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$K \to \pi \ell^+ \ell^-$ form factor in the large-N_c and cutoff regularization method

Estefania Coluccio Leskow,[1,*] Giancarlo D'Ambrosio,[1,†] David Greynat,[1,2,‡] and Atanu Nath[2,§]

[1]INFN-Sezione di Napoli, Via Cintia, 80126 Napoli, Italia

[2]Dip. di Scienze Fisiche, Universitá di Napoli "Federico II", Via Cintia, 80126 Napoli, Italia

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In a series of papers, Bardeen, Buras and Gérard have proposed a large-N_c method to evaluate hadronic weak matrix elements to attack, for instance, the determination of the $\Delta I = 1/2$ rule and $\mathrm{Re}(\epsilon'/\epsilon)$. Here we test this method to the determination of the form factor parameters a_+ and b_+ in the decays $K^+ \to \pi^+ \ell^+ \ell^-$ and $K_S \to \pi^0 \ell^+ \ell^-$. The results are encouraging, in particular after a complete treatment of vector meson dominance.

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I. INTRODUCTION

Rare kaon decays play a crucial role in particle physics [1–8], particularly now with the beautiful physics program of NA62 [9], where 100 events of $K^+ \to \pi^+ \nu\bar{\nu}$ are expected, and the J-PARC KOTO experiment with the goal of a few $K_L \to \pi^0 \nu\bar{\nu}$ SM events in a 3- to 4-year run with a signal-to-noise ratio of about 2 [10].

Similar to $K_L \to \pi^0 \nu\bar{\nu}$, the short-distance (SD) part of $K_L \to \pi^0 e^+ e^-$ gives information on $V_{ts}V_{td}^*$ and thus measures the height of the unitarity triangle. The measurement of this decay may also lead to new physics tests [11]. There is also an indirect CP-violating contribution from $K_S \to \pi^0 e^+ e^-$, the magnitude of which can be obtained from the measured BR for the corresponding K_S decay [12,13]. A theoretical determination is also needed, and recently the lattice RBC and UKQCD Collaborations [14] addressed this issue. The related $K^\pm \to \pi^\pm \ell^+ \ell^-$ decay may also help with this goal; the experimental form factor here has been measured well by NA48/2 [9,12,15]. The appearance of chiral unknown constants [16,17] brings up the crucial question of determining them either by lattice [14] or in a model-dependent manner [1–8,18], as we will do in this paper. Since one can measure $K^\pm \to \pi^\pm e^+ e^-$ and $K^\pm \to \pi^\pm \mu^+ \mu^-$ separately, the question of lepton flavor universality violation is also interesting [19–58].

In this paper, we evaluate the $K^\pm \to \pi^\pm \ell^+ \ell^-$ form factor in the theoretical framework suggested by Bardeen, Buras, and Gerard (BBG) [59–65]; the authors of this approach have successfully applied the method to the explanation of the $\Delta I = 1/2$ rule and $\pi^+ - \pi^0$-mass difference. We think it is interesting to apply it here as well.

The recent lattice results from the RBC and UKQCD Collaborations [14] report on the $K \to \pi\pi$ matrix elements $\mathrm{Re}(A_0)$ and $\mathrm{Im}(A_0)$, their findings lead to 2σ–3σ deviation from the experimental world average of $\mathrm{Re}(\epsilon'/\epsilon)$. This has motivated the authors of Refs. [55,56] to evaluate the same weak matrix elements B_6 and B_8. Finding consistency with lattice results they conclude [56] that new physics seems to be required to accommodate the present experimental value of $\mathrm{Re}(\epsilon'/\epsilon)$. Using large N_c and minimal hadronic ansatz, in Ref. [66] Hambye *et al.* still find agreement with experimental values. The final-state interaction is not accurately described by the lattice [the lattice result [14] for the $I = 0$ phase shift $\delta_0 = 23.8(4.9)(1.2)^\circ$ is about 3σ smaller than the value obtained in dispersive treatments of Ref. [67–69]], and a good theoretical description could lead to agreement with experiments as in the approach of Refs. [70,71]; for an alternative solution, see Ref. [72,73].

Nevertheless, we think it is interesting to check the BBG method in $K^+ \to \pi^+ \ell^+ \ell^-$ decay. We dedicate Secs. II and III to a model-independent discussion, Sec. IV to the BBG method, Sec. V to the form factor evaluation, Sec. VI to the addition of vectors, and Sec. VII to the $K_S \to \pi^0 \ell^+ \ell^-$ form factor.

II. MODEL-INDEPENDENT ANALYSIS

The decay $K \to \pi \bar{\ell} \ell$ is dominated by a virtual photon exchange [16,17],

$$\mathcal{A}[K(k) \to \pi(p)\gamma^*(q)] = \frac{W_+(z)}{(4\pi)^2}[z(k+p)_\mu - (1 - r_\pi^2)q_\mu], \tag{1}$$

where $r_x \doteq \frac{M_x}{M_K}$ and $z \doteq \frac{q^2}{M_K^2}$, with q^2 being the photon transferred momentum. With these conventions the decay amplitude takes the form ($\alpha \doteq e^2/4\pi$)

$$\begin{aligned}\mathcal{A}[K(k) &\to \pi(p)\ell^+(p_+)\ell^-(p_-)] \\ &= -\frac{\alpha}{4\pi M_K^2} W_+(z)(k+p)_\mu \bar{u}_\ell(p_-)\gamma^\mu v_\ell(p_+).\end{aligned} \tag{2}$$

*coluccio@na.infn.it
†gdambros@na.infn.it
‡greynat@na.infn.it
§nath@na.infn.it

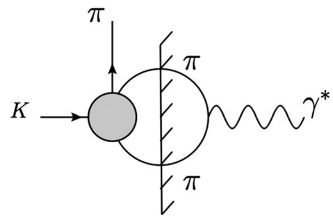

FIG. 1. Pion loop contribution to $K^+ \to \pi^+\gamma^*$. The unitary cut used is represented, too. The blob represents the $K \to 3\pi$ vertex.

The form factor $W_+(z)$ can be decomposed into two parts: one coming from the dominant pion loop contribution $W_+^{\pi\pi}(z)$ and another one, $W_+^{\rm pol}(z)$, that accounts for the contributions of higher mass intermediate states (like K^+K^-, for instance, and local pieces). $W_+^{\rm pol}(z)$ can be well approximated by a linear polynomial for small values of z, $W_+^{\rm pol}(z) \sim a_+ + b_+ z$. In this way, $W_+(z)$ can be written as [17]

$$W_+(z) = G_F M_K^2 (a_+ + b_+ z) + W_+^{\pi\pi}(z), \qquad (3)$$

with *a priori* unknown low-energy constants contributing to a_+ and b_+, which have to be experimentally determined [9,13,15].

$W_+^{\pi\pi}(z)$ is obtained from the analytic structure of the diagram in Fig. 1 [17].

In Ref. [17], the behavior of $W_+(z)$ at $z \to 0$ is entirely fixed up to $W_+^{\rm pol}(z)$,

$$W_+(z) \underset{z\to 0}{\sim} G_F M_K^2 a_+ + \left(G_F M_K^2 b_+ + \frac{3r_\pi^2(\alpha_+ - \beta_+) - \beta_+}{180 r_\pi^6} \right) z, \qquad (4)$$

where $\alpha_+ = (-20.6 \pm 0.5) \times 10^{-8}$ and $\beta_+ = (-2.6 \pm 1.2) \times 10^{-8}$ are the $K \to 3\pi$ parameters from Refs. [74–79].

The local counterterm structures at $\mathcal{O}(p^4)$ are

$$a_+^{(4)} = \frac{G_8}{G_F}\left(\frac{1}{3} - w_+\right), \qquad (5)$$

where w_+ is given [16] in terms of N_i's [80] and L_9 [81] by

$$w_+ = \frac{64\pi^2}{3}(N_{14}^r - N_{15}^r + 3L_9^r) + \frac{1}{3}\ln\frac{\mu^2}{M_K M_\pi}. \qquad (6)$$

Since w_+ is scale independent, the μ dependence of the combination among the N_i's and L_9 is exactly compensated by the $\log \mu^2$ from the chiral loop in Eq. (6). From the loop contribution, one also has

$$b_+^{(4)} = -\frac{G_8}{G_F}\frac{1}{60}. \qquad (7)$$

Experimentally [15], we have

$$a_+^{\rm exp} = -0.578 \pm 0.016, \qquad (8)$$

$$b_+^{\rm exp} = -0.779 \pm 0.066. \qquad (9)$$

As we can see, the experimental values for a_+ and b_+ are the same order of magnitude. We then have to understand why this is so. Indeed, a_+, L_9 and the N_i's have large contributions from the vector meson dominance (VMD) [80,82,83]; since b_+ is mostly an $\mathcal{O}(p^6)$ observable, it should have an important enhancement.

The K_S decay is discussed in Sec. VII.

III. AMPLITUDE ANALYSIS AND SHORT-DISTANCE RESULTS

The behavior of the amplitude in Eq. (2) can be studied by distinguishing two different contributions: (i) the long-distance (LD) one described by chiral perturbation theory (χPT) [16,17] and (ii) the short-distance (SD) one described by an effective four-quark Hamiltonian [84–89]. The complete description of the amplitude then implies a continuation through both regions.

The dominant $\Delta S = 1$, SD effective four-quark Hamiltonian is given by [84–92]

$$\mathcal{H}_{\rm eff}^{\Delta S=1} = -\frac{G_F V_{us}^* V_{ud}}{\sqrt{2}}[C_-(\mu^2)Q_-(\mu^2) + C_7(\mu^2)Q_7], \qquad (10)$$

where $C_-(\mu^2)$ and $C_7(\mu^2)$ are the Wilson coefficients (see Appendix C for their expressions) associated with the four-quark operators $Q_-(\mu^2)$ and Q_7, respectively, given by

$$Q_- = 4(\bar{s}_L\gamma^\nu u_L)(\bar{u}_L\gamma_\nu d_L) - 4(\bar{s}_L\gamma^\nu d_L)(\bar{u}_L\gamma_\nu u_L), \qquad (11)$$

$$Q_7 = 2\alpha(\bar{s}_L\gamma^\nu d_L)(\bar{e}\gamma_\nu e). \qquad (12)$$

The SD amplitude then takes the form

$$\mathcal{A}(K \to \pi\ell^+\ell^-) = -\frac{G_F V_{us}^* V_{ud}}{\sqrt{2}} \times \langle \pi\ell^+\ell^- | C_-(\mu^2)Q_-(\mu^2) + C_7(\mu^2)Q_7 | K\rangle. \qquad (13)$$

Both the Wilson coefficients and the four-quark operators depend on the renormalization scale μ that separates the two regimes. Nevertheless, the physical amplitude cannot depend on μ.Q_7 in Eq. (13) is a μ^2-independent operator, so in order for the amplitude to be μ^2 independent, the Wilson coefficient $C_7(\mu^2)$ has to cancel the μ^2 dependence in $C_-(\mu^2)Q_-(\mu^2)$. Some of the consequences of this SD property will be considered in a model-independent form in Ref. [93].

IV. BARDEEN-BURAS-GÉRARD FRAMEWORK

In Refs. [59–65], the authors use an order p^2 chiral Lagrangian and a physical cutoff M to regularize the contributions beyond tree level instead of the usual local counterterms (e.g., the L_i and N_i constants). Consequently, their results exhibit a quadratic dependence on the physical cutoff M which according to them is a crucial ingredient in the matching of the meson and quark pictures. They argue that one can obtain a parametrization of nonperturbative QCD effects by matching a low-energy Lagrangian, valid up to the scale M, to the logarithmic behavior of relevant Wilson coefficients at high energy. In this work we refer to this computational method as the BBG framework.

In this context, the function $W_+(z)$ becomes a function of q^2 and M^2,

$$W_+(z) \mapsto W_+(z, M^2). \tag{14}$$

Our goal is to predict the values of the a_+ and b_+ coefficients using the BBG framework.

At the matching scale M, the description for low and high energy must coincide; this means that the LD quadratic divergence in M has to be numerically equal to the SD logarithmic divergence. Therefore, at $\mu^2 = M^2$ the SD Hamiltonian,

$$\mathcal{H}_{\mathrm{eff}}^{\Delta S=1} = -\frac{G_F V_{us}^* V_{ud}}{\sqrt{2}} \times [C_-(M^2) Q_-(M^2) + C_7(M^2) Q_7], \tag{15}$$

must coincide with its chiral representation at LD.

A. Amplitude properties

The BBG approach considers only a chiral $\mathcal{O}(p^2)$ effective Lagrangian below the scale M; thus, since the loop calculations are regularized by the cutoff M, higher order Lagrangians (i.e., with L_i and N_i constants) do not appear at all. Following their prescriptions, one then has

$$\mathcal{A}(K \to \pi \ell^+ \ell^-) = -\frac{G_F V_{us}^* V_{ud}}{\sqrt{2}} \times \langle \pi \ell^+ \ell^- | C_-(M^2) Q_-(M^2) + C_7(M^2) Q_7 | K \rangle. \tag{16}$$

The chiral loop calculation of the matrix element of Q_- with the $\mathcal{O}(p^2)$ chiral Lagrangian does not provide any quadratic divergences (of course, not in dimensional regularization) even in the cutoff regularization (see Appendix B). The $\ln M^2$ appearing here at the chiral scale is canceled by local counterterms in Eq. (6) in usual χPT. Now, this role is played by $C_7(\mu^2 = M^2) Q_7$.

The matching between SD and LD should be around 1 GeV; then $C_-(M^2) Q_-(M^2)$ and $C_7(\mu^2 = M^2) Q_7$ have to evolve from the chiral scale to 1 GeV. But this evolution implies a mixing between the operators Q_- and Q_7 according to renormalization group equation (RGE) [84–92]. In the BBG framework, this mixing is captured by the quadratic divergences [59–65], which in our case can come only from the $K \to 3\pi$ vertex (chirally related to $K \to 2\pi$ studied by BBG; see below). In other words, the authors of Refs. [59–65] have extended the usual renormalization flow of the SD sector (from M_W^2 to M^2) to a flow in the LD sector from M^2 to 0 (as depicted in Fig. 2) through the relation

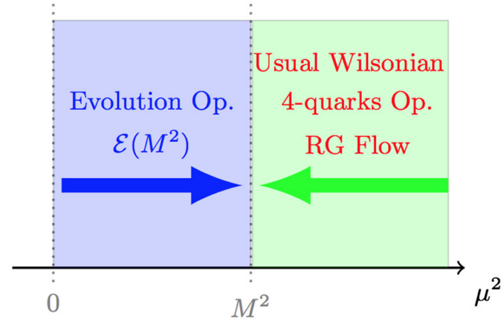

FIG. 2. The usual Wilsonian renormalization flow is represented above the scale M^2. The extended renormalization flow defined in Eq. (18) is shown below M^2.

$$Q_-(M^2) = \mathcal{E}(M^2) Q_-(0), \tag{17}$$

where $\mathcal{E}(M^2)$ is the evolution operator given by [65]

$$\mathcal{E}(M^2) \doteq 1 + \frac{3}{16\pi^2} \left[\frac{M^2}{f_\pi^2} + \frac{M_K^2}{4 f_\pi^2} \ln\left(1 + \frac{M^2}{\tilde{m}^2}\right) \right], \tag{18}$$

with $\tilde{m} \approx 0.3$ GeV. Here, $\mathcal{E}(M^2)$ comes from the $K \to \pi\pi$ analysis in Ref. [65], and soft-pion theorem tells us that it can be applied to the $K \to 3\pi$ vertex [1–8]. The amplitude is then given by

$$C_-(M^2) \langle 3\pi | Q_-(M^2) | K \rangle = C_-(M^2) \mathcal{E}(M^2) \langle 3\pi | Q_-(0) | K \rangle. \tag{19}$$

The authors of Ref. [65] find that the range of numerical values for M that leaves the amplitude invariant is

$$0.6 \text{ GeV} \leq M < 1 \text{ GeV}, \tag{20}$$

with a preferred value at 0.7 GeV (without vector contribution).

V. DETERMINATION OF a_+ AND b_+ (NO VECTORS)

Equation (4) uniquely determines the coefficients a_+ and b_+ as we will see here. Writing

$$W_+(z, M^2) \underset{z \to 0}{\sim} M_K^2 G_F a_+(M^2) + M_K^2 G_F b_+(M^2) z, \tag{21}$$

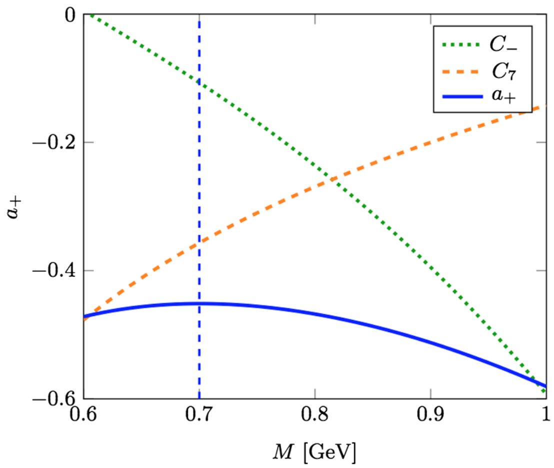

FIG. 3. In blue, we show the variation of a_+ as a function of M in GeV. The dotted green curve represents the contribution proportional to $C_-(M^2)$, and the dashed orange curve represents the one proportional to $C_7(M^2)$. The vertical dashed line stands for the matching scale.

we identify (the wave function renormalization factors Z_π and Z_K are given in Appendix B)

$$a_+(M^2) = -\frac{V_{us}^* V_{ud}}{\sqrt{2}} \sqrt{Z_\pi Z_K} \bigg\{ -4\pi C_7(M^2) + C_-(M^2) \left[-\frac{5}{9} + \frac{1}{3} \ln \frac{M^2}{M_\pi M_K} \right] \mathcal{E}(M^2) \bigg\}. \quad (22)$$

Compared to the analysis of $K \to 2\pi$ in Refs. [59–65], we have a further cancellation of the log in $[-\frac{5}{9} + \frac{1}{3} \ln \frac{M^2}{M_\pi M_K}]$ and the log in $C_7(M^2)$. This fixes M and then a_+ as shown in Fig. 3. We also have

$$b_+(M^2) = -\frac{V_{us}^* V_{ud}}{\sqrt{2}} \frac{\sqrt{Z_\pi Z_K}}{60 r_\pi^2} C_-(M^2) \mathcal{E}(M^2) - \frac{1}{M_K^2} \frac{3 r_\pi^2 (\alpha_+ - \beta_+) - \beta_+}{180 G_F r_\pi^6}. \quad (23)$$

In order to find the value of M where there is a compensation between the LD quadratic dependence [including both terms in Eq. (22), the constant and the novel logarithmic one) and the SD logarithm, we look for the solution of $\partial_{M^2} a_+ = 0$. We find that this equation is satisfied when $M = 0.7$ GeV, and numerically one gets

$$a_+((0.7 \text{ GeV})^2) = -0.5, \quad (24)$$

$$b_+((0.7 \text{ GeV})^2) = -0.12. \quad (25)$$

Comparing with the experimental values in Eq. (9), we find good agreement for a_+ but not for b_+. Figure 3 shows a_+ as a function of M, together with the contributions coming from C_- and C_7, separately. These are the expected behaviors from LD physics. The dashed vertical line corresponds to the scale where $\partial_{M^2} a_+ = 0$.

In the following section we study the inclusion of vectors in the BBG approach.

VI. VECTOR CONTRIBUTIONS IN THE BBG FRAMEWORK

Vector contributions increase the range of validity of M^2 and smooth over the transitions between short- and long-distance continuation [59–65]. We have to consider two counterterm structures from Eq. (6) which are shown in Fig. 4, the structure coming from L_9 [diagram (a)] and the $N_{14} - N_{15}$ contributions [diagram (b)].

Diagram (a) includes vectors with mass M_V in the evolution operator of Q_- in Eq. (18) as explained in [65],

$$\mathcal{E}(M^2) \mapsto \mathcal{E}(M^2, M_V^2) = \mathcal{E}(M^2) + \Delta(M^2, M_V^2), \quad (26)$$

where

$$\Delta(M^2, M_V^2) = \frac{3}{16\pi^2} \left[-\frac{9}{16} \frac{M^2}{f_\pi^2} + \frac{3}{8} \frac{M^2}{f_\pi^2} \frac{M_V^2}{M^2 + M_V^2} + \frac{3}{16} \frac{M_V^2}{f_\pi^2} \ln \left(1 + \frac{M^2}{M_V^2} \right) \right], \quad (27)$$

and the electromagnetic form factor changes as

$$1 \mapsto 1 + z \frac{M_K^2}{M_V^2}. \quad (28)$$

Diagram (b) in Fig. 4, corresponding to the $N_{14} - N_{15}$ local counterterms, implies a modification of the mixing between Q_- and Q_7 in the RGE by adding an extra contribution $C_-(M^2) \eta_V(M^2, z)$. This contribution is not present in $K \to \pi\pi$ processes and so does not affect the results in Refs. [59–65]. The complete calculation with vectors can be done using the hidden local symmetry framework [94–98]. We have to be careful, the counting in large N_c must be respected by including all terms up to $1/N_c$ corrections with the same argument as in Sec. IV. One can evaluate this contribution as

$$\eta_V(M^2, z) = 4\pi \left[\frac{f_\pi^2}{M_V^2} - z \frac{2}{3} \frac{M_K^2}{M_V^2} \ln \frac{M^2}{M_K^2} \right]. \quad (29)$$

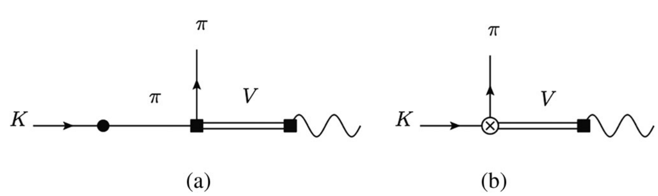

FIG. 4. Diagram (a) represents the VMD contribution to L_9 [the circle vertex is the $\mathcal{O}(p^2)$ $\Delta S = 1$ vertex]. Analogously, diagram (b) represents the VMD contribution to the $N_{14} - N_{15}$ one (the times vertex is the $\Delta S = 1$ vertex coming from Q_-).

Therefore, one gets

$$a_+(M^2, M_V^2) = -\frac{V_{us}^* V_{ud}}{\sqrt{2}} \sqrt{Z_\pi Z_K} \left\{ -4\pi C_7(M^2) + C_-(M^2) \left[-\frac{5}{9} + \frac{1}{3} \ln \frac{M^2}{M_\pi M_K} \right] \mathcal{E}(M^2, M_V^2) + C_-(M^2) 4\pi \frac{f_\pi^2}{M_V^2} \right\}, \quad (30)$$

and

$$b_+(M^2, M_V^2) = \frac{M_K^2}{M_V^2} a_+(M^2, M_V^2) - \frac{1}{M_K^2} \frac{3r_\pi^2(\alpha_+ - \beta_+) - \beta_+}{180 G_F r_\pi^6} - \frac{V_{us}^* V_{ud}}{\sqrt{2}} \sqrt{Z_\pi Z_K} C_-(M^2) \times \left[\frac{1}{60 r_\pi^2} \mathcal{E}(M^2, M_V^2) - \frac{8\pi}{3} \frac{M_K^2}{M_V^2} \ln \frac{M^2}{M_K^2} \right]. \quad (31)$$

In the same manner as before, we evaluate the scale M by requiring $\partial_{M^2} a_+ = 0$ in Eq. (30), and we obtain that for $M = 0.7$ GeV

$$a_+((0.7\ \text{GeV})^2, (0.775\ \text{GeV})^2) = -0.54, \quad (32)$$

$$b_+((0.7\ \text{GeV})^2, (0.775\ \text{GeV})^2) = -0.72. \quad (33)$$

The interplay between strong amplitudes (L_9) with external weak transitions [Fig. 4(b)] have already been noticed by the authors of Ref. [99] for the VMD $\mathcal{O}(p^6)$ contribution to $K_L \to \pi^0 \gamma\gamma$.

In Fig. 5, we show a_+ as a function of M in the three different scenarios: "BBG no vect." is the framework where no vectors are included, and "BBG(vect)(a)" is the one where only diagram (a) in Fig. 4 is considered. We refer to "BBG(vect) (a) + (b)" as the last case where both diagrams in Fig. 4 have been included.

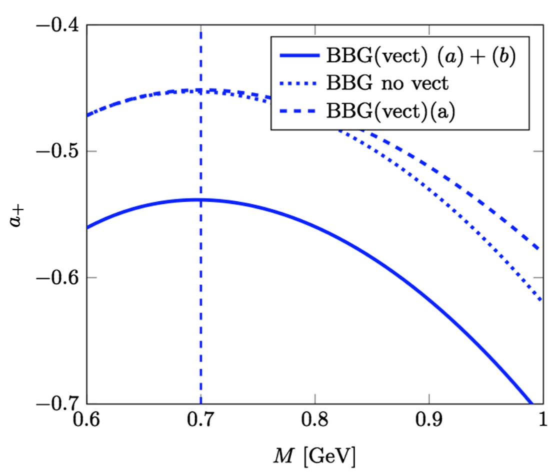

FIG. 5. Here, we show a_+ as a function of M in the three different frameworks: "BBG no vect.," where vectors are not included; "BBG(vect)(a)," with the contribution coming only from Fig. 4(a); and "BBG(vect) (a) + (b)," where both Figs. 4(a) and 4(b) are included. The vertical line indicates the value $M = 0.7$ GeV.

Following Buchalla *et al.* in Ref. [100], we investigate our predictions of what the authors call a_+^{VMD} and a_+^{nVMD}. Under the general hypothesis that the b_+ term in Eq. (3) is generated by the expansion of a vector-meson propagator, $W_+(z)$ can be written as

$$W_+(z) = G_F M_K^2 \left[(a_+^{\text{VMD}} + a_+^{\text{nVMD}}) + a_+^{\text{VMD}} \frac{M_K^2}{M_V^2} z \right], \quad (34)$$

where a_+^{nVMD} denotes z-independent non-VDM contributions. The introduction of the η_V contribution is necessary to recover this separation between a_+^{VMD} and a_+^{nVMD}. Indeed, we find

$$a_+^{\text{VMD}} = \frac{M_V^2}{M_K^2} (b_+|_{\text{BBG(vect)}(a)+(b)} - b_+|_{\text{BBG}}) \quad (35)$$

$$= \frac{M_V^2}{M_K^2} [-0.72 - (-0.12)] = -1.5, \quad (36)$$

which is in good agreement with $a_+^{\text{VMD}} = \frac{M_K^2}{M_V^2} b_+^{\text{exp}} = -1.6 \pm 0.1$ [100].

VII. ANALYSIS OF $K_S \to \pi^0 \ell^+ \ell^-$

The analysis of $K_S \to \pi^0 \bar{\ell}\ell$ can be directly deduced from the previous one, $K^+ \to \pi^+ \bar{\ell}\ell$. Indeed,

$$\langle \pi^0 \gamma^*(q) | Q_-(0) | K_S \rangle = -\langle \pi^0 \gamma^*(q) | Q_-(0) | K^+ \rangle |_{M_\pi = M_K}. \quad (37)$$

And, in this case, the local counterterm structures at $\mathcal{O}(p^4)$ are [17]

$$a_s^{(4)} = \frac{G_8}{G_F} \left(\frac{1}{3} - w_s \right), \quad (38)$$

where w_s is [16,80,81]

$$w_S = \frac{32\pi^2}{3} (2N_{14}^r + N_{15}^r) + \frac{1}{3} \ln \frac{\mu^2}{M_K^2}. \quad (39)$$

Given that the decay $K_S \to 3\pi$ ($\Delta I = 1/2$ transitions) is not allowed ($\Delta I = 3/2$ transitions are permitted), only kaons are present in the loop (see Appendix B). Using the same identification as in Eq. (4) and following the same procedure as in the case of the decay $K^+ \to \pi^+ \bar{\ell}\ell$, we find that $a_S = 1.2$ ($a_S^{\text{exp}} = |1.08|^{+0.26}_{-0.21}$ [15]) for the same scale $M = 0.7$ GeV established from Eq. (30). This value is in agreement with the fitted w_S value obtained in Ref. [101].

VIII. CONCLUSION

We have evaluated the $K^+ \to \pi^+ \ell^+ \ell^-$ form factor parameters a_+ and b_+ in the BBG framework.

Regarding a_+ the theoretical dependence or uncertainty in this framework on the matching scale seems small (see Fig. 3): comparison with phenomenology seems very successful [see Eq. (24)]. Consistency with the full chiral structure of the weak counterterms required a more general discussion on vector contributions (see Sec. VI and Fig. 4), which led to an extension of the Q_- evolution studied by the authors of Refs. [59–65] in the context of $K \to 2\pi$. This extension met nicely with the experimental values [15]. We have applied our method to $K_S \to \pi^0 e^+ e^-$ in Sec. VII and found a good agreement with experimental results, too.

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APPENDIX A: EVALUATION OF SCALAR INTEGRALS

The loop integral with a cutoff M^2 has the form

$$I(\alpha, R^2, M^2) \doteq -i \int^{M^2} \frac{\mathrm{d}^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - R^2)^\alpha} \qquad \text{(A1)}$$

and gives

$$I(\alpha, R^2, M^2) = \frac{M^4 R^{-2\alpha}}{6(4\pi)^2}\, {}_2\mathrm{F}_1\left(\begin{matrix}\alpha, 2\\ 3\end{matrix}\middle| -\frac{M^2}{R^2}\right), \qquad \text{(A2)}$$

where ${}_2\mathrm{F}_1$ is the Gauss hypergeometric function. In the one-loop case, for example, the integral is given by

$$\begin{aligned}\mathrm{A}_0(m^2) &= I(1, m^2, M^2)\\ &= \frac{1}{3(4\pi)^2}\left[M^2 - m^2 \ln\left(1 + \frac{M^2}{m^2}\right)\right]. \end{aligned} \qquad \text{(A3)}$$

All the scalar integrals can be evaluated using Eq. (A2).

APPENDIX B: AMPLITUDE FORMULAS

1. $K^+ \to \pi^+\gamma^*$

The form factor defined in Eq. (14) is obtained from

$$\begin{aligned}W_+(z, M^2) = \frac{M_K^2 G_F V_{us}^* V_{ud}}{\sqrt{2}} \sqrt{Z_\pi Z_K}\\ \times [C_-(M^2)\langle\pi^+\gamma^*(q)|Q_-(M^2)|K^+\rangle\\ + 4\pi C_7(M^2)],\end{aligned} \qquad \text{(B1)}$$

where

$$\langle\pi^+\gamma^*(q)|Q_-(M^2)|K^+\rangle = \mathcal{E}(M^2)\langle\pi^+\gamma^*(q)|Q_-(0)|K^+\rangle, \qquad \text{(B2)}$$

and

$$\begin{aligned}\sqrt{Z_\pi Z_K} = 1 + \frac{1}{16\pi^2}\left[\frac{M^2}{f_\pi^2} - \frac{5}{12}\frac{M_K^2}{f_\pi^2}\ln\left(1 + \frac{M^2}{M_K^2}\right)\right.\\ \left. - \frac{1}{8}\frac{M_\eta^2}{f_\pi^2}\ln\left(1 + \frac{M^2}{M_\eta^2}\right)\right].\end{aligned} \qquad \text{(B3)}$$

From a pure χPT loop calculation using the cutoff prescription in Eq. (A2), one has

$$\langle\pi^+\gamma^*(q)|Q_-(0)|K^+\rangle = \chi\left(\frac{z}{r_\pi^2}\right) + \chi(z) - \frac{5}{9} + \frac{1}{3}\ln\frac{M^2}{M_\pi M_K}. \qquad \text{(B4)}$$

The χ function is the one defined in Ref. [17], and it is related to the Φ in Ref. [16] as $\chi(z) = \Phi(z) + 1/6$. Numerically, the kaon loop contribution, the $\chi(z)$ term in Eq. (B4), is negligible. The extra constant term and $\ln(M^2)$ in (B4) come from the cutoff regularization. It is from this formula that one can extract the expressions for $a_+(M^2)$ and $b_+(M^2)$.

2. $K_S \to \pi^0\gamma^*$

For this decay, the form factor $W_S(z, M^2)$ is

$$\begin{aligned}W_S(z, M^2) = \frac{M_K^2 G_F V_{us}^* V_{ud}}{\sqrt{2}} \sqrt{Z_\pi Z_K}\\ \times [C_-(M^2)\langle\pi^0\gamma^*(q)|Q_-(M^2)|K_S\rangle - 4\pi C_7(M^2)].\end{aligned} \qquad \text{(B5)}$$

The evolution operator in Eq. (18) is exactly the same as in the K_S case, so

$$\langle\pi^0\gamma^*(q)|Q_-(M^2)|K_S\rangle = \mathcal{E}(M^2)\langle\pi^0\gamma^*(q)|Q_-(0)|K_S\rangle, \qquad \text{(B6)}$$

where

$$\langle\pi^0\gamma^*(q)|Q_-(0)|K_S\rangle = 2\chi(z) - \frac{5}{9} + \frac{1}{3}\ln\frac{M^2}{M_K^2}. \qquad \text{(B7)}$$

APPENDIX C: EXPRESSIONS FOR $C_-(\mu^2)$ AND $C_7(\mu^2)$

The expressions for $C_-(\mu^2)$ and $C_7(\mu^2)$ are [89]

$$C_-(\mu^2) = \frac{1}{2}\left[\frac{\alpha_s(\mu^2, 4)}{\alpha_s(M_c^2, 3)}\right]^{\frac{12}{27}}\left[\frac{\alpha_s(M_c^2, 4)}{\alpha_s(M_W^2, 4)}\right]^{\frac{12}{25}} \qquad \text{(C1)}$$

and

$$C_7(\mu^2) = \frac{16}{99\alpha_s(M_c^2,3)} \left\{ \left[\frac{\alpha_s(M_c^2,4)}{\alpha_s(M_W^2,4)}\right]^{-\frac{6}{25}} \left[1 - \left(\frac{\alpha_s(\mu^2,3)}{\alpha_s(M_c^2,3)}\right)^{-\frac{33}{27}}\right] - \frac{11}{10}\left[\frac{\alpha_s(M_c^2,4)}{\alpha_s(M_W^2,4)}\right]^{\frac{12}{25}} \left[1 - \left(\frac{\alpha_s(\mu^2,3)}{\alpha_s(M_c^2,3)}\right)^{-\frac{15}{27}}\right] \right\}, \tag{C2}$$

where

$$\alpha_s(\mu^2,n) = \frac{12\pi}{33-2n} \frac{1}{\ln(\frac{\mu^2}{(0.3\ \mathrm{GeV})^2})}. \tag{C3}$$

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