $\vec{\mu} = \vec{n}AI$, see Fig. 3(a). It then follows:

$$-F^{\alpha}_{\rm EM}U_{\alpha} = (\nabla\times\vec{E})\cdot\vec{n}AI = I\oint_{\rm loop}\vec{E} \equiv \mathcal{P}_{\rm ind}, \qquad (73)$$

where in the second equality we first used the fact that the loop is (by definition) infinitesimal, so $(\nabla\times\vec{E})\cdot\vec{n}A = \int_{\Sigma^{(2)}}(\nabla\times\vec{E})\cdot d\vec{\Sigma}$ for a 2-surface $\Sigma^{(2)}$ enclosed by the loop, and then applied the Stokes theorem in the 3D local rest space of the dipole. Here $\vec{E}$ is the *induced electric field*, coming from the induction law[7] (23).

Thus $-F^{\alpha}_{\rm EM}U_{\alpha} \equiv \mathcal{P}_{\rm ind}$ is the rate of work transferred to the dipole by Faraday's law of induction. Using Eqs. (36) and (37), we see that it consists of the variation of the proper mass m, *minus* the projection along U^{α} of the derivative of the hidden momentum (to which only the electromagnetic hidden momentum contributes):

$$\mathcal{P}_{\rm ind} = \frac{dm}{d\tau} - \frac{DP^{\alpha}_{\rm hid}}{d\tau}U_{\alpha} = \frac{dm}{d\tau} - \frac{DP^{\alpha}_{\rm hidEM}}{d\tau}U_{\alpha}. \qquad (74)$$

Note, from Eq. (68), that $\mathcal{P}_{\rm ind}$ is the variation of the dipole's energy $E = -P_0$ *as measured in a momentarily comoving inertial frame*.

The induction phenomenon in Eq. (73) has no counterpart in gravity. Since $\mathbb{H}_{\alpha\beta}$ is spatial relative to U^{α}, we *always have*

$$F^{\alpha}_{\rm G}U_{\alpha} = 0, \qquad (75)$$

and the proper mass m is a constant [since also $P^{\alpha}a_{\alpha} = -U_{\alpha}DP^{\alpha}_{\rm hid}/d\tau = 0$, cf. Eqs. (36) and (37)]. That is, the *energy* of the gyroscope, *as measured in its CM rest frame*, is constant. We see thus that the spatial character of the gravitational tidal tensors *precludes* induction effects analogous to the electromagnetic ones.

B. Time components as measured by static observers

1. Electromagnetism

With respect to an arbitrary congruence of observers of 4-velocity u^{α}, the time projection of the force exerted on a magnetic dipole is, cf. Eq. (69),

$$-F^{\alpha}_{\rm EM}u_{\alpha} = -\gamma F^{\alpha}_{\rm EM}U_{\alpha} + F^{\alpha}_{\rm EM\perp}v_{\alpha} = \gamma\mathcal{P}_{\rm ind} + F^{\alpha}_{\rm EM\perp}v_{\alpha}, \qquad (76)$$

where, in accordance with the discussion above, we identify $\mathcal{P}_{\rm ind} = -F^{\alpha}_{\rm EM}U_{\alpha}$ as the power transferred to the dipole by Faraday's induction, and $F^{\alpha}_{\perp\rm EM}v_{\alpha}$ is the power transferred by the component of $F^{\alpha}_{\rm EM}$ orthogonal to the particle's worldline. Consider now a congruence of observers along whose worldlines the fields are covariantly constant, $F^{\alpha\beta}{}_{;\gamma}u^{\gamma} = 0$; the time projection of the force with respect to them vanishes:

$$-F^{\alpha}_{\rm EM}u_{\alpha} \equiv -\frac{DP_{\alpha}}{d\tau}u^{\alpha} = -\star F_{\gamma\beta;\alpha}U^{\beta}\mu^{\gamma}u^{\alpha} = 0. \qquad (77)$$

This tells us that the total work done on the dipole, *as measured by such observers*, is zero. Take these observers to form, moreover, an inertial frame; these will be dubbed in this context *static* or "laboratory"[8] observers. In this case

[7]This generalization of the Maxwell-Faraday law for accelerated frames is needed if one is to deal with the electric and magnetic fields measured in the test particle's frame, which in general accelerates. One could instead base the analysis in the inertial frame *momentarily* comoving with it, as done in Sec. V of [7], where $\partial\vec{B}/\partial\tau = -\nabla\times\vec{E}$ holds; the two treatments are equivalent.

[8]The reason for such denominations is that, in the electromagnetic setups herein [the magnet in Fig. 3(b), the spinning/nonspinning charges of Secs. III and V], only the observers *at rest* relative to the sources obey the condition $F^{\alpha\beta}{}_{;\gamma}u^{\gamma} = 0$. Note that even for e.g. observers u'^{α} in circular motion around a Coulomb charge we have $F^{\alpha\beta}{}_{;\gamma}u'^{\gamma} \neq 0$ [as can be seen replacing $U^{\alpha} \to u'^{\alpha}$ in Eq. (51), which implies $\star F^{\alpha\beta}{}_{;\gamma}u'^{\gamma} = 2(B^{u'})_{[\alpha\beta]} \neq 0$ when $u'^{i} \neq 0$], even though u'^{α} is in that case a symmetry of $F_{\alpha\beta}$, $\mathcal{L}_{u'}F_{\alpha\beta} = 0$, and $F_{\alpha\beta}$ is time independent in the corotating frame.

$u_{\alpha;\beta} = 0$ and the second term of Eq. (68) vanishes; therefore, the energy of the particle, $E = -P_\alpha u^\alpha$, is a conserved quantity in such frame. Using Eq. (30), we can write it in the form

$$E = m + T + E_{\text{hid}} = \text{constant}, \tag{78}$$

where we dub $E_{\text{hid}} \equiv -P^\alpha_{\text{hid}} u_\alpha$ the "hidden energy" (i.e., the time component of the hidden momentum), and $T \equiv (\gamma - 1)m$ is the kinetic energy of translation of the center of mass, as measured in this frame (in the Newtonian regime, $T \approx mv^2/2$). In the (very scarce, to the authors' knowledge) literature addressing this problem, a cancellation between the variations of T and m is suggested in [28], or, for the case of a spherical spinning charged body, of T and kinetic energy of rotation about the CM [83–85] (which agrees with the former assertion, since for such a body, $dm/d\tau$ is essentially a variation of kinetic energy of rotation, as we shall see in Sec. VI A 3). Equation (78) shows however that, in the general case when $P^\alpha_{\text{hid}} \neq 0$, the energy exchange occurs between three components, with E_{hid} also playing a role. A suggestive example are the bobbings of a particle with magnetic dipole moment orbiting a cylindrical charge considered in Sec. III. B. 1 of [20] (and illustrated in Fig. 1 of [75]).

In this paper we are especially interested in the case $P^\alpha_{\text{hid}} = 0$ ($\Rightarrow E_{\text{hid}} = 0$), so that $m + T = $ constant; i.e., the energy exchange, due to the action of the force F^α_{EM}, occurs only between proper mass and translational kinetic energy. It follows also that $\mathcal{P}_{\text{ind}} = dm/d\tau$. Therefore, from (68) and (76) (and since $u_{\alpha;\beta} = 0$),

$$\frac{dE}{d\tau} = -F^\alpha_{\text{EM}} u_\alpha = \mathcal{P}_{\text{ind}} + \mathcal{P}_{\text{trans}} = 0, \tag{79}$$

where

$$\mathcal{P}_{\text{trans}} \equiv \frac{dT}{d\tau} = F^\alpha_{\text{EM}\perp} v_\alpha + (\gamma - 1)\frac{dm}{d\tau} \tag{80}$$

is the rate of variation of *translational kinetic energy*, and we noted that $F^\alpha_{\text{EM}\perp} v_\alpha = m d\gamma/d\tau$. An example is the problem depicted in Fig. 3(b): a magnetic dipole falling along the symmetry axis of the field generated by a strong magnet. (We have $P^\alpha_{\text{hid}} = 0$ for this configuration.[9]) From the point of view of the static observers, $\vec{E}(u) = 0$ and only magnetic field $\vec{B}(u)$ is present; we know that the latter can do no work, because if we think about the dipole as a current loop (cf. Fig. 3) and consider the force exerted in each of its individual moving charges, we see that the magnetic force $\vec{F} = q(\vec{v} \times \vec{B})$ is always orthogonal to the velocity $\vec{v}$ of the charges, so that no work can be done. It is thus quite natural that $F^\alpha_{\text{EM}} u_\alpha = 0$. According to Eq. (79), this arises from an exact cancellation between $\mathcal{P}_{\text{trans}}$ and $\mathcal{P}_{\text{ind}}$: on the one hand there is an attractive spatial force $\vec{F}_{\text{EM}}$ causing the dipole to gain translational kinetic energy; on the other hand there is a variation of its internal energy (proper mass m) by induction, which allows for the total work to vanish (in agreement with the reasoning in [28], page 21). Further remarks on this issue are given in Secs. VI C and Appendix B 4.

It is worth mentioning that this cancellation solves an apparent paradox that has for long been discussed in the literature [28,82,83,85]—that on the one hand a force is exerted on a magnetic dipole placed in a nonhomogeneous magnetic field, causing it to move, whilst on the other hand $\vec{B}$ can do no work in any charge/current distribution. The analysis above generalizes and reformulates, in a relativistic covariant framework, the arguments in [82–85], and supports the claim in [28] that the solution of the apparent paradox lies on the variation of m. It is also useful, to make these points more clear, to compare with the cases of a monopole charged particle, and of an electric dipole subject to an electromagnetic field: there is also a force on the particle, which is set into motion gaining kinetic energy; but, in these cases, *the electric field is doing work*, there is a potential energy involved, and the gain in translational kinetic energy is *not* canceled out by a variation of the particle's proper mass (m is constant for a monopole particle, and also for an electric dipole if one assumes that the dipole vector is parallel transported). These cases are discussed in detail in Appendix B 4.

2. Gravity

In gravity, where $F^\alpha_{\text{G}} U_\alpha = 0$ (i.e., the induction effects are absent), we have, for arbitrary observers u^α,

$$-F^\alpha_{\text{G}} u_\alpha = F^\alpha_{\text{G}} v_\alpha. \tag{81}$$

This implies that a cancellation similar to the one in Eq. (79) does not occur. Except when $v^\alpha \perp F^\alpha_{\text{G}}$, F^α_{G} does work whenever the particle moves relative to the reference frame; in particular it is so from the point of view of static observers in a stationary spacetime (i.e., a stationary gravitomagnetic tidal field does work on mass currents), *by contrast with its electromagnetic counterpart*. There is a potential energy associated with such work, as we shall now show.

A conserved quantity for a spinning particle in a stationary spacetime is (e.g. [19,20,27,97])

$$E_{\text{tot}} = -P^\alpha \xi_\alpha + \frac{1}{2}\xi_{\alpha;\beta} S^{\alpha\beta} = \text{constant}, \tag{82}$$

[9] That this is a solution of the equations of motion supplemented with Mathisson-Pirani condition can be seen by arguments analogous to the ones given in Appendix C 1 for the gravitational counterpart.

where $\boldsymbol{\xi} \equiv \partial/\partial t$ is the time *Killing vector field*. Consider the congruence of static observers,[10] of 4-velocity parallel to ξ^α: $u^\alpha = \xi^\alpha/\xi$, where $\xi \equiv \sqrt{-\xi^\alpha \xi_\alpha}$ is their lapse, or redshift factor (see e.g. [12,13]). The first term of (82), $-P^\alpha \xi_\alpha = E\xi$, is the "Killing energy," a conserved quantity for the case of a nonspinning particle ($S^{\alpha\beta} = 0$) in geodesic motion, which yields its energy with respect to thestatic observers *at infinity*.[11] It can be interpreted as its "total energy" (rest mass + kinetic + "Newtonian potential energy") in a gravitational field (e.g. [67]). The energy E_{tot} can likewise be interpreted as the energy at infinity for the case of a spinning particle. To see the interpretation of the second term in (82),

$$V \equiv \frac{1}{2}\xi_{\alpha;\beta} S^{\alpha\beta}, \tag{83}$$

consider the case when $P^\alpha_{\text{hid}} = 0$. We have

$$0 = \frac{dE_{\text{tot}}}{d\tau} = -F^\alpha_{\text{G}}\xi_\alpha - mU^\alpha U^\beta \xi_{\alpha;\beta} + \frac{dV}{d\tau} \tag{84}$$

$$\Leftrightarrow \xi F^\alpha_{\text{G}} u_\alpha = \frac{dV}{d\tau}, \tag{85}$$

where we used the Killing equation $\xi_{(\alpha;\beta)} = 0$. The quantity $-\xi F^\alpha_{\text{G}} u_\alpha = \xi F^\alpha_{\text{G}} v_\alpha$ is the rate work (per unit of particle's proper time τ) of F^α_{G}, as measured by the static observers *at infinity*, and thus V is the spin-curvature *potential energy* associated with that work.[12]

In order to compare with the electromagnetic equation (78), note that $d\xi/d\tau = -\gamma G(u)_\alpha v^\alpha$, and that for $P^\alpha_{\text{hid}} = 0$ we have $E = \gamma m = m + T$. Thus we can write $dE_{\text{tot}}/d\tau = d(\xi E + V)/d\tau$ in the form

$$\xi \frac{dT}{d\tau} - \xi m\gamma^2 G_\alpha v^\alpha + \frac{dV}{d\tau} = 0. \tag{86}$$

The second term accounts for the "power" of the gravitoelectric "force" $m\gamma^2 \vec{G}(u)$ (which is *not* a physical force, arising, as explained above, from the observers' acceleration); it reduces to the variation of Newtonian potential energy in the weak field slow motion limit. Equation (86) tells us that the variation of translational kinetic energy T comes from the spin-curvature potential energy V, and from the power transferred by $m\gamma^2\vec{G}(u)$ (m being constant); this contrasts with the case of the magnetic dipole discussed above, where (again for $P^\alpha_{\text{hid}} = 0$) the variation of kinetic energy comes from the variation of proper mass m, with no potential energy being involved. In terms of the work done on the particle, F^α_{G} is thus more similar to the electromagnetic forces exerted on a monopole charge or on an electric dipole (for $Dd^\alpha/d\tau = 0$), where the proper mass is likewise constant and the energy exchange is between T and potential energy (see Appendix B 4 for more details).

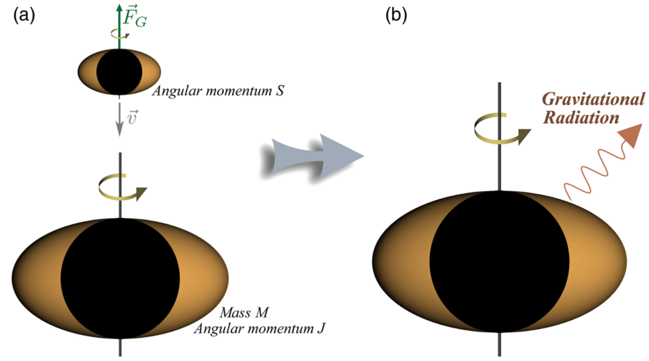

FIG. 4. (a) Gyroscope (small Kerr black hole) in the field of a large Kerr black hole; (b) black hole merger. Evidence that, unlike its electromagnetic counterpart, the gravitomagnetic tidal field *does* work: the spin-dependent part of the energy released is the work (as measured by the static observers at infinity) of F^α_{G}.

There is a known consequence of the fact that F^α_{G} does work (and of the interaction energy V): the spin dependence of the upper bounds for the energy released by gravitational radiation when two black holes collide [Fig. 4(b)], obtained by Hawking [22] from the area law.

In order to see this, consider the apparatus in Fig. 4: two Kerr black holes with their spins aligned, a large one (mass M, spin $J = aM$), which is the source, and a small one (4-velocity U^α, spin $S \equiv \sqrt{S^\alpha S_\alpha}$), which we take to be the test particle, falling into the former along the symmetry axis (how this is set up with the Mathisson-Pirani spin condition is discussed in Appendix C 1). For this setup $\mathbf{U} = U^0\mathbf{e}_0 + U^r\mathbf{e}_r$, $\mathbf{S} = S^0\mathbf{e}_0 + S^r\mathbf{e}_r$, where $\mathbf{e}_\alpha \equiv \partial/\partial_\alpha$ are Boyer-Lindquist coordinate basis vectors; and $P^\alpha_{\text{hid}} = 0$. Moreover, V becomes a *pure spin-spin* potential energy, since, for radial motions, $F^\alpha_{\text{G}} = 0$ if $J = 0$, cf. Eq. (50). Using $S_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta}U^\delta S^\gamma$ (as follows from the condition $S^{\alpha\beta}U_\beta = 0$, cf. Sec. II A), and noting that $S^r U^0 - S^0 U^r = S$ (as follows from $S^\alpha U_\alpha = 0$, and, along the axis, $g_{00} = -1/g_{rr}$), one obtains, for Eq. (83),

$$V(r) = \pm \frac{2aMSr}{(a^2 + r^2)^2} = \int_\infty^{\tau(r)} \xi F^\alpha_{\text{G}} u_\alpha d\tau,$$

[10]See point 7 of Sec. I D. In stationary asymptotically flat spacetimes, such as the Kerr metric studied below, these are observers rigidly fixed to the asymptotic inertial rest frame of the source. They are thus the closest analogue of the flat spacetime notion of observers at rest relative to the source in the electromagnetic systems above.

[11]If the particle is in a bounded orbit, one can imagine this measurement process as follows: let $E_{\text{tot}}(\tau_1)$ be the total energy of the particle at τ_1; if, at that instant, the particle was by some process converted into light and sent to infinity, the resulting radiation would reach infinity with an energy $E = -u^\alpha P_\alpha = E_{\text{tot}}(\tau_1)$.

[12]One may check explicitly that $dV/d\tau = \xi_{\alpha;\beta\gamma}S^{\alpha\beta}U^\gamma = \xi F^\alpha_G v_\alpha$, noting that $DS^{\alpha\beta}/d\tau = 0$ if $P^\alpha_{\text{hid}} = 0$, and using the general relation for a Killing vector $\xi_{\mu;\nu\lambda} = R_{\lambda\sigma\mu\nu}\xi^\sigma$.

the $\pm$ sign applying to the case when $\vec{S}$ and $\vec{J}$ are parallel/antiparallel. The second equality follows from Eq. (85); this result can be checked noting that, in Boyer-Lindquist coordinates, $\xi F^{\alpha}_{\rm G} u_{\alpha} = (F_{\rm G})_0$, and computing explicitly the time component $(F_{\rm G})_0$ for axial fall, Eq. (37) of [1]. Thus we see that $V(r)$ is *minus* the work done by $F^{\alpha}_{\rm G}$ as the particle goes from infinity to r. Let us comment on the presence of the lapse factor ξ in the integral above. Computing the work of $F^{\alpha}_{\rm G}$ does not amount to integrating the power measured by the local static observers, $-F^{\alpha}_{\rm G} u_{\alpha} = F^{\alpha}_{\rm G} v_{\alpha}$ (i.e., summing up the work elements $dW \equiv F^{\alpha}_{\rm G} v_{\alpha} d\tau$), as that would mean adding energies measured by different observers; instead, we should integrate quantity $\xi F^{\alpha}_{\rm G} v_{\alpha}$, which can be thought as summing up work elements measured by the static observer *at infinity*.

Let us now analyze the problem of the black hole merger. The increase of translational kinetic energy of the small black hole during the fall is given by Eq. (86). The term $m\gamma^2 G_{\alpha} v^{\alpha}$ is the gain in kinetic energy due to the "Newtonian" attraction, and exists regardless of S^{α}; the term involving the spin-spin potential energy V, however, will cause the test particle's kinetic energy and, therefore, the energy available to be released by gravitational radiation in the collision, to depend on S. Upper bounds for this energy, which are, accordingly, spin dependent, were obtained in [22] by a totally independent method. From such limits, and for the setup in Fig. 4, Wald [1] obtained an expression [Eq. (35) therein] for the amount of energy ΔE_s by which the upper bound is increased/reduced when $\vec{S}$ is parallel/antiparallel to $\vec{J}$, comparing with the case $S = 0$ (fall along a geodesic). It turns out that this energy is precisely *minus* the value of $V(r)$ at the horizon r_+, $\Delta E_s = -V(r_+)$; that is, it is *the work done by* $F^{\alpha}_{\rm G}$ on the small black hole as it comes from infinity to the horizon: $\Delta E_s = \int_{\infty}^{\tau(r_+)} (-\xi F^{\alpha}_{\rm G} u_{\alpha}) d\tau$.

We close this section with some remarks on the meaning of the work done by the gravitomagnetic tidal field. One can associate to the static observers in the Kerr spacetime a gravitomagnetic "vector" field $\vec{H}$ (see Sec. II C, and e.g. [3,6,12,14]; in the weak field regime this field is well known to be very similar to its electromagnetic counterpart, e.g. [2,3,98]), causing inertial (i.e., fictitious) "forces" on test particles of the type $\vec{v} \times \vec{H}$, formally similar to the magnetic force $q\vec{v} \times \vec{B}$. Namely, the force is orthogonal to the velocity; hence this analogy might lead one to believe that, similarly to its magnetic counterpart, the gravitomagnetic field cannot do work on test particles. One must bear in mind, however, that $\vec{H}$ (by contrast with $\vec{B}$) has no local existence, as it is a mere artifact of the reference frame; hence it would never be involved in a covariant quantity like the 4-force $DP^{\alpha}/d\tau$, or the work done by it. Moreover, both in electromagnetism and in gravity, *it is the tidal fields that yield the force*; that is manifest in force Eqs. (I.1) of Table I. The electromagnetic tidal tensors herein are essentially derivatives of the fields; for this reason we were able to argue in terms of the fields in the applications depicted in Fig. 3 (even though it is their derivatives that show up in the equations). But the gravitational tidal tensors cannot be cast as derivatives of the GEM fields, even in the weak field regime, except under very special conditions (see Sec. III. 5 of [14]); the force $F^{\alpha}_{\rm G}$ is thus in general very different from its electromagnetic counterpart. Namely, it is so whenever the test particle moves relative to the source—so that the work of $F^{\alpha}_{\rm G}$ can dramatically differ from that of $F^{\alpha}_{\rm EM}$, which is well exemplified by the contrast herein: as measured in the test particle's frame, we have $F^{\alpha}_{\rm EM} U_{\alpha} \neq 0$, $F^{\alpha}_{\rm G} U_{\alpha} = 0$; as measured by the static observers u^{α}, we have precisely the *opposite* situation: $F^{\alpha}_{\rm EM} u_{\alpha} = 0$, $F^{\alpha}_{\rm G} u_{\alpha} \neq 0$.

Section IV in brief.—The work done on the particle (magnetic dipole vs gyroscope):

(i) The time projection of the force, $-F^{\alpha} u_{\alpha}$, is the rate at which it does work on the particle, as measured by an observer of 4-velocity u^{α}.

Time projections along the particle's worldline (U^{α}):

(i) Electromagnetic is nonvanishing, $F^{\alpha}_{\rm EM} U_{\alpha} \neq 0$; it is the rate of work done by Faraday's induction law, arising from $E_{[\alpha\beta]}$ (or equivalently, from $B_{\alpha\beta} U^{\beta}$); reflected in a variation of m.

(ii) Gravitational is zero, $F^{\alpha}_{\rm G} U_{\alpha} = 0$; the gyroscope's proper mass m is constant; no analogous induction effect (as $\mathbb{H}_{\alpha\beta} U^{\beta} = 0$).

Time projections relative to *static observers* (u^{α}):

(i) Electromagnetic is zero, $F^{\alpha}_{\rm EM} u_{\alpha} = 0 \Rightarrow$ a stationary electromagnetic field does no work on magnetic dipoles.

(ii) Gravitational is nonzero, $F^{\alpha}_{\rm G} u_{\alpha} \neq 0 \Rightarrow$ gravitomagnetic (tidal) field does work—there is a spin-curvature potential energy; embodies Hawking-Wald spin-spin interaction energy.

V. WEAK FIELD REGIME AND GRAVITATIONAL SPIN-SPIN FORCE

In the previous two sections we discussed the crucial differences between the gravitational and electromagnetic forces on a spinning particle that are manifest in the symmetries and time projections of the tidal tensors. However, in the literature (e.g. [2,3,5,20]) concerning the weak field, slow motion regime—where the nonlinearities of the gravitational field are negligible, and one might indeed expect a similarity between the gravitational and electromagnetic interactions—they are usually portrayed as being very similar. In this section we will study this regime, and dissect the impact of the aforementioned differences. We shall consider the basic example of analogous physical systems: a magnetic dipole in the electromagnetic field of a spinning charge (charge Q, magnetic moment μ_s), and a gyroscope in the gravitational field of a spinning mass

(mass M, angular momentum J), asymptotically described by the Kerr solution.

We start by briefly describing the approximations that we will use. The electromagnetic potentials are, exactly, $\phi \equiv Q/r$ and $\vec{A} \equiv \vec{\mu}_s \times \vec{r}/r^3$; for the gravitational field we take the linearized Kerr metric

$$ds^2 = -(1+2\Phi)dt^2 + 2\mathcal{A}_j dt dx^j + (1-2\Phi)\delta_{ij}dx^i dx^j, \tag{87}$$

with the gravitational "potentials" $\Phi \equiv -M/r$, $\vec{\mathcal{A}} \equiv -2\vec{J} \times \vec{r}/r^3$. The gravitational tidal tensors, are, consistently, linearized in the potentials. The electromagnetic tidal tensors are linear in the potentials, hence no weak field assumption is made in the forces (90), (92) and (93). The expression for the acceleration (42), however, involves a term of second order in the electromagnetic fields, which is to be neglected in a coherent comparison with linearized gravity. In the computation of the electromagnetic and gravitational tidal tensors involved in the forces (92)–(95), exerted on slowly moving test particles (velocity v), only terms up to first order in v are kept (as usual in slow motion approximations, e.g. [1]). The relationship with the post-Newtonian approximations in e.g. [99–105] is established in [26].

Let us first consider stationary setups, where the test particle is at rest relative to the central source (or singularity, for the case of a black hole); i.e., at rest with respect to the static observers u^α (cf. point 7 of Sec. I D). For these observers, the *linearized* gravitational tidal tensors match the electromagnetic ones, identifying the appropriate parameters:

$$(\mathbb{E}^u)_{ij} \simeq \frac{M}{r^3}\delta_{ij} - \frac{3Mr_ir_j}{r^5} \overset{M\leftrightarrow Q}{=} (E^u)_{ij}, \tag{88}$$

$$(\mathbb{H}^u)_{ij} \simeq 3\left[\frac{(\vec{r}\cdot\vec{J})}{r^5}\delta_{ij} + 2\frac{r_{(i}J_{j)}}{r^5} - 5\frac{(\vec{r}\cdot\vec{J})r_ir_j}{r^7}\right] \overset{J\leftrightarrow\mu_s}{=} (B^u)_{ij} \tag{89}$$

(all the time components vanish identically for these observers). Therefore, the force exerted on a gyroscope *whose center of mass is at rest* relative to the central mass is similar (apart for a minus sign) to its electromagnetic counterpart, identifying $\mu_s \leftrightarrow J$ and $\mu \leftrightarrow S$,

$$F^i_{\rm G} = -\mathbb{H}^{ji}S_j \overset{\{J,S\}\leftrightarrow\{\mu_s,\mu\}}{\simeq} -F^i_{\rm EM}. \tag{90}$$

In other words, there is, for stationary setups, a gravitational spin-spin force similar to its electromagnetic counterpart; this result is due to Wald [1].

Manifestation of the different symmetries.—In the general case, where the dipole/gyroscope is allowed to move, however, Table I makes clear that the two forces differ, because $\mathbb{H}_{\alpha\beta}$ remains symmetric, whereas $B_{\alpha\beta}$ acquires an antisymmetric part. This leads to key differences in the dynamics (already exemplified in Sec. III), which are non-negligible in the weak fieldand slow motion approximation, as we shall now see. Consider the test particles to be moving with 3-velocity $\vec{v}$ relative to the central sources. The magnetic tidal tensor as seen by the moving dipole, $B_{\alpha\beta}$, can be obtained in terms of the tidal tensors $(E^u)_{\alpha\beta}$, $(B^u)_{\alpha\beta}$ measured by the static observers, using the decomposition

$$\star F_{\alpha\beta;\gamma} = 2u_{[\alpha}(B^u)_{\beta]\gamma} - \epsilon_{\alpha\beta\mu\sigma}u^\sigma(E^u)^\mu{}_\gamma. \tag{91}$$

The force (I.2a) exerted on the magnetic dipole reads, *to first order* in v,

$$F^i_{\rm EM} \simeq B^{ji}\mu_j \simeq (B^u)^{ji}\mu_j - (E^u)^{li}\epsilon^j{}_{kl}v^k\mu_j, \tag{92}$$

$$F^0_{\rm EM} = B^{i0}\mu_i = 0. \tag{93}$$

The gravitomagnetic tidal tensor as seen by the moving gyroscope, $\mathbb{H}_{\alpha\beta}$, can analogously be obtained in terms of the tidal tensors $(\mathbb{E}^u)_{\alpha\beta}$, $(\mathbb{H}^u)_{\alpha\beta}$ measured by the static observers, using the dual of decomposition (44),

$$\begin{aligned}\star R_{\alpha\beta}{}^{\gamma\delta} = {} & 4\epsilon^{\lambda\tau}{}_{\alpha\beta}u_\lambda u^{[\gamma}(\mathbb{E}^u)_\tau{}^{\delta]} - 2\epsilon^\tau{}_{\alpha\beta}{}^{[\gamma}(\mathbb{E}^u)_\tau{}^{\delta]} \\ & + 4(\mathbb{H}^u)_{[\beta}{}^{[\delta}u^{\gamma]}u_{\alpha]} + \epsilon^{\lambda\tau}{}_{\alpha\beta}\epsilon^{\gamma\delta\mu\nu}(\mathbb{H}^u)_{\mu\tau}u_\lambda u_\nu.\end{aligned}$$

The force exerted on the gyroscope reads, to linear order in the fields, and to first order in v,

$$F^i_{\rm G} \simeq -\mathbb{H}^{ji}S_j \simeq -(\mathbb{H}^u)^{ji}S_j + 2(\mathbb{E}^u)^{l(i}\epsilon^{j)}{}_{kl}v^kS_j, \tag{94}$$

$$F^0_{\rm G} \simeq -\mathbb{H}^{i0}S_i \simeq -(\mathbb{H}^u)^{ji}v_jS_i. \tag{95}$$

We note that, to this accuracy, the spatial part of the forces, apart from global signs and a factor of 2 in the second term of (94) as compared to (92), differ essentially in the fact that the former expression is *symmetrized* in $\{i,j\}$, whereas the latter is *not*. Thus the differences in the symmetries of the tidal tensors, discussed in Sec. III, *are manifest to leading order*. (Explicit expressions for $\vec{F}_{\rm G}$ and $\vec{F}_{\rm EM}$ are given in [26].)

Also the differences in the time components, studied in Sec IV, are manifest in Eqs. (95) and (93) herein: $F^0_{\rm G} \simeq -F^\alpha_{\rm G}u_\alpha \neq 0$, telling us that, from the point of view of the static observers $\mathcal{O}(u)$, non-negligible work is done on the gyroscope; but $F^0_{\rm EM} = -F^\alpha_{\rm EM}u_\alpha = 0$ [an exact result, cf. Eq. (77)], telling us that no work is done on the dipole. One may also check that whereas $F^\alpha_{\rm G}U_\alpha = 0$, $F^\alpha_{\rm EM}$ has a nonvanishing time projection in the particle's frame, which, to first order in v, reads $F^{\alpha}_{\rm EM}U_{\alpha} \simeq (B^u)^{ji}\mu_j v_i$.

It should however be noted that the forces above do not translate in a trivial fashion into accelerations; F^{α} is in general not even parallel to a^{α}, as the test particles possess hidden momentum. Assuming $\mu^{\alpha} = \sigma S^{\alpha}$, from Eq. (42) we have, in the electromagnetic case,

$$m_0 a^i \simeq F^i_{\rm EM} + 2B^{[ij]}\mu_j \simeq B^{ij}\mu_j \simeq (B^u)^{ij}\mu_j - (E^u)^{lj}\epsilon^i{}_{kl}v^k\mu_j. \tag{96}$$

The last two terms of (42) are herein neglected. As for the term $\star F_{\beta}{}^{i}D\mu^{\beta}/d\tau$, it follows from Eq. (25), for $\mu^{\alpha} = \sigma S^{\alpha}$, that it is of second order in the fields, thus to be neglected in a coherent comparison with linearized gravity. The term $DP^{\alpha}_{\rm hidI}/d\tau = \epsilon^{\alpha}{}_{\beta\gamma\delta}U^{\delta}D(S^{\gamma}a^{\beta})/d\tau$ is also negligible in this approximation if, among the many possible solutions [21] allowed by the condition $S^{\alpha\beta}U_{\beta} = 0$, we choose the nonhelical one; actually, imposing $P^{\alpha}_{\rm hidI} \approx 0$ amounts, *in this application* (not in general), to picking such a solution, as we explain in detail in [26]. The explicit result, substituting $(E^u)_{ij}$ and $(B^u)_{ij}$ from Eqs. (88) and (89), reads

$$m_0 a^i \simeq \frac{3}{r^5}\left[(\vec{r}\cdot\vec{\mu}_{\rm s})\mu^i + 2r^{(j}\mu_{\rm s}^{i)}\mu_j - 5\frac{(\vec{r}\cdot\vec{\mu}_{\rm s})(\vec{r}\cdot\vec{\mu})r^i}{r^2}\right] + \frac{Q}{r^3}\left[2\vec{v}\times\vec{\mu} + \frac{3\vec{r}[(\vec{v}\times\vec{r})\cdot\vec{\mu}]}{r^2} + 3\frac{(\vec{v}\cdot\vec{r})\vec{\mu}\times\vec{r}}{r^2}\right]^i. \tag{97}$$

In the gravitational system we have, from Eq. (42),

$$m_0 a^i \simeq F^i_{\rm G} \simeq -(\mathbb{H}^u)^{ji}S_j + 2(\mathbb{E}^u)^{l(i}\epsilon^{j)}{}_{kl}v^k S_j. \tag{98}$$

Again the last term of (42) is negligible for the nonhelical representation (in the purely gravitational case, to this accuracy, taking $P^{\alpha}_{\rm hidI} \approx 0 \Rightarrow P^{\alpha} \approx mU^{\alpha}$ works generically as a means of picking such a representation [106]), as explained in detail in [26]. The explicit result reads

$$m_0 a^i \simeq -\frac{3}{r^5}\left[(\vec{r}\cdot\vec{J})S^i + 2r^{(j}J^{i)}S_j - 5\frac{(\vec{r}\cdot\vec{J})(\vec{r}\cdot\vec{S})r^i}{r^2}\right] - \frac{3M}{r^3}\left[\vec{v}\times\vec{S} + \frac{2\vec{r}[(\vec{v}\times\vec{r})\cdot\vec{S}]}{r^2} + \frac{(\vec{v}\cdot\vec{r})\vec{S}\times\vec{r}}{r^2}\right]^i. \tag{99}$$

Comparing with (97) we note that all the terms in the gravitational equation have an electromagnetic counterpart. However, the spin-orbit interaction terms [second lines of Eqs. (97) and (99)] all have differing factors; these factors reflect, in this regime, the consequences of the different symmetries of the tidal tensors, and account for the contrasting effects studied in Sec. III. One may check, for instance, why (99), but not (97), allows for radial motion in the field of *static sources* ($\vec{\mu}_{\rm s} = \vec{J} = 0$): if $\vec{v}$ is radial, $\vec{v}\times\vec{r} = 0$ and $\vec{v}\times\vec{S} = -(\vec{v}\cdot\vec{r})\vec{S}\times\vec{r}/r^2$, so the first and third terms of the second line of Eq. (99) cancel out, yielding $m_0 a^i = 0$. But such cancellation does not occur in the electromagnetic equation (97), which yields $m_0 a^i \neq 0$.

To conclude, from Eqs. (I.3) of Table I we expected that if the fields do not vary along the test particle's worldline (so that $F_{\alpha\beta;\gamma}U^{\gamma} = 0$) then $F^{\alpha}_{\rm EM}$ and $F^{\alpha}_{\rm G}$ should be similar in the weak field approximation, since $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ have the same symmetries and the nonlinearities of the later are negligible; and that otherwise, when $F_{\alpha\beta;\gamma}U^{\gamma} \neq 0$, differences should arise, due to the differing symmetries of the tidal tensors. In the application herein, this translates into the following: when the test particles are at rest with respect to the sources the two forces indeed are similar; however, in the general dynamical case where the particles move, the two forces differ significantly even to first order in the velocity (and in the fields), cf. Eqs. (97) and (99). Thus the tidal tensor formalism makes transparent an aspect that can be rephrased as in [1]: the spin-spin interactions in gravity and electromagnetism are very similar (in this regime), but the "spin-orbit" interactions are substantially different.

In the literature concerning the weak field gravitoelectromagnetic analogy (e.g. [2–5]), the gravitational force acting on a gyroscope is commonly cast in the form $\vec{F}_{\rm G} = K\nabla(\vec{H}\cdot\vec{S})/2$ (where K is some constant depending on the convention, and $H^i \equiv \epsilon^{ijk}g_{0k,j}/K$ is the gravitomagnetic field), similar to its electromagnetic counterpart $\vec{F}_{\rm EM} = \nabla(\vec{B}\cdot\vec{\mu})$, seemingly implying a similarity between the two interactions. We emphasize that such expressions are not suited to describe dynamics, as they hold only if the gyroscope's center of mass is at *rest* in a *stationary* field (this is usually overlooked in the literature, despite the assertion in [1], where this analogy was originally presented, that it was derived under these conditions). A detailed discussion of these issues and comparison with the results in the literature is given in [26].

Section V in brief.—

(i) In the *stationary*, weak field regime, when the particles are at rest with respect to the sources, the gravitational and electromagnetic interactions are very similar, having a similar spin-spin force.

(ii) When the test particles move, the differences (made clear in the symmetries of the tidal tensors) *are of leading order*, thus non-negligible in any slow motion approximation.

VI. BEYOND POLE-DIPOLE—THE TORQUE ON THE SPINNING PARTICLE

In the pole-dipole approximation, as we have seen in Sec. II C, it follows from Eq. (25) that purely magnetic dipoles with $\vec{\mu} = \sigma\vec{S}$ have S^2 as a constant of the motion. This might be somewhat surprising. If one imagines the magnetic dipole as a spinning charged body, one expects, in a time-varying magnetic field, the induced electric field to exert in general (due to its curl) a net torque on it, which will accelerate[13] its rotation. Indeed, we have seen in Sec. IV that the induced electric field does work on the spinning body, causing a variation $dm/d\tau = -\vec{\mu}\cdot D\vec{B}/d\tau$ of its proper mass m. Such variation is known, from the nonrelativistic treatments in [83,85], where a rigid spherical body is considered, to be a variation of rotational kinetic energy.[14] Thus we expect it to be associated to a variation of the spinning angular velocity, and hence of S^2. However, the dipole torque in Eqs. (25) and (26) consists only of the term $\vec{\mu}\times\vec{B}$ (which is there even if the field is constant, and conserves S^2); there is no term coupling to the *derivatives* of the electromagnetic fields, i.e., no sign of induction phenomena.

As we shall see below, this apparent inconsistency is an artifact inherent to the pole-dipole approximation, where terms $\mathcal{O}(a^2)$ ($a \equiv$ size of the particle), which are of quadrupole order, are neglected. Indeed, whereas the contribution of the work done by the induced electric field to the body's energy is of the type $\vec{\mu}\cdot\vec{B}$, i.e., of dipole order, the associated torque involves *the trace of* the quadrupole moment of the charge distribution. Moreover, there is no analogous torque in the gravitational case, confirming the absence of an analogous gravitational induction effect.

For clarity, we will treat the two interactions (electromagnetic and gravitational) separately.

A. Electromagnetic torque

We start by the electromagnetic case in flat spacetime. The equation for the spin evolution of an extended spinning charged body subject to an electromagnetic field is, up to quadrupole order [e.g. Eq. (8.5) of [30]],

$$\frac{DS^{\alpha\beta}_{\rm can}}{d\tau} = 2P^{[\alpha}_{\rm Dix}U^{\beta]} + 2Q^{\theta[\beta}F^{\alpha]}{}_{\theta} + 2m^{[\alpha}{}_{\rho\mu}F^{\beta]\mu;\rho}, \quad (100)$$

where $P^{\alpha}_{\rm Dix}$ and $S^{\alpha\beta}_{\rm can}$ are defined by Eqs. (A7) and (A8), and consist on the sum of the *physical* momenta $P^{\alpha}, S^{\alpha\beta}$, Eqs. (4) and (5), plus *electromagnetic terms*, see Appendix A. It is shown in [107] that $S^{\alpha\beta}_{\rm can}$ and $P^{\alpha}_{\rm Dix} + qA^{\alpha} \equiv P^{\alpha}_{\rm can}$ are the canonical momenta associated to the Lagrangian of the system. In the equation above $Q^{\alpha\beta}$ is the electromagnetic dipole moment as defined in (A5), and $m^{\alpha\beta\gamma}$ is an electromagnetic quadrupole moment, defined as

$$m^{\alpha\beta\gamma} \equiv \frac{4}{3}Q^{(\alpha\beta)\gamma}; \qquad Q^{\alpha\beta\gamma} \equiv \mathcal{J}^{\alpha[\beta\gamma]} + \frac{1}{2}q^{\alpha[\beta}U^{\gamma]}, \quad (101)$$

where $\mathcal{J}^{\alpha\beta\gamma}$ and $q^{\alpha\beta}$ are, respectively, the current and charge "quadrupole moments,"[15] see Eqs. (3.8) and (3.9) of [30][16]:

$$q^{\hat{\alpha}\hat{\beta}} \equiv \int_{\Sigma(\tau,U)} x^{\hat{\alpha}}x^{\hat{\beta}}j^{\hat{\gamma}}d\Sigma_{\hat{\gamma}}; \quad (102)$$

$$\mathcal{J}^{\hat{\alpha}\hat{\beta}\hat{\nu}} \equiv \int_{\Sigma(\tau,U)} x^{\hat{\alpha}}x^{\hat{\beta}}j^{\hat{\nu}}d\Sigma. \quad (103)$$

In flat spacetime, the normal coordinates $\{x^{\hat{\alpha}}\}$ are just a rectangular coordinate system originating at $z^{\alpha}(\tau)$. Decomposing $\mathcal{J}^{\alpha\beta\nu}$ into its projections parallel and orthogonal to U^{ν}, we obtain

$$\mathcal{J}^{\alpha\beta\nu} = q^{\alpha\beta}U^{\nu} + \mathcal{J}^{\alpha\beta\gamma}(h^U)^{\nu}_{\gamma}, \quad (104)$$

where we noted that, in flat spacetime, $\Sigma(\tau,U)$ is a *hyperplane* orthogonal to $n^{\alpha} = U^{\alpha}$, thus $-j^{\nu}U_{\nu}d\Sigma = j^{\gamma}d\Sigma_{\gamma}$.

Using Eq. (A11ii), we may rewrite Eq. (100) explicitly in terms of the physical angular momentum $S^{\alpha\beta}$:

$$\frac{DS^{\alpha\beta}}{d\tau} = \frac{DS^{\alpha\beta}_{\rm can}}{d\tau} - \frac{DS'^{\alpha\beta}}{d\tau}; \qquad S'^{\alpha\beta} = F^{[\alpha}{}_{\sigma}q^{\beta]\sigma}. \quad (105)$$

Note that $DS'^{\alpha\beta}/d\tau$ is a quadrupole type contribution.

We are interested in the torque τ^{α}, i.e., the vector that measures the rate of deviation of the spin vector from Fermi-Walker transport, Eq. (17):

$$\tau^{\alpha} \equiv \frac{D_F S^{\alpha}}{d\tau} \Rightarrow \tau^{\sigma} = \frac{1}{2}\epsilon_{\alpha\beta}{}^{\sigma\delta}U_{\delta}\frac{DS^{\alpha\beta}}{d\tau}. \quad (106)$$

Using Eqs. (100) and (105), it follows that

$$\frac{D_F S^{\alpha}}{d\tau} = \tau^{\alpha}_{\rm DEM} + \tau^{\alpha}_{\rm QEM}; \quad (107)$$

[13] Unlike the dipole torque $\vec{\tau} = \vec{\mu}\times\vec{B}$, the torque due to the induced electric field will not in general be orthogonal to $\vec{S}$, and hence will change its magnitude. For instance, in the example in Fig. 5(a) below, $\vec{E}_{\rm ind}$ has circular lines around $\vec{S}$, so that $\vec{\tau}_{\rm ind}\|\vec{S}$.

[14] It is not cast therein as a variation of proper mass m (as those are nonrelativistic treatments), but of the Hamiltonian term $-\vec{\mu}\cdot\vec{B}$.

[15] Following the convention in e.g. [20,35,36], we dub integrals of the type (102) and (103) quadrupole moments. However, frequently in the literature the term "charge quadrupole moment" refers to the traceless part of $q^{\alpha\beta}$. Note that $q^{\alpha\beta} \neq 0$ for a uniform spherical body, contrary with its traceless part, which measures a type of deviation from spherical symmetry (more consistent with the actual picture of a quadrupole of charges). Sometimes (e.g. [23], p. 977) $q^{\alpha\beta}$ is called the "second moment of the charge."

[16] In Eq. (3.8) of [30], $w^{\gamma}d\Sigma_{\gamma}$, instead of $d\Sigma$, appears; however, $w^{\hat{\gamma}} = n^{\hat{\gamma}} + \mathcal{O}(x^2)$, cf. Eq. (A1), yielding a correction to the integrand of order $\mathcal{O}(x^4)$, negligible to quadrupole order [where only terms up to $\mathcal{O}(x^2)$ are to be kept [31]]. Hence we can take therein $w^{\gamma}d\Sigma_{\gamma} \simeq n^{\gamma}d\Sigma_{\gamma} = d\Sigma$, cf. Eq. (9).

$$\tau^\sigma_{\rm DEM} \equiv \epsilon^\sigma{}_{\alpha\beta\nu} U^\nu (d^\alpha E^\beta + \mu^\alpha B^\beta); \tag{108}$$

$$\tau^\sigma_{\rm QEM} \equiv \tau^\sigma_{\rm QEMcan} - \tau'^\sigma; \tag{109}$$

$$\tau^\sigma_{\rm QEMcan} \equiv \epsilon^\sigma{}_{\alpha\beta\nu} U^\nu m^{[\alpha}{}_{\rho\mu} F^{\beta]\mu;\rho}; \tag{110}$$

$$\tau'^\sigma = \frac{1}{2}\epsilon^\lambda{}_{\alpha\beta\nu} U^\nu E^{[\alpha\beta]}(q^\gamma{}_\gamma \delta^\sigma_\lambda - q^\sigma{}_\lambda) + \frac{1}{2}\epsilon^\sigma{}_{\alpha\beta\nu} U^\nu F^\alpha{}_\gamma \frac{Dq^{\beta\gamma}}{d\tau}. \tag{111}$$

Here $\tau^\alpha_{\rm DEM}$ is the dipole torque already present in Eq. (16), i.e., just a covariant form for $\vec{\tau} = \vec{\mu} \times \vec{B} + \vec{d} \times \vec{E}$. We split the quadrupole torque $\tau^\sigma_{\rm QEM}$ into two parts. The first part, $\tau^\sigma_{\rm QEMcan}$, which we may dub the "canonical electromagnetic quadrupole torque," is the torque[17] coming from the third term of (100) (i.e. the quadrupole contribution to $DS^{\alpha\beta}_{\rm can}/d\tau$). The second part, $\tau'^\sigma \equiv \frac{1}{2}\epsilon_{\alpha\beta}{}^{\sigma\delta} U_\delta DS'^{\alpha\beta}/d\tau$, plays a crucial role in this discussion, since the first term of (111) is minus the torque due to the *electric field induced* in the CM frame by the Maxwell-Faraday law (23). This is what we shall now see.

1. The induction torque

Consider the rectangular coordinates $\{x^{\hat\alpha}\}$ to be comoving with the particle's CM, $\partial_{\hat 0} = \mathbf{U}$. In such a frame, the torque (about the CM) due to the induced electric field is $\vec{\tau}_{\rm ind} = \int \rho_c \vec{x} \times \vec{E}_{\rm ind} d^3x$, where $\rho_c \equiv -j^\alpha U_\alpha$ is the charge density in the CM frame. Let us expand $\vec{E}$ in a Taylor series around the CM: $E^{\hat i} = E^{\hat i}_{\rm CM} + E^{\hat i,\hat j}_{\rm CM} x_{\hat j} + \cdots$ (for the integral above, to quadrupole order, only terms up to linear order in $\vec{x}$ are to be kept in this expansion), which, splitting $E^{\hat i,\hat j}_{\rm CM} = E^{[\hat i,\hat j]}_{\rm CM} + E^{(\hat i,\hat j)}_{\rm CM}$, we may write as

$$E^{\hat i} = E^{\hat i}_{\rm CM} - \frac{1}{2}[\vec{x} \times (\nabla \times \vec{E})_{\rm CM}]^{\hat i} + E^{(\hat i,\hat j)}_{\rm CM} x_{\hat j}.$$

The second term is the part of $\vec{E}$ that has a curl, that is, the induced electric field: $\vec{E}_{\rm ind}(x) \approx -\vec{x} \times (\nabla \times \vec{E})_{\rm CM}/2$. (The third term may be cast as a gradient of some scalar function, thus not related with induction.) Therefore, recalling the definition of $q^{\alpha\beta}$, Eq. (102) above, we have

$$\tau^{\hat i}_{\rm ind} = -\frac{1}{2}(\nabla \times \vec{E}_{\rm CM})^{\hat j} \int \rho_c [x^{\hat i} x_{\hat j} - \delta^{\hat i}{}_{\hat j} x^2] d^3x = -\frac{1}{2}(\nabla \times \vec{E}_{\rm CM})^{\hat j} [q^{\hat i}{}_{\hat j} - \delta^{\hat i}{}_{\hat j} q^\gamma{}_\gamma], \tag{112}$$

which, by relations (22), is a noncovariant form for

$$\tau^\alpha_{\rm ind} = \frac{1}{2}\epsilon^\sigma{}_{\mu\nu\lambda} U^\lambda E^{[\mu\nu]}[q^\alpha{}_\sigma - \delta^\alpha{}_\sigma q^\gamma{}_\gamma] \tag{113}$$

$$= \frac{1}{2} B^\sigma{}_\beta U^\beta [q^\alpha{}_\sigma - \delta^\alpha{}_\sigma q^\gamma{}_\gamma], \tag{114}$$

i.e., the first term of (111). In the second equality we used Eqs. (I.4a) of Table I.

2. Rigid spinning charged body

Consider the case when the test particle is a charged, "quasirigid" body [27,47,108], rotating with an angular velocity Ω^α, defined as follows. If A^α is a spatial vector with origin at the CM, orthogonal to the CM 4-velocity U^α, and *fixed* to the body, then

$$\frac{D_F A^\alpha}{d\tau} = \Omega^\alpha{}_\beta A^\beta; \qquad \Omega_{\alpha\beta} = \epsilon_{\beta\alpha\mu\nu}\Omega^\mu U^\nu. \tag{115}$$

Let $U^\alpha_{\rm p}$ be the 4-velocity field of the points in the body; we may write $U^\alpha_{\rm p} = \gamma_{\rm p}(U^\alpha + v^\alpha_{\rm p})$; $\gamma_{\rm p} \equiv -U^\alpha_{\rm p} U_\alpha$, cf. Eqs. (67), where $v^\alpha_{\rm p} = \Omega^\alpha{}_\beta x^\beta$ is the velocity of a point in the body relative to the CM frame. It follows that the charge 4-current density is

$$j^\alpha = \rho_c(U^\alpha + v^\alpha_{\rm p}) = \rho_c(U^\alpha + \Omega^\alpha{}_\beta x^\beta),$$

whose space part reads, in the CM frame, $j(\vec{r}) = \rho_c \vec{\Omega} \times \vec{x}$; here $\rho_c = -j^\alpha U_\alpha$ is the charge density as measured in the CM frame. The magnetic dipole moment, Eq. (8), then becomes

$$\mu^{\hat\alpha} = \frac{\Omega^{\hat\beta}}{2}[\delta^{\hat\alpha}{}_{\hat\beta} q^\gamma{}_\gamma - q^{\hat\alpha}{}_{\hat\beta}], \tag{116}$$

where we used (102), and noted that $\rho_c d\Sigma = j^\gamma d\Sigma_\gamma$. Thus the rate of work done on this body by the induction torque $\tau^\alpha_{\rm ind}$, $\mathcal{P} = \tau^\alpha_{\rm ind}\Omega_\alpha$, is, from Eqs. (113) and (114),

$$\tau^\alpha_{\rm ind}\Omega_\alpha = -\epsilon_{\beta\mu\nu\lambda} U^\lambda E^{[\mu\nu]} \mu^\beta = -B^\alpha{}_\beta U^\beta \mu_\alpha = -F^\alpha_{\rm EM} U_\alpha. \tag{117}$$

That is, we obtain precisely the work $\mathcal{P}_{\rm ind} = -F^\alpha_{\rm EM} U_\alpha$ of Sec. IV A, Eqs. (72) and (73). This is the result we seek: we have just proved that the work transferred to the body by Faraday's law of induction, which, to pole-dipole order, is manifest in the projection along U^α of the dipole force $F^\alpha_{\rm EM}$ (and in the variation of the proper mass $dm/d\tau$), is indeed

[17] In the literature concerning Dixon's multipole scheme, $\tau^\sigma_{\rm QEMcan}$ is commonly portrayed as the quadrupole torque, see e.g. [20]. However, it is clear from Eq. (107) that it is not the *total* quadrupole torque $\tau^\sigma_{\rm QEM}$, and the results below show how crucial this distinction is.

associated to an induction torque, which causes S^2 to vary as expected (since $\tau^\alpha_{\rm ind}$ is not orthogonal to S^α in general). This torque was known to exist from some nonrelativistic treatments [83–85] dealing with the special case of spinning *spherical* charged bodies. It just happens that it is *not* manifest to pole-dipole order, as it involves the second moment of the charge $q_{\alpha\beta}$, which is of quadrupole order.[18] But the rate of work that this torque does, $\tau^\alpha_{\rm ind}\Omega_\alpha$, is manifest to dipole order, since, for a rigid body, $q_{\alpha\beta}$ and Ω^α combine into the magnetic dipole moment μ^α, by virtue of Eq. (116).

3. Torque on spherical charged body

In this context, and in view of a comparison with the gravitational problem, it is interesting to consider the case of a uniform, spherical charged body, whose quadrupole moments of j^α reduce to the trace of $q_{\alpha\beta}$, so that we expect the total quadrupole torque on the particle τ^α to come essentially from $\tau^\alpha_{\rm ind}$.

First let us explicitly compute the quadrupole moments for this type of body. It is clear that the charge quadrupole, Eq. (102), is such that, in rectangular coordinates $\{x^{\hat\alpha}\}$ originating at the center of mass and comoving with it ($\partial_{\hat 0}=\mathbf{U}$), its time components are zero, $q^{\hat 0\hat 0}=q^{\hat 0\hat i}=0$, and its spatial part reduces to its trace, $q^{\hat i\hat j}=\delta^{\hat i\hat j}q^{\hat k}{}_{\hat k}/3$.

Such tensor is covariantly written as

$$q^{\alpha\beta}=\frac{1}{3}q^\tau{}_\tau(h^U)^{\alpha\beta}. \tag{118}$$

As for the tensor $\mathcal{J}^{\alpha\beta\gamma}$, due to the axisymmetry and the reflection symmetry with respect to the equatorial plane, all of its spatial components $\mathcal{J}^{\hat i\hat j\hat k}$ in the CM frame vanish. The only nonvanishing components are $\mathcal{J}^{\hat i\hat j\hat 0}=\int_{x^{\hat 0}=0}x^{\hat i}x^{\hat j}j^{\hat 0}d^3x=q^{\hat i\hat j}$. Hence $\mathcal{J}^{\alpha\beta\gamma}(h^U)^\nu_\gamma=0$; thus, by virtue of Eq. (104), $\mathcal{J}^{\alpha\beta\nu}=q^{\alpha\beta}U^\nu$, and therefore

$$\mathcal{J}^{\alpha\beta\nu}=\frac{1}{3}q^\sigma{}_\sigma U^\nu(h^U)^{\alpha\beta}. \tag{119}$$

Substituting (118) and (119) into (101), we have

$$Q^{\alpha\beta\gamma}=\frac{1}{2}q^\tau{}_\tau(h^U)^{\alpha[\beta}U^{\gamma]}=\frac{1}{2}q^\tau{}_\tau g^{\alpha[\beta}U^{\gamma]}. \tag{120}$$

Let us now compute the quadrupole torque exerted on the body, Eq. (109). Substituting (120), (101) into Eq. (110), we obtain

$$\tau^\sigma_{\rm QEMcan}=\frac{1}{3}q^\gamma{}_\gamma U^{[\alpha}F^{\beta]\lambda}{}_{;\lambda}\epsilon_{\alpha\beta}{}^{\sigma\delta}U_\delta=0, \tag{121}$$

the second equality holding in vacuum (which is the problem at hand) by virtue of Maxwell's equations $F^{\alpha\beta}{}_{;\beta}=4\pi j^\alpha$. This means that $\tau^\sigma_{\rm QEM}=-\tau'^\sigma$. In order to compute τ'^σ, Eq. (111), we must give a law of evolution for $q_{\alpha\beta}$. Equation (118) guarantees that the body is spherical; we also demand $dq^\alpha_\alpha/d\tau=0$, so that it has constant size (in a comoving frame). Together, these relations imply that $q_{\alpha\beta}$ is Fermi-Walker transported, $D_Fq_{\alpha\beta}/d\tau=0$, i.e., it has constant components in an orthonormal tetrad comoving with the body's CM, as expected. It then follows from Eqs. (109), (111), (113) and (114) that the quadrupole torque reduces to

$$\tau^\sigma_{\rm QEM}=-\tau'^\sigma=\tau^\sigma_{\rm ind}+\frac{1}{6}\epsilon^\sigma{}_{\alpha\beta\lambda}U^\lambda a^\alpha E^\beta q^\gamma{}_\gamma; \tag{122}$$

$$\tau^\sigma_{\rm ind}=-\frac{q^\gamma{}_\gamma}{3}\epsilon^\sigma{}_{\alpha\beta\lambda}U^\lambda E^{[\alpha\beta]}=-\frac{q^\gamma{}_\gamma}{3}B^\sigma{}_\beta U^\beta. \tag{123}$$

In other words, up to an acceleration dependent term (arising from the Fermi-Walker transport of $q_{\alpha\beta}$), $\tau^\sigma_{\rm QEM}$ is *the torque due to the induced electric field.*

To compare with the results known in the literature, consider a body with uniform charge and mass densities. For such a body we may write $2\sigma I^\alpha{}_\beta=(q_\gamma{}^\gamma(h^U)^\alpha_\beta-q^\alpha{}_\beta)$, where $\sigma\equiv q/2m$ is the classical gyromagnetic ratio and $I_{\alpha\beta}$ the moment of inertia tensor (cf. footnote 18); in the spherical case we have $q_\sigma{}^\sigma/3=\sigma I_\sigma{}^\sigma/3=\sigma I$, where $I=I_{zz}=I_{xx}=I_{yy}$ denotes the moment of inertia of the sphere with respect to any axis of rotation passing through its center. Thus $\tau^\alpha_{\rm ind}=-\sigma I\epsilon^\alpha{}_{\mu\nu\lambda}U^\lambda E^{[\mu\nu]}=-\sigma IB^\alpha{}_\beta U^\beta$. In the CM frame, and in vector notation, the total torque (106) on such a body reads

$$\vec\tau\equiv\frac{D\vec S}{d\tau}=\vec\tau_{\rm DEM}+\vec\tau_{\rm QEM}=\vec\mu\times\vec B-\sigma I\frac{D\vec B}{d\tau}-\frac{\sigma I}{2}\vec a\times\vec E, \tag{124}$$

which is the relativistic generalization of Eq. (1) of [83], or Eq. (6) of [84] [those nonrelativistic results follow from Eq. (124) above by replacing $\tau\to t$, and neglecting the acceleration dependent term].

Work done on the particle and rotational kinetic energy.—Let us now compute the work, $\tau^\sigma\Omega_\sigma$, done by the total torque $\tau^\alpha=\tau^\alpha_{\rm DEM}+\tau^\alpha_{\rm QEM}$ on the particle. First note that, for a quasirigid body, the relation $S^\alpha=I^{\alpha\beta}\Omega_\beta$ holds [27]; which, for a uniform spherical body, becomes

$$S^\alpha=I\Omega^\alpha. \tag{125}$$

[18]Note also that in order to assign a moment of inertia $I_{\alpha\beta}$ and an angular velocity to a spinning particle one must go beyond dipole order, as $I_{\alpha\beta}=(h^U)_{\alpha\beta}(m_{\rm Q})^\tau{}_\tau-(m_{\rm Q})_{\alpha\beta}$ (cf. e.g. [23]), where $(m_{\rm Q})_{\alpha\beta}$ is the mass quadrupole, Eq. (136).

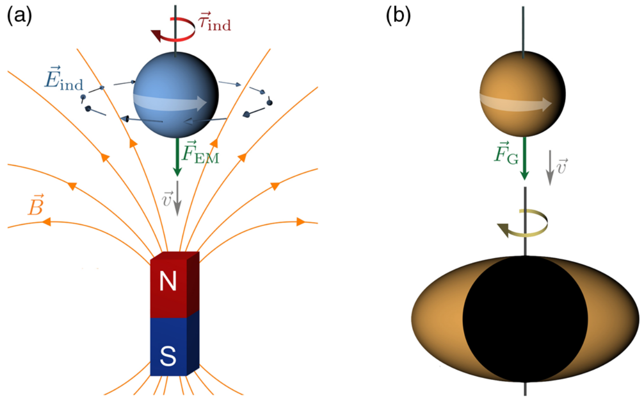

FIG. 5. (a) A spinning, positively charged spherical body being pulled by a strong magnet; $\vec{E}_{\rm ind} \equiv$ electric field induced in the body's CM frame. (b) A spinning spherical body falling into a Kerr black hole. As the spinning charge moves in the inhomogeneous magnetic field $\vec{B}$, a torque $\tau^\alpha_{\rm ind}$, Eq. (123), is exerted on it due to $\vec{E}_{\rm ind}$, i.e., due to the skew part $E_{[\alpha\beta]}$ of the electric tidal tensor. This causes S, and the body's angular velocity $\Omega = S/I$, to vary. The torque $\tau^\alpha_{\rm ind}$ does work at a rate $\tau^\alpha_{\rm ind}\Omega_\alpha = \mathcal{P}_{\rm ind}$, which exactly matches the time projection of the dipole force $F^\alpha_{\rm EM}$ it its rest frame, cf. Eq. (117). This causes the body's kinetic energy of rotation to decrease, manifest in a decrease of proper mass m, and canceling out the gain in translational kinetic energy ($\mathcal{P}_{\rm trans}$), so that the total work transfer, as *measured in the laboratory frame*, is *zero* (cf. Sec. IV B). In the gravitational case no analogous induction effects occur (as expected, since $\mathbb{E}_{[\alpha\beta]} = 0$): no torque is exerted on the spinning particle; its angular momentum S, angular velocity Ω, and proper mass m, are constant; and there is a *net* work done on it by $F^\alpha_{\rm G}$ at a rate $\mathcal{P}_{\rm tot} = F^\alpha_{\rm G} v_\alpha$, corresponding to an increase of translational kinetic energy.

Hence, assuming the proportionality $\mu^\alpha = \sigma S^\alpha$, it follows from Eq. (108) (with $d^\alpha = 0$, which is the problem at hand) that the work of the dipole torque is zero, $\tau^\alpha_{\rm DEM}\Omega_\alpha = 0$. Thus, $\tau^\sigma\Omega_\sigma = \tau^\alpha_{\rm QEM}\Omega_\alpha$. From Eqs. (116) and (118), we have

$$\mu^\alpha = \frac{1}{3}\Omega^\alpha q^\gamma{}_\gamma, \tag{126}$$

and therefore, from Eqs. (122) and (33),

$$\tau^\sigma\Omega_\sigma = \tau^\sigma_{\rm ind}\Omega_\sigma + \frac{1}{2}\epsilon^\sigma{}_{\alpha\beta\lambda}U^\lambda a^\alpha E^\beta \mu_\sigma = \tau^\sigma_{\rm ind}\Omega_\sigma - \frac{1}{2}P^\alpha_{\rm hidEM}a_\alpha.$$

Now consider the case when there is no electromagnetic hidden momentum ($P^\alpha_{\rm hidEM} = 0$), as is the case of the setup in Fig. 5(a); then $\tau^\sigma\Omega_\sigma = \tau^\sigma_{\rm ind}\Omega_\sigma$. On the other hand, from Eqs. (106) and (125), we have that $\tau^\sigma = ID_F\Omega^\sigma/d\tau$ and $2\tau^\sigma\Omega_\sigma = Id(\Omega^2)/d\tau$. Therefore, using (117), we obtain

$$\frac{1}{2}I\frac{d(\Omega^2)}{d\tau} = \tau^\sigma_{\rm ind}\Omega_\sigma = -F^\alpha_{\rm EM}U_\alpha. \tag{127}$$

Note that $I\Omega^2/2$ is the body's kinetic energy of rotation about its CM, see e.g. [27,47]; hence Eq. (127) tells us that, for this setup, the rate of variation of the body's kinetic energy of rotation equals the rate of work, *as measured in the CM frame*, done by the dipole force $F^\alpha_{\rm EM}$ on the particle (that is, its projection $-F^\alpha_{\rm EM}U_\alpha$).

Observing, from Eqs. (I.1a) of Table I, (2), and (33), that $P^\alpha_{\rm hidEM} = 0$ implies $F^\alpha_{\rm EM}U_\alpha = \mu_\alpha DB^\alpha/d\tau$, and using $\mu^\alpha = \sigma S^\alpha$, together with Eqs. (107), (108), (122) and (123), we can rewrite Eq. (127) as

$$\begin{aligned}\frac{I}{2}\frac{d(\Omega^2)}{d\tau} &= -\frac{d(B^\alpha\mu_\alpha)}{d\tau} + B^\alpha\frac{D\mu_\alpha}{d\tau}\\ &= -\frac{d(B^\alpha\mu_\alpha)}{d\tau} - \frac{\sigma q^\gamma_\gamma}{6}\left[\frac{d(B^2)}{d\tau} + \epsilon^\sigma{}_{\alpha\beta\delta}U^\delta a^\alpha E^\beta B_\sigma\right],\end{aligned} \tag{128}$$

which is the relativistic generalization of Eq. (10) of [84] (the acceleration dependent term is absent therein). From this equation we see that, for this setup, the varying part of the mass, $-B^\alpha\mu_\alpha$, present in the dipole approximation, Eq. (41), is *kinetic energy of rotation* (*not* potential energy, as claimed in some literature, e.g. [27,38,81]). This establishes, in a relativistic covariant formulation, and in the context of Dixon's multipole approach, the claims in [82–85]. The second terms in the right members of Eq. (128), of quadrupole order, are not manifest in the *dipole order* mass equation (40), since to that accuracy $B^\alpha D\mu_\alpha/d\tau = 0$, by virtue of Eq. (25).

B. Gravitational torque

The equation for the spin evolution of an extended body in a gravitational field is, up to quadrupole order [20,36]

$$\frac{DS^{\kappa\lambda}}{d\tau} = 2P^{[\kappa}U^{\lambda]} + \frac{4}{3}J^{\mu\nu\rho[\kappa}R^{\lambda]}{}_{\rho\mu\nu}, \tag{129}$$

leading to the torque [cf. Eq. (106)]

$$\frac{D_F S^\sigma}{d\tau} = \tau^\sigma_{\rm QG}, \qquad \tau^\sigma_{\rm QG} \equiv \frac{4}{6}J^{\mu\nu\rho[\kappa}R^{\lambda]}{}_{\rho\mu\nu}\epsilon_{\kappa\lambda}{}^{\sigma\delta}U_\delta. \tag{130}$$

Here [cf. Eqs. (9.12) of [36] or (5.29) of [30]]

$$J^{\alpha\beta\gamma\delta} = \frac{1}{2}(t^{\gamma[\alpha\beta]\delta} - t^{\delta[\alpha\beta]\gamma}) - U^{[\alpha}p^{\beta][\gamma\delta]} - U^{[\gamma}p^{\delta][\alpha\beta]}, \tag{131}$$

where the moments $t^{\alpha\beta\gamma\delta}$ and $p^{\alpha\beta\gamma}$ can be written, in Riemann normal coordinates $\{x^{\hat\alpha}\}$, as

$$t^{\hat\alpha\hat\beta\hat\gamma\hat\delta} \equiv \int_{\Sigma(\tau,U)} x^{\hat\alpha}x^{\hat\beta}T^{\hat\gamma\hat\delta}d\Sigma; \tag{132}$$

$$p^{\hat\alpha\hat\beta\hat\gamma} \equiv \int_{\Sigma(\tau,U)} x^{\hat\alpha}x^{\hat\beta}J^{\hat\gamma}d\Sigma, \tag{133}$$

where $J^\gamma \equiv -T^{\gamma\delta}n_\delta$ is the mass/energy current as measured by the observers orthogonal to $\Sigma(\tau, U)$ [so that $T^{\gamma\delta}d\Sigma_\delta = J^\gamma d\Sigma$, cf. Eq. (9)]. Expressions (132) and (133) correspond,[19] in flat spacetime, to Eqs. (5.2) and (5.3) of [30], and, in curved spacetime, to Eqs. (9.4) and (9.11) of [36]. They are *tensors* (similarly to the expressions in [36]), since the use of Riemann normal coordinates $\{x^{\hat{\alpha}}\}$ amounts to defining the moments in terms of the exponential map (see [31,33]).

The tensor $p^{\alpha\beta\gamma}$ has the interpretation of the quadrupole moment of the mass current, analogous to the quadrupole moment of the charge current $\mathcal{J}^{\alpha\beta\gamma}$, Eq. (103). Note moreover that $-t^{\alpha\beta\gamma\delta}U_\delta = p^{\alpha\beta\gamma}$, since $n_{\hat{\alpha}} = U_{\hat{\alpha}} + \mathcal{O}(x^2)$, cf. Eq. (A2), and therefore, to quadrupole order, we may take $J^\gamma \simeq -T^{\gamma\delta}U_\delta$ in (133). We may thus decompose $t^{\alpha\beta\gamma\delta}$ as

$$t^{\alpha\beta\gamma\delta} = p^{\alpha\beta\gamma}U^\delta + p^{\alpha\beta\sigma}(h^U)^\delta{}_\sigma U^\gamma + t^{\alpha\beta\lambda\sigma}(h^U)^\gamma{}_\lambda (h^U)^\delta{}_\sigma. \tag{134}$$

Similarly, $p^{\alpha\beta\gamma}$ may also be decomposed as

$$p^{\alpha\beta\gamma} = (m_{\rm Q})^{\alpha\beta}U^\gamma + p^{\alpha\beta\lambda}(h^U)^\gamma_\lambda, \tag{135}$$

analogous to (104), where

$$(m_{\rm Q})^{\hat{\alpha}\hat{\beta}} = \int_{\Sigma(\tau,U)} x^{\hat{\alpha}}x^{\hat{\beta}}J^{\hat{\gamma}}d\Sigma_{\hat{\gamma}} \tag{136}$$

is the mass quadrupole (or "second moment of the mass" see [20,23,109] and footnote 15), analogous to the charge quadrupole (102).

1. Torque on "spherical" body

Our goal in this section is to consider the gravitational analogue of the problem in Sec. VI A 3. Therein we considered a spherical charged body in flat spacetime, whose charge quadrupole moment was shown to reduce to its trace, $q^\alpha{}_\beta = q^\tau{}_\tau (h^U)^\alpha{}_\beta/3$, and whose current quadrupole was $\mathcal{J}^{\alpha\beta\nu} = q^\sigma{}_\sigma U^\nu (h^U)^{\alpha\beta}/3$. We prescribe the analogous test body for the gravitational problem by demanding it to have *an analogous multipole structure* (i.e., an analogous "gravitational skeleton" [29]), rather than demanding its shape to be spherical, which in a general curved spacetime is not a well defined notion. (A body with such multipole structure will of course be a sphere in the case of flat spacetime; and otherwise may be thought of as one if the field is not too strong.) As shown above, the quadrupole moment $p^{\alpha\beta\gamma}$, Eq. (133), has an analogous definition to $\mathcal{J}^{\alpha\beta\gamma}$, Eq. (103), only with J^α in the place of j^α; hence its structure must be [analogously to Eq. (119)]:

$$p^{\alpha\beta\gamma} = \frac{1}{3}(m_{\rm Q})^\tau{}_\tau (h^U)^{\alpha\beta}U^\gamma. \tag{137}$$

The last term of (134) is the quadrupole moment of the space part of $T^{\gamma\delta}$, $(h^U)^\gamma{}_\lambda (h^U)^\delta{}_\sigma T^{\lambda\sigma} \equiv T_\perp^{\gamma\delta}$, which has no electromagnetic analogue. For a quasirigid spinning body, we have (e.g. [108,110]) $T^{\alpha\beta}({\rm p}) = \rho U^\alpha_{\rm p} U^\beta_{\rm p} + s^{\alpha\beta}$, where $s^{\alpha\beta}$ are the stresses, $U^\alpha_{\rm p} = \gamma_{\rm p}(U^\alpha + v^\alpha_{\rm p})$ is the 4-velocity of the (rotating) mass element at the point p of the body, $v^\alpha_{\rm p}$ is the spatial velocity of p relative to the center of mass frame, and $\gamma_{\rm p} = -U^\alpha_{\rm p}U_\alpha$, see decomposition (67). Hence $T_\perp^{\alpha\beta} = \rho\gamma_{\rm p}^2 v^\alpha_{\rm p} v^\beta_{\rm p} + s_\perp^{\alpha\beta}$, its two terms being of the same order of magnitude $\sim\rho v_{\rm p}^2$ (e.g. [110]). For nonrelativistic rotation speeds $v_{\rm p} \ll 1$, we have $\|T_\perp^{\alpha\beta}\| \ll \rho$, and therefore the last term of (134) is negligible compared to the others. It then follows:

$$J^{\alpha\beta\gamma\delta} \approx -(m_{\rm Q})^\tau{}_\tau U^{[\alpha}g^{\beta][\gamma}U^{\delta]}, \tag{138}$$

[in agreement with Eq. (7.31) of [27]]. Substituting in Eq. (130), we obtain the gravitational torque:

$$\tau^\sigma_{\rm QG} = -\frac{1}{3}(m_{\rm Q})^\tau{}_\tau U^{[\alpha}R^{\beta]}{}_\mu U^\mu \epsilon_{\alpha\beta}{}^{\sigma\delta}U_\delta = 0,$$

the second equality holding for vacuum ($R^{\mu\nu} = 0$), which (as in the electromagnetic case) is the problem at hand. Thus, no gravitational torque is exerted, up to quadrupole order, on a spinning spherical body.[20] This means that there is no gravitational counterpart to the electromagnetic torque $\tau^\alpha_{\rm ind}$ exerted on the spherical charged body of Sec. VI A 3 (generated, from the viewpoint of the particle's frame, by the induced electric field). This is the result we expected from the discussion in Sec. II B: $\tau^\alpha_{\rm ind}$ comes from the antisymmetric part of $E_{\alpha\beta}$, or, equivalently, from the (time) projection along U^β of $B_{\alpha\beta}$, cf. Eqs. (113) and (114). Since the gravitoelectric tidal tensor $\mathbb{E}_{\alpha\beta}$ is symmetric, and $\mathbb{H}_{\alpha\beta}$ is spatial with respect to U^β, the absence of an analogous torque in gravity is thus natural.

C. Summarizing with a simple realization

The results in Secs. VI A and VI B entirely corroborate the discussion in Sec. IV (and Sec. II E); namely, the manifestation of electromagnetic induction and the absence

[19]Noting that $w^{\hat{\sigma}}d\Sigma_{\hat{\sigma}} = d\Sigma + \mathcal{O}(x^2)$, cf. footnote 16, and that, in the system $\{x^{\hat{\alpha}}\}$, the bitensors in [36] read $\sigma^{\hat{\alpha}} = -x^{\hat{\alpha}}$, $\sigma^{\hat{\alpha}}{}_{\hat{\beta}} = \delta^{\hat{\alpha}}{}_{\hat{\beta}} + \mathcal{O}(x^2)$, $\Theta^{\hat{\kappa}\hat{\lambda}\hat{\mu}\hat{\nu}} = \delta^{\hat{\kappa}(\hat{\mu}}\delta^{\hat{\nu})\hat{\lambda}} + \mathcal{O}(x^2)$, $H_{\hat{\alpha}\hat{\beta}} = \delta_{\hat{\alpha}\hat{\beta}} + \mathcal{O}(x^2)$; so the corrections due to them in (132) and (133) are integrands of order $\mathcal{O}(x^4)$, negligible to quadrupole order [where only terms up to $\mathcal{O}(x^2)$ are to be kept].

[20]This is consistent with the results from the post-Newtonian treatment in e.g. [109], where the approximate vacuum expression $\tau^i_{\rm QG} \approx \epsilon^i{}_{jk}[\mathbb{E}^j{}_l \mathcal{J}^{lk} + 4\mathbb{H}^j{}_l \mathcal{S}^{lk}/3]$ (Eq. (1.9c) therein) is derived. In our notation, $\mathcal{S}^{jk} = \epsilon^{(k}{}_{lm}p^{j)lm}$, $\mathcal{J}_{ij} = (m_{\rm Q})_{ij} - (m_{\rm Q})^k{}_k\delta_{ij}/3$; it then follows from the analysis above that, for a spherical body, $\mathcal{S}_{ij} = \mathcal{J}_{ij} = 0 \Rightarrow \vec{\tau}_{\rm QG} = 0$.

of an analogous phenomenon in the *physical* gravitational forces and torques. In this context, it is interesting to consider the analogous setups in Fig. 5: a spinning spherical charge moving in the field of a strong magnet (or another spinning charged body), and a spinning "spherical" mass moving in Kerr spacetime.

Let us start by the electromagnetic case. A force $F^\alpha_{\rm EM}$, Eq. (I.1a), will be exerted on the particle, causing it to move [thereby gaining translational kinetic energy, at a rate $\mathcal{P}_{\rm trans}$, Eq. (80)]. As it moves in an inhomogeneous magnetic field, a torque $\tau^\alpha_{\rm ind}$ is exerted upon it; from the viewpoint of the observer comoving with the particle, this is due to the electric field induced by the time-varying magnetic field. That torque will cause a variation of the particle's angular momentum S^α, and therefore of its angular velocity $\Omega^\alpha = S^\alpha/I$ [measured with respect to the comoving Fermi-Walker transported tetrad, cf. Eq. (115)]. Clearly, S^2 *is not conserved*, since $d(S^\alpha S_\alpha)/d\tau = 2\tau^\alpha S_\alpha \neq 0$, as we see from Eqs. (122) and (123) or (124). The variation of Ω also implies a variation of the particle's rotational kinetic energy, equal to the work of the torque $\tau^\alpha_{\rm ind}$, which in turn is *exactly* the work done by the dipole force $F^\alpha_{\rm EM}$ as measured in the *frame comoving* with the particle, cf. Eq. (127). (This is reflected in a variation of the particle's proper mass m.) From the point of view of the laboratory frame (i.e., the static observers u^α), no net work is done on the particle, $F^\alpha_{\rm EM}u_\alpha = 0$, and its total energy, $E = -P_\alpha u^\alpha$, is conserved, cf. IV B. That means that the rate of variation in translational kinetic energy $\mathcal{P}_{\rm trans}$ of the center of mass is exactly canceled out by the variation of rotational kinetic energy $\mathcal{P}_{\rm ind}$ (the work of $\tau^\alpha_{\rm ind}$), guaranteeing that a stationary magnetic field does not do work.

In the gravitational case, there is also a net force $F^\alpha_{\rm G}$ on the body, cf. Eq. (I.1b) of Table I, causing it to gain kinetic energy at a rate $\mathcal{P}_{\rm trans} = F^\alpha_{\rm G} v_\alpha$. But no torque is exerted on it; up to quadrupole order we have

$$\frac{D_F S^\alpha}{d\tau} = 0; \qquad S^2 = \text{constant}$$

(i.e., the spin vector of the spinning spherical mass is Fermi-Walker transported), implying $\Omega = $ constant. This is consistent with the constancy of the proper mass (manifest in the fact that $F^\alpha_{\rm G}$ is orthogonal to U^α), because, since there is no torque, the kinetic energy of rotation is constant. Thus in this case the gain in translational kinetic energy is not canceled out by a variation of rotational kinetic energy, and therefore a stationary gravitomagnetic field will do a net rate of work $-F^\alpha_{\rm G}u_\alpha = \mathcal{P}_{\rm trans}$ on the particle.

We close this section with a few additional remarks. The application in Fig. 5 illustrates an important aspect of the frame dragging effect, and the contrast with the electromagnetic analogue. For clarity, let us consider the case when the test balls are initially nonspinning. In the electromagnetic case, Fig. 5(a), as the ball moves towards the magnet, it starts spinning, increasingly faster (relative to the Fermi-Walker transported tetrad) due to the torque $\tau^\alpha_{\rm ind}$. In the frame comoving with the ball, $\vec{\tau}_{\rm ind}$ is due to the induced electric field $\vec{E}_{\rm ind}$; and from the point of view of the laboratory frame (static observers), where the field is stationary (thus there is no induced electric field therein), $\vec{\tau}_{\rm ind}$ comes from the overall effect of the Lorentz force $dq\vec{v}\times\vec{B}$ applied to each charge element dq of the ball. In the gravitational case, Fig. 5(b), no such rotation arises. *If* initially $\Omega^\alpha = 0$, the ball in Fig. 5(b) will never gain any rotation *relative to the local compass of inertia*; S^α remains always zero. Indeed, an observer sitting firmly with his tetrad on top of the ball will not detect any sign of rotation: he will not measure any Coriolis forces acting on any test particle that he may throw, and will see gyroscope axes fixed. However, from the point of view of a frame adapted to the static observers (which is anchored to the "distant stars," see Sec. II C), the ball indeed starts spinning increasingly faster as it approaches the black hole. This is because, due to frame dragging, a system of axes which is *locally nonrotating* (i.e., Fermi-Walker transported) close to the black hole, is seen to be rotating from a frame fixed to the distant stars. The effect is larger the closer one gets to the black hole, and is quite analogous to the electromagnetic situation *as viewed by the static observers*: in the linear limit, it is well known [2,3,14,68,69,111] that the gravitomagnetic field $\vec{H}$ is very similar to its electromagnetic analogue; then the gravitomagnetic "force" $\vec{v}\times\vec{H}$, acting on each mass element, seemingly leads to an analogous "torque". These are not, however, real forces or torques, but artifacts of the reference frame, not measurable in any local experiment (only by locking the frame to the distant stars, e.g. by means of a telescope); it is therefore no surprise that they are not manifest in the torque equation (130). For indeed *it is the static observers that rotate* relative to the local compass of inertia, which is manifest in the fact that they have vorticity, and measure a nonzero $\vec{H}$ (causing, in their frame, test particles in geodesic motion to be deflected by fictitious Coriolis forces $\vec{v}\times\vec{H}$, and gyroscopes to precess, cf. Sec. II C; for more details, see e.g. [14] Secs. 3.2 and 3.3). This contrasts with the situation in the electromagnetic analogue, where $\tau^\alpha_{\rm ind}$ is a physical, covariant torque, causing the particle to indeed have an accelerated rotation with respect to the local compass of inertia.

Section VI in brief.—

(1) The electromagnetic quadrupole torque contains the torque $\tau^\alpha_{\rm ind}$ due to Faraday's law of induction; it is a coupling of $E_{[\alpha\beta]}$ to $q_{\alpha\beta}$ (the charge quadrupole).
  (a) Dipole approximation ignores $q_{\alpha\beta}$; hence $\tau^\alpha_{\rm ind}$ is not manifest to dipole order;
  (b) but the rate of work it does, $\tau^\alpha_{\rm ind}\Omega_\alpha$, *is of dipole order* ($\Omega^\alpha$ and $q_{\alpha\beta}$ combining into μ^α). For a rigid body, it equals the projection of the dipole force along its worldline, $-F^\alpha_{\rm EM}U_\alpha$.

(2) The torque $\tau^\alpha_{\rm ind}$ has no gravitational analogue (consistent with $\mathbb{E}_{[\alpha\beta]}=0$).
(3) A time-varying electromagnetic field torques a spherical charged body, changing its angular momentum S, angular velocity Ω, and kinetic energy of rotation (manifest in m). The gravitational field *never* torques a "spherical" body; S, Ω, and m, are *constant*.

VII. CONCLUSION

In this paper we studied the dynamics of spinning test particles in general relativity, in the framework of *exact* gravitoelectromagnetic analogies. A detailed summary of the main results and realizations is given in Sec. I; herein we conclude with some additional remarks.

Both equations of motion—force and spin evolution—of a spinning particle in a gravitational field are related to their electromagnetic counterparts by *exact* analogies, valid for generic fields. Moreover, a third analogy arises, for the so-called "hidden momentum," first obtained in [20] as an approximate result, and introduced herein in its exact form. All these analogies are shown to emerge from the rigorous equations of motion for pole-dipole particles if the Mathisson-Pirani spin condition is employed.

The first remark we want to make is that it is important to realize that the existence of these analogies does *not* mean that the interactions are similar. These are *functional* analogies: the magnetic tidal tensor $B_{\alpha\beta}$ plays in Eq. (I.1a) of Table I, for the force exerted on a magnetic dipole, the same role as the gravitomagnetic tidal tensor $\mathbb{H}_{\alpha\beta}$ in Eq. (I.1b) for the gravitational force exerted on a gyroscope. The analogy extends to the Maxwell and Einstein field equations, as manifest in Table I. Moreover, in the appropriate frame, the gravitomagnetic field $\vec{H}$ plays in the precession of the gyroscope an analogous role to $\vec{B}$ in the precession of a magnetic dipole, cf. Eq. (26) (the analogy also extends, under certain conditions, to the equations for the geodesics, for the force on the test particle, and to the field equations, see [6,12,14]). But the analogies *do not imply*, even in seemingly analogous setups, that the objects are similar. First, $\vec{H}$ and $\mathbb{H}_{\alpha\beta}$, unlike their electromagnetic counterparts, are nonlinear. Second, even in the weak field regime (where the nonlinearities of the gravitational field can be neglected), the symmetries and the time projections of the tidal tensors $B_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ continue to differ crucially. The apparent similarity suggested by the usual linear approaches in the literature, e.g. [2–5], can be misleading, as the differing terms in the force/acceleration equations are of leading order, as shown in Sec. V. We have actually seen (Sec. IV) cases where the electromagnetic and gravitational effects are *opposite*: in a frame comoving with the test particle, the work done by the spin-curvature force $F^\alpha_{\rm G}$ is zero ($F^\alpha_{\rm G}U_\alpha=0$) whereas the work of its electromagnetic counterpart $F^\alpha_{\rm EM}$ is nonzero ($F^\alpha_{\rm EM}U_\alpha\neq 0$); from the point of view of static observers u^α, the situation is reversed: it is the electromagnetic force that does no work, $F^\alpha_{\rm EM}u_\alpha=0$ (stationary electromagnetic fields cannot do work on a magnetic dipole) whereas the gravitational one does, $F^\alpha_{\rm G}u_\alpha\neq 0$.

The analogies are instead suited for a comparison between the two interactions, as this amounts to comparing mathematical objects that play analogous dynamical roles in both theories. It is the main point of this work that one can learn a lot (about *both* of them) from such a comparison. The differences in the structure of the gravitational and electromagnetic tidal tensors encode fundamental differences in the interactions, namely the phenomenon of electromagnetic induction, and the way *it manifests itself in the electromagnetic tidal forces and torques*, which has no analogue in gravity. We have seen in Sec. III that $B_{\alpha\beta}$ has an antisymmetric part, reading, in vacuum, $2B_{[\alpha\beta]}=\star F_{\alpha\beta;\gamma}U^\gamma$. This equation (which encodes the Maxwell equation $\nabla\times\vec{B}=\partial\vec{E}/\partial t$) tells us that whenever the field varies along the particle's worldline (e.g. when it moves in a nonuniform electric field), $B_{[\alpha\beta]}\neq 0$, hence $B_{\alpha\beta}$ is nonvanishing, and so a force $F^\alpha_{\rm EM}=B_\beta{}^\alpha\mu^\beta\neq 0$ is exerted on the magnetic dipole (except for some special orientations of $\vec{\mu}$). Such induction effect has no counterpart in gravity, since, in vacuum, $\mathbb{H}_{\alpha\beta}$ is always symmetric; indeed, it is possible for particles moving in a (nonuniform) gravitational field to measure $\mathbb{H}_{\alpha\beta}=0$, so that no force is exerted on them, $F^\alpha_{\rm G}=-\mathbb{H}_\beta{}^\alpha S^\beta=0$. This leads to the existence of geodesic motions for spinning particles, as exemplified in Secs. III A and III B by radial geodesics in Schwarzschild spacetimes, and circular geodesics in Kerr-dS. Reinforcing the insight of the analogy, the velocity fields for which $\mathbb{H}_{\alpha\beta}=0$ mirror the ones where, in the electromagnetic analogue, $B_{\alpha\beta}$ reduces to its antisymmetric part.

Likewise, the results in Sec. IV, concerning the time components of the force, and in Sec. VI, concerning the torque exerted on the spinning particle, are manifestations of the antisymmetric part of the electric tidal tensor $E_{\alpha\beta}$ (or, equivalently, to the projection of $B_{\alpha\beta}$ along U^α), and of the absence of a gravitational counterpart. The antisymmetric part $E_{[\alpha\beta]}$ encodes the Maxwell-Faraday law $\nabla\times\vec{E}=-\partial\vec{B}/\partial t$; the gravitoelectric tidal tensor by contrast is symmetric, $\mathbb{E}_{[\alpha\beta]}=0$, translating in an absence of analogous induction effects in the *physical*[21] gravitational forces and

[21]In the framework of inertial forces, the fact that the time-dependent gravitoelectric $\vec{G}$ and gravitomagnetic $\vec{H}$ fields have a curl, in analogy with their electromagnetic counterparts, can be interpreted as analogous to the electromagnetic induction laws, see e.g. [112]. These, however, are reference frame artifacts; such curls do not contribute to the tidal tensors $\mathbb{E}_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$ (i.e., to the tidal forces, which are the only locally measurable forces of gravity), *only the symmetrized derivatives* of $\vec{G}$ and $\vec{H}$ do. For more details see Secs. 3.5 and 4 of [14].

torques. In this framework, we understood the variation of proper mass m of a classical particle with magnetic dipole moment—it arises from the work done on it by the induced electric field (at a rate $\mathcal{P}_{\rm ind} = -F^{\alpha}_{\rm EM} U_{\alpha}$), encoded in the projection of $B_{\alpha\beta}$ along the particle's 4-velocity U^{α}—and why m is conserved for a gyroscope in a gravitational field—it is because $\mathbb{H}^{\alpha\beta}$ is spatial with respect to U^{α}, signaling the absence of an analogous effect. We have also understood the contrast between the work of these forces as measured by static observers, and the spin dependence of Hawking's upper bound [22] for the energy released when two black holes collide: if one considers a magnetic dipole falling into a strong magnet [Fig. 3(b)], there is no net gain in the particle's energy (from the point of view of static observers); any gain in translational kinetic energy is exactly canceled out by the work transferred to the dipole by Faraday's induction law (i.e., by a loss in proper mass $dm/d\tau = \mathcal{P}_{\rm ind}$), ensuring that the stationary magnetic field does no net work on it. In gravity, however, since $\mathcal{P}_{\rm ind}$ has no counterpart (m is constant), such cancellation does not occur, and therefore a net work $-F^{\alpha}_{\rm G} u_{\alpha} = F^{\alpha}_{\rm G} v_{\alpha}$ is done on a gyroscope; there is a potential energy associated with it, of which the Hawking-Wald spin interaction energy [1] is a special case. In other words, the gravitational spin interaction energy, and the spin dependence of the black hole collision energy (at least in the case where one black hole is much smaller than the other, so that it can be treated as a test particle moving in a stationary field), are justified by the fact that, *unlike its electromagnetic counterpart*, a stationary gravitational (tidal) field does work on mass currents.

The analogies and formalism herein also provide useful tools and intuition for practical applications, which is exemplified in Sec. III. From the *formal* analogy between the quadratic invariants of the Maxwell and Weyl tensors, we guessed that $\mathbb{H}_{\alpha\beta}$ should vanish for observers at rest or moving radially in the Schwarzschild spacetime, in analogy with the situation for B^{α} in a Coulomb field. The tidal tensor form of the spin-curvature force, $F^{\alpha}_{\rm G} = -\mathbb{H}_{\beta}{}^{\alpha} S^{\beta}$, then tells us that no force is exerted on gyroscopes comoving with such observers; for instance, a gyroscope dropped from rest will fall along a geodesic towards the singularity. In the same framework, we predicted that in the equatorial plane of the Kerr or Kerr-dS spacetimes there should be velocity fields for which $\mathbb{H}_{\alpha\beta} = 0$ (because it is so for B^{α} in the equatorial plane of a spinning charge), and from that the existence of circular geodesics for spinning particles in Kerr-dS (which were not known in the literature, to our knowledge). Note that even the problem of the radial fall in the Schwarzschild spacetime (the simplest in this work) could be a complex problem outside the tidal tensor formalism/the Mathisson-Pirani spin condition (involving possibly complicated descriptions, and difficulties in setting up its initial conditions, see Appendix C 1). As for the geodesics for gyroscopes in Kerr-dS, it would be very difficult to ever notice the effect otherwise.

In the course of this paper a number of issues concerning the dynamics of spinning particles in general relativity were clarified. First, the problem of the equations of motion for pole-dipole particles; the gravitational part is well established, but difficulties exist in the electromagnetic part, as there are different versions of the equations in the literature, and inconsistencies in their physical interpretation, whose clarification is the purpose of Appendix A 2. Moreover, the time projections of the forces, their physical content, and relationship with the mass of the particle and the work done by the fields, is ignored in most literature, or misunderstood (e.g. [27,38,81,113,114]); they are thoroughly discussed in Sec. IV and (for particles with electric dipole moment) in Appendix B. Another important clarification was made in Sec. VI A, concerning the quadrupole order torque according to Dixon's equations [20,30,36], and the physical meaning of the quantities involved therein. In their usual form they are equations for the "canonical" angular momentum $S^{\alpha\beta}_{\rm can}$, Eq. (A4), *not* for the physical angular momentum $S^{\alpha\beta}$, Eq. (5); failing to notice this leads one to overlook the torque ($\tau^{\alpha}_{\rm ind}$) exerted on the body due to the curl of the electric field (i.e., to the antisymmetric part of the electric tidal tensor), and to incorrectly conclude e.g. that the electromagnetic field cannot torque a spherical body—which is known, from basic electromagnetism [83–85], to be false, and would be at odds with the variation of the particle's mass discussed in Secs. II E and IV (which, for a rigid body, is essentially a variation of rotational kinetic energy, cf. Sec. VI A 3).

As a future direction, we plan an investigation of the gravitoelectromagnetic analogies in the equations of motion for spinning particles to quadrupole and higher orders in the multipole expansion.

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APPENDIX A: THE EQUATIONS OF MOTION FOR SPINNING PARTICLES IN THE LITERATURE

It is perhaps surprising that the problem of the covariant equations describing the motion of spinning particles subject to gravitational and electromagnetic fields is still not generally well understood, with different methods and derivations leading to different versions of the equations, whose relation is not always clear. Curiously, it is the electromagnetic field that has been posing more problems (some authors [113,114] have even concluded that such covariant description is not possible). The equations of motion for pole-dipole particles in electromagnetic fields are derived in unambiguous forms in [28], for special relativity, and in [34] in the context of general relativity. Rigorous derivations are also given in [20,30,36]; in this case, however, one must be aware of the subtleties involved in their interpretation. These equations (unlike the ones in [28,34,38]) are symmetric with respect to electric and magnetic dipoles; this is actually the most common form of the equations, appearing in many other works, e.g. [27,79,80,115,116]. If not properly interpreted, that would lead to physically inconsistent predictions (given the different nature of the two dipole models), as we shall see below and in Appendix B. Moreover, if one takes the "angular momentum" tensor defined in [30,36] as the physical one, the torque equations therein would, at quadrupole order, seemingly contradict well-known results from elementary electromagnetism (and experimental evidence), as discussed in Sec. VI A. Herein we will dissect these issues and explain how the different versions of the equations relate to each other, and to the ones used in this paper.

1. Relation with the equations used in this paper

Equations (11) and (12) correspond to Dixon's equations (6.31) and (6.32) of [34] [cf. also (3.1) and (3.2) of [38]], with the following simplifications in the definitions of the moments:

(1) instead of the bitensors in [34], we use (following [31]) the exponential map to define the moments in curved spacetime [which amounts to using Riemann normal coordinates $\{x^{\hat{\alpha}}\}$ in the integrals (4)–(8)]. The bitensor $-\sigma^{;\alpha}$ of [34], which is the vector at z^α tangent to the geodesic connecting z^α to the point of integration x^α, and whose length equals that of the geodesic, has, in the system $\{x^{\hat{\alpha}}\}$, coordinates given simply by $-\sigma^{;\hat{\alpha}} = x^{\hat{\alpha}}$. The bitensor of geodesic displacement $\bar{g}^\kappa{}_\alpha$ of [34] reads, in the system $\{x^{\hat{\alpha}}\}$, $\bar{g}^{\hat{\kappa}}{}_{\hat{\alpha}} = \delta^{\hat{\kappa}}{}_{\hat{\alpha}} + \mathcal{O}(x^2)$ (see Appendix of [33]); thus to dipole order (which is linear in x), $\bar{g}^{\hat{\kappa}}{}_{\hat{\alpha}} \simeq \delta^{\hat{\kappa}}{}_{\hat{\alpha}}$, and indeed our definitions of P^α, $S^{\alpha\beta}$ ($\equiv J^{\alpha\beta}$ in [34]), and d^α ($\equiv q^\alpha$ in [34]) agree with [34].

(2) The vector w^γ involved in the definition of $\mu_{\alpha\beta}$ ($\equiv m_{\alpha\beta}$ in [34]) via the moment $j^{\alpha\beta}$ therein, which is a vector such that displacement of every point by $w^\gamma d\tau$ maps $\Sigma(\tau)$ into $\Sigma(\tau + d\tau)$, can, to pole-dipole order, be taken as $w^\gamma \simeq n^\gamma$. That is, $w^\gamma d\Sigma_\gamma \simeq d\Sigma$, cf. Eq. (9). This is easily seen in the case of flat spacetime[22] [30,117], where we have [for $\Sigma(U)$ orthogonal to U^α, and noting that $n^{\hat{\alpha}} = U^{\hat{\alpha}}$]

$$w^{\hat{\gamma}} = n^{\hat{\gamma}}\left(1 - \frac{x_{\hat{\alpha}}}{n_{\hat{\beta}} n^{\hat{\beta}}} \frac{Dn^{\hat{\alpha}}}{d\tau}\right) = n^{\hat{\gamma}}(1 + x_{\hat{\alpha}} a^{\hat{\alpha}}). \quad \text{(A1)}$$

Hence $j^{\hat{\alpha}\hat{\beta}} \equiv \int_{\Sigma(\tau,U)} x^{\hat{\alpha}} j^{\hat{\beta}} w^{\hat{\gamma}} d\Sigma_{\hat{\gamma}}$, Eq. (6.8) of [34], reads

$$j^{\hat{\alpha}\hat{\beta}} = \int_{\Sigma(\tau,U)} x^{\hat{\alpha}} j^{\hat{\beta}} d\Sigma - a_{\hat{\sigma}} \int_{\Sigma(\tau,U)} x^{\hat{\sigma}} x^{\hat{\alpha}} j^{\hat{\beta}} d\Sigma,$$

the second term being negligible to pole-dipole order.

(3) The 1-form n_α normal to $\Sigma(\tau, U)$ reads, in the coordinates $\{x^{\hat{\alpha}}\}$, $n_{\hat{\alpha}} = (-1, 0, 0, 0)(-g^{\hat{0}\hat{0}})^{-1/2}$. Since $g^{\hat{0}\hat{0}} = -1 + \mathcal{O}(x^2)$ (see e.g. [23]), and, at the reference worldline z^α, $n_{\hat{\alpha}} = (-1, 0, 0, 0) = U_{\hat{\alpha}}$, we have

$$n_{\hat{\alpha}} = U_{\hat{\alpha}} + \mathcal{O}(x^2); \quad \text{(A2)}$$

hence, to dipole order, we may take (when of interest) $d\Sigma_{\hat{\delta}} \equiv -n_{\hat{\delta}} d\Sigma \simeq -U_{\hat{\delta}} d\Sigma$. It follows that $-j^{\alpha\beta} U_\beta = d^\alpha$, cf. Eq. (7), and therefore the magnetic dipole tensor $m^{\alpha\beta}$ defined in [34] as $m^{\alpha\beta} = j^{[\alpha\beta]} - d^{[\alpha} U^{\beta]}$ matches ours: $m^{\alpha\beta} = (h^U)^\alpha{}_\gamma (h^U)^\beta{}_\delta j^{[\gamma\delta]} = \mu^{\alpha\beta}$, cf. Eqs. (10) and (8).

(4) The moments are defined relative to an hypersurface of integration $\Sigma(\tau, U)$ normal to U^α at z^α, as done in [28,30], whereas in e.g. [27,34,36] hypersurfaces $\Sigma(\tau, P)$ orthogonal to P^α are used. That does not change the shape of the equations to dipole order, as one can check[23] comparing the equations in [27,34,36] with the ones in [30] (identifying the appropriate quantities, as explained in Sec. A 2 below), or in the independent derivation in [28].

2. Dixon's "symmetric" equations

In later works by Dixon [27,30,36] the equations of motion for spinning particles are presented in a different form, e.g. Eqs. (1.33) and (1.34) of [36], symmetric with respect to the electric and magnetic dipoles. Taking into account the different signature and conventions, they read, to dipole order,

$$\frac{DP^\alpha_{\text{Dix}}}{d\tau} = qF^{\alpha\beta} U_\beta + \frac{1}{2} F^{\mu\nu;\alpha} Q_{\mu\nu} - \frac{1}{2} R^\alpha{}_{\beta\mu\nu} S^{\mu\nu} U^\beta, \quad \text{(A3)}$$

$$\frac{DS^{\alpha\beta}_{\text{can}}}{d\tau} = 2P^{[\alpha}_{\text{Dix}} U^{\beta]} + 2Q^{\theta[\beta} F^{\alpha]}{}_\theta, \quad \text{(A4)}$$

[22] It suffices for this purpose to work in flat spacetime; a generalization of w^α to curved spacetime only amounts to small corrections to something already negligible in special relativity.

[23] In the purely gravitational case ($F^{\alpha\beta} = 0$), the integrals (4) and (5), defined at $z^\alpha(\tau)$ over an hypersurface $\Sigma(\tau, u)$ orthogonal to u^α, are actually, to pole-dipole order, independent of u^α, see [33].

where $Q^{\alpha\beta}$ is the *electromagnetic dipole* moment tensor about $z^\alpha(\tau)$ [Eq. (5.62) of [35], or, for flat spacetime, Eq. (3.44) of [30]], which reads, in the system $\{x^{\hat{\alpha}}\}$,

$$Q^{\hat{\alpha}\hat{\beta}} \equiv \int_{\Sigma(\tau,U)} x^{[\hat{\alpha}} j^{\hat{\beta}]} d\Sigma + U^{[\hat{\beta}} \int_{\Sigma(\tau,U)} x^{\hat{\alpha}]} j^{\hat{\gamma}} d\Sigma_{\hat{\gamma}}. \qquad \text{(A5)}$$

Since $d\Sigma_{\hat{\delta}} \simeq -U_{\hat{\delta}} d\Sigma$, cf. Eq. (A2), this tensor embodies the intrinsic electric and magnetic dipoles d^α and $\mu_{\alpha\beta}$, Eqs. (7)–(10), as its time and space projections with respect to U^α,

$$d^\alpha = -Q^{\alpha\beta} U_\beta, \qquad \mu^{\alpha\beta} = (h^U)^\alpha{}_\gamma (h^U)^\beta{}_\delta Q^{\gamma\delta}, \qquad \text{(A6)}$$

in terms of which it has the decomposition

$$Q^{\alpha\beta} = 2 d^{[\alpha} U^{\beta]} + \epsilon^{\alpha\beta\gamma\delta} \mu_\gamma U_\delta.$$

It must be noted that $P^\alpha_{\rm Dix}$ and $S^{\alpha\beta}_{\rm can}$ (P^α, $S^{\alpha\beta}$ in the notation of [27,30,35,36]) are *not the physical* momentum and angular momentum given by Eqs. (4) and (5) above, but instead contain additional electromagnetic terms, cf. [27,30]. In our framework, they can be written as

$$P^\alpha_{\rm Dix} = P^\alpha + P'^\alpha, \qquad P'^{\hat{\alpha}} \equiv \int_{\Sigma(z,U)} \Psi^{\hat{\alpha}} j^{\hat{\beta}} d\Sigma_{\hat{\beta}}, \qquad \text{(A7)}$$

$$S^{\alpha\beta}_{\rm can} = S^{\alpha\beta} + S'^{\alpha\beta}, \qquad S'^{\alpha\beta} \equiv 2 \int_{\Sigma(z,U)} x^{[\hat{\alpha}} \Phi^{\hat{\beta}]} j^{\hat{\gamma}} d\Sigma_{\hat{\gamma}}, \qquad \text{(A8)}$$

with

$$\Psi^{\hat{\alpha}}(z,x) \equiv -\int_0^1 F^{\hat{\alpha}}{}_{\hat{\beta}}(u) x^{\hat{\beta}} du, \qquad \text{(A9)}$$

$$\Phi^{\hat{\alpha}}(z,x) \equiv -\int_0^1 u F^{\hat{\alpha}}{}_{\hat{\beta}}(u) x^{\hat{\beta}} du. \qquad \text{(A10)}$$

Equations (A9) and (A10) are integrals along the geodesic $\eta^\alpha(u)$ connecting z^α and x^α, parametrized by u so that $\eta^\alpha(0) = z^\alpha$, $\eta^\alpha(1) = x^\alpha$. In flat spacetime, these expressions are exactly[24] Eqs. (7.1), (7.2), (7.6) and (7.7) of [30]. In curved spacetime, they match, to the accuracy at hand, Eqs. (3.14), (3.15), (5.1) and (5.2) of [27] [corrections due to the bitensors therein are of order $\mathcal{O}(a^3)$ for P'^α, and $\mathcal{O}(a^4)$ for $S'^{\alpha\beta}$, where $a \equiv$ size of the body, hence both negligible to quadrupole order, $\mathcal{O}(a^2)$].

[24]Therein Cartesian coordinates are used, and $F^{\alpha\beta}$ has argument $F^{\alpha\beta}(\mathbf{z} + u\mathbf{r})$, where $r^\alpha = x^\alpha - z^\alpha$ is the vector connecting the reference worldline to the point x^α. Since $\eta^\alpha(u)$ is in this case a straightline, indeed $\eta^\alpha(u) = z^\alpha + u r^\alpha$. Noting moreover that $z^{\hat{\alpha}} = 0$, $r^{\hat{\alpha}} = x^{\hat{\alpha}}$ in the system $\{x^{\hat{\alpha}}\}$, one obtains (A7)–(A10).

The lowest order approximation to these integrals is to take only the zeroth order term in the expansion of $F^{\alpha\beta}$ around z^α, i.e., to take $F^{\alpha\beta} \approx$ *constant along the body*; this is sufficient for our purposes, as higher terms in the expansion of $F^{\alpha\beta}$ lead to contributions of higher multipole moments to P'^α and $S'^{\alpha\beta}$. We obtain

$$P'^\alpha = -F^\alpha{}_\gamma d^\gamma, \qquad (i) \qquad S'^{\alpha\beta} = F^{[\alpha}{}_\sigma q^{\beta]\sigma}, \qquad (ii), \qquad \text{(A11)}$$

where d^α and $q^{\alpha\beta}$ are the charge dipole and quadrupole moments, Eqs. (7) and (102). As such, $S'^{\alpha\beta}$ is negligible to pole-dipole order, but it is of crucial importance in Sec. VI, where terms up to quadrupole order are kept.

Note now the following: substituting (A3), (A4), and (A11) into Eqs. (A3) and (A4) (and noting that, to dipole order, $S^{\alpha\beta} \simeq S^{\alpha\beta}_{\rm can}$), we obtain Eqs. (11) and (12); hence indeed the two sets of equations *are equivalent*.

As shown in [107], $P^\alpha_{\rm Dix} + qA^\alpha \equiv P^\alpha_{\rm can}$ and $S^{\alpha\beta}_{\rm can}$ have the interpretation of canonical momenta associated to the Lagrangian of the system. $P^\alpha_{\rm can}$ is the quantity conserved in collisions [118], and its time component $P^0_{\rm can} = -\mathbf{P}_{\rm can} \cdot \partial_0$ is the scalar conserved under stationary fields in flat spacetime, cf. Eq. (B9) below. The quantity $S^{\alpha\beta}_{\rm can}$ generalizes the canonical angular momentum of some nonrelativistic treatments [83–85]; in [85], a canonical angular momentum vector, Eq. (31) therein, is obtained differentiating $\partial\mathcal{L}/\partial\vec{\Omega}$ ($\mathcal{L} \equiv$ Lagrangian of the system, $\vec{\Omega} \equiv$ angular velocity of the body). Such 3-vector is but a noncovariant form for the spatial[25] vector $S^\gamma_{\rm can} \equiv \epsilon^\gamma{}_{\mu\alpha\beta} S^{\alpha\beta}_{\rm can} U^\mu/2$, as can be easily shown. From (A8), $S^\gamma_{\rm can} = S^\gamma + S'^\gamma$, with

$$S'^\gamma \equiv \frac{1}{2} \epsilon^\gamma{}_{\mu\alpha\beta} U^\mu S'^{\alpha\beta} = \frac{B^\alpha}{2} [\delta^\gamma{}_\alpha q_\sigma{}^\sigma - q^\gamma{}_\alpha], \qquad \text{(A12)}$$

where we used Eq. (1) and the orthogonality condition $q^{\alpha\beta} U_\alpha = q^{\alpha\beta} U_\beta = 0$. If the body has uniform mass and energy density, $S'^\gamma = (q/2m) B^\alpha I_\alpha{}^\gamma$, where $I^{\alpha\beta}$ is the moment of inertia (see footnote 18). In this case we have, in the particle's CM frame (where $U^i = 0$), $S^\gamma_{\rm can} = (0, \vec{S}_{\rm can})$, with $\vec{S}_{\rm can} = \vec{S} + \vec{S}'$ matching expression (31) of [85].

The distinction between $P^\alpha_{\rm Dix}$ in Eqs. (A3) and (A4) and the physical momentum P^α should not be overlooked when the particle possesses electric dipole moment. Since those equations are essentially symmetric with respect to d^α and μ^α, failing to make that distinction would lead one to believe that the two dipoles are *dynamically* similar. Given their different nature, as defined by Eqs. (7) and (8) (the magnetic dipole is modeled by a current loop, the electric

[25]The definition of $S^\gamma_{\rm can}$ is not a dualization of $S^{\alpha\beta}_{\rm can}$, as neither $S^{\alpha\beta}_{\rm can}$ nor $S'^{\alpha\beta}$ are spatial with respect to U^α under the Mathisson-Pirani condition $S^{\alpha\beta} U_\beta = 0$. Hence S'^γ and $S^\gamma_{\rm can}$ do not contain the same information as $S^{\alpha\beta}_{\rm can}$ and $S'^{\alpha\beta}$ (only their spatial part).

dipole by a pair of opposite charges), that would be physically inconsistent: (i) the electric dipole would have a hidden momentum (just like a magnetic dipole), cf. Eq. (B4), which would violate the conservation equations; (ii) a static electric field would do no work on the dipole (regardless of its motion), which is well known to be false; (iii) the particle's proper mass m would vary in a way consistent with a dipole arising from a current of magnetic monopoles, not a pair of charges; (iv) the spatial part of the force would not be consistent with the results known from classical electromagnetism. A detailed account of these issues is given in the next section.

At quadrupole order, it is also crucial to not confuse $S^{\alpha\beta}_{\mathrm{can}}$ with the physical angular momentum $S^{\alpha\beta}$ (the one which is proportional to the angular velocity in the case of a rigid body). Otherwise, as discussed in Sec. VI A, one would erroneously conclude that in vacuum the electromagnetic field does not couple to the trace of $q_{\alpha\beta}$, implying e.g. that no torque (besides the dipole torque $\vec{\tau} = \vec{\mu} \times \vec{B}$, if it spins) could be exerted on a spherical charged body, which is well known, both from elementary electromagnetism and from experiment, to be false.

APPENDIX B: THE ELECTRIC DIPOLE

In order to better understand some key issues in this work—the physical meaning of the time projection of the force on a magnetic dipole, the variation of its proper mass, the work done on it by the external fields, and the hidden momentum—it is useful to make the contrast with the case of an electric dipole.

It is clear from Eqs. (11) and (12) that both the force and the spin evolution equations are different for electric and magnetic dipoles. This is due to the intrinsic differences of the two types of dipole: d^α, Eq. (7), is the dipole moment of the charge density, which can be modeled by a pair of two (close) opposite charges; μ^α, Eq. (8), is the dipole moment of the spatial current, modeled by a (small) current loop. For a particle possessing only electric dipole moment ($\mu^{\alpha\beta} = 0$, $q = 0$) in flat spacetime, Eqs. (11) and (12) read

$$\frac{DP^\alpha}{d\tau} \equiv F^\alpha_{\mathrm{el}} = E^\alpha{}_\beta d^\beta + F^\alpha{}_\beta \frac{Dd^\beta}{d\tau}, \tag{B1}$$

$$\frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} + 2d^{[\alpha}F^{\beta]}{}_\gamma U^\gamma, \tag{B2}$$

where $E_{\alpha\beta} \equiv F_{\alpha\gamma;\beta}U^\gamma$ is the electric tidal tensor [7].

First we note that, unlike its magnetic counterpart Eq. (I.1a) of Table I, the force on an electric dipole is not (generically) given by a contraction of a tidal tensor with the dipole vector (only if $Dd^\alpha/d\tau = 0$). Indeed, it is not entirely a tidal effect, due to the extra term $F^\alpha{}_\beta Dd^\beta/d\tau$ (overlooked in most literature), which does not involve derivatives of $F_{\alpha\beta}$. This term is physically interpreted as follows. From Eq. (3.23a) of [30] we have

$$\frac{Dd^\gamma}{d\tau} = \mathcal{J}^\gamma - U^\gamma q,$$

where $\mathcal{J}^{\hat{\alpha}} \equiv \int_{\Sigma(U,\tau)} j^{\hat{\alpha}} w^{\hat{\gamma}} d\Sigma_{\hat{\gamma}}$. q is the particle's *total* charge, and $\mathcal{J}^\alpha$ is roughly its total current. Then $\mathcal{J}^\gamma - U^\gamma q$ is essentially the particle's spatial current with respect to U^α. For an electric dipole ($q = 0$), Eq. (B1) can be rewritten as

$$F^\alpha_{\mathrm{el}} = E^\alpha{}_\beta d^\beta + F^\alpha{}_\beta \mathcal{J}^\beta. \tag{B3}$$

The term $F^\alpha{}_\gamma \mathcal{J}^\gamma$ has a straightforward interpretation: if the dipole vector d^α varies with τ (e.g., if the dipole rotates) then it generates a net electric current in the CM frame; therefore, a magnetic force $F^\alpha{}_\gamma \mathcal{J}^\gamma$ is exerted on it, in addition to the tidal force $E^\alpha{}_\beta d^\beta$. As a simple example, consider a rotating electric dipole under a uniform magnetic field; a net force arises from the magnetic forces (with the same direction) that act on each of its charge poles, due to their circular motion about the CM.

Second, we note that in the term $E^{\alpha\beta} d_\beta$ the indices of the tidal tensor are reversed as compared to the force on the magnetic dipole, Eq. (I.1a). In Secs. B 2 and B 3 below we shall see some consequences.

For an electric dipole at rest in an inertial frame (where $E_{ij} = \nabla_j E_i$), the space part of (B1) reads $\vec{F}_{\mathrm{el}} = (\vec{d} \cdot \nabla)\vec{E} - \vec{B} \times D\vec{d}/d\tau$, matching the result from classical treatments, e.g. [72]. Note also that $D\vec{P}_{\mathrm{Dix}}/d\tau = \nabla(\vec{E} \cdot \vec{d})$ [analogous to the force on a magnetic dipole, $\vec{F}_{\mathrm{EM}} = \nabla(\vec{B} \cdot \vec{\mu})$], which differs from the physical force $\vec{F}_{\mathrm{el}} = D\vec{P}/d\tau$.

1. No hidden momentum for electric dipole

Unlike the current loop, the two-charge type of dipole cannot store hidden momentum of electromagnetic origin, see e.g. [72]. The expression for the momentum of an electric dipole is obtained contracting Eq. (B2) with U_β, leading to (using $U^\alpha d_\alpha = 0$) $P^\alpha = mU^\alpha + S^{\alpha\beta} a_\beta$, showing that the only hidden momentum present is the pure gauge term $P^\alpha_{\mathrm{hidI}} = S^{\alpha\beta} a_\beta$ arising from the spin condition (which exists regardless of the electromagnetic multipole structure of the particle). This was expected from conservation arguments. Unlike its magnetic counterpart, the electric dipole does *not* generate electromagnetic *field* momentum (cross momentum $P^\alpha_\times$, see [26]) when placed in an electromagnetic field [119]. Now consider a *stationary* configuration; in this case the conservation equations $(T_{\mathrm{tot}})^{\alpha\beta}{}_{;\beta} = 0$ imply that the total spatial momentum vanishes, $\vec{P}_{\mathrm{tot}} = 0$; if the dipole were to have any hidden momentum, it would *not* be canceled out by the field momentum, violating the conservation equations.

This shows the importance of distinguishing between the physical momentum P^α and Dixon's momentum $P^\alpha_{\rm Dix} = P^\alpha + P'^\alpha$ of Eqs. (A3) and (A4); as can be seen from (A7) and (A11), $P^\alpha_{\rm Dix}$ includes a term $-\epsilon^\alpha{}_{\theta\mu\sigma} d^\theta B^\mu U^\sigma$, analogous to the hidden momentum $P^\alpha_{\rm hidEM} = \epsilon^\alpha{}_{\theta\mu\sigma}\mu^\theta E^\mu U^\sigma$ of the magnetic dipole (but of opposite sign):

$$P^\alpha_{\rm Dix} = P^\alpha - E^\beta d_\beta U^\alpha - \epsilon^\alpha{}_{\theta\mu\sigma} d^\theta B^\mu U^\sigma. \tag{B4}$$

Thus, confusing $P^\alpha_{\rm Dix}$ with P^α would lead one to believe that the electric dipole has a hidden momentum just like a magnetic dipole, which not only would make no sense for the dipole model at stake, as it would violate the conservation equations.

2. Proper mass and time projection of the force in the CM frame

Contracting (B1) with U^α one obtains

$$F^\alpha_{\rm el} U_\alpha = -E_\gamma \frac{Dd^\gamma}{d\tau} = -E_\gamma \mathcal{J}^\gamma, \tag{B5}$$

where $E^\alpha \equiv F^{\alpha\beta} U_\beta$ is the electric field as measured by the test particle. Hence, like the force on a magnetic dipole ($F^\alpha_{\rm EM}$), $F^\alpha_{\rm el}$ has in general a (time) projection along the particle's worldline. They are very different, however. As noticed above, the order of the indices in the tidal tensor of (B1) is reversed compared to $F^\alpha_{\rm EM} = B^{\beta\alpha}\mu_\beta$; since $E_{\alpha\beta}$ and $B_{\alpha\beta}$ are spatial relative to U^α in the first, but *not* in the second index, then, *by contrast* with $F^\alpha_{\rm EM}$, the projection of the tidal force $E^{\alpha\beta} d_\beta$ along U^α is zero. This means that, as measured *in the particle's CM frame*, the tidal force does *no work*. Thus $F^\alpha_{\rm el} U_\alpha$ reduces to the projection of the second term of (B1), arising from the variation of *the dipole vector* d^α along the particle's worldline. This contrasts with its magnetic counterpart $F^\alpha_{\rm EM} U_\alpha = U^\beta \mu^\gamma D\star F_{\gamma\beta}/d\tau$, cf. Eq. (I.1a) of Table I, which comes from the variation of *the field*.

Equation (B5) makes sense: $\mathcal{J}^\gamma$ is *essentially* the total current as measured in the dipole's frame; when it is nonvanishing (for instance, due to a rotation of the dipole), a nonvanishing work, in this frame, is done on the dipole by the electric field. Noting from (34) that $P^\alpha a_\alpha = 0$, we have

$$\frac{dm}{d\tau} = -F^\alpha_{\rm el} U_\alpha = E_\gamma \frac{Dd^\gamma}{d\tau}. \tag{B6}$$

Hence, if $Dd^\alpha/d\tau = 0$, the particle's proper mass is constant, which contrasts with the situation for a magnetic dipole, where $dm/d\tau$ is zero only if $DB^\alpha/d\tau = 0$ (*not* $D\mu^\alpha/d\tau = 0$), cf. Eq. (39).

Consider now the special case of a rigid dipole which is allowed to rotate: $D_F d^\alpha/d\tau = \Omega^\alpha{}_\beta d^\beta$, with $\Omega_{\alpha\beta}$ defined by Eqs. (115). In this case, using (17),

$$\frac{dm}{d\tau} = -F^\alpha_{\rm el} U_\alpha = \epsilon_{\gamma\beta\mu\nu} U^\nu E^\gamma \Omega^\beta d^\mu = \tau^\beta \Omega_\beta; \tag{B7}$$

this is the rate of work done by the torque $\tau^\beta = \epsilon^\beta{}_{\mu\gamma\nu} U^\nu d^\mu E^\gamma$ exerted on the dipole by virtue of Eqs. (B2) and (106). The torque τ^β causes an accelerated rotation of the dipole; the corresponding variation of rotational kinetic energy reflects itself in a variation of m.

Note that Eqs. (B6) and (B7) yield, e.g., the well-known work done on an electric dipole whose CM is at rest in a static, uniform electric field, from the point of view of the rest frame. Thus again we see the importance of not confusing $P^\alpha_{\rm Dix}$ in Eqs. (A3) and (A4) with the physical momentum P^α: overlooking the distinction would lead to the conclusion that, just like for a magnetic dipole, a static field does no work on a rotating electric dipole, which we know from basic electromagnetism to be false.

3. Time component of the force as measured by generic observers

With respect to a congruence of observers $\mathcal{O}(u)$ of 4-velocity u^α, the time projection of the force exerted on the electric dipole is

$$-F^\alpha_{\rm el} u_\alpha = \gamma (E^u)_{\beta\gamma} d^\gamma v^\beta + (E^u)_\alpha \frac{Dd^\alpha}{d\tau}, \tag{B8}$$

where $(E^u)^\alpha \equiv F^{\alpha\beta} u_\beta$ and $(E^u)_{\beta\gamma} \equiv F_{\beta\mu;\gamma} u^\mu$ are, respectively, the electric field and electric tidal tensor measured by $\mathcal{O}(u)$, and v^α [the particle's velocity relative to $\mathcal{O}(u)$] and γ are defined in Eqs. (67). As discussed in Sec. IV, this is the rate of work done by the force as measured by $\mathcal{O}(u)$. The first term is a natural result: in a nonuniform electric field $[(E^u)_{\alpha\beta} \neq 0]$, a force is in general exerted on an electric dipole; if it is allowed to move ($v^\alpha \neq 0$) that force does work. The second term contributes when $Dd^\alpha/d\tau \neq 0$, and is nonzero even if the fields are uniform. It is the work done by the electric field when the dipole rotates or oscillates, discussed in the previous section.

The power $-F^\alpha_{\rm el} u_\alpha$ differs significantly from its magnetic counterpart Eq. (76). Consider (when they exist) observers along whose worldlines the field is covariantly constant, $F^{\alpha\beta}{}_{;\gamma} u^\gamma = 0$ (e.g. the static observers of Sec. IV B 1, cf. footnote 8); as we have seen in Sec. IV B, relative to such observers, the field does no work on a magnetic dipole, $F^\alpha_{\rm EM} u_\alpha = 0$, cf. Eq. (77). But it *does work* on an electric dipole, both terms of (B1) contributing to it (regarding the tidal term, the reason why $E^\alpha{}_\beta d^\beta$ does work, $E^\alpha{}_\beta d^\beta u_\alpha \neq 0$, whereas $F^\alpha_{\rm EM} = B_\beta{}^\alpha \mu^\beta$ does not, is again due to the order of the indices in the tidal tensor). This was to be expected given the different nature of the dipoles: in the magnetic case, the total work is zero due to (in the simplest case when there is no hidden momentum) a cancellation between the variation of translational kinetic energy and the work done on the current loop by the electric field induced in it; the latter has no counterpart in the electric dipole, since it does not consist of a current of magnetic monopoles; therefore such cancellation does not occur.

4. Conserved quantities, proper mass and work done by the fields

In order to better elucidate the relationship between the work done by the fields and the variation of the proper mass, we will compare, in a *static* electromagnetic field, three different test particles: a point monopole charge, an electric dipole, and a magnetic dipole. Let u^α be the 4-velocity of the *inertial* frame $\mathcal{O}(u)$ relative to which the fields are static. Then u^α preserves the electromagnetic field, $\mathcal{L}_u F^{\alpha\beta} = 0$, and, therefore, from the constancy of expressions (5.3) of [27], or (29) of [20], we have

$$P^\alpha_{\rm Dix} u_\alpha + qA^\alpha u_\alpha = P^\alpha u_\alpha + (E^u)^\alpha d_\alpha - q\phi = \text{constant}, \tag{B9}$$

where $\phi \equiv -A^\alpha u_\alpha$ is the electric potential measured in $\mathcal{O}(u)$. Using Eq. (30), it is useful to rewrite (B9) as

$$m + T + V + E_{\rm hid} = \text{constant}, \tag{B10}$$

where $V = -(E^u)^\alpha d_\alpha + q\phi$ is the potential energy of the particle under the field, $T \equiv (\gamma - 1)m$ is the kinetic energy associated to the translation of its center of mass, $\gamma \equiv -U^\alpha u_\alpha$, and $E_{\rm hid} = -P^\alpha_{\rm hid} u_\alpha$ the hidden energy [i.e., the time component of the hidden momentum relative to $\mathcal{O}(u)$, see Sec. IV B]. In this section we shall ignore the inertial hidden momentum $P^\beta_{\rm hidI}$, as in the applications below it either vanishes or is made negligible by appropriate choices of the reference worldline (e.g. Tulczyjew-Dixon, or Mathisson-Pirani nonhelical centroids). Thus, $P^\beta_{\rm hid} = P^\beta_{\rm hidEM}$ herein.

Point monopole charge ($d^\alpha = P^\alpha_{\rm hid} = 0$).—In this case, condition (B10) reads $m + T + q\phi =$ constant. There is no exchange of energy with the proper mass of the particle, which is a constant:

$$\frac{dm}{d\tau} = -\frac{DP_\alpha}{d\tau} U^\alpha = -qF_{\alpha\beta} U^\alpha U^\beta = 0.$$

This just tells us that, in a stationary electromagnetic field, the "total mechanical energy" of the particle—kinetic energy T, plus electric potential energy $V = q\phi$—is a constant of the motion, as is well known. Every gain in T must come from the potential energy V, so there is no doubt that the field doing work, at a rate given by the time projection of the Lorentz force $F^\alpha_{\rm L} = qF^{\alpha\beta} U_\beta$ relative to $\mathcal{O}(u)$, cf. Eq. (68):

$$\frac{dE}{d\tau} = -F^\alpha_{\rm L} u_\alpha = q\gamma (E^u)^\alpha v_\alpha = -\frac{dV}{d\tau} = F^\alpha_{\rm L} v_\alpha.$$

In vector notation, $dE/d\tau = q\gamma \vec{E}(u) \cdot \vec{v}$, with $\vec{E}(u) = -\nabla\phi = -\nabla V/q$.

Electric dipole ($q = P^\alpha_{\rm hid} = 0$).—Condition (B10) reads $m + T - (E^u)^\alpha d_\alpha =$ constant. From Eq. (B6), the proper mass m is not constant; this means that energy is exchanged between the three forms: potential energy $V = -(E^u)^\alpha d_\alpha$, translational kinetic energy T, and m. Two special subcases are particularly enlightening:

(1) Dipole vector covariantly constant, $Dd^\alpha/d\tau = 0$, implying $dm/d\tau = 0$. In this case the energy exchange is similar to the monopole charge: every gain in translational kinetic energy comes from the potential energy V. It is clear that the electric tidal field is doing work, at a rate [cf. Eq. (B8)]

$$\frac{dE}{d\tau} = -F^\alpha_{\rm el} u_\alpha = \gamma (E^u)_{\beta\gamma} d^\gamma v^\beta = -\frac{dV}{d\tau} = F^\alpha_{\rm el} v_\alpha.$$

(2) Dipole's CM at rest ($U^\alpha = u^\alpha$, $v^\alpha = 0$), i.e., $T = 0$. In this case, $m - E^\alpha d_\alpha =$ constant, and the energy exchange occurs between the potential energy $V = -E^\alpha d_\alpha$ and proper mass m (which includes rotational kinetic energy of the particle). The work of the field thus equals the mass variation,

$$\frac{dE}{d\tau} = -F^\alpha_{\rm el} u_\alpha = \frac{dm}{d\tau} = -\frac{dV}{d\tau}.$$

Magnetic dipole ($q = d^\alpha = 0$).—Condition (B10) means in this case $m + T + E_{\rm hid} =$ constant; if we take $\mu^\alpha = \sigma S^\alpha$, from Eq. (41) we have $m = m_0 - \mu^\alpha B_\alpha$, and thus the condition becomes $T - \mu^\alpha B_\alpha + E_{\rm hid} =$ constant. The energy exchange is between translational kinetic energy, proper mass and $E_{\rm hid}$. There is no potential energy involved (cf. [82–85]), which is consistent with the fact that the static field does no work on the magnetic dipole: $dE/d\tau = -F^\alpha_{\rm EM} u_\alpha = 0$, cf. Eq. (77). A case of interest in the context of this work is the one depicted in Fig. 3(b), a magnetic dipole falling towards a magnet along the field's axis of symmetry. In this case $P^\alpha_{\rm hid} = E_{\rm hid} = 0$, implying $T + m =$ constant. The energy exchange is only between translational kinetic energy and proper mass; every gain in the former comes at the expense of latter (which, for a rigid body, consists essentially of a variation of *rotational kinetic energy*, cf. Sec. VI A 3 and [82–85]). Hence what the field does is to interconvert translational kinetic energy into rotational or other forms of internal energy.

APPENDIX C: COMPARISON OF THE DIFFERENT SPIN CONDITIONS

In this paper we have so far been using equations of motion supplemented by the Mathisson-Pirani (MP) spin condition, as it is the one that makes explicit the analogies used. As we shall see below, it is also the one that leads to the simplest description of the force/center of mass motion in the applications in Secs. III and IV B. However, other spin conditions (14) can be used; as explained in Sec. II A, the (infinite) possible choices of u^α correspond to different, but equivalent, ways of describing the motion of a spinning body, they differ just in the choice of its representative point. Below we compare some best known spin conditions in the applications in this paper, and explore the gravitoelectromagnetic analogies that emerge using them.

1. Comparison of the spin conditions in the applications in this paper

We start with the problem of the falling motion along the symmetry axis ($\theta = 0$ in Boyer-Lindquist coordinates, hereafter the "z-axis") of a gyroscope in a Kerr spacetime, discussed in Sec. IV B. Setting its initial position, velocity $\vec{U}$ and spin $\vec{S}$ all along the axis, one expects, at first sight, from symmetry arguments, an axial fall. It turns out, however, that such naive prescription of initial conditions does not completely determine the problem, nor does it ensure its axial symmetry. One needs also to prescribe the field of unit timelike vectors u^α relative to which the CM is computed (i.e., the field entering the spin condition $S^{\alpha\beta}u_\beta = 0$), which, for an arbitrary choice, breaks the axial symmetry. The momentum-velocity relation also depends on this choice, cf. Eq. (29), implying that $\vec{U}$ will not in general be parallel to $\vec{P}$ (hidden momentum), so that they do not both lie along the z-axis. Note that, as explained in Sec. II D, the acceleration of the CM does not originate solely from the force, but also from the variation of field u^α along the CM worldline.

In order to prescribe an axisymmetric problem, we start by demanding, as initial conditions, $\vec{U}_{\rm in} = U^z\vec{e}_z$ ($\vec{e}_z \equiv \vec{e}_r = \partial/\partial r$ in Boyer-Lindquist coordinates, for $\theta = 0$), $\vec{u} = u^z\vec{e}_z$, and an initial CM position $z^\alpha_{\rm in} = x^\alpha_{\rm CM}(u)|_{\rm in}$ also along the z-axis. The MP condition, $u^\alpha = U^\alpha$, clearly allows for these initial conditions, so let us start with it. The momentum reads, cf. Eq. (34),

$$P^\alpha = mU^\alpha - \epsilon^\alpha{}_{\beta\gamma\delta}S^\beta a^\gamma U^\delta, \tag{C1}$$

and the spatial part of the equation of motion $F^\beta_{\rm G} \equiv DP^\beta/d\tau = -\mathbb{H}^{\alpha\beta}S_\alpha$ [cf. Eq. (I.1b)] reads

$$m\vec{a} - \frac{D(\vec{S}\times_U \vec{a})}{d\tau} = \vec{F}_{\rm G} = -\mathbb{H}^{i\alpha}S_\alpha\vec{e}_i, \tag{C2}$$

where $\vec{S}\times_U\vec{a}$ denotes the space components of $\epsilon^\alpha{}_{\beta\gamma\delta}S^\beta a^\gamma U^\delta$. Initially, with $\vec{U}_{\rm in} = U^z\vec{e}_z$, one obtains $\vec{F}_{\rm G}|_{\rm in} = -\mathbb{H}^{z\alpha}S_\alpha\vec{e}_z$ (it is straightforward to check that along the axis we have $\mathbb{H}^{i\alpha} = 0$ if $i \neq z$); thus the force is along z, as expected from symmetry arguments, given the axial symmetry of the initial setup and the fact that $\mathbb{H}_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\nu}U^\mu U^\nu$ depends only on U^α. It is clear from the equation above that one[26] of the possible solutions of (C2) is the most natural result, namely motion along the z-axis, with the body accelerating in the direction of the force (and of $\vec{S}$): $\vec{a} = a^z\vec{e}_z \Rightarrow \vec{S}\times_U\vec{a} = 0$, implying $P^\alpha = mU^\alpha$, and $F^\alpha_{\rm G} = ma^\alpha$. It is a nonhelical solution (since it is a straightline), and therefore the description we seek. Hence we have solved the axial fall problem, and a unique relation between P^α and U^α was naturally established (for this solution) in the course of the analysis.

Now let us compare with the equivalent descriptions for this problem given by other spin conditions. For a generic field u^α with $\vec{u}$ not lying along the z-axis, we no longer have axial symmetry, therefore we should not expect to obtain a centroid moving in straightline along the axis; what we expect, in general, is a different (possibly exotic) but equivalent description of the same physical motion, using a different representative worldline. The problem, however, is how to prescribe its initial conditions. If one naively sets up an initial position $z^\alpha_{\rm in} = x^\alpha_{\rm CM}(u)|_{\rm in}$ lying on the z-axis, and then $\vec{P}$ or $\vec{U}$ [there is an ambiguity on this choice, as they are not parallel in general, cf. Eq. (29)] also along the z-axis, the solution in general will not be an axial fall; in fact, it will not even be a different description for it, but a different physical motion.

So first we must establish how we make sure that we are dealing with the same particle. A pole-dipole particle is characterized by its two moments: P^α and $S^{\alpha\beta}$. These are defined with respect to a reference worldline $z^\alpha(\tau)$ and a hypersurface of integration $\Sigma(\tau, u)$, cf. Eqs. (4) and (5); different representations of the same particle must yield the same moments with respect to the *same* point and $\Sigma(\tau, u)$. To dipole order, P^α is independent of the spin condition (see [33]), but $S^{\alpha\beta} \equiv S^{\alpha\beta}(z)$ depends on it. Let $S^{\alpha\beta}$ and $\bar{S}^{\alpha\beta}$ be the angular momentum taken about, respectively, the centroids $z^\alpha = x^\alpha_{\rm CM}(u)$ and $\bar{z}^\alpha = x^\alpha_{\rm CM}(\bar{u})$; i.e., $S^{\alpha\beta}u_\beta = 0$, and $\bar{S}^{\alpha\beta}\bar{u}_\beta = 0$, cf. Sec. II A. The integral expressions for $S^{\hat{\alpha}\hat{\beta}}$ and $\bar{S}^{\hat{\alpha}\hat{\beta}}$, in normal coordinates $\{x^{\hat{\alpha}}\}$ originating *at* z^α, are given, to dipole order,[27] by Eq. (5) (in the case of $\bar{S}^{\hat{\alpha}\hat{\beta}}$, replacing therein $x^{\hat{\alpha}}$ by $x^{\hat{\alpha}} - \bar{z}^{\hat{\alpha}}$, so that it is taken about the point $\bar{z}^{\hat{\alpha}}$). We obtain

[26] Other solutions are possible, because the set of initial conditions $\{z^\alpha, S^{\alpha\beta}, U^\alpha, m\}|_{\rm in}$ is not sufficient to uniquely specify a solution under the MP condition, see [33]. Note however that, since $U^\alpha_{\rm in}$ is fixed, such solutions correspond to different values of $P^\alpha_{\rm in}$, therefore they are *not* representations of the same *physical motion* (i.e., those will be "helical" representations but of different motions).

[27] This is because both the dependence of $S^{\alpha\beta}$ on the argument u^α of Σ (see [33]), and the nonlinearity, due to the curvature, of the transformation between normal coordinates originating at z^α and $\bar{z}^\alpha$ (denote the latter by $\{x^{\tilde{\alpha}}\}$), are negligible to dipole order: $x^{\tilde{\alpha}} = x^{\hat{\alpha}} - \bar{z}^{\hat{\alpha}} + \mathcal{O}(\|x^{\hat{\alpha}} - \bar{z}^{\hat{\alpha}}\|^2\Delta x)$, cf. e.g. Eq. (11.12) of [120]; hence, in the computation of $\bar{S}^{\alpha\beta}$, one can use $x^{\tilde{\alpha}} \simeq x^{\hat{\alpha}} - \bar{z}^{\hat{\alpha}}$, as the correction is of order $\mathcal{O}(a^4)$, whereas to dipole order only terms of $\mathcal{O}(a)$ are kept ($a \equiv$ size of the body).

$$\bar{S}^{\hat{\alpha}\hat{\beta}} = S^{\hat{\alpha}\hat{\beta}} + 2P^{[\hat{\alpha}}\Delta x^{\hat{\beta}]}, \tag{C3}$$

where $\Delta x^{\hat{\alpha}} = \bar{z}^{\hat{\alpha}} - z^{\hat{\alpha}} = \bar{z}^{\hat{\alpha}}$; this is similar to the flat spacetime transformation (e.g. [21,23]). Hence, to obtain a solution corresponding to the same physical motion above, we must prescribe the same momentum $\vec{P} = P^z\vec{e}_z$, and correct the spin tensor and initial position of the centroid using Eq. (C3). As can be seen contracting (C3) with $\bar{u}_{\hat{\beta}}$ (taking $u^\alpha = U^\alpha$), the condition $\bar{S}^{\alpha\beta}\bar{u}_\beta = 0$ yields, in general, a centroid $\bar{z}^\alpha = x^\alpha_{\rm CM}(\bar{u})$ at a different point compared to the MP centroid $z^\alpha = x^\alpha_{\rm CM}(U)$, not on the z-axis, manifesting that the problem is no longer axisymmetric. Since, in general, $U^\alpha \nparallel P^\alpha$, cf. Eq. (29), the centroid $\bar{z}^\alpha$ does not even move parallel to the axis. Writing $\bar{S}_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu}\bar{S}^\mu\bar{u}^\nu$, where $\bar{S}^\alpha$ denotes the new spin vector, the force now reads

$$\frac{DP^\alpha}{d\tau} = -\frac{1}{2}R^\alpha{}_{\mu\nu\lambda}U^\mu\bar{S}^{\nu\lambda} = -\star R^{\sigma\tau\alpha}{}_\mu U^\mu\bar{u}_\tau\bar{S}_\sigma, \tag{C4}$$

which depends both on U^α and $\bar{u}^\alpha$, and, in general, will also not be parallel to the axis. This clearly leads to a more complicated description of the same problem.

The case of the Tulczyjew-Dixon (TD) condition, $\bar{u}^\alpha = P^\alpha/M$, exemplifies some of these difficulties. First, we face the complicated equation relating P^α and U^α [19,108,121,122],

$$U^\alpha = \frac{m}{M^2}\left(P^\alpha + \frac{2\bar{S}^{\alpha\nu}R_{\nu\tau\kappa\lambda}\bar{S}^{\kappa\lambda}P^\tau}{4M^2 + R_{\alpha\beta\gamma\delta}\bar{S}^{\alpha\beta}\bar{S}^{\gamma\delta}}\right), \tag{C5}$$

which in general are not parallel; and to obtain the force, given by Eq. (C4), one needs to know both (not just U^α, as with the MP condition). Based only on these equations, it would not be clear that an axial fall (of the physical body) is possible, what kind of solution represents it in this gauge, and how to set up its initial conditions. Using the knowledge of the MP solution (which is an axial fall), we know that, for this problem, $\vec{P}$ is parallel to $\vec{e}_z$; then, tentatively setting $\bar{\mathbf{S}} = \bar{S}^0\mathbf{e}_0 + \bar{S}^z\mathbf{e}_z$, and $\bar{z}^\alpha$ along the z-axis, it can eventually be shown from (C5) (see e.g. [123]) that, for such a setup, $P^\alpha = mU^\alpha$, and therefore the solution coincides with the one obtained using the MP condition. We thus end up (in this case) with the same solution, but taking a more complicated route.

In Sec. III B 3 we concluded that in the equatorial plane of Kerr-dS, for suitable r and $\vec{v}$, spinning particles move in prograde circular geodesics; we were able to do it only because we used the MP condition. With this condition, the force is given by a contraction of $\mathbb{H}_{\alpha\beta}$ with S^α, cf. Eq. (I.1b). From the curvature invariants, we deduced that in the equatorial plane there is a velocity field for which $\mathbb{H}_{\alpha\beta} = 0$, Eq. (60); for certain $r = r_{\rm geo}$ [solution of Eq. (66)], it matches the velocity of a circular geodesic. Along such a circle, the equation of motion reduces to

$$\frac{DP^\alpha}{d\tau} = 0 \Leftrightarrow ma^\alpha - \epsilon^\alpha{}_{\beta\gamma\delta}U^\delta\frac{D(S^\beta a^\gamma)}{d\tau} = 0, \tag{C6}$$

admitting $a^\alpha = 0$ as trivial solution (obviously a nonhelical one); the spinning particle will thus move along the circular geodesic. We would not be able to reach this conclusion using other spin conditions: for $\bar{u}^\alpha \neq U^\alpha$, the force is no longer governed by the magnetic part of the Riemann tensor $\mathbb{H}_{\alpha\beta}$ [but instead by a tensor $\mathcal{H}_{\alpha\beta} = \star R_{\alpha\mu\beta\nu}\bar{u}^\mu U^\nu$ involving *both* $\bar{u}^\beta$ and U^β, cf. Eq. (C4)], and therefore a similar analysis in terms of curvature invariants is not possible. In particular, in the framework of the TD condition $\bar{u}^\alpha = P^\alpha/M$, we doubt that it would ever be possible to notice this effect using the system formed by Eqs. (C4) and (12), coupled with the momentum-velocity relation (C5).

As for the application in Sec. III A, the motion in the Schwarzschild spacetime of a particle with radial initial velocity, first notice that, for a particle with *generic* spin S^α, the problem does not have spherical symmetry [regardless of the spin condition; indeed, a force orthogonal to $\vec{e}_r$ arises in the analogous electromagnetic setup, cf. Eq. (52)]. Using the MP condition, setting $\vec{U} = U^r\vec{e}_r$, we have, cf. Eqs. (50), $\mathbb{H}_{\alpha\beta} = 0 \Rightarrow DP^\alpha/d\tau = 0$. Hence we have (C6) as the equation of motion, with trivial solution $a^\alpha = 0 \Rightarrow P^\alpha = mU^\alpha$, i.e., the gyroscope moves along a radial geodesic. In the case of the TD condition, again we face the complicated Eqs. (C4) and (C5), not being transparent what occurs if one sets initially $\vec{U}|_{\rm in} = U^r\vec{e}_r$, or if the solution thereby obtained corresponds to the same physical motion above (a radial fall; in this framework it is not even obvious that it occurs). From the analysis with the MP condition, we know that, in order to represent the same problem, $\vec{P} = P^z\vec{e}_z =$ constant. It is useful to rewrite Eq. (C5) in terms of tidal tensors,

$$U^\alpha = \frac{m}{M^2}\left(P^\alpha + \frac{\epsilon^{\alpha\gamma}{}_{\tau\delta}\bar{S}^\tau P^\delta(\mathbb{H}^P)_{\sigma\gamma}\bar{S}^\sigma}{M^2 + (\mathbb{F}^P)^{\lambda\sigma}\bar{S}_\lambda\bar{S}_\sigma}\right), \tag{C7}$$

where $(\mathbb{H}^P)_{\alpha\gamma} \equiv \star R_{\alpha\beta\gamma\delta}P^\beta P^\delta/M^2$ and $(\mathbb{F}^P)_{\alpha\gamma} \equiv \star R\star_{\alpha\beta\gamma\delta}P^\beta P^\delta/M^2$ are, respectively, the gravitomagnetic tidal tensor and the "$\mathbb{F}$ tensor" [14,124] measured by an observer of 4-velocity $\bar{u}^\alpha = P^\alpha/M$. Noting, from Eq. (50), that, for radial $\vec{P}$, $(\mathbb{H}^P)_{\alpha\beta} = 0$, Eq. (C7) yields $P^\alpha = mU^\alpha$, and Eq. (C4) gives $DP^\alpha/d\tau = 0$; i.e., we end up with the same solution obtained with the MP condition. Other spin conditions, in general, will lead to $DP^\alpha/d\tau \neq 0$, and $U^\alpha \nparallel P^\alpha$ (see Figs. 6(c) and 6(d) of [33]), thus more complicated descriptions for this motion.

In the case of the analogous electromagnetic problem, a magnetic dipole with initial radial velocity in the Coulomb field, first we note that, due to the electromagnetic hidden momentum $P^\alpha_{\rm hidEM}$, in general P^α cannot be parallel to U^α. Furthermore, since $F^\alpha_{\rm EM} \neq 0$ and $a^\alpha \neq 0$, it is not trivial to (exactly) prescribe the initial conditions for the MP nonhelical solution (which in the previous examples was ensured by $a^\alpha = 0$). To first order in S, we can impose it by taking $S^{\alpha\beta}a_\beta \approx 0$, see [26]. With the TD condition, we face again a complicated equation relating P^α with U^α (and therefore $F^\alpha_{\rm EM}$ with a^α), Eq. (35) of [20]. An interesting choice for this system is the Corinaldesi-Papapetrou condition [54] $\bar{S}^{\alpha\beta}\bar{u}_\beta = 0$, where $\bar{\mathbf{u}} = \partial/\partial t$ corresponds to the static observers. In this case $\bar{S}^{\alpha\beta}D\bar{u}_\beta/d\tau = 0$, thus $P^\alpha_{\rm hidI} = 0$, cf. Eq. (31), leading to $P^\alpha = mU^\alpha + P^\alpha_{\rm hidEM}$, which is the simplest momentum-velocity relation possible for this problem.

More generally, in arbitrarily curved spacetimes, the inertial hidden momentum $P^\alpha_{\rm hidI}$ can *always* be made to vanish by choosing a $\bar{u}^\alpha$ parallel transported along the reference worldline, cf. Eq. (31). This choice may actually be cast as a spin supplementary condition [39] (for its detailed discussion, see [33,39,125]). It is especially favored for pole-dipole particles in purely gravitational systems, because it leads to particularly simple equations: the momentum-velocity relation is simply $P^\alpha = mU^\alpha$, and $\bar{S}^{\alpha\beta}$ is parallel transported, $D\bar{S}^{\alpha\beta}/d\tau = 0$, cf. Eq. (12). On the other hand, in some treatments spin conditions for which $P^\alpha_{\rm hid} \neq 0$ are preferred; that is the case of the Newton-Wigner [55,56] condition $\bar{u}^\alpha \propto P^\alpha/M + u^\alpha_{\rm lab}$, where $u^\alpha_{\rm lab}$ is the 4-velocity of some "laboratory" observer [58] (it may thus be cast as a combination of the Tulczyjew-Dixon and Corinaldesi-Papapetrou conditions). It is of advantage in some Hamiltonian and effective field theory approaches [57–63] (see also [126,127]) because it leads to canonical Dirac brackets (to linear order in the spin, in the case of curved spacetime [57,61]). The bottom line is that the spin condition is *gauge freedom*, and as such one should choose, in each application, the one that suits it the most. For the ones in this paper (where we have been exploring exact analogies that rely on it), it is the MP condition that is of clear advantage, as explained above.

2. Analogies under other spin conditions

The *exact* gravitoelectromagnetic analogies studied so far in this paper were obtained by employing, in the equations of motion, the Mathisson-Pirani (MP) spin condition. In this section we will study how the situation changes by choosing other spin conditions.

a. Analogy based on tidal tensors

For an arbitrary spin condition $\bar{S}^{\alpha\beta}\bar{u}_\beta = 0$, it is natural to define, as above, the spin vector $\bar{S}^\mu$ by $\bar{S}_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu}\bar{S}^\mu\bar{u}^\nu$, in terms of which the spin-curvature force reads $DP^\alpha/d\tau = -\mathcal{H}_\gamma{}^\alpha\bar{S}^\gamma$, where $\mathcal{H}_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\nu}\bar{u}^\mu U^\nu$, cf. Eq. (C4). Thus the force is still given by a contraction of a rank 2 tensor $\mathcal{H}_{\alpha\beta}$ with $\bar{S}^\alpha$; this new tensor, however, does not coincide with the magnetic part of the Riemann tensor $(\mathbb{H}^u)_{\alpha\beta} = \star R_{\alpha\mu\beta\nu}u^\mu u^\nu$ as measured by any observer u^α, because it results from a contraction of $\star R_{\alpha\mu\beta\nu}$ with *two different* vectors ($\bar{u}^\mu$ and U^ν). It does not obey the field equations in Table I, since the trace and antisymmetric parts of $\mathcal{H}_{\alpha\beta}$ no longer yield projections of the Einstein field equations, nor equations of the type (I.2b) and (I.3b) of Table I.[28] Instead, another analogy can be drawn here. First note that by choosing, as reference worldline, the centroid $x^\alpha_{\rm CM}(\bar{u})$ given by the condition $\bar{S}^{\alpha\beta}\bar{u}_\beta = 0$, that generates a mass dipole $d^\alpha_{\rm G} = -\bar{S}^{\alpha\beta}U_\beta$ in the centroid rest frame, cf. Eq. (13). Decomposing $\bar{S}^{\alpha\beta}$ into its time and space projections relative to the centroid 4-velocity $U^\alpha = dx^\alpha_{\rm CM}(\bar{u})/d\tau$, we have

$$\bar{S}^{\alpha\beta} = 2d_{\rm G}^{[\alpha}U^{\beta]} + \epsilon^{\alpha\beta}{}_{\mu\lambda}U^\lambda(\bar{S}^U)^\mu, \tag{C8}$$

where we used Eq. (4) of [14], and the vector

$$(\bar{S}^U)^\mu \equiv \frac{1}{2}\epsilon^\mu{}_{\alpha\beta\gamma}\bar{S}^{\alpha\beta}U^\gamma \tag{C9}$$

encodes the components of $\bar{S}^{\alpha\beta}$ spatial with respect to U^α, that is, what one would physically interpret as the classical angular momentum 3-vector (cf. e.g. [128]) about $x^\alpha_{\rm CM}(\bar{u})$, as measured *in the centroid frame* (i.e., as measured by the observer of 4-velocity U^α). Substituting Eq. (C8) into the second member of Eq. (C4) yields

$$\frac{DP^\alpha}{d\tau} = -\mathbb{H}_\beta{}^\alpha(\bar{S}^U)^\beta - \mathbb{E}_\beta{}^\alpha d^\beta_{\rm G}. \tag{C10}$$

This resembles the electromagnetic force exerted on a particle possessing both magnetic and electric dipole moments (as measured in the centroid frame). Indeed, the right-hand member of Eq. (C10) is formally analogous to the second and third terms of Eq. (15); however the last term of (15) (which is also part of the force on an electric dipole), has no counterpart in (C10). Since this term is not a tidal term, it is natural that it has no gravitational counterpart. An exact analogy exists however between Eq. (C10) and the canonical electromagnetic force on a particle with electric and magnetic dipole moments (and zero charge),

$$\frac{DP^\alpha_{\rm Dix}}{d\tau} = B_\beta{}^\alpha\mu^\beta + E_\beta{}^\alpha d^\beta \tag{C11}$$

obtained by substituting Eq. (A5) into (A3).

Tulczyjew-Dixon (TD) condition ($\bar{u}^\alpha = P^\alpha/M$).— Noting that $U^\alpha = (P^\alpha - P^\alpha_{\rm hid})/m$, we have in this case

[28] Namely those will not be equations involving only tidal tensors and sources, by contrast with both their magnetic counterparts (I.2a) and (I.3a) of Table I, and also with the gravitoelectric counterparts Eqs. (1.3b) and (1.7b) of Table 1 of [14]. Moreover, the tensorial structure of $\mathcal{H}_{\alpha\beta}$ (unlike $\mathbb{H}_{\alpha\beta}$) is not similar to its gravitoelectric counterpart $\mathbb{E}_{\alpha\beta}$, i.e., it is not spatial in both indices with respect to the same timelike vector, nor does it have to be symmetric in vacuum.

$d_{\rm G}^\alpha = -\bar{S}^{\alpha\beta}U_\beta = \bar{S}^\alpha{}_\beta P^\beta_{\rm hid}/m = \mathcal{O}$, and $(\bar{S}^U)^\mu = \bar{S}^\mu + \mathcal{O}$, where $\mathcal{O}$ is of order $\mathcal{O}(S^2)$ if electromagnetic hidden momentum is present ($P^\alpha_{\rm hidEM} \neq 0$), or $\mathcal{O}(S^3)$ otherwise. Therefore, to a good approximation (in particular in a pole-dipole approximation), Eq. (C10) becomes $F^\alpha_{\rm G} = -\mathbb{H}_\beta{}^\alpha \bar{S}^\beta$, and the analogy in Table I holds.

Corinaldesi-Papapetrou (CP) condition ($\bar{u}_\alpha = u^\alpha_{\rm lab}$).—This condition was introduced, for the case of Schwarzschild spacetime, in the noncovariant form $\bar{S}^{i0} = 0$ [54], where it states that the reference worldline is the centroid as measured by the observers at rest in Schwarzschild coordinates. It can be generalized [33,39] to arbitrary coordinate systems in arbitrary spacetimes in the covariant form $\bar{S}^\alpha{}_\beta u^\beta_{\rm lab} = 0$, where $u^\beta_{\rm lab}$ is the 4-velocity of the observers at rest in the chosen coordinate system (the laboratory observers $u^i_{\rm lab} = 0$ [20,33]). In this case $d^\alpha_{\rm G} = -\bar{S}^{\alpha\beta}U_\beta = -\bar{S}^{\alpha\beta}v_\beta(U,u_{\rm lab})\gamma(U,u_{\rm lab})$, where $v^\beta(U,u_{\rm lab})$ is the velocity of the centroid relative to the laboratory observers, cf. decomposition (67). Therefore the second term of (C10) is of first order in S and cannot in general be neglected (for instance, in the Schwarzschild spacetime, the two terms are typically of the same magnitude, see Sec. 3.4.2 of [33]). So the analogy that holds is between Eqs. (C10) and (C11) (not the one in Table I, between the spin curvature and the force on a magnetic dipole).

Newton-Wigner (NW) condition ($\bar{u}^\alpha \propto u^\alpha_{\rm lab} + P^\alpha/M$).—In this case the reference worldline is chosen as the centroid $x^\alpha_{\rm CM}(u_{\rm NW})$ as measured by observers of 4-velocity (cf. [57,58,61,125])

$$u^\alpha_{\rm NW} = K\left(u^\alpha_{\rm lab} + \frac{P^\alpha}{M}\right); \qquad K \equiv \sqrt{\frac{M}{2(M + m_{\rm lab})}} \tag{C12}$$

($m_{\rm lab} \equiv -u^\alpha_{\rm lab}P_\alpha$), that is, an even-weighted combination of the 4-velocity of the laboratory and the zero 3-momentum observers. Due to that, the situation with this spin condition is essentially similar to with the CP condition; it resembles more the electromagnetic force on a particle possessing both electric and magnetic moments (as measured in the centroid frame), and is closely analogous to the canonical electromagnetic force on such particle (except that the mass dipole $d^\alpha_{\rm G} = -\bar{S}^{\alpha\beta}U_\beta$ is different from the CP one, as $\bar{S}^{\alpha\beta}$ is now a different tensor, obeying $\bar{S}^\alpha{}_\beta u^\beta_{\rm NW} = 0$).

Parallel condition ($D\bar{u}^\alpha/d\tau = 0$).—This condition chooses as reference worldline some timelike vector $\bar{u}^\alpha$ parallel transported along the reference worldline $\bar{z}^\alpha$. Since the initial vector $\bar{u}^\alpha_{\rm in}$ is arbitrary [39], we may choose it as $\bar{u}^\alpha_{\rm in} = U^\alpha$, so that initially one obtains exactly the analogy in Table I, just like for the MP condition. Since, in general, the motion is nongeodesic, $\bar{u}^\alpha$ will progressively diverge from U^α, so at later instants that analogy will be only approximate, whilst the analogy between Eqs. (C10) and (C11) remains exact.

b. Spin precession

The analogy found in Eq. (26) of Sec. II C using the MP condition holds in an orthonormal frame comoving with the centroid, for a spin vector S^α which represents the angular momentum, as measured in the centroid frame, taken about the centroid $x^\alpha_{\rm CM}(U)$ measured, again, in is own rest frame. Other spin conditions $\bar{S}^{\alpha\beta}\bar{u}_\beta = 0$ correspond to different angular momentum tensors $\bar{S}^{\alpha\beta}$, taken about the centroids $x^\alpha_{\rm CM}(\bar{u})$ measured by the observer of 4-velocity $\bar{u}^\alpha$ (not U^α). The vector which encodes the angular momentum about $x^\alpha_{\rm CM}(\bar{u})$, and *as measured in the centroid frame* is, as explained above, $(\bar{S}^U)^\alpha$, see Eqs. (C8) and (C9). To compute its evolution equation, one first notes that $\epsilon_{\alpha\beta\gamma\delta}U^\delta D\bar{S}^{\alpha\beta}/d\tau = 2D(\bar{S}^U)_\gamma/d\tau - \epsilon_{\alpha\beta\gamma\delta}a^\delta\bar{S}^{\alpha\beta}$; then, using (27) (with $S^{\alpha\beta} = \bar{S}^{\alpha\beta}$) and (C8), we have

$$\frac{D(\bar{S}^U)^\gamma}{d\tau} = (\bar{S}^U)_\mu a^\mu U^\gamma + \epsilon^\gamma{}_{\alpha\beta\delta}U^\delta\left[d^\beta_{\rm G}a^\alpha + \frac{1}{2}\tau^{\alpha\beta}\right].$$

In an orthonormal tetrad $\mathbf{e}_{\hat{\alpha}}$ comoving with the centroid, this equation reads, using (28) (see Sec. II C),

$$\frac{d\vec{\bar{S}}^U}{d\tau} = \vec{\bar{S}}^U \times \vec{\Omega} + \vec{d}_{\rm G} \times \vec{G} + \vec{\mu} \times \vec{B} + \vec{d} \times \vec{E}, \tag{C13}$$

where $G^\alpha = -a^\alpha$ is the gravitoelectric field as measured in the centroid frame, cf Sec. II D. This equation manifests that, for *an arbitrary* spin condition, an exact analogy always exists, with $\{\vec{\bar{S}}^U, \vec{d}_{\rm G}\}$ playing a role analogous to the magnetic and electric dipole moment vectors $\{\vec{\mu}, \vec{d}\}$, and the inertial fields $\{\vec{\Omega}, \vec{G}\}$ playing a role analogous to the electromagnetic fields $\{\vec{B}, \vec{E}\}$ (all quantities measured in *the centroid frame*). As discussed in Sec. II C, if $\mathbf{e}_{\hat{\alpha}}$ is adapted to a congruence of observers, then $\vec{\Omega} = \vec{H}/2$, and the analogy deepens. The term $\vec{d}_{\rm G} \times \vec{G} \equiv -\vec{d}_{\rm G} \times \vec{a}$ is the exact version of the "instrumental torque" discussed in [128][29] in the weak field and slow motion regime. If one chooses $\bar{u}^\alpha = U^\alpha$ (MP condition) then $\vec{d}_{\rm G} = 0$, $\vec{\bar{S}}^U = \vec{S}$, and, taking also particles with no electric dipole moment in the centroid frame ($\vec{d} = 0$), Eq. (C13) reduces to Eq. (26). Under the TD condition $\bar{S}^{\alpha\beta}P_\beta = 0$ the situation is similar to a good approximation: as we have seen in Sec. C 2 a, $d^\alpha_{\rm G}$ is of order $\mathcal{O}(S^2)$ if $P^\alpha_{\rm hidEM} \neq 0$, or $\mathcal{O}(S^3)$ otherwise.

[29]To make the connection with [128], we note that: therein the CP condition is considered, so $d^\alpha_{\rm G} = -\bar{S}^\alpha{}_\beta U^\beta = \epsilon^\alpha{}_{\gamma\delta\beta}U^\beta v^\delta \bar{S}^\gamma$ with $v^\gamma \equiv v^\gamma(U,u_{\rm lab})$, reading, in the centroid frame, $\vec{d}_{\rm G} = \vec{\bar{S}} \times \vec{v}$; $\vec{\bar{S}} \equiv \mathbf{S}$, $\vec{\bar{S}}^U \equiv \mathbf{S}_0$ in their notation; and $\vec{a} = \vec{F}/m + \mathcal{O}(S)$.

Under the CP condition ($\bar{S}^{\alpha}{}_{\beta}u^{\beta}_{\rm lab}=0$), $\vec{d}_{\rm G}$ is of order $\mathcal{O}(S)$ (cf. footnote 29), hence the situation depends on the type of force applied on the body. If $q=\vec{d}=0$, and only gravitational and electromagnetic forces are present, $\vec{d}_{\rm G}\times\vec{a}\sim\mathcal{O}(S^2)$, and one recovers, to a good approximation, the analogy in Eq. (26) (with $\vec{\bar{S}}^U$ in the place of $\vec{S}$). Otherwise, for a generic force (or if $q\neq 0$), $\vec{d}_{\rm G}\times\vec{a}\sim\mathcal{O}(S)$, non-negligible in pole-dipole, nor in weak-field slow motion approximations [128], thus in this case it is only the analogy in Eq. (C13) that holds. With the NW condition, $\bar{S}^{\alpha}{}_{\beta}u^{\beta}_{\rm NW}=0$, the situation is very similar, due to the contribution of $u^{\alpha}_{\rm lab}$ to $u^{\alpha}_{\rm NW}$ in Eq. (C12).

c. Hidden momentum

Under an arbitrary spin condition neither $P^{\alpha}_{\rm hidI}$ nor $P^{\alpha}_{\rm hidEM}$ take the forms (33), and there is no longer a close analogy between the two. For instance, under the parallel condition $D\bar{u}^{\alpha}/d\tau=0$, one has simply $P^{\alpha}_{\rm hidI}=0$; moreover $P^{\alpha}_{\rm hidEM}$ [Eq. (32)] takes in general a complicated form, encoding not only the hidden momentum modeled in e.g. Fig. 9 of [71] (which is physical), but also a pure gauge part that is due solely to the choice of centroid, see Sec. 3.5.1 of [33]. An exception is the TD condition, under which Eq. (33) is still obtained to a good approximation [namely by neglecting terms of order $\mathcal{O}(S^2)$ and $\mathcal{O}(Sd)$, consistent with a dipole approximation]; it was actually in such approximate form that this analogy was first introduced in [20].

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