

**Renormalization of the quark mass matrix**S. H. Chiu<sup>1,\*</sup><sup>1</sup>*Physics Group, CGE, Chang Gung University, Taoyuan 33302, Taiwan*T. K. Kuo<sup>2,†</sup><sup>2</sup>*Department of Physics, Purdue University, West Lafayette, Indiana 47907, USA*

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Using a set of rephasing-invariant variables, it is shown that the renormalization group equations for quark mixing parameters can be written in a form that is compact, in addition to having simple properties under flavor permutation. We also found approximate solutions to these equations if the quark masses are hierarchical or nearly degenerate.

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With the recent discovery of the Higgs boson, the last “missing piece” of the standard model (SM) was finally found. However, the long-standing mystery, that the Higgs couplings (mass matrices) appear to be rather arbitrary, remains to be resolved. A commonly held view posits that the SM is but an effective theory originating from some other theory valid at high energies, and that more regularity can be found there. To bridge these two energy regimes, one makes use of the renormalization group equations (RGEs). Such RGEs for the mass matrices have been around for a long time (see, e.g., Refs. [1–9]). They are relatively simple when written in terms of the mass matrices themselves. However, these matrices contain a large number of unphysical degrees of freedom, which must be stripped away to reveal the values of the physical variables, viz., the masses and the mixing matrices. The procedure is by no means easy, and it is hard to correlate the variables in the two energy regions. For this reason a lot of efforts have gone into recasting the RGEs into equations containing only physical variables [6–9]. With these equations the physical variables at different energies can be directly related. Thus, for instance, one may test possible scenarios for mass patterns at high energies, using the RGEs to see if they could evolve into the existing low-energy values. The challenge here comes from the complexity of the RGEs, which are lengthy, nonlinear, partial differential equations, so that the relations of variables at different energy scales are often obscure, and one can have only a partial view with the use of various approximation schemes. This difficulty, one would hope, can be mitigated to some extent by a judicious choice of the physical variables. Indeed, in this paper we propose to cast the one-loop quark RGEs in terms of a set of rephasing-invariant variables introduced earlier [10]. It is found that these RGEs can be written in a

compact form. In addition, they exhibit manifest symmetries which, as a consequence of the permutation properties of the chosen variables, give these equations a very simple structure. As it turns out, this set of equations is still too complicated to be solved analytically. However, under reasonable assumptions (hierarchy, degeneracy, etc.), approximate solutions are available. These will be presented in this paper. As more properties are found about these equations, one may hope that they will help in the search for a viable high-energy theory.

**II. REPHASING-INVARIANT PARAMETRIZATION**

It is well known that physical observables are independent of rephasing transformations on the mixing matrices of quantum-mechanical states. Thus, instead of individual elements of the mixing matrix, only rephasing-invariant combinations thereof are physical. Whereas there is nothing wrong with using these elements in intermediate steps of a calculation, at the end of the day, they must form rephasing-invariant combinations in physical quantities. This situation is similar to that in gauge theory, where one often resorts to a particular gauge choice for certain problems. The final results, however, must be gauge invariant. In this paper, we propose to use, from the outset, parameters that are rephasing invariant. As we will demonstrate in Sec. III, in terms of these, the quark RGEs become quite simple in structure, making it easier to analyze the properties of their solutions.

We turn now to Ref. [10], where it was pointed out that six rephasing-invariant combinations can be constructed from elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix,  $V$ :

$$\Gamma_{ijk} = V_{1i}V_{2j}V_{3k} = R_{ijk} - iJ, \quad (1)$$

where  $(i, j, k)$  is a cyclic permutation of  $(1, 2, 3)$  and  $\det V = +1$  is imposed. The common imaginary part is

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identified with the Jarlskog invariant [11], and the real parts are defined as

$$(R_{123}, R_{231}, R_{312}; R_{132}, R_{213}, R_{321}) = (x_1, x_2, x_3; y_1, y_2, y_3). \quad (2)$$

The  $(x_i, y_j)$  parameters are bounded,  $-1 \leq (x_i, y_j) \leq 1$ , with  $x_i \geq y_j$  for any pair of  $(i, j)$ . It is also found that the six parameters satisfy two conditions,

$$\det V = (x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 1, \quad (3)$$

$$(x_1x_2 + x_2x_3 + x_3x_1) - (y_1y_2 + y_2y_3 + y_3y_1) = 0, \quad (4)$$

leaving four independent parameters for the mixing matrix. They are related to the Jarlskog invariant,

$$J^2 = x_1x_2x_3 - y_1y_2y_3 \quad (5)$$

and the squared elements of  $V$ ,

$$W = [|V_{ai}|^2] = \begin{pmatrix} x_1 - y_1 & x_2 - y_2 & x_3 - y_3 \\ x_3 - y_2 & x_1 - y_3 & x_2 - y_1 \\ x_2 - y_3 & x_3 - y_1 & x_1 - y_2 \end{pmatrix}. \quad (6)$$

The matrix of the cofactors of  $W$ , denoted as  $w$  with  $w^T W = (\det W)I$ , is given by

$$w = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \\ x_3 + y_2 & x_1 + y_3 & x_2 + y_1 \\ x_2 + y_3 & x_3 + y_1 & x_1 + y_2 \end{pmatrix}. \quad (7)$$

The elements of  $w$  are also bounded,  $-1 \leq w_{ai} \leq +1$ , and

$$\sum_i w_{ai} = \sum_\alpha w_{ai} = \det W, \quad (8)$$

$$\det W = \sum x_i^2 - \sum y_j^2 = \sum x_i + \sum y_j. \quad (9)$$

The relations between  $(x_i, y_j)$  and the standard parametrization can be found in Ref. [12].

There are some useful expressions for the rephasing-invariant combinations. One first considers the product of four mixing elements [11]

$$\pi_{ij}^{\alpha\beta} = V_{ai}V_{\beta j}V_{\alpha j}^*V_{\beta i}^*, \quad (10)$$

which can be reduced to

$$\begin{aligned} \pi_{ij}^{\alpha\beta} &= |V_{ai}|^2|V_{\beta j}|^2 - \sum_{\gamma k} \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} V_{ai}V_{\beta j}V_{\gamma k} \\ &= |V_{\alpha j}|^2|V_{\beta i}|^2 + \sum_{\gamma k} \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} V_{\alpha j}^*V_{\beta i}^*V_{\gamma k}. \end{aligned} \quad (11)$$

In addition, for  $\alpha \neq \beta \neq \gamma$  and  $i \neq j \neq k$ , we define

$$\pi_{ij}^{\alpha\beta} \equiv \pi_{\gamma k} = \Lambda_{\gamma k} + iJ. \quad (12)$$

Since  $\text{Re}(\pi_{ij}^{\alpha\beta})$  takes the forms,

$$\text{Re}(\pi_{ij}^{\alpha\beta}) = |V_{ai}|^2|V_{\beta j}|^2 - x_a = |V_{\beta i}|^2|V_{\alpha j}|^2 + y_b, \quad (13)$$

we have

$$\Lambda_{\gamma k} = \frac{1}{2}(|V_{ai}|^2|V_{\beta j}|^2 + |V_{\alpha j}|^2|V_{\beta i}|^2 - |V_{\gamma k}|^2). \quad (14)$$

In terms of the  $(x, y)$  variables,

$$\Lambda_{\gamma k} = x_a y_j + x_b x_c - y_j(y_k + y_l), \quad (15)$$

where  $(x_a, y_j)$  comes from  $|V_{\gamma k}|^2 = x_a - y_j$ , and  $a \neq b \neq c$ ,  $j \neq k \neq l$ .

### III. RGEs FOR QUARKS

The one-loop RGEs for the quark mass matrices have been developed and widely studied [5–7]. In terms of the mass-squared matrices for the  $u$ -type quarks,  $M_u = Y_u Y_u^\dagger$ , and that for the  $d$ -type quarks,  $M_d = Y_d Y_d^\dagger$ , where  $Y$  is the Yukawa coupling matrices of the Higgs boson to the quarks, the RGEs take a simple form:

$$\mathcal{D}M_u = a_u M_u + b M_u^2 + c \{M_u, M_d\}, \quad (16)$$

$$\mathcal{D}M_d = a_d M_d + b M_d^2 + c \{M_u, M_d\}. \quad (17)$$

Here,  $\mathcal{D} \equiv (16\pi^2) \frac{d}{dt}$  and  $t = \ln(\mu/M_W)$ , where  $\mu$  is the energy scale and  $M_W$  is the  $W$  boson mass. The model dependence of the RGEs is implanted in  $a_u, a_d, b$ , and  $c$ .

Although the RGEs are simple in their matrix forms, one must extract the physical variables (masses and mixing parameters) from these matrices. This is complicated because they contain a large number of unphysical degrees of freedom and it is not easy to infer the evolution of the physical variables from that of the mass matrices. For this reason it is useful to deduce from Eqs. (16)–(17) the RGEs in terms of the physical variables, which can then yield direct information on the evolution of these variables. This procedure results in the following equations for the masses and CKM elements  $V_{ij}$ :

$$\mathcal{D} \ln(f_i^2) = a_u + b f_i^2 + 2c \sum_j h_j^2 |V_{ij}|^2, \quad (18)$$

$$\mathcal{D} \ln(h_i^2) = a_d + b h_i^2 + 2c \sum_j f_j^2 |V_{ij}|^2, \quad (19)$$

$$\mathcal{D} V_{ij} = c \left[ \sum_{l, k \neq i} F_{ik} h_l^2 V_{il} V_{kl}^* V_{kj} + \sum_{m, k \neq j} H_{jk} f_m^2 V_{mk}^* V_{mj} V_{ik} \right], \quad (20)$$

where  $f_i^2$  and  $h_i^2$  are the eigenvalues of  $M_u$  and  $M_d$ , respectively, and

$$F_{ik} = \frac{f_i^2 + f_k^2}{f_i^2 - f_k^2}, \quad H_{jk} = \frac{h_j^2 + h_k^2}{h_j^2 - h_k^2}. \quad (21)$$

It should be emphasized that Eq. (20), as it stands, is not rephasing invariant. The physical part thereof is obtained by using it only on rephasing-invariant combinations of  $V_{ij}$ , such as  $|V_{ij}|^2$  or the  $(x, y)$  variables defined in Eq. (2). In Ref. [13], we obtained the evolution equations of  $x_i$  and  $y_j$  in the form

$$-\mathcal{D}x_i/c = [\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][A_i][H_{23}, H_{31}, H_{12}]^T + [\Delta h_{23}^2, \Delta h_{31}^2, \Delta h_{12}^2][B_i][F_{23}, F_{31}, F_{12}]^T, \quad (22)$$

$$-\mathcal{D}y_i/c = [\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][A'_i][H_{23}, H_{31}, H_{12}]^T + [\Delta h_{23}^2, \Delta h_{31}^2, \Delta h_{12}^2][B'_i][F_{23}, F_{31}, F_{12}]^T, \quad (23)$$

where  $\Delta f_{ij}^2 = f_i^2 - f_j^2$  and  $\Delta h_{ij}^2 = h_i^2 - h_j^2$ . In terms of  $(x_i, y_j)$ , the explicit forms of the matrices  $[A_i]$ ,  $[A'_i]$ ,  $[B_i]$ , and  $[B'_i]$  are given in Table II of Ref. [13]. Since  $\sum \Delta f_{ij}^2 = \sum \Delta h_{ij}^2 = 0$ , to the matrices  $[A_i]$ ,  $[B_i]$ ,  $[A'_i]$ , and  $[B'_i]$ , we can add arbitrary matrices of the form

$$\begin{pmatrix} \delta_1 & \delta_2 & \delta_3 \\ \delta_1 & \delta_2 & \delta_3 \\ \delta_1 & \delta_2 & \delta_3 \end{pmatrix}.$$

Thus, for instance, from Table II in Ref. [13]

$$\begin{aligned} [A_1] &= \begin{pmatrix} 2x_1y_1 & x_1x_2 + y_2y_3 & x_1x_3 + y_2y_3 \\ x_1x_3 + y_1y_2 & 2x_1y_3 & x_1x_2 + y_1y_2 \\ x_1x_2 + y_1y_3 & x_1x_3 + y_1y_3 & 2x_1y_2 \end{pmatrix} \\ &= 2[Z_1] - [Z_0] + \left( J^2 + 3 \sum x_i x_j - x_2 x_3 \right) \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} y_2y_3 & y_1y_2 & y_1y_3 \\ y_2y_3 & y_1y_2 & y_1y_3 \\ y_2y_3 & y_1y_2 & y_1y_3 \end{pmatrix}, \end{aligned} \quad (24)$$

where we have used the relations  $W_{KL}\Lambda_{KL} = J^2 + x_a y_j$ ,  $W_{KL} = x_a - y_j$ . It follows that

$$[\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][A_1] = [\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2](2[Z_1] - [Z_0]). \quad (25)$$

Similarly, all the  $[A]$  and  $[B]$  matrices can be so transformed and we may recast Eqs. (22)–(23) in a more suggestive form,

$$-\mathcal{D}x_i/c = [\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2](2[Z_i] - [Z_0])[H_{23}, H_{31}, H_{12}]^T + [\Delta h_{23}^2, \Delta h_{31}^2, \Delta h_{12}^2](2[Z_i] - [Z_0])^T[F_{23}, F_{31}, F_{12}]^T, \quad (26)$$

$$-\mathcal{D}y_i/c = [\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2](2[Z'_i] - [Z_0])[H_{23}, H_{31}, H_{12}]^T + [\Delta h_{23}^2, \Delta h_{31}^2, \Delta h_{12}^2](2[Z'_i] - [Z_0])^T[F_{23}, F_{31}, F_{12}]^T. \quad (27)$$

TABLE I. The explicit expressions of the matrices  $[Z_i]$ ,  $[Z'_i]$ , and  $[Z_0]$ . Here  $\Lambda_{pk}$  is defined in Eq. (14).

$$\begin{aligned} Z_1 &= \begin{pmatrix} \Lambda_{11} & 0 & 0 \\ 0 & \Lambda_{22} & 0 \\ 0 & 0 & \Lambda_{33} \end{pmatrix}, & Z_2 &= \begin{pmatrix} 0 & \Lambda_{12} & 0 \\ 0 & 0 & \Lambda_{23} \\ \Lambda_{31} & 0 & 0 \end{pmatrix}, & Z_3 &= \begin{pmatrix} 0 & 0 & \Lambda_{13} \\ \Lambda_{21} & 0 & 0 \\ 0 & \Lambda_{32} & 0 \end{pmatrix} \\ Z'_1 &= \begin{pmatrix} \Lambda_{11} & 0 & 0 \\ 0 & 0 & \Lambda_{23} \\ 0 & \Lambda_{32} & 0 \end{pmatrix}, & Z'_2 &= \begin{pmatrix} 0 & \Lambda_{12} & 0 \\ \Lambda_{21} & 0 & 0 \\ 0 & 0 & \Lambda_{33} \end{pmatrix}, & Z'_3 &= \begin{pmatrix} 0 & 0 & \Lambda_{13} \\ 0 & \Lambda_{22} & 0 \\ \Lambda_{31} & 0 & 0 \end{pmatrix} \end{aligned}$$

$$[Z_0] = \begin{pmatrix} (1 - |V_{11}|^2)\Lambda_{11} & (1 - |V_{12}|^2)\Lambda_{12} & (1 - |V_{13}|^2)\Lambda_{13} \\ (1 - |V_{21}|^2)\Lambda_{21} & (1 - |V_{22}|^2)\Lambda_{22} & (1 - |V_{23}|^2)\Lambda_{23} \\ (1 - |V_{31}|^2)\Lambda_{31} & (1 - |V_{32}|^2)\Lambda_{32} & (1 - |V_{33}|^2)\Lambda_{33} \end{pmatrix}$$

TABLE II. The explicit expressions of the matrix  $[S_{ij}]$ .

$S_{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda_{22} & -\Lambda_{23} \\ 0 & -\Lambda_{32} & \Lambda_{33} \end{pmatrix},$	$S_{12} = \begin{pmatrix} 0 & 0 & 0 \\ -\Lambda_{21} & 0 & \Lambda_{23} \\ \Lambda_{31} & 0 & -\Lambda_{33} \end{pmatrix},$	$S_{13} = \begin{pmatrix} 0 & 0 & 0 \\ \Lambda_{21} & -\Lambda_{22} & 0 \\ -\Lambda_{31} & \Lambda_{32} & 0 \end{pmatrix}$
$S_{21} = \begin{pmatrix} 0 & -\Lambda_{12} & \Lambda_{13} \\ 0 & 0 & 0 \\ 0 & \Lambda_{32} & -\Lambda_{33} \end{pmatrix},$	$S_{22} = \begin{pmatrix} \Lambda_{11} & 0 & -\Lambda_{13} \\ 0 & 0 & 0 \\ -\Lambda_{31} & 0 & \Lambda_{33} \end{pmatrix},$	$S_{23} = \begin{pmatrix} -\Lambda_{11} & \Lambda_{12} & 0 \\ 0 & 0 & 0 \\ \Lambda_{31} & -\Lambda_{32} & 0 \end{pmatrix}$
$S_{31} = \begin{pmatrix} 0 & \Lambda_{12} & -\Lambda_{13} \\ 0 & -\Lambda_{22} & \Lambda_{23} \\ 0 & 0 & 0 \end{pmatrix},$	$S_{32} = \begin{pmatrix} -\Lambda_{11} & 0 & \Lambda_{13} \\ \Lambda_{21} & 0 & -\Lambda_{23} \\ 0 & 0 & 0 \end{pmatrix},$	$S_{33} = \begin{pmatrix} \Lambda_{11} & -\Lambda_{12} & 0 \\ -\Lambda_{21} & \Lambda_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The matrices  $[Z_i]$ ,  $[Z'_i]$ , and  $[Z_0]$  are listed in Table I. It is noteworthy that the matrix structures of  $[Z_i]$  and  $[Z'_i]$  mirror those of  $x_i$  and  $y_i$ , when written as products of  $V_{ij}$ , e.g.,  $x_1 = \text{Re}(V_{11}V_{22}V_{33})$ . It is also satisfying to establish  $[B_i] = [A_i]^T$  and  $[B'_i] = [A'_i]^T$ , which is a consequence of the conjugate roles played by the  $u$ -type and  $d$ -type quarks. The RGEs of  $W_{ij}(|V_{ij}|^2)$  and  $J^2$  can be obtained:

$$-\frac{1}{2c} \mathcal{D}W_{ij} = [\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][S_{ij}][H_{23}, H_{31}, H_{12}]^T + [\Delta h_{23}^2, \Delta h_{31}^2, \Delta h_{12}^2][S_{ij}]^T[F_{23}, F_{31}, F_{12}]^T, \quad (28)$$

$$-\frac{1}{2c} \mathcal{D} \ln J^2/c = [\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][w][H_{23}, H_{31}, H_{12}]^T + [\Delta h_{23}^2, \Delta h_{31}^2, \Delta h_{12}^2][w]^T[F_{23}, F_{31}, F_{12}]^T. \quad (29)$$

Although  $[S_{ij}]$  can be directly written down from  $[Z_i]$  and  $[Z'_i]$ , we list them explicitly in Table II, since it will be used for the analyses of  $\mathcal{D}W_{ij}$  in the next section.

The simple and compact form of Eqs. (26)–(29) can be contrasted with the RGEs written in terms of the standard parametrization (see, e.g., Ref. [14]), for which it is hard to find any regularity in the structure. It is seen that these equations clearly exhibit symmetries under permutation of the indices, owing to the same properties inherent in the definition of the  $(x, y)$  variables. The situation here can be compared to a familiar one in electricity and magnetism. While the wave equations take a simple form for the (gauge-invariant)  $\vec{E}$  and  $\vec{B}$  fields, depending on the choice

of gauge, the corresponding equations for the potential  $A_\mu$  can be very complicated. Another salient feature of them is the prominent role played by the rephasing invariants  $\Lambda_{\gamma k}$ , which are the same Jarlskog invariants that appear in formulas of the neutrino oscillation probabilities,  $P(\nu_\alpha \rightarrow \nu_\beta)$ . Without them the RGEs would look rather cumbersome, as written in Ref. [13]. In addition, they facilitate the calculation of approximate solutions of the RGEs, as we will see in the next section. Last, from Eqs. (26) and (27), and Table I, it can be verified that  $\sum \mathcal{D}(x_i) - \sum \mathcal{D}(y_j) = 0$  and  $\sum \mathcal{D}(x_i x_j) - \sum \mathcal{D}(y_i y_j) = 0$ , as one expects from the constraint equations [Eqs. (3) and (4)].

Notice that the evolution equations of  $\Lambda_{\gamma k}$  can also be cast in compact forms similar to that of  $W_{ij}$  and  $J^2$ :

$$-\frac{1}{2c} \mathcal{D}\Lambda_{\gamma k} = [\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2][Y_{\gamma k}][H_{23}, H_{31}, H_{12}]^T + [\Delta h_{23}^2, \Delta h_{31}^2, \Delta h_{12}^2][Y_{\gamma k}]^T[F_{23}, F_{31}, F_{12}]^T. \quad (30)$$

Here the matrix  $[Y_{\gamma k}]$  takes the form

$$[Y_{\gamma k}] = \begin{pmatrix} c_{11}\Lambda_{11} & c_{12}\Lambda_{12} & c_{13}\Lambda_{13} \\ c_{21}\Lambda_{21} & c_{22}\Lambda_{22} & c_{23}\Lambda_{23} \\ c_{31}\Lambda_{31} & c_{32}\Lambda_{32} & c_{33}\Lambda_{33} \end{pmatrix}, \quad (31)$$

where the coefficients  $c_{ij}$  are functions of  $|V_{ij}|^2$ . As an example,

$$[Y_{11}] = \begin{pmatrix} (|V_{23}|^2 + |V_{32}|^2 - |V_{22}|^2 - |V_{33}|^2)\Lambda_{11} & (|V_{22}^2 - |V_{32}|^2)\Lambda_{12} & (|V_{33}|^2 - |V_{23}|^2)\Lambda_{13} \\ (|V_{22}|^2 - |V_{23}|^2)\Lambda_{21} & (1 - |V_{22}|^2)\Lambda_{22} & (-1 + |V_{23}|^2)\Lambda_{23} \\ (|V_{33}|^2 - |V_{32}|^2)\Lambda_{31} & (-1 + |V_{32}|^2)\Lambda_{32} & (1 - |V_{33}|^2)\Lambda_{33} \end{pmatrix}. \quad (32)$$

It is seen that

$$\sum_i c_{Ii} = \sum_I c_{Ii} = 0, \quad (33)$$

and the  $2 \times 2$  submatrix (indices 2 and 3) has a simple structure,  $c_{\gamma k} = \pm 1 \pm |V_{\gamma k}|^2$ ,  $(\gamma k) = (2, 3)$ . With the condition (33), one can construct the  $3 \times 3$  matrix from the known  $2 \times 2$  matrix. Finally, the evolution equations for the

combinations of  $\Lambda_{\gamma k}$ , such as  $\mathcal{D}(\sum_{\gamma} \Lambda_{\gamma k})$ ,  $\mathcal{D}(\sum_k \Lambda_{\gamma k})$ , and  $\mathcal{D}(\sum_{\gamma, k} \Lambda_{\gamma k})$ , can also be cast in similar forms, in which  $c_{ij}$  are functions of the elements of  $W_{ij}$  and  $w_{ij}$ . We will not show the details here.

#### IV. ANALYSIS OF THE RGEs

Although the solutions to the quark RGEs are not available, it turns out that, under certain reasonable assumptions, one can find approximate solutions for them. Before embarking on this analysis, it should be noticed that, with the observed values in the mass matrices, the parameter  $c/16\pi^2$  and all  $\Lambda_{ij}$ 's are small. This means that renormalization effects are generally small if one starts from low energy using the SM and the known values of the physical variables. However, it is interesting to entertain the possibility that, at some point, a new theory can intervene with a fast-paced renormalization evolution. It is then relevant to consider RGE evolution from high to low  $t$  values, with other assumed parameters at high energies. To do this we consider various scenarios of the mass parameters: A)  $f_3^2 \gg f_2^2 \gg f_1^2$  and  $h_3^2 \gg h_2^2 \gg h_1^2$ ; B)  $f_3^2 \gg f_2^2 \approx f_1^2$  and  $h_3^2 \gg h_2^2 \approx h_1^2$ ; C)  $f_3^2 \gg f_2^2 \gg f_1^2$  and  $h_3^2 \gg h_2^2 \approx h_1^2$ . While case A) corresponds to the mass patterns at low energy, the other choices are possibilities which may prevail at some high energy scale. These considerations are useful for model building, so that one can bridge the mixing patterns between the high and low energy scales. We will now present the detailed results for case A), but leave the discussion of the other cases to the Appendix.

For the hierarchical case in A), one may simplify the matrices so that  $[F_{23}, F_{31}, F_{12}] \approx [-1, 1, -1]$  and  $[H_{23}, H_{31}, H_{12}] \approx [-1, 1, -1]$ . In addition,

$$[\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2] \approx f_3^2[-1, 1, 0], \quad (34)$$

$$[\Delta h_{23}^2, \Delta h_{31}^2, \Delta h_{12}^2] \approx h_3^2[-1, 1, 0]. \quad (35)$$

The approximations lead to

$$-\frac{1}{2c} \mathcal{D}W_{ij} \approx \left( f_3^2 \left[ \sum_{p,q} (-1)^{p+q} S_{ij}^{pq} \right] + h_3^2 \left[ \sum_{p,q} (-1)^{p+q} S_{ij}^{pqT} \right] \right), \quad (36)$$

where  $S_{ij}^{pq}$  is the  $(p, q)$  element of  $S_{ij}$  with  $p = 1, 2$  and  $q = 1, 2, 3$ . We show the explicit expressions of  $\mathcal{D}W_{ij}$  in the Appendix.

Note that out of the nine equations, six of them can be cast in the following forms:

$$\frac{1}{2c} \mathcal{D} \ln W_{11} = f_3^2 W_{31} + h_3^2 W_{13}, \quad (37)$$

$$\frac{1}{2c} \mathcal{D} \ln W_{13} = -f_3^2 W_{33} - h_3^2 (1 - W_{13}), \quad (38)$$

$$\frac{1}{2c} \mathcal{D} \ln W_{23} = -f_3^2 W_{33} - h_3^2 (W_{33} - W_{13}), \quad (39)$$

$$\frac{1}{2c} \mathcal{D} \ln W_{31} = -f_3^2 (1 - W_{31}) - h_3^2 W_{33}, \quad (40)$$

$$\frac{1}{2c} \mathcal{D} \ln W_{32} = -f_3^2 (W_{33} - W_{31}) - h_3^2 W_{33}, \quad (41)$$

$$\frac{1}{2c} \mathcal{D} \ln W_{33} = f_3^2 (1 - W_{33}) + h_3^2 (1 - W_{33}). \quad (42)$$

A RGE invariant can then be derived directly,

$$\mathcal{D} \ln \left( \frac{W_{13} W_{31} W_{33}}{W_{23} W_{32}} \right) = 0. \quad (43)$$

Since from the theoretical point of view there is no preferred scenario concerning the relative magnitudes of  $f_i^2$  and  $h_i^2$  at high energies, it would be interesting to further pursue possible invariants under the following assumptions about the couplings. (i) If  $f_3^2 \gg h_3^2$ , we obtain three more approximate invariants:

$$\mathcal{D} \ln \left( \frac{W_{13}}{W_{23}} \right) \approx 0, \quad (44)$$

$$\mathcal{D} \ln \left( \frac{W_{11} W_{13}}{W_{32}} \right) \approx 0, \quad (45)$$

$$\mathcal{D} \ln \left( \frac{W_{31} W_{33}}{W_{32}} \right) \approx 0. \quad (46)$$

(ii) If on the other hand,  $f_3^2 \ll h_3^2$ , we have

$$\mathcal{D} \ln \left( \frac{W_{31}}{W_{32}} \right) \approx 0, \quad (47)$$

$$\mathcal{D} \ln \left( \frac{W_{13} W_{33}}{W_{23}} \right) \approx 0, \quad (48)$$

$$\mathcal{D} \ln \left( \frac{W_{11} W_{31}}{W_{23}} \right) \approx 0. \quad (49)$$

Despite the complexity of its original forms, the RGEs of  $W_{ij}$  can be solved approximately. With  $c' = 16\pi^2/[2c(f_3^2 + h_3^2)]$  and  $a_{ij}$  the initial value of  $W_{ij}$ , Eq. (A9) yields

$$W_{33} \approx \frac{1}{(a_{33}^{-1} - 1)e^{-(t-t_0)/c'} + 1}. \quad (50)$$

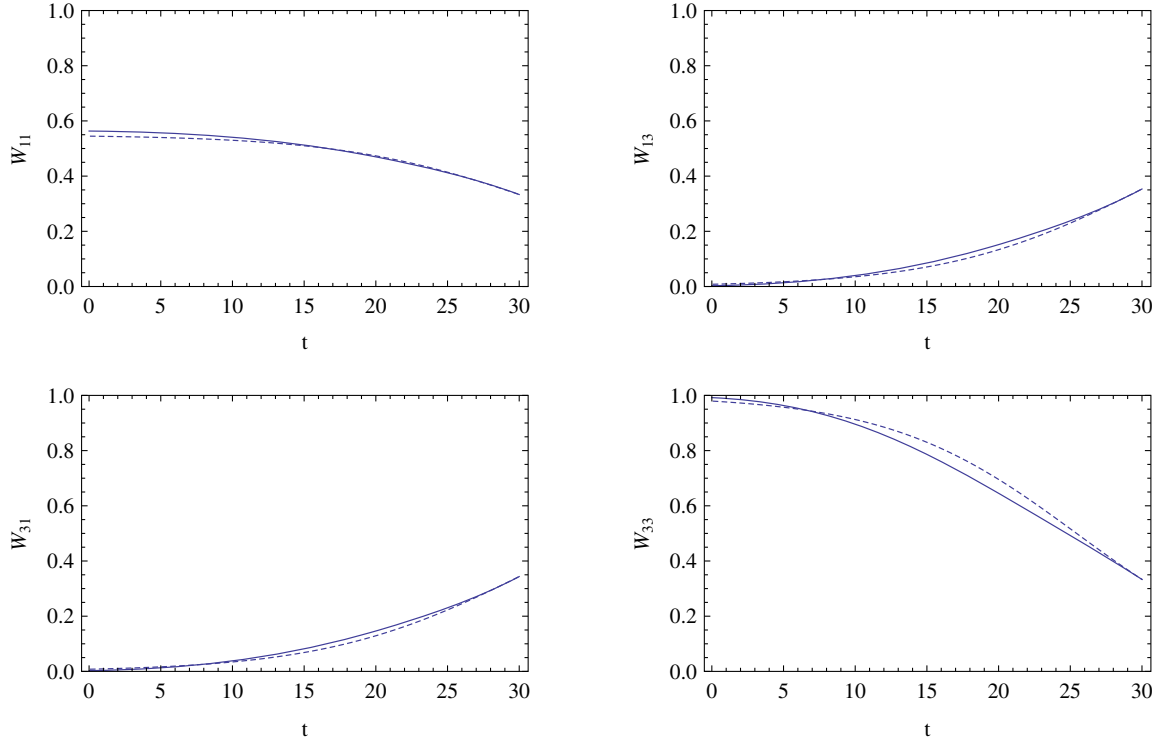


FIG. 1. The approximate solutions (dashed) are compared with the full, numerical solutions (solid) for the hierarchical scenario with  $f_3^2 = h_3^2 = 4$ , where  $f_3^2 \gg f_2^2 \gg f_1^2$  and  $h_3^2 \gg h_2^2 \gg h_1^2$ . Here  $(b, c) = (3, -3/2)$  under the standard model. The initial values of  $(x, y)$  at  $t = 30$  are taken to be  $x_1 = (1/6) + \varepsilon$ ,  $x_2 = (1/6) - \varepsilon$ ,  $y_1 = -(1/6) + \varepsilon$ , and  $-(1/6) - \varepsilon$ , where  $\varepsilon = 0.01$ .

With the solution of  $W_{33}$ , one may in principle solve for  $W_{13}$ ,  $W_{33}$ , and  $W_{11}$ . However, we will not show the long expressions here, but instead further assume the following scenarios of the couplings to obtain simple, approximate solutions for the rest of the  $W_{ij}$ . Note that  $f_3^2$  and  $h_3^2$  are treated as constants here, i.e., the approximate solutions are only valid for a range of  $t$  values in which the variations of  $f_3^2$  and  $h_3^2$  are negligible.

(i) If  $f_3^2 \gg h_3^2$ , it leads to

$$W_{13} \approx \frac{a_{13}}{(1 - a_{33}) + a_{33}e^{(t-t_0)/c_f}}, \quad (51)$$

$$W_{31} \approx \frac{a_{31}}{a_{31} + (1 - a_{31})e^{(t-t_0)/c_f}}, \quad (52)$$

$$W_{11} \approx \frac{a_{11}}{(1 - a_{31}) + a_{31}e^{-(t-t_0)/c_f}}, \quad (53)$$

where  $c_f = 16\pi^2/(2cf_3^2) \approx c'$ .

(ii) If  $f_3^2 \ll h_3^2$ ,

$$W_{13} \approx \frac{a_{13}}{a_{13} + (1 - a_{13})e^{(t-t_0)/c_h}}, \quad (54)$$

$$W_{31} \approx \frac{a_{31}}{(1 - a_{33}) + a_{33}e^{(t-t_0)/c_h}}, \quad (55)$$

$$W_{11} \approx \frac{a_{11}}{(1 - a_{13}) + a_{13}e^{-(t-t_0)/c_h}}, \quad (56)$$

where  $c_h = 16\pi^2/(2ch_3^2) \approx c'$ .

(iii) If  $f_3^2 \approx h_3^2$ , then  $c_f \approx c_h \approx 2c'$ , and

$$W_{13} \approx \frac{a_{13}(1 - a_{33})}{\frac{a_{13}K}{1-L} + (1 - a_{13} - a_{33})K}, \quad (57)$$

$$W_{31} \approx \frac{a_{31}(1 - a_{33})}{\frac{a_{31}K}{1-L} + (1 - a_{31} - a_{33})K}, \quad (58)$$

$$W_{11} \approx a_{11}(1 - a_{33})[-1 + 2a_{13} + a_{33} - a_{13}^2L]^{-1/2}[-1 + 2a_{31} + a_{33} - a_{31}^2L]^{-1/2} \cdot \left[ \frac{a_{13}(K+1) - (1 - a_{33})}{a_{13}(K-1) + (1 - a_{33})} \right]^{1/2} \cdot \left[ \frac{a_{31}(K+1) - (1 - a_{33})}{a_{31}(K-1) + (1 - a_{33})} \right]^{1/2}, \quad (59)$$

where  $L = 1 - \exp[-(t-t_0)/c']$  and  $K = \sqrt{1-L + a_{33}L}$ .

For the purpose of illustration, we show a numerical example in Fig. 1, in which the approximate solutions for  $W_{11}$ ,  $W_{13}$ ,  $W_{31}$ , and  $W_{33}$  are compared with the full numerical solutions. It is seen that although  $f_3^2$  and  $h_3^2$  are treated as constants in the approximation, the resultant solutions agree well with the full numerical solutions in which  $f_3^2$  and  $h_3^2$  vary by a factor of 4. Note that due to a lack of details at the high-energy regimes, the chosen input at high energy in this example only leads to  $W_{11} \approx 3/5$  at low energy.

## V. CONCLUSION

One of the cornerstones of quantum field theories is the RGE of coupling ‘‘constants,’’ which describe the change of couplings as functions of energy scales. When applied to gauge couplings, they led to the well-established phenomenon of asymptotic freedom, in addition to the concept of unification, which is a most interesting conjecture for high-energy theories. Given the plethora of masses and mixing parameters, one would hope that RGEs can introduce some regularity, or at least certain insights, into this set of seemingly random observables. However, so far this goal remains largely unfulfilled. One obvious obstacle comes from the complexity of the RGEs, when written in terms of the variables of the standard parametrization. In this paper we obtained evolution equations for a set of rephasing-invariant mixing parameters. They exhibit compact and simple structures, with manifest permutation symmetry. Although a full analysis of these equations is still lacking, they are simple enough for one to find approximate solutions under a number of reasonable assumptions for possible mass parameters. They should be helpful in assessing the viability of proposed theories at high energies. Hopefully, as we learn more about these equations, we can have a clear picture of the relations of Higgs couplings between low and high energies.

## ACKNOWLEDGMENTS

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## APPENDIX

Following the discussions in Sec. IV, in this appendix we collect the explicit RGEs under various assumptions about the quark masses, whether hierarchical or nearly degenerate, when appropriate, we also present approximate solutions for the individual cases.

### 1. Case A): $f_3^2 \gg f_2^2 \gg f_1^2$ and $h_3^2 \gg h_2^2 \gg h_1^2$

In this case, the explicit expressions of  $DW_{ij}$  following Eq. (36) are given by

$$\frac{1}{2c} DW_{11} \approx f_3^2 W_{11} W_{31} + h_3^2 W_{11} W_{13}, \quad (\text{A1})$$

$$\frac{1}{2c} DW_{12} \approx f_3^2 (W_{13} W_{33} - W_{11} W_{31}) + h_3^2 W_{12} W_{13}, \quad (\text{A2})$$

$$\frac{1}{2c} DW_{13} \approx -f_3^2 W_{13} W_{33} - h_3^2 W_{13} (1 - W_{33}), \quad (\text{A3})$$

$$\frac{1}{2c} DW_{21} \approx f_3^2 W_{21} W_{31} + h_3^2 (W_{31} W_{33} - W_{11} W_{13}), \quad (\text{A4})$$

$$\begin{aligned} \frac{1}{2c} DW_{22} \approx & -f_3^2 (W_{21} W_{31} - W_{23} W_{33}) \\ & - h_3^2 (W_{12} W_{13} - W_{32} W_{33}), \end{aligned} \quad (\text{A5})$$

$$\frac{1}{2c} DW_{23} \approx -f_3^2 W_{23} W_{33} - h_3^2 W_{23} (W_{33} - W_{13}), \quad (\text{A6})$$

$$\frac{1}{2c} DW_{31} \approx -f_3^2 W_{31} (1 - W_{31}) - h_3^2 W_{31} W_{33}, \quad (\text{A7})$$

$$\frac{1}{2c} DW_{32} \approx -f_3^2 W_{32} (W_{33} - W_{31}) - h_3^2 W_{32} W_{33}, \quad (\text{A8})$$

$$\frac{1}{2c} DW_{33} \approx f_3^2 W_{33} (1 - W_{33}) + h_3^2 W_{33} (1 - W_{33}). \quad (\text{A9})$$

Here, use has been made of the identities such as  $\Lambda_{11} + \Lambda_{12} = -W_{23} W_{33}$ , etc. Also, it can be verified that  $\sum_a DW_{ai} = \sum_i DW_{ai} = 0$ .

### 2. Case B): $f_3^2 \gg f_2^2 \approx f_1^2$ and $h_3^2 \gg h_2^2 \approx h_1^2$

In this case,  $[F_{23}, F_{31}, F_{12}] \approx (2f_2^2/\epsilon_f)[0, 0, -1]$  and  $[H_{23}, H_{31}, H_{12}] \approx (2h_2^2/\epsilon_h)[0, 0, -1]$ , where  $\epsilon_f = f_2^2 - f_1^2$  and  $\epsilon_h = h_2^2 - h_1^2$ . In addition,

$$[\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2] \approx f_3^2[-1, 1, 0], \quad (\text{A10})$$

$$[\Delta h_{23}^2, \Delta h_{31}^2, \Delta h_{12}^2] \approx h_3^2[-1, 1, 0]. \quad (\text{A11})$$

The general expression for  $DW_{ij}$  becomes

$$\frac{1}{2c} DW_{ij} = -\eta(S_{ij}^{13} + S_{ij}^{23}) + \eta'(S_{ij}^{23}), \quad (\text{A12})$$

with  $S_{ij}^{pq}$  the  $(p, q)$  element of  $S_{ij}$ ,  $\eta = 2f_3^2 h_2^2 / \epsilon_h$ , and  $\eta' = 2h_3^2 f_2^2 / \epsilon_f$ . Their explicit forms are given by

$$\frac{1}{2c} DW_{11} \approx -(\eta \Lambda_{23} + \eta' \Lambda_{32}), \quad (\text{A13})$$

$$\frac{1}{2c} DW_{12} \approx \eta \Lambda_{23} - \eta' \Lambda_{31}, \quad (\text{A14})$$

$$\frac{1}{2c} \mathcal{D}W_{13} \approx -\eta' W_{13} W_{23}, \quad (\text{A15})$$

$$\frac{1}{2c} \mathcal{D}W_{21} \approx -\eta \Lambda_{13} + \eta' \Lambda_{32}, \quad (\text{A16})$$

$$\frac{1}{2c} \mathcal{D}W_{22} \approx \eta \Lambda_{13} + \eta' \Lambda_{31}, \quad (\text{A17})$$

$$\frac{1}{2c} \mathcal{D}W_{23} \approx \eta' W_{13} W_{23}, \quad (\text{A18})$$

$$\frac{1}{2c} \mathcal{D}W_{31} \approx -\eta W_{31} W_{32}, \quad (\text{A19})$$

$$\frac{1}{2c} \mathcal{D}W_{32} \approx \eta W_{31} W_{32}, \quad (\text{A20})$$

$$\frac{1}{2c} \mathcal{D}W_{33} \approx 0. \quad (\text{A21})$$

It is seen that  $\mathcal{D}(W_{13} + W_{23}) \approx 0$ ,  $\mathcal{D}(W_{31} + W_{32}) \approx 0$ ,  $W_{33} \approx \text{constant}$ , and  $W_{11} + W_{12} + W_{21} + W_{22} \approx \text{constant}$ .

With the immediate solution for  $\mathcal{D}W_{33}$ ,

$$W_{33} \approx a_{33}, \quad (\text{A22})$$

and the condition  $W_{13} + W_{23} = W_{31} + W_{32} = 1 - a_{33}$ , we obtain the following:

$$W_{13} \approx \frac{1 - a_{33}}{1 - \left(1 - \frac{1 - a_{33}}{a_{13}}\right) e^{(1 - a_{33})(t - t_0)/a_{\eta'}}}, \quad (\text{A23})$$

$$W_{23} \approx \frac{1 - a_{33}}{1 - \left(1 - \frac{1 - a_{33}}{a_{23}}\right) e^{-(1 - a_{33})(t - t_0)/a_{\eta'}}}, \quad (\text{A24})$$

$$W_{31} \approx \frac{1 - a_{33}}{1 - \left(1 - \frac{1 - a_{33}}{a_{31}}\right) e^{(1 - a_{33})(t - t_0)/a_{\eta}}}, \quad (\text{A25})$$

$$W_{32} \approx \frac{1 - a_{33}}{1 - \left(1 - \frac{1 - a_{33}}{a_{32}}\right) e^{-(1 - a_{33})(t - t_0)/a_{\eta}}}, \quad (\text{A26})$$

where  $a_{\eta} = 16\pi^2/(2c\eta)$  and  $a_{\eta'} = 16\pi^2/(2c\eta')$ .

### 3. Case C): $f_3^2 \gg f_2^2 \gg f_1^2$ and $h_3^2 \gg h_2^2 \approx h_1^2$

In this case,  $[F_{23}, F_{31}, F_{12}] \approx [-1, 1, -1]$  and  $[H_{23}, H_{31}, H_{12}] \approx (2h_2^2/\epsilon_h)[0, 0, -1]$ . In addition,

$$[\Delta f_{23}^2, \Delta f_{31}^2, \Delta f_{12}^2] \approx f_3^2[-1, 1, 0], \quad (\text{A27})$$

$$[\Delta h_{23}^2, \Delta h_{31}^2, \Delta h_{12}^2] \approx h_3^2[-1, 1, 0]. \quad (\text{A28})$$

The general expression for  $\mathcal{D}W_{ij}$  becomes

$$\frac{1}{2c} \mathcal{D}W_{ij} = \eta[-S_{ij}^{13} + S_{ij}^{23}] - h_3^2 \left[ \sum_{p,q \neq 3}^3 S_{ij}^{pq} (-1)^{p+q} \right]. \quad (\text{A29})$$

The explicit expressions are

$$\frac{1}{2c} \mathcal{D}W_{11} \approx -\eta \Lambda_{23} + h_3^2 W_{11} W_{13}, \quad (\text{A30})$$

$$\frac{1}{2c} \mathcal{D}W_{12} \approx \eta \Lambda_{23} + h_3^2 W_{12} W_{13}, \quad (\text{A31})$$

$$\frac{1}{2c} \mathcal{D}W_{13} \approx -h_3^2 W_{13}(1 - W_{13}), \quad (\text{A32})$$

$$\frac{1}{2c} \mathcal{D}W_{21} \approx -\eta \Lambda_{13} + h_3^2 (W_{31} W_{33} - W_{11} W_{13}), \quad (\text{A33})$$

$$\frac{1}{2c} \mathcal{D}W_{22} \approx \eta \Lambda_{13} + h_3^2 (W_{32} W_{33} - W_{12} W_{13}), \quad (\text{A34})$$

$$\frac{1}{2c} \mathcal{D}W_{23} \approx h_3^2 W_{23} (W_{13} - W_{33}), \quad (\text{A35})$$

$$\frac{1}{2c} \mathcal{D}W_{31} \approx -\eta W_{31} W_{32} - h_3^2 W_{31} W_{33}, \quad (\text{A36})$$

$$\frac{1}{2c} \mathcal{D}W_{32} \approx \eta W_{31} W_{32} - h_3^2 W_{32} W_{33}, \quad (\text{A37})$$

$$\frac{1}{2c} \mathcal{D}W_{33} \approx h_3^2 W_{33} (1 - W_{33}). \quad (\text{A38})$$

The approximate solutions of  $W_{33}$ ,  $W_{13}$ , and  $W_{23}$  are given by

$$W_{33} \approx \frac{1}{1 + (a_{33}^{-1} - 1) e^{-(t - t_0)/c_h}}, \quad (\text{A39})$$

$$W_{13} \approx \frac{1}{1 + (a_{13}^{-1} - 1) e^{(t - t_0)/c_h}}, \quad (\text{A40})$$

$$W_{23} \approx \frac{a_{23}}{[(1 - a_{13}) + a_{13} e^{-(t - t_0)/c_h}][(1 - a_{33}) + a_{33} e^{(t - t_0)/c_h}]}, \quad (\text{A41})$$

where  $c_h = 16\pi^2/(2ch_3^2)$ . A special case when  $\eta = 2f_3^2 h_2^2/\epsilon_h \ll h_3^2$ , leads to

$$W_{11} \approx \frac{a_{11}}{(1 - a_{13}) + a_{13} e^{-(t - t_0)/c_h}}, \quad (\text{A42})$$

$$W_{12} \approx \frac{a_{12}}{(1 - a_{13}) + a_{13} e^{-(t - t_0)/c_h}}, \quad (\text{A43})$$

$$W_{31} \approx \frac{a_{31}}{(1 - a_{33}) + a_{33} e^{(t - t_0)/c_h}}, \quad (\text{A44})$$



$$W_{32} \simeq \frac{a_{32}}{(1 - a_{33}) + a_{33}e^{(t-t_0)/c_h}}. \quad (\text{A45})$$

The RGEs and their solutions for the case of  $f_3^2 \gg f_2^2 \approx f_1^2$  and  $h_3^2 \gg h_2^2 \gg h_1^2$  can be obtained from that for case C) by replacing  $f \leftrightarrow h$ . One notes that in the literature, there exist solutions for the RGEs under different approximate schemes; see, e.g., Refs. [14,15].

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