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Limits on nonlinear electrodynamics

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In this paper we set a framework in which experiments whose goal is to test QED predictions can be used in a more general way to test nonlinear electrodynamics (NLED) which contains low-energy QED as a special case. We review some of these experiments and we establish limits on the different free parameters by generalizing QED predictions in the framework of NLED. We finally discuss the implications of these limits on bound systems and isolated charged particles for which QED has been widely and successfully tested.

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I. INTRODUCTION

Interactions between electromagnetic fields in vacuum, absent from Maxwell's classical field equations, were first predicted in 1934 by Born and Infeld [1] in the framework of a new field theory. The main goal of this intrinsically nonlinear theory was to solve the difficulty related to the fact that the self-energy of a point charge is infinite by assuming the existence of an *absolute field* [1] in nature. Born and Infeld chose the absolute field amplitude as the amplitude of the electric field created by an electron at a distance equivalent to its classical radius, in other words, by equating the classical self-energy of the electron with its mass energy at rest.

In the following years (1935 and 1936), Euler and Kockel [2] and then Heisenberg and Euler [3] established their own nonlinear electromagnetic theory, based on Dirac's vacuum model [4]. The related effective Lagrangian was validated in 1951 by Schwinger [5] in the framework of QED field theory, and it is nowadays accepted as the mathematical description of field interactions.

Born-Infeld and Heisenberg-Euler theories are two different forms of what is called nonlinear electrodynamics (NLED). NLED is a general framework of theories that describe field-field interactions and predict a large panel of phenomena, from variations of the velocity of light in vacuum in the presence of electromagnetic fields to photon-photon scattering, but also changes in the long-range electromagnetic potential induced by charged particles, as discussed in this paper.

QED is considered as a very well tested theory. It is indisputable that some of QED's numerical predictions have been experimentally verified with an astonishing precision (see, e.g., Ref. [6]). Thus, it is legitimate to wonder whether alternative NLED forms have been definitively ruled out. Moreover, in the framework of QED itself, it is worthwhile to understand the impact of QED tests for bound or isolated particles into the photon sector, where tests are hardly found. In other words, do complex experiments looking for photon-photon interactions still have an impact on QED, or can they be considered as a somewhat useless technological prowess whose results are known in advance?

In this paper we set a framework in which experiments whose goal is to test QED predictions can be used in a more general way to test different NLED theories, which contain low-energy QED [7] as a special case. This can be done by properly parametrizing effective Lagrangians. Actually, assuming that Lorentz invariance holds in vacuum, the mathematical description of all forms of Lorentz-invariant NLED—also known as NLED theories of the Plebański class [8–10]—are given by a Lagrangian depending only on the two Lorentz invariants $\mathcal{F}$ and $\mathcal{G}$:

$$\mathcal{F} = \epsilon_0 E^2 - \frac{B^2}{\mu_0}, \tag{1}$$

$$\mathcal{G} = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{E}\cdot\mathbf{B}, \tag{2}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, ϵ_0 is the vacuum permittivity, and μ_0 is the vacuum permeability. For weak electromagnetic fields, the Lagrangian can be written as a power expansion of $\mathcal{F}$ and $\mathcal{G}$ [11]:

$$\mathcal{L} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{i,j}\mathcal{F}^i\mathcal{G}^j. \tag{3}$$

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The number of free parameters $c_{i,j}$ is infinite, but it is generally accepted that the lowest orders in the fields are sufficient to describe the phenomena induced in most experiments. The Lagrangian becomes

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm NL}, \tag{4}$$

$$\text{with} \quad \mathcal{L}_0 = \frac{1}{2}\mathcal{F}, \tag{5}$$

$$\text{and} \quad \mathcal{L}_{\rm NL} \simeq c_{0,1}\mathcal{G} + c_{2,0}\mathcal{F}^2 + c_{0,2}\mathcal{G}^2 + c_{1,1}\mathcal{F}\mathcal{G}. \tag{6}$$

The lowest-order term $\mathcal{L}_0$ gives the classical Maxwell Lagrangian, with $c_{1,0} = 1/2$. The nonlinear correction $\mathcal{L}_{\rm NL}$ depends on four parameters: $c_{0,1}$, $c_{2,0}$, $c_{0,2}$, and $c_{1,1}$.

To describe the nonlinear response of vacuum, we treat it as a polarizable medium. One can use the Maxwell equations together with the constitutive equations related to the Lagrangian as follows [12]:

$$\mathbf{P} = \frac{\partial \mathcal{L}}{\partial \mathbf{E}} - \epsilon_0 \mathbf{E}, \tag{7}$$

$$\mathbf{M} = \frac{\partial \mathcal{L}}{\partial \mathbf{B}} - \frac{\mathbf{B}}{\mu_0}. \tag{8}$$

$\mathbf{P}$ is the polarization and $\mathbf{M}$ is the magnetization. Using Eqs. (4), (5), and (6), one obtains

$$\mathbf{P} = c_{0,1}\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{B} + 4c_{2,0}\epsilon_0\mathcal{F}\mathbf{E} + 2c_{0,2}\sqrt{\frac{\epsilon_0}{\mu_0}}\mathcal{G}\mathbf{B} + c_{1,1}\left(2\epsilon_0\mathcal{G}\mathbf{E} + \sqrt{\frac{\epsilon_0}{\mu_0}}\mathcal{F}\mathbf{B}\right), \tag{9}$$

$$\mathbf{M} = c_{0,1}\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{E} - 4c_{2,0}\mathcal{F}\frac{\mathbf{B}}{\mu_0} + 2c_{0,2}\sqrt{\frac{\epsilon_0}{\mu_0}}\mathcal{G}\mathbf{E} - c_{1,1}\left(2\mathcal{G}\frac{\mathbf{B}}{\mu_0} - \sqrt{\frac{\epsilon_0}{\mu_0}}\mathcal{F}\mathbf{E}\right). \tag{10}$$

Starting from these constitutive equations, one can study the phenomenology associated with the four parameters $c_{0,1}$, $c_{2,0}$, $c_{0,2}$, and $c_{1,1}$. Corresponding experiments are then able to discriminate different forms of nonlinear electrodynamics.

The scope of our work is not to provide a review of theoretical activities and experimental proposals regarding NLED. Our main goal is to use some existing experimental results to set limits on NLED in a unified framework. In particular, we aim to give a unified approach to compare the results on light propagation in vacuum and experiments on bound systems and isolated particles.

In the following we first give some examples of NLED Lagrangians, in particular the Heisenberg and Euler Lagrangian predicted in the framework of QED. Then, experimental constraints on the $c_{i,j}$ parameters are reviewed. We start with photon-photon interaction experiments. Discussing vacuum magnetic birefringence and photon-photon scattering, we show that a limit on vacuum magnetic birefringence cannot directly give a limit on the photon-photon scattering cross section as claimed in several papers [13–15]. We finally discuss the implications of this type of Lagrangian on bound systems and isolated charged particles for which QED has been widely and successfully tested.

II. SOME EFFECTIVE NONLINEAR LAGRANGIANS

To illustrate the general form of the nonlinear Lagrangian given in Eq. (6), we focus on some of the most well-known ones.

A. Heisenberg and Euler effective Lagrangian

The generally accepted effective Lagrangian is the one established in 1936 by Heisenberg and Euler [3] in the framework of QED. It generalized at all orders the previous work of Euler and Kockel in 1935 [2]. The vacuum is assumed to be C, P, and T invariant. This implies that the coefficients $c_{i,j}$ with an odd index j are null, in particular, $c_{0,1} = 0$ and $c_{1,1} = 0$. The nonlinear correction of the Lagrangian is then

$$\mathcal{L}_{\rm NL} = c_{2,0}\mathcal{F}^2 + c_{0,2}\mathcal{G}^2. \tag{11}$$

Following the Euler and Kockel result [2], the values of $c_{2,0}$ and $c_{0,2}$ can be written as

$$c_{2,0} = \frac{2\alpha^2\hbar^3}{45 m_e^4 c^5} \tag{12}$$

$$= \frac{\alpha}{90\pi}\frac{1}{\epsilon_0 E_{\rm cr}^2} = \frac{\alpha}{90\pi}\frac{\mu_0}{B_{\rm cr}^2} \tag{13}$$

$$\simeq 1.66 \times 10^{-30}\left[\frac{m^3}{J}\right], \tag{14}$$

$$c_{0,2} = 7c_{2,0}, \tag{15}$$

and therefore

$$\mathcal{L}_{\rm NL} = \frac{\alpha}{90\pi}\frac{1}{\epsilon_0 E_{cr}^2}[\mathcal{F}^2 + 7\mathcal{G}^2], \tag{16}$$

where $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine-structure constant, e is the elementary charge, and $\hbar$ is Planck's constant h divided by 2π. $E_{\rm cr} = m_e^2 c^3/e\hbar$ is a quantity obtained by combining the fundamental constant m_e, the electron mass, c, e, and $\hbar$. It has the dimensions of an electric field, and it is called the critical electric field. Its value is $E_{\rm cr} = 1.3 \times 10^{18}$ V/m. A critical magnetic field can also be defined in the same manner: $B_{\rm cr} = E_{\rm cr}/c = m_e^2 c^2/e\hbar = 4.4 \times 10^9$ T.

The existence of several phenomena can be predicted using this Lagrangian, as detailed in Ref. [11]. As long as QED is assumed to be correct in the presently accepted form, the values of the $c_{i,j}$ coefficients are fixed. Therefore, there are no predictions that contain free parameters. The values of the physical quantities to be measured simply correspond to linear combinations of powers of the fundamental constants α, $\hbar$, m_e, and c.

B. Born-Infeld effective Lagrangian

The Born-Infeld effective Lagrangian [1] is a well known example of a NLED theory that was developed in 1934, even before the Heisenberg-Euler one. It was introduced to remove the problem of the classical self-energy of elementary particles, which is infinite. The Lagrangian is established from the postulate that there exists an "absolute field" E_{abs} corresponding to the upper limit of a purely electric field. The Lagrangian is

$$\mathcal{L} = \epsilon_0 E_{\mathrm{abs}}^2 \left(-\sqrt{1 - \frac{\mathcal{F}}{\epsilon_0 E_{\mathrm{abs}}^2} - \frac{\mathcal{G}^2}{(\epsilon_0 E_{\mathrm{abs}}^2)^2}} + 1 \right). \tag{17}$$

E_{abs} is a free parameter corresponding to a new fundamental constant to be determined. If we assume that $(\frac{\mathcal{F}}{\epsilon_0 E_{\mathrm{abs}}^2} - \frac{\mathcal{G}^2}{\epsilon_0 E_{\mathrm{abs}}^4}) \ll 1$, the Lagrangian can be developed and, at the lowest orders in the fields, it can be written as

$$\mathcal{L} \simeq \frac{1}{2}\mathcal{F} + \frac{1}{8\epsilon_0 E_{\mathrm{abs}}^2}\mathcal{F}^2 + \frac{1}{2\epsilon_0 E_{\mathrm{abs}}^2}\mathcal{G}^2. \tag{18}$$

The corresponding $c_{i,j}$ parameters are

$$c_{1,0} = \frac{1}{2}, \tag{19}$$

$$c_{0,1} = c_{1,1} = 0, \tag{20}$$

$$c_{2,0} = \frac{1}{8\epsilon_0 E_{\mathrm{abs}}^2}, \tag{21}$$

$$c_{0,2} = \frac{1}{2\epsilon_0 E_{\mathrm{abs}}^2} = 4c_{2,0}. \tag{22}$$

Comparing these terms with the ones obtained in Eqs. (12) and (15) with the Heisenberg-Euler Lagrangian, one can see that no value of E_{abs} allows the parameters to coincide. Both Lagrangians are essentially different and will lead to different nonlinear properties. Experimental tests are thus crucial to establish which one is valid. Some examples of possible experiments will be presented in the following section, but other configurations can be found, for instance, in Refs. [16,17].

The absolute field constant was estimated in Ref. [1]. It was related to the "radius" of the electron r_0 as follows: $E_{\mathrm{abs}} = e/4\pi\epsilon_0 r_0^2$. Using the classical electron radius $r_0 = e^2/4\pi\epsilon_0 m_e c^2$, one finds $E_{\mathrm{abs}} \simeq 2 \times 10^{20}$ V/m, which corresponds to a $c_{2,0}$ about 4 times smaller than the one of Heisenberg and Euler.

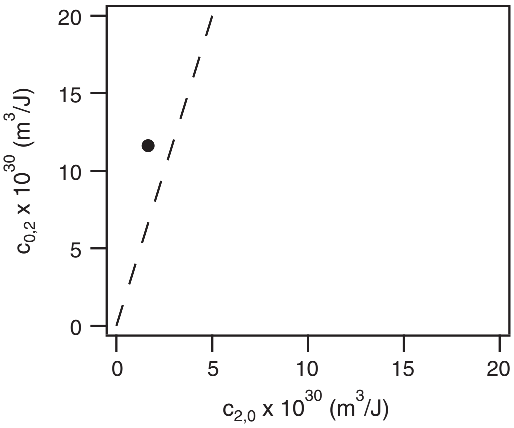

FIG. 1. Born-Infeld prediction and Heisenberg-Euler prediction in the $(c_{2,0}, c_{0,2})$ parameter space. The Born-Infeld prediction is represented by a straight line, while the Heisenberg-Euler one is a point.

Let us recall that the Born and Infeld choice of the absolute field is arbitrarily related to the pointlike particle known in their time, the electron. The absolute field is therefore a free parameter of the Born-Infeld theory that can be experimentally constrained or measured. The ratio between $c_{2,0}$ and $c_{0,2}$ is however fixed. In the $(c_{2,0}, c_{0,2})$ parameter space, the Born-Infeld prediction is thus represented by a straight line, while the Heisenberg-Euler one is represented by a point, as shown in Fig. 1.

C. Lagrangian in the string theory framework

Both the Heisenberg-Euler and Born-Infeld Lagrangians at the lowest orders in the fields can be considered as special cases of a more general one obtained in the framework of string theory [18], which gives a more general interest to the field of NLED. This is discussed in detail in Ref. [19]. This Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}\mathcal{F} + \frac{\gamma}{4}[(1-b)\mathcal{F}^2 + 6\mathcal{G}^2], \tag{23}$$

where γ and b are two free parameters. The corresponding $c_{i,j}$ parameters are

$$c_{1,0} = \frac{1}{2}, \tag{24}$$

$$c_{0,1} = c_{1,1} = 0, \tag{25}$$

$$c_{2,0} = \frac{\gamma}{4}(1-b), \tag{26}$$

$$c_{0,2} = \frac{3}{2}\gamma. \tag{27}$$

The Born-Infeld Lagrangian is recovered with $b = -1/2$ and $\gamma = 1/3\epsilon_0 E_{\rm abs}^2$. For the Heisenberg-Euler prediction, one has $b = 1/7$ and $\gamma = 7\alpha/135\pi\epsilon_0 E_{\rm cr}^2$.

III. LIGHT PROPAGATION IN VACUUM

The expected nonlinear optical phenomena in vacuum are reviewed in Ref. [11]. It goes from birefringence effects induced by electric or magnetic fields, to vacuum dichroism, photon splitting, photon-photon scattering, and second harmonic generation. In the following, we will focus on the two nonlinear effects whose experimental observation has been sought quite recently: the magnetic birefringence and photon-photon scattering.

A. Magnetic birefringence

Birefringence can be induced by an electric field, a magnetic field, or a combination of both. However, experiments are mostly devoted to magnetically induced effects. This is due to the fact that the same level of effect is obtained in the presence of a B field or an electric field E equal to cB. From a technological point of view, magnetic fields of several tesla are easier to produce than electric fields of about 1 $\mathrm{GV\,m^{-1}}$.

1. Expected birefringence

The calculation of the birefringence induced by a transverse static magnetic field, using the general Lagrangian given by Eqs. (4)–(6), can be found in Ref. [20]. In the following, we only briefly give the main steps.

The total magnetic field corresponds to the sum of the static magnetic field $\mathbf{B}_0$ and that of the propagating wave $\mathbf{B}_\omega$: $\mathbf{B} = \mathbf{B}_\omega + \mathbf{B}_0$. The electric field associated to the propagating wave is $\mathbf{E}_\omega$. Introducing these quantities in Eqs. (9) and (10) and keeping only the ω component, we obtain

$$\begin{aligned}\mathbf{P}_\omega = &-\frac{4\epsilon_0 c_{2,0}}{\mu_0}B_0^2\mathbf{E}_\omega + \frac{2\epsilon_0 c_{0,2}}{\mu_0}(\mathbf{E}_\omega\cdot\mathbf{B}_0)\mathbf{B}_0\\ &+\sqrt{\frac{\epsilon_0}{\mu_0}}\left(c_{0,1} - \frac{c_{1,1}}{\mu_0}B_0^2\right)\mathbf{B}_\omega\\ &-\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{2c_{1,1}}{\mu_0}(\mathbf{B}_\omega\cdot\mathbf{B}_0)\mathbf{B}_0,\end{aligned} \tag{28}$$

$$\begin{aligned}\mathbf{M}_\omega = &\frac{4c_{2,0}}{\mu_0^2}B_0^2\mathbf{B}_\omega + \frac{8c_{2,0}}{\mu_0^2}(\mathbf{B}_\omega\cdot\mathbf{B}_0)\mathbf{B}_0\\ &-\sqrt{\frac{\epsilon_0}{\mu_0}}\left(-c_{0,1} + \frac{c_{1,1}}{\mu_0}B_0^2\right)\mathbf{E}_\omega\\ &-\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{2c_{1,1}}{\mu_0}(\mathbf{E}_\omega\cdot\mathbf{B}_0)\mathbf{B}_0.\end{aligned} \tag{29}$$

We define the static magnetic field direction as the x direction. This magnetic field is transverse to the light propagation, which is assumed to be along the z direction. We assume the existence of plane-wave eigenmodes with refractive index n:

$$\mathbf{E}_\omega(\mathbf{r},t) = \mathbf{E}_0 e^{i\omega(\frac{n}{c}\mathbf{e}_z\cdot\mathbf{r}-t)}. \tag{30}$$

Injected into the Maxwell equations, one gets in the polarization plane (x, y)

$$\begin{pmatrix} n^2\left(\frac{4c_{2,0}}{\mu_0}B_0^2 - 1\right) + 2 + \frac{2(c_{0,2}-2c_{2,0})}{\mu_0}B_0^2 & \frac{2nc_{1,1}}{\mu_0}B_0^2 \\ \frac{2nc_{1,1}}{\mu_0}B_0^2 n^2 & \left(\frac{12c_{2,0}}{\mu_0}B_0^2 - 1\right) + 2 - \frac{4c_{2,0}}{\mu_0}B_0^2 \end{pmatrix}\mathbf{E}_\omega = \mathbf{E}_\omega.$$

We can first note that the $c_{0,1}$ term has canceled out and thus does not contribute to the propagation of light. The diagonal terms correspond to the Cotton-Mouton effect. In this case, the eigenmodes are parallel and perpendicular to the magnetic field. The corresponding indices of refraction are

$$n_\parallel = 1 + \frac{c_{0,2}}{\mu_0}B_0^2, \tag{31}$$

$$n_\perp = 1 + \frac{4c_{2,0}}{\mu_0}B_0^2, \tag{32}$$

where $n_\parallel$ is the index of refraction for light polarized parallel to the external magnetic field and $n_\perp$ is the index of refraction for light polarized perpendicular to the external magnetic field. While the refractive index $n_\parallel$ depends only on $c_{0,2}$, $n_\perp$ depends only on $c_{2,0}$. Since dispersive effects can be neglected, $n_\parallel$ and $n_\perp$ always have to be greater than 1 and $c_{0,2}$ and $c_{2,0}$ have to be greater than 0.

The anisotropy Δn is equal to

$$\Delta n_{\rm CM} = n_\parallel - n_\perp = \frac{c_{0,2} - 4c_{2,0}}{\mu_0}B_0^2 \tag{33}$$

and depends on both parameters. Let us note that in the case of the Heisenberg-Euler Lagrangian, one gets

$$\Delta n_{\rm CM,HE} = \frac{3c_{2,0}}{\mu_0}B_0^2 = \frac{2\alpha^2\hbar^3}{15\mu_0 m_e^4 c^5}B_0^2. \tag{34}$$

On the other hand, with the Born-Infeld Lagrangian, no Cotton-Mouton effect is expected [9,10,21] since we get

$$\Delta n_{\mathrm{CM,BI}} = 0. \tag{35}$$

The nondiagonal terms can be interpreted as a magnetic Jones birefringence, with a linear birefringence along the axes which are at $\pm 45°$ relative to the direction of the static magnetic field. The corresponding difference between the refractive indices is [22]

$$\Delta n_{\mathrm{J}} = n_{+45°} - n_{-45°} = \frac{2c_{1,1}}{\mu_0} B_0^2. \tag{36}$$

2. Experimental limits

Two types of experiments have been realized to measure this variation of the light velocity in the presence of a transverse magnetic field [11]. The first one is based on interferometers with separated arms, such as the Michelson-Morley interferometer. The basic idea is to look at the interference displacement when a magnetic field is applied on one of the arms. This type of configuration has the advantage of directly measuring one of the parameters $c_{0,2}$ or $c_{2,0}$ if the magnetic field is oriented parallel or perpendicular to the light polarization.

In 1940, Farr and Banwell reported results obtained using an interferometer where one of the two arms is immersed in a 2 T magnetic field. The measured relative variation of the light velocity was less than 2×10^{-9} [23]. The light polarization with respect to the magnetic field was not clearly stated. For the sake of argument, assuming that one can infer limits on the $c_{i,j}$ parameters from their measurements, we obtain

$$c_{2,0} < 1.6 \times 10^{-16}\ \mathrm{m^3\,J^{-1}}, \tag{37}$$

$$c_{0,2} < 6.3 \times 10^{-16}\ \mathrm{m^3\,J^{-1}}, \tag{38}$$

$$c_{1,1} < 6.3 \times 10^{-16}\ \mathrm{m^3\,J^{-1}}. \tag{39}$$

Anyway, these limits are 14 orders of magntiude higher than the QED predictions [see Eqs. (15) and (14)].

The second type of experiments is based on polarimetry. The principle is to measure the magnetic birefringence via the ellipticity induced on a linearly polarized laser beam propagating in a transverse magnetic field [24]. In this case, one measures the difference between the refractive indices and not the refractive index directly. Therefore, concerning the Cotton-Mouton configuration, the measurement cannot by itself constrain both $c_{0,2}$ and $c_{2,0}$ but only a particular linear combination of the two free parameters: $c_{0,2} - 4c_{2,0}$. Let us note finally that, even if one measures the value predicted by the Heisenberg and Euler Lagrangian for Δn_{CM}, i.e., $3c_{2,0}^{\mathrm{HE}} B_0^2/\mu_0$, this cannot be considered in principle the definitive demonstration that this Lagrangian is correct. Any Lagrangian with $c_{0,2} - 4c_{2,0} = 3c_{2,0}^{\mathrm{HE}}$ predicts the same value.

The most advanced experiments in this domain are those operated by the PVLAS Collaboration [15] and the BMV group [25]. The direction of the static magnetic field is at 45° compared to the light polarization, corresponding to the Cotton-Mouton configuration. Experiments measure Δn_{CM} with an error $\delta\Delta n_{\mathrm{CM}}$. This corresponds in the $(c_{0,2}, c_{2,0})$ parameter plane to two regions of exclusion:

$$c_{0,2} < 4c_{2,0} + \mu_0(\Delta n_{\mathrm{CM}} + \delta\Delta n_{\mathrm{CM}}), \tag{40}$$

$$c_{0,2} > 4c_{2,0} + \mu_0(\Delta n_{\mathrm{CM}} - \delta\Delta n_{\mathrm{CM}}). \tag{41}$$

The best limit is given in Ref. [15] with $\Delta n = (0.4 \pm 2.0) \times 10^{-22} B_0^2$ at 1σ, corresponding to

$$c_{0,2} < 4c_{2,0} + 3 \times 10^{-28}\ \mathrm{m^3\,J^{-1}}, \tag{42}$$

$$c_{0,2} > 4c_{2,0} - 2 \times 10^{-28}\ \mathrm{m^3\,J^{-1}}. \tag{43}$$

These limits are summarized in Fig. 2.

Finally, to give a limit on the $c_{1,1}$ parameter, one should use the Jones configuration, with the light polarization parallel or perpendicular to the magnetic field as discussed in Refs. [20] and [26]. In the last reference, Millo and Faccioli have also estimated the magnitude of this effect within the standard model using quantum chromodynamics chiral perturbation theory, obtaining that $c_{1,1}$ is expected to be at least 20 orders of magnitude smaller than $c_{2,0}^{\mathrm{HE}}$. Anyway, no one has ever done such a measurement.

B. Photon-photon scattering

Testing low-energy QED with ultra-intense lasers is widely discussed in the literature, in particular with the direct observation of photon-photon scattering. Recent reviews can be found in Refs. [27–31]. In the following, we will focus on the experiment which has reported the best experimental limit up to now [32].

The simplest experiment to look at photon-photon scattering in vacuum consists in two colliding laser beams, as proposed in Ref. [33]. The calculation of the corresponding total photon-photon scattering cross section for unpolarized light with the Heisenberg-Euler or the Born-Infeld Lagrangian can be found, for example, in Ref. [17]. The number of scattered photons can be enhanced by using a third beam which stimulates the reaction [34]. In this configuration, the link between the $c_{i,j}$ coefficients and the measurement of the number of scattered photons can be established following the approach proposed in Refs. [35] and [32] where a third-order nonlinear effective susceptibility χ_v^3 was introduced, as in classical nonlinear optics. Here, we only present the main steps of the calculations.

In elastic scattering, the energy and momentum conservation holds, corresponding to

$$\mathbf{k}_4 = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3, \tag{44}$$

$$\omega_4 = \omega_1 + \omega_2 - \omega_3, \tag{45}$$

where $\mathbf{k}_i$ is the wave vector of laser beam number i and ω_i is its frequency multiplied by 2π. The three incoming beams are 1, 2, and 3, while beam number 4 is the scattered one. Using Eqs. (9) and (10) and keeping only the ω_4 component, we obtain

$$\mathbf{P}_{\omega_4} = \epsilon_0^2 E_1 E_2 \bar{E}_3 \left[2c_{2,0}\mathbf{K}_{P_{20}} + \frac{c_{02}}{2}\mathbf{K}_{P_{02}} + \frac{c_{11}}{2}(\mathbf{K}_{P_{11,1}} + \mathbf{K}_{P_{11,2}})\right] \tag{46}$$

$$= \epsilon_0^2 E_1 E_2 \bar{E}_3 \mathbf{K}_P, \tag{47}$$

$$\mathbf{M}_{\omega_4} = c\epsilon_0^2 E_1 E_2 \bar{E}_3 \left[-2c_{2,0}\mathbf{K}_{P_{11,2}} + \frac{c_{02}}{2}\mathbf{K}_{P_{11,1}} - \frac{c_{11}}{2}(-\mathbf{K}_{P_{02}} + \mathbf{K}_{P_{20}})\right] \tag{48}$$

$$= c\epsilon_0^2 E_1 E_2 \bar{E}_3 \mathbf{K}_M, \tag{49}$$

where $\mathbf{E}_i$ is the electric field of beam number i. The geometrical factors are

$$\mathbf{K}_{P_{20}} = \mathbf{u_1}(\mathbf{u_2}.\mathbf{u_3} - \mathbf{v_2}.\mathbf{v_3}) + \mathbf{u_2}(\mathbf{u_1}.\mathbf{u_3} - \mathbf{v_1}.\mathbf{v_3}) + \mathbf{u_3}(\mathbf{u_1}.\mathbf{u_2} - \mathbf{v_1}.\mathbf{v_2}), \tag{50}$$

$$\mathbf{K}_{P_{02}} = \mathbf{v_1}(\mathbf{u_2}.\mathbf{v_3} + \mathbf{v_2}.\mathbf{u_3}) + \mathbf{v_2}(\mathbf{u_1}.\mathbf{v_3} + \mathbf{v_1}.\mathbf{u_3}) + \mathbf{v_3}(\mathbf{u_1}.\mathbf{v_2} + \mathbf{v_1}.\mathbf{u_2}), \tag{51}$$

$$\mathbf{K}_{P_{11,1}} = \mathbf{u_1}(\mathbf{u_2}.\mathbf{v_3} + \mathbf{v_2}.\mathbf{u_3}) + \mathbf{u_2}(\mathbf{u_1}.\mathbf{v_3} + \mathbf{v_1}.\mathbf{u_3}) + \mathbf{u_3}(\mathbf{u_1}.\mathbf{v_2} + \mathbf{v_1}.\mathbf{u_2}), \tag{52}$$

$$\mathbf{K}_{P_{11,2}} = \mathbf{v_1}(\mathbf{u_2}.\mathbf{u_3} - \mathbf{v_2}.\mathbf{v_3}) + \mathbf{v_2}(\mathbf{u_1}.\mathbf{u_3} - \mathbf{v_1}.\mathbf{v_3}) + \mathbf{v_3}(\mathbf{u_1}.\mathbf{u_2} - \mathbf{v_1}.\mathbf{v_2}). \tag{53}$$

The unit vectors $\mathbf{u_i}$ and $\mathbf{v_i}$ indicate the direction of the electric field (i.e., the polarization) of the photon beam i and the direction of the corresponding magnetic field. The geometrical factors depend on the directions of the incident beam and on their polarizations.

The propagation equation for the electric field $\mathbf{E}_4$ is obtained thanks to Maxwell's equations in the slow-varying wave approximation [32,35],

$$\nabla^2\mathbf{E}_4 - \frac{1}{c^2}\frac{\partial^2\mathbf{E}_4}{\partial t^2} = \mu_0\left(\frac{\partial}{\partial t}\nabla\otimes\mathbf{M}_{\omega_4} + \frac{\partial^2\mathbf{P}_{\omega_4}}{\partial t^2} - c^2\nabla(\nabla\cdot\mathbf{P}_{\omega_4})\right), \tag{54}$$

which gives, in the paraxial formulation with beam 4 propagating in the z direction, the following growth of the amplitude E_4:

$$\left(\frac{\partial E_4}{\partial z} + \frac{1}{c}\frac{\partial E_4}{\partial t}\right)\mathbf{u_4} = -\frac{i\mu_0\omega_4}{2}[(cP_{\omega_4,x} + M_{\omega_4,y})\mathbf{u_x} + (cP_{\omega_4,y} - M_{\omega_4,x})\mathbf{u_y}]. \tag{55}$$

The x and y subscripts stand for the x and y components. The same type of growth is obtained in four-wave mixing in a standard medium where an effective susceptibility χ_v^3 is defined and where we get

$$\left(\frac{\partial E_4}{\partial z} + \frac{1}{c}\frac{\partial E_4}{\partial t}\right)\mathbf{u_4} = -\frac{i\omega_4}{2c}\chi_v^3 E_1 E_2 \bar{E}_3\mathbf{u_4}. \tag{56}$$

The vacuum effective susceptibility thus corresponds to

$$\chi_v^3 = \frac{c\mu_0}{E_1 E_2 \bar{E}_3}\sqrt{(cP_{\omega_4,x} + M_{\omega_4,y})^2 + (cP_{\omega_4,y} - M_{\omega_4,x})^2}, = \epsilon_0\sqrt{(K_{P,x} + K_{M,y})^2 + (K_{P,y} - K_{M,x})^2}. \tag{57}$$

It depends on the $c_{i,j}$ parameters through the $\mathbf{P}$ and $\mathbf{M}$ vectors given in Eqs. (46) and (48), or the $\mathbf{K}_P$ and $\mathbf{K}_M$ vectors given in Eqs. (47) and (49). The scattered photon polarization is given by

$$\mathbf{u_4} = \frac{(cP_{\omega_4,x} + M_{\omega_4,y})\mathbf{u_x} + (cP_{\omega_4,y} - M_{\omega_4,x})\mathbf{u_y}}{\sqrt{(cP_{\omega_4,x} + M_{\omega_4,y})^2 + (cP_{\omega_4,y} - M_{\omega_4,x})^2}}. \tag{58}$$

It also depends on the $c_{i,j}$ parameters.

Finally, the expected number of scattered photons is obtained by integrating Eq. (56). The result depends on the beams' profile (plane wave, Gaussian beam, etc.), but it is always proportional to the square of χ_v^3 and proportional to the total cross section of the process.

Experimentally, the choice of the laser setup and geometry is important to maximize the number of scattered photons and to maximize the signal-to-noise ratio. But, to see more clearly the link between the $c_{i,j}$ coefficients and the number of scattered photons, let us take some simple configurations with beams 2 and 3 counterpropagating with respect to beam 1.

If $\mathbf{u_1} = \mathbf{u_2} = \mathbf{u_3}$ and $\mathbf{v_1} = -\mathbf{v_2} = -\mathbf{v_3}$, one gets $\mathbf{K}_P = 8c_{2,0}\mathbf{u_1} - 2c_{1,1}\mathbf{v_1}$ and $\mathbf{K}_M = 8c_{2,0}\mathbf{v_1} - 2c_{1,1}\mathbf{u_1}$. The effective susceptibility is then

$$\chi^3_{v,\mathrm{first}} = 16\epsilon_0 c_{2,0}. \tag{59}$$

The $c_{1,1}$ parameter cancels out and χ^3_v only depends on $c_{2,0}$. A measurement in this configuration thus allows us to constrain this parameter independently from the others.

If $\mathbf{u_1} = -\mathbf{v_2} = -\mathbf{v_3}$ and $\mathbf{v_1} = -\mathbf{u_2} = -\mathbf{u_3}$, we get $\mathbf{K}_P = 2c_{0,2}\mathbf{u_1} + 2c_{1,1}\mathbf{v_1}$ and $\mathbf{K}_M = 2c_{0,2}\mathbf{v_1} + 2c_{1,1}\mathbf{u_1}$. The effective susceptibility is then

$$\chi^3_{v,\mathrm{second}} = 4\epsilon_0 c_{0,2}. \tag{60}$$

It only depends on $c_{0,2}$.

Finally, if $\mathbf{u_1} = \mathbf{v_2} = \mathbf{u_3}$ and $\mathbf{v_1} = \mathbf{u_2} = -\mathbf{v_3}$, we get $\mathbf{K}_P = (4c_{2,0} - c_{0,2})\mathbf{v_1} + 2c_{1,1}\mathbf{u_1}$ and $\mathbf{K}_M = -(4c_{2,0} - c_{0,2})\mathbf{u_1} - 2c_{1,1}\mathbf{v_1}$. The effective susceptibility is then

$$\chi^3_{v,\mathrm{third}} = \sqrt{2}\epsilon_0(4c_{2,0} - c_{0,2}). \tag{61}$$

It now depends on a linear combination of $c_{2,0}$ and $c_{0,2}$.

For more complicated laser beam configurations, the number of scattered photons $N_{\gamma\gamma}$ is of the form

$$N_{\gamma\gamma} \propto (\chi^3_v)^2 \tag{62}$$

$$\propto ac^2_{2,0} + bc^2_{0,2} + cc^2_{1,1} + 2dc_{2,0}c_{0,2} + 2ec_{0,2}c_{1,1} + 2fc_{2,0}c_{1,1}. \tag{63}$$

The $c_{0,1}$ parameter is absent. No limit or measurement on this coefficient can thus be given by photon-photon scattering experiments. In principle, studying the scattered photon polarization, given by Eq. (58), would allow us to extract further information on the different parameters $c_{2,0}$, $c_{0,2}$, and $c_{1,1}$.

The best experimental limit was reported in 2000 [32]. The value is compatible with zero. The error is about 18 orders of magnitude higher than the QED prediction, which corresponds to $c_{2,0}$ and $c_{0,2}$ given in Eqs. (14) and (15), and $c_{1,1} = 0$.

C. Magnetic birefringence versus photon-photon scattering

Among experiments on light propagation in vacuum, the most sensitive one concerns the measurement of magnetic birefringence using polarimetry. While the others are more than 14 orders of magnitude higher than the QED (Heisenberg-Euler) prediction (14 orders of magnitude for the magnetic birefringence using a separated arms interferometer; 18 orders of magnitude for the photon-photon scattering cross section), the measurement of the Cotton-Mouton effect is less than 2 orders of magnitude higher than the QED prediction.

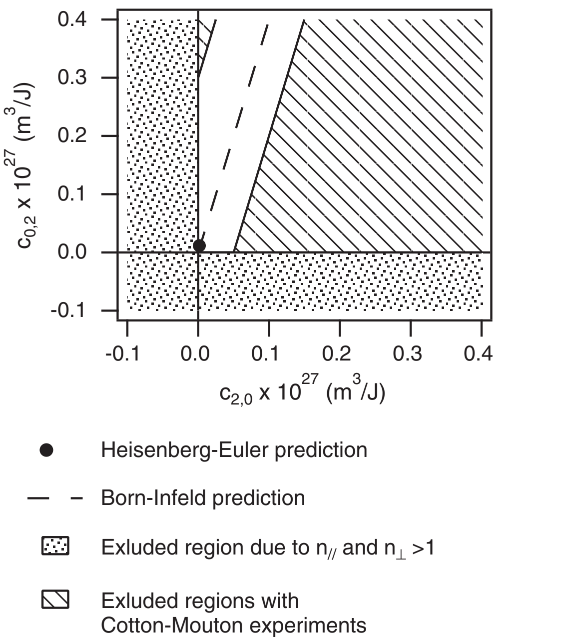

FIG. 2. Best experimental limits on $c_{0,2}$ and $c_{2,0}$ parameters. Striped areas: Excluded region obtained with the result of Ref. [15]. Point: Heisenberg-Euler prediction. Dashed line: Born-Infeld prediction. The point seems superimposed on the dashed line due to the scale. Dotted areas: Excluded regions due to the fact that $n_\parallel$ and $n_\perp > 1$.

One could then envisage using the most sensitive measurement to put a constraint on the others, and more particularly on the photon-photon scattering cross section. As said before, the measurement of the vacuum magnetic birefringence cannot by itself constrain $c_{0,2}$ and $c_{2,0}$ separately. On the other hand, we have shown through simple examples that the χ^3_v dependance on the $c_{i,j}$ coefficients depends on the laser beam configuration. Limits on vacuum magnetic birefringence cannot therefore be translated into limits on photon-photon scattering since the dependence of the effects from the NLED free parameters are generally different. However, photon-photon scattering limits can be represented as exclusion regions, as done in Fig. 2 for vacuum magnetic birefringence measurements, closing further the allowed range in the parameter space. Experiments whose goal is to measure the vacuum magnetic birefringence or the photon-photon scattering cross section, far from being redundant, are complementary to tests of NLED theories.

This point, although apparently simple, is not always fully understood. As a matter of fact, the authors of Refs. [13,15] declared that a measurement of vacuum magnetic birefringence can constrain the Heisenberg-Euler Lagrangian parameters and consequently the photon-photon scattering cross section, which is not correct, as we just explained.

IV. POINTLIKE PARTICLES

For the moment, the experiments devoted to the study of light propagation in vacuum have not been able to test the Heisenberg-Euler Lagrangian. However, experiments on vacuum magnetic birefringence are only at 2 orders of magnitude from the QED prediction, and one can hope that they will be gained in the near future. Does this mean that the Heisenberg-Euler Lagrangian has not yet been tested? It is admitted that QED is widely and successfully tested on bound systems (for example, in the hydrogen atom) and on isolated charged particles (for example, the measurement of the anomalous magnetic dipole moment of the electron). Does it correspond to a test of the Heisenberg-Euler Lagrangian? Is there any space still open for alternative NLED theories?

A. General expressions

In the presence of external electric and magnetic fields, the vacuum reacts. It becomes polarized and magnetized and thus modifies the electric and magnetic fields. Let us first calculate the **P** and **M** vectors induced by a pointlike particle of charge Q and magnetic moment $\boldsymbol{\mu} = \mu \mathbf{e}_z$. The corresponding external electric and magnetic fields are

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{e}_r, \tag{64}$$

$$\mathbf{B} = \frac{\mu_0\mu}{4\pi r^3}[3(\mathbf{e}_z.\mathbf{e}_r)\mathbf{e}_r - \mathbf{e}_z] \tag{65}$$

$$= \frac{\mu_0\mu}{4\pi r^3}(3\cos\theta\mathbf{e}_r - \mathbf{e}_z). \tag{66}$$

To keep the validity of our nonlinear Lagrangian development, we only consider an electric field and a magnetic field well below the critical ones defined in the Heisenberg-Euler Lagrangian. We therefore assume that $r \gg r_{\mathrm{cr}}^E$ and $r \gg r_{\mathrm{cr}}^B$, with $r_{\mathrm{cr}}^E = \sqrt{Q/4\pi\epsilon_0 E_{\mathrm{cr}}}$ and $r_{\mathrm{cr}}^B = (\mu_0\mu/4\pi B_{\mathrm{cr}})^{1/3}$. For a proton, $Q = 1.6\times10^{-19}$ C and $\mu = 1.41\times10^{-26}\ \mathrm{JT}^{-1}$, and one obtains $r_{\mathrm{cr}}^E \sim 3\times10^{-14}$ m and $r_{\mathrm{cr}}^B \sim 7\times10^{-15}$ m.

Injecting the previous electric and magnetic fields into the Lorentz invariants given by Eqs. (1) and (2), we get

$$\mathcal{F} = \frac{Q^2}{(4\pi)^2\epsilon_0 r^4}\left[1 - \left(\frac{\mu}{cQr}\right)^2(1+3\cos^2\theta)\right], \tag{67}$$

$$\mathcal{G} = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{Q2\mu\cos\theta}{(4\pi)^2 r^5}. \tag{68}$$

The corresponding **P** and **M** vectors are

$$\begin{aligned}\mathbf{P} = {} & c_{0,1}\sqrt{\epsilon_0\mu_0}\frac{\mu}{4\pi r^3}(3\cos\theta\mathbf{e}_r - \mathbf{e}_z) + c_{2,0}\epsilon_0\mathbf{E}\frac{Q^2}{4\pi^2\epsilon_0 r^4}\left[1-\left(\frac{\mu}{cQr}\right)^2(1+3\cos^2\theta)\right] \\ & + c_{0,2}\epsilon_0 E\frac{\mu_0\mu^2\cos\theta}{4\pi^2 r^6}(3\cos\theta\mathbf{e}_r - \mathbf{e}_z) + c_{1,1}\epsilon_0\mathbf{E}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{Q\mu\cos\theta}{4\pi^2 r^5} \\ & + c_{1,1}\epsilon_0 E\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{Q\mu_0\mu}{(4\pi)^2\epsilon_0 r^5}\left[1-\left(\frac{\mu}{cQr}\right)^2(1+3\cos^2\theta)\right](3\cos\theta\mathbf{e}_r - \mathbf{e}_z),\end{aligned} \tag{69}$$

$$\begin{aligned}\mathbf{M} = {} & c_{0,1}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{Q}{4\pi\epsilon_0 r^2}\mathbf{e}_r - c_{2,0}\frac{\mathbf{B}}{\mu_0}\frac{Q^2}{4\pi^2\epsilon_0 r^4}\left[1-\left(\frac{\mu}{cQr}\right)^2(1+3\cos^2\theta)\right] + c_{0,2}\frac{B(\theta=0)}{\mu_0}\frac{Q^2\cos\theta}{8\pi^2\epsilon_0 r^4}\mathbf{e}_r \\ & - c_{1,1}\frac{\mathbf{B}}{\mu_0}\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{Q\mu\cos\theta}{4\pi^2 r^5} + c_{1,1}\frac{B(\theta=0)}{\mu_0}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{Q^3}{32\pi^2\epsilon_0^2\mu r^3}\left[1-\left(\frac{\mu}{cQr}\right)^2(1+3\cos^2\theta)\right]\mathbf{e}_r,\end{aligned} \tag{70}$$

with $B(\theta=0) = \mu_0\mu/2\pi r^3$.

The electric and magnetic fields are slightly modified by the polarization and magnetization of the vacuum and become

$$\mathbf{E}_{\mathrm{V}} = \mathbf{E} - \frac{\mathbf{P}}{\epsilon_0}, \tag{71}$$

$$\mathbf{B}_{\mathrm{V}} = \mathbf{B} + \mu_0\mathbf{M}. \tag{72}$$

Some of the corrections to the fields given in the previous equations have a form that is very unusual, like, for example, the radial correction to **M**. These unusual corrections are related to $(\mathbf{E}\cdot\mathbf{B})$ and $c_{0,2}$.

B. Electric dipole moment and magnetic monopole

We first focus on the first term of Eqs. (69) and (70) proportional to the $c_{0,1}$ coefficient:

TABLE I. Constraints on the electric dipole moment of charged particles and corresponding constraints on the $c_{0,1}$ coefficient.

Particle	d (e cm)	References	$c_{0,1}$
Electron	$<10.5 \times 10^{-28}$	[38]	$<5.43 \times 10^{-17}$
Muon	$(-0.1 \pm 0.9) \times 10^{-19}$	[39]	$(1.1 \pm 9.6) \times 10^{-7}$
Tau	-0.22 to 0.45×10^{-16}	[40]	-8.1 to 4×10^{-3}
Proton	$<7.9 \times 10^{-25}$	[41]	$<2.69 \times 10^{-11}$

$$\mathbf{P}_{01} = c_{0,1}\sqrt{\epsilon_0\mu_0}\frac{\mu}{4\pi r^3}(3\cos\theta\mathbf{e}_r - \mathbf{e}_z) \tag{73}$$

$$= c_{0,1}\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{B}, \tag{74}$$

$$\mathbf{M}_{01} = c_{0,1}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{Q}{4\pi\epsilon_0 r^2}\mathbf{e}_r \tag{75}$$

$$= c_{0,1}\sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{E}. \tag{76}$$

If $c_{0,1}$ is not zero, as soon as an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ are superimposed in a vacuum, a nonlinear term appears inducing a correction to $\mathbf{E}$ proportional to $\mathbf{B}$ and a correction to $\mathbf{B}$ proportional to $\mathbf{E}$. So, for the case of an isolated particle of charge Q and magnetic moment μ, if the $c_{0,1}$ parameter is not zero, the magnetic dipole field should also appear as an electric dipole field so that the particle acquires an electric dipolar moment:

$$\mathbf{d} = \frac{c_{0,1}}{c}\boldsymbol{\mu}. \tag{77}$$

On the other hand, the radial electric field should induce a radial magnetic field so that the particle acquires a magnetic monopole:

$$m = c_{0,1}Qc, \tag{78}$$

where we write the monopole radial field $\mathbf{B}_\mathrm{m}$ as $\mathbf{B}_\mathrm{m} = \mu_0 m/4\pi r^2\mathbf{e}_r$.

The standard model predicts a nonzero electric dipole moment (EDM) for the electron, muon, or tau particles due to CP violation. The predicted value is however well below the current experimental sensitivities. For example, for the electron one expects $d_e \simeq 10^{-38}$ e cm [36]. As far as we understand, a $c_{0,1} \simeq 10^{-28}$ would therefore mimic the standard model EDM for the electron. No experiment has ever detected this deviation, but constraints can be found. Some of them are listed in Table I with the corresponding limit on $c_{0,1}$ (see also the particle data book [37]).

Concerning magnetic monopoles, they were first introduced by Dirac in 1931 [42]. The goal was to explain the quantization of electric charge by postulating the existence of an elementary magnetic charge, $Q_\mathrm{M}^\mathrm{D} = 2\pi\hbar/e$, which is now called the Dirac charge. More recently, it was understood that in the framework of grand unification theories the electric and magnetic charges are naturally quantized [43].

From an experimental point of view, limits exist for the electron and proton magnetic charge [37,44]. The electron magnetic charge Q_M, inducing a Coulomb magnetic field $\mathbf{B} = Q_M/4\pi r^2\mathbf{e}_r$, has been found to be

$$Q_\mathrm{M} < 4 \times 10^{-24} Q_\mathrm{M}^\mathrm{D}. \tag{79}$$

This corresponds to

$$c_{0,1} < 3 \times 10^{-22}, \tag{80}$$

which is a stronger limit than the one obtained from the EDM search.

C. Bound system and Lamb shift

For the sake of simplicity, we now consider $c_{0,1}$ and $c_{1,1}$ to be zero, or at least negligible. Using Eqs. (69) and (70), the $\mathbf{E}_\mathrm{V}$ and $\mathbf{B}_\mathrm{V}$ vectors can be approximated, at the leading order, to

$$\mathbf{E}_\mathrm{V} = \mathbf{E}\left[1 - c_{2,0}\frac{Q^2}{4\pi^2\epsilon_0 r^4}\right], \tag{81}$$

$$\mathbf{B}_\mathrm{V} = \mathbf{B}\left[1 - c_{2,0}\frac{Q^2}{4\pi^2\epsilon_0 r^4}\right] + c_{0,2}\frac{B(\theta=0)}{\mu_0}\frac{Q^2\cos\theta}{8\pi^2\epsilon_0 r^4}\mathbf{e}_r. \tag{82}$$

Let us first discuss the implications of Eq. (81). The correction in the Coulomb potential energy is proportional to $1/r^5$:

$$\delta V = -c_{2,0}\frac{Q^3}{80\pi^3\epsilon_0^2 r^5}. \tag{83}$$

In the QED framework, one obtains

$$\delta V_\mathrm{QED} = -\frac{Q}{4\pi\epsilon_0 r}\frac{2\alpha^3}{225\pi}\left(\frac{\hbar}{m_e c r}\right)^4. \tag{84}$$

This correction has been studied since 1956 [45] and it is called the Wichmann-Kroll potential.

This correction, proportional to $c_{2,0}$, induces an energy shift in bound systems and it is indeed part of the well-known Lamb shift. In Table II, we give some examples of the contribution of the Wichmann-Kroll correction to the leading term for different energy transitions and different systems. We also add the corresponding experimental precision.

In the case of the Lamb shift of the 1S and 2S level in atomic hydrogen, the Wichmann-Kroll correction has been calculated to be 0.3 ppm of the leading term [46,49], while the corresponding measurements have a precision of about

TABLE II. Examples of the contribution of the Wichmann-Kroll correction to the Lamb shift leading term for two different energy transitions and systems, to be compared to the relative uncertainties obtained on the Lamb shift measurements [46].

System and energy levels	Wichmann-Kroll contribution to the leading term	Experimental relative uncertainty	Remarks
H 1S	0.3 ppm	3 ppm [47]	
H muonic 2S-2P	5 ppm	15 ppm [48]	Proton charge radius puzzle

3 ppm [46,47]. All these calculations has been performed in the accepted QED framework with $c_{2,0}$ given by Eq. (14). It is worth stressing that $c_{0,2}$ cannot be constrained by bound systems studies.

Now the $c_{2,0}$ dependence of the Wichmann-Kroll correction to the Lamb shift is linear [45]. This means that the measurement of the 1S-2S Lamb shift in hydrogen, presented in Table II, constrains the value as follows: $c_{2,0} < 10 \times c_{2,0}^{\mathrm{HE}}$. We add the corresponding excluded region in Fig. 3.

In the case of the 2S-2P lamb shift of muonic hydrogen, the correction is evaluated at 5 ppm [50], but the measurement is at 15 ppm [48]. Furthermore, the proton radius extracted from this measurement is not in agreement with the one inferred from the hydrogen measurement. This is an important issue that is now called the "proton charge radius puzzle." This means that the Wichmann-Kroll correction has not been tested and therefore there is not yet further information on $c_{2,0}$ coming from QED tests in bound systems.

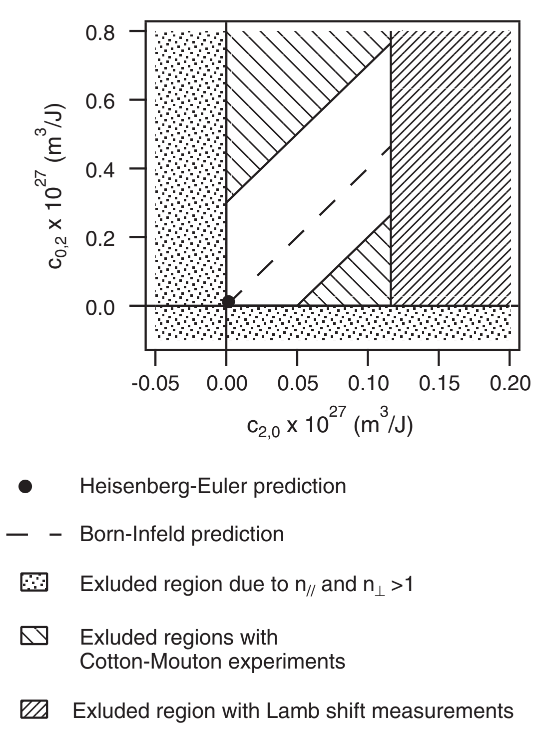

FIG. 3. Best experimental limits on $c_{0,2}$ and $c_{2,0}$ parameters. The excluded region due to Lamb shift measurements is added.

Let us discuss the modification to the magnetic field. It looks like nobody has ever considered it except for Heyl as a modification of a macroscopic magnetic dipole [51], but without the term proportional to $c_{0,2}$ coming from the coupling between the electric and the magnetic field. This term has a very unusual form. No calculation of the energy shift induced by this correction exists, although this modification of the magnetic field of a pointlike charge should affect at least the atomic hyperfine splitting. In fact, the leading term in this energy splitting, called the Fermi term [52], is proportional to the field due to the bound particle at the position of the nucleus. In the case of the hydrogen atom, the correction of the electron magnetic field at a distance of a Bohr radius is of the order of 2×10^{-17}, when the precision of the hydrogen ground-state hyperfine splitting measurement is of the order of 10^{-13} [53]. For the muonic hydrogen the correction of the muon magnetic field at the position of the proton is of the order of 4×10^{-8}, but the ground-state hyperfine splitting of muonic hydrogen has not yet been measured (see, e.g., Ref. [54]).

D. Limits on the Born-Infeld E_{abs} free parameters

The Born-Infeld NLED is constructed on the assumption that an absolute electric field exists in nature. Atomic energy levels should therefore be different from the ones predicted without such a field limitation. The natural way to constrain such a free parameter is therefore to look for the predicted energy variation in high atomic number atoms where nonlinearities should be more important. This was done in 1973 by Soff, Rafelski, and Greiner [55] who reported that E_{abs} has to be greater than 1.7×10^{22} V/m. More recently, their results have been questioned [56] even if the authors agree that the value proposed by Born and Infeld is not physically viable. For the sake of argument, let us note that an $E_{\mathrm{abs}} = 1.7 \times 10^{22}$ V/m corresponds to a $c_{2,0}$ that is about 5 orders of magnitude smaller than the one predicted by QED.

V. CONCLUSION

In this paper we developed a framework in which experiments whose goal is to test QED predictions can be used in a more general way to test NLED, which contains low-energy QED as a special case. We reviewed some of these experiments and we established limits on the different free parameters $c_{0,1}$, $c_{2,0}$, $c_{0,2}$, and $c_{1,1}$, generalizing QED predictions in the framework of NLED. Actually, only $c_{0,1}$, $c_{2,0}$, and $c_{0,2}$ can be constrained. As far as we know, no experiment constraining $c_{1,1}$ exists.

The parametrization of the photon-photon interaction Lagrangian is also very useful to understand the mutual impact of QED tests of different nature. In particular, we showed that $c_{2,0}$ can be limited by measurements of Wichmann-Kroll potential corrections, as in the case of the 1S-2S Lamb-shift in atomic hydrogen, at a level that is compatible with limits coming from vacuum magnetic birefringence.

The Heisenberg-Euler Lagrangian is a special case of NLED. In bound systems it is related to the Wichmann-Kroll potential which is the correction to the Coulomb potential at large distances. The leading term to the Coulomb potential corrections is given by the Uehling potential representing the short-distance corrections. The Wichmann-Kroll potential induces lower-order corrections than the Uehling ones and that is why, while in general one can say that QED in bound systems is well tested, this is not true specifically for the long-distance corrections where the direct tests of NLED come into play. Of course, the Wichmann-Kroll potential and the Uehling one both come from the same theoretical framework and it is difficult to imagine that the short-range regime is well treated while the long-range regime is not; nevertheless, one has to test whether some new physics appears in the long range that induces corrections not predicted by standard QED. This looks like an important task largely justifying NLED direct tests.

Let us finish with the anomalous magnetic moment $(g-2)$ of isolated particles which is one of the best tested quantities in QED [6]. As discussed, for example, in Ref. [57], photon-photon scattering contributes as a subdiagram to the $g-2$ and the Lamb shift. At first sight, it thus seems feasible to use $g-2$ measurements to constrain the $c_{0,2}$ or $c_{2,0}$ parameters, as was done in the previous section with the Lamb shift. However, the $g-2$ of isolated particles corresponds to a physical quantity that is related to short-range physics (as far as we understand), or at least the long-range corrections have never been stated explicitly as in the case of the Wichmann-Kroll corrections for the Lamb shift in bound systems. Furthermore, the $g-2$ corrections change the value of the magnetic moment, not the shape of the dipolar field. But the correction given in Eq. (70) indicates a change in the shape of the field. It is not clear to us how to combine them. This is certainly an important point to clarify.

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