ℭ

F-theory and the classification of little strings

Lakshya Bhardwaj,[1,*] Michele Del Zotto,[2,†] Jonathan J. Heckman,[3,4,5,‡] David R. Morrison,[6,7,§]
Tom Rudelius,[2,¶] and Cumrun Vafa[2,‖]

[1]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*
[2]*Jefferson Physical Laboratory, Harvard University, Cambridge, Massachusetts 02138, USA*
[3]*Department of Physics, University of North Carolina, Chapel Hill, North Carolina 27599, USA*
[4]*Department of Physics, Columbia University, New York, New York 10027, USA*
[5]*CUNY Graduate Center, Initiative for the Theoretical Sciences, New York, New York 10016, USA*
[6]*Department of Mathematics, University of California, Santa Barbara, California 93106, USA*
[7]*Department of Physics, University of California, Santa Barbara, California 93106, USA*

Little string theories (LSTs) are UV complete nonlocal six-dimensional (6D) theories decoupled from gravity in which there is an intrinsic string scale. In this paper, we present a systematic approach to the construction of supersymmetric LSTs via the geometric phases of F-theory. Our central result is that all LSTs with more than one tensor multiplet are obtained by a mild extension of 6D superconformal field theories in which the theory is supplemented by an additional, nondynamical tensor multiplet, analogous to adding an affine node to an ADE quiver, resulting in a negative semidefinite Dirac pairing. We also show that all 6D superconformal field theories naturally embed in a LST. Motivated by physical considerations, we show that in geometries where we can verify the presence of two elliptic fibrations, exchanging the roles of these fibrations amounts to T-duality in the 6D theory compactified on a circle.

I. INTRODUCTION

One of the concrete outcomes from the postduality era of string theory is the wealth of insights it provides into strongly coupled quantum systems. In the context of string compactification, this has been used to argue, for example, for the existence of novel interacting conformal field theories in spacetime dimensions $D > 4$. In a suitable gravity-decoupling limit, the nonlocal ingredients of a theory of extended objects such as strings are instead captured by a quantum field theory with a local stress energy tensor.

String theory also predicts the existence of novel nonlocal theories. Our focus in this work will be on six-dimensional (6D) theories known as little string theories (LSTs).[1] For a partial list of LST constructions, see, e.g., Refs. [1–7]. In these systems, 6D gravity is decoupled, but an intrinsic string scale M_{string} remains. At energies far below M_{string}, we have an effective theory which is well approximated by the standard rules of quantum field

theory with a high scale cutoff. However, this local characterization breaks down as we reach the scale M_{string}. The UV completion, however, is not a quantum field theory.[2]

The mere existence of little string theories leads to a tractable setting for studying many of the essential features of string theory—-such as the presence of extended objects—but with fewer complications (such as coupling to quantum gravity). It also raises important conceptual questions connected with the UV completion of low energy quantum field theory. For example, in known constructions, these theories exhibit T-duality upon toroidal compactification [4,5,9] and a Hagedorn density of states [10], properties which are typical of closed string theories with tension set by M_{string}^2.

Several families of LSTs have been engineered in the context of superstring theory by using various combinations of branes probing geometric singularities. The main idea in many of these constructions is to take a gravity-decoupling limit where the 6D Planck scale $M_{pl} \to \infty$ and the string coupling $g_s \to 0$, but with an effective string scale M_{string} held fixed. Even so, an overarching picture of how to construct (and study) LSTs has remained somewhat elusive.

Our aim in this work is to give a systematic approach for realizing LSTs via F-theory and to explore its

---

[*]lbhardwaj@perimeterinstitute.ca
[†]delzotto@physics.harvard.edu
[‡]jheckman@email.unc.edu
[§]drm@physics.ucsb.edu
[¶]rudelius@physics.harvard.edu
[‖]vafa@physics.harvard.edu
[1]We leave open the question of whether nonsupersymmetric little string theories exist.

[2]In fact, all known properties of LSTs are compatible with the axioms for quasilocal quantum field theories [8].

consequences.[3] To do this, we will use both a bottom up characterization of little string theories on the tensor branch (i.e., where all effective strings have picked up a tension) as well as a formulation in terms of compactifications of F-theory. To demonstrate UV completeness of the resulting models, we will indeed need to use the F-theory characterization.

Recall that in F-theory, we have a noncompact base B of complex dimension 2, which is supplemented by an elliptic fibration to reach a noncompact Calabi-Yau threefold. In the resolved phase, the intersection pairing of the base coincides with the Dirac pairing for 2-form potentials of the theory on its tensor branch. For an superconformal field theory (SCFT), we demand that the Dirac pairing is negative definite. For a LST, we instead require that this pairing is negative semidefinite; i.e., we allow for a non-trivial null space.

F-theory also imposes the condition that we can supplement this base by an appropriate elliptic fibration to reach a noncompact Calabi-Yau threefold. In field theory terms, this is usually enforced by the condition that all gauge theoretic anomalies are cancelled on the tensor branch of the theory. For 6D gauge theories which complete to LSTs, this condition was discussed in detail in Ref. [14]. Even when no gauge theory interpretation is available, this means that in the theory on the tensor branch, some linear combination of tensor multiplets is nondynamical and instead defines a dimensionful parameter (effectively a UV cutoff) for the 6D effective field theory.

In F-theory terms, classifying LSTs thus amounts to determining all possible elliptic Calabi-Yau threefolds which support a base B with negative semidefinite intersection pairing. One of our results is that all LSTs are given by a small extension of 6D SCFTs; i.e. they can always be obtained by adding just one more curve to the base of an SCFT so that the resulting base has an intersection pairing with a null direction. Put in field theory terms, we find that the string charge lattice of any LST with more than one tensor multiplet is an affine extension of the string charge lattice of an SCFT, with the minimal imaginary root of the lattice corresponding to the little string charge. Hence, much as in the case of Lie algebras, all LSTs arise from an affine extension of SCFTs. See Fig. 1 for a depiction of this process.

In fact, the related classification of 6D SCFTs has already been successfully carried out. See, e.g., the partial list of Refs. [15–21]. What this means is that we can freely borrow this structure to establish a classification of LSTs. Much as in Ref. [20], we establish a similar "atomic

classification" of how LSTs are built up from smaller constituent elements. We find that the base of an F-theory geometry is organized according to a single spine of "nodes" which are decorated by possible radicals, i.e., links which attach to these nodes. As opposed to the case of SCFTs, however, the topology of a LST can be either a tree or a loop.

Using this characterization of LSTs, we also show that all 6D SCFTs can be embedded in some LST by including additional curves and 7-branes:

$$6D\,SCFTs \rightarrow 6D\,LSTs. \tag{1.1}$$

Deformations in both Kähler and complex structure moduli for the LST then take us back to the original SCFT. It is curious to note that, although many 6D SCFTs cannot be coupled to 6D supergravity, they *can* always be embedded in another theory with an intrinsic length scale.

A hallmark of all known LSTs is T-duality; that is, by compactifying on a small circle,[4] we reach another 6D LST compactified on a circle of large radius. This motivates a physical conjecture that all LSTs exhibit such a T-duality. In geometries where we can verify the presence of two elliptic fibrations, we find that exchanging the roles of these fibrations amounts to T-duality in the 6D theory compactified on a circle.[5] In some cases, we find that T-duality takes us to the *same* LST. For a recent application of this double elliptic fibration structure in the study of the correspondence between instantons and monopoles via compactifications of little string theory, see Ref. [23].

The rest of this paper is organized as follows. In Sec. II, we state necessary bottom up conditions to realize a LST. This includes the core condition that the Dirac pairing for a LST is a negative semidefinite matrix. After establishing some of the conditions this enforces, we then turn in Sec. III to the rules for constructing LSTs in F-theory. We also explain the (small) differences between the rules for constructing LSTs vs SCFTs. Section IV gives some
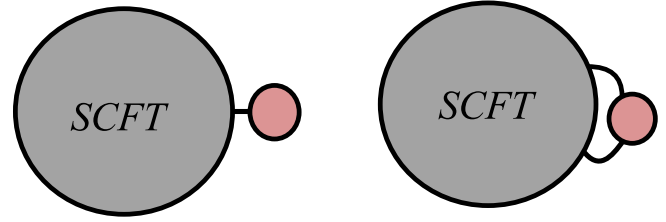

FIG. 1. Depiction of how to construct the base of an F-theory model for a LST. All LST bases are obtained by adding one additional curve to the base for a 6D SCFT. This additional curve can intersect either one or two curves of the SCFT base. Much as in the study of Lie algebras, LSTs should be viewed as an "affine extension" of SCFTs.

---

[3]More precisely, we focus on the case of geometric phases of F-theory, ignoring the (small) list of possible models with "frozen" singularities (see, e.g., Refs. [11–13]). We return to this point later in Sec. X when we discuss the mismatches between field theory motivated LST constructions and their possible lifts to string constructions.

---

[4]That is, small when compared with the effective string scale.
[5]This has been independently observed by Daniel Park [22].

examples of known constructions of LSTs and their embedding in F-theory. In Sec. V, we show how decoupling a tensor multiplet to reach an SCFT leads to strong constraints on possible F-theory models. In Sec. VI, we present an atomic classification of bases, and in Sec. VII, we turn to the classification of possible elliptic fibrations over a given base. In Sec. VIII, we demonstrate that every 6D SCFT constructed in F-theory can be embedded into at least one 6D LST constructed in F-theory. In Sec. IX, we show how T-duality of the LST shows up as the existence of a double elliptic fibration structure and the exchange in the roles of the elliptic fibers. As a consequence, we show that LSTs can acquire discrete gauge symmetries for particular values of their moduli. In Sec. X, we discuss the small mismatch with possible LST constructions suggested by field theory and their potential embedding in a nongeometric phase of an F-theory model. Section XI contains our conclusions, and some additional technical material is deferred to a set of Appendixes.

II. LSTS FROM THE BOTTOM UP

In this section, we state some of the conditions necessary to realize a supersymmetric little string theory.

We consider 6D supersymmetric theories which admit a tensor branch (which can be zero dimensional, as will be the case for many LSTs), that is, we will have a theory with some dynamical tensor multiplets, and vacua parametrized (at low energies) by vacuum expectation values (vevs) of scalars in these tensor multiplets. We will tune the vevs of the dynamical scalars to zero to reach a point of strong coupling. Our aim will be to seek out theories in which this region of strong coupling is not described by an SCFT but rather by a LST. In addition to dynamical tensor multiplets, we will allow the possibility of nondynamical tensor multiplets which set mass scales for the 6D supersymmetric theory.

Recall that in a theory with T tensor multiplets, we have scalars S^I and their bosonic superpartners $B_{\mu\nu}^{-,I}$, with anti-self-dual field strengths. The vevs of the S^I govern, for example, the tension of the effective strings which couple to these 2-form potentials. In a theory with gravity, one must also include an additional 2-form potential $B_{\mu\nu}^+$ coming from the graviton multiplet. Given this collection of 2-form potentials, we get a lattice of string charges Λ_{string} and a Dirac pairing,[6]

$$\Lambda_{\text{string}} \times \Lambda_{\text{string}} \to \mathbb{Z}, \tag{2.1}$$

in which we allow for the possibility that there may be a mull space for this pairing. It is convenient to describe the pairing in terms of a matrix A in which all signs have been reversed. Thus, we can write the signature of A as (p, q, r)

---

[6]Here, we ignore possible torsional contributions to the pairing.

for q self-dual field strengths, p anti-self-dual field strengths, and r the dimension of the null space.

Now, in a 6D theory with q self-dual field strengths and p anti-self-dual field strengths, the signature of A is $(p, q, 0)$. For a 6D supergravity theory with T tensor multiplets, the signature is $(T, 1, 0)$. In fact, even more is true in a 6D theory of gravity: diffeomorphism invariance enforces the condition found in Ref. [24] that $\det A = -1$.

Now, since we are interested in supersymmetric theories decoupled from gravity, we arrive at the *necessary* condition that the signature of A is $(p, 0, r)$. In this special case, each of our 2-form potentials has a real scalar superpartner, which we denote as S^I. The kinetic term for these scalars is

$$\mathcal{L}_{\text{eff}} \supset A_{IJ} \partial S^I \partial S^J. \tag{2.2}$$

Observe that if A has a zero eigenvector, some linear combinations of the scalars will have a trivial kinetic term. When this occurs, these tensor multiplets define parameters of the effective theory on the tensor branch (i.e., they are nondynamical fields).

This leaves us with two general possibilities. Either A is positive definite (i.e., $A > 0$) or it is positive semidefinite (i.e., $A \geq 0$). Recall, however, that to reach a 6D SCFT, a necessary condition is $A0$ [14,15,20,21]. We summarize the various possibilities for self-consistent 6D theories:

	6D SUGRA	6D LST	6D SCFT
Signature	$(T, 1, 0)$	$(p, 0, r)$	$(T, 0, 0)$
$\det A$:	$\det A = -1$	$\det A = 0$	$\det A > 0$

$$\tag{2.3}$$

For now, we have simply indicated a LST as any theory where $\det A = 0$.

As already mentioned, when $\det A = 0$, some linear combinations of the scalar fields for tensor multiplets will have trivial kinetic term. This means that they are better viewed as defining dimensionful parameters. For example, in the case of a 6D theory with a single gauge group factor and no dynamical tensor multiplets, this parameter is just the overall value $S_{\text{null}} = 1/g_{\text{YM}}^2$, with g_{YM} the Yang-Mills coupling of a gauge theory. Indeed, this Yang-Mills theory contains solitonic solutions which we can identify with strings,

$$F = - *_4 F; \tag{2.4}$$

that is, we dualize in the four directions transverse to an effective string. More generally, we can expect A to contain some general null space, and with each null direction, a nondynamical tensor multiplet of parameters,

$$\vec{v}_{\text{null}} \equiv N_1 \vec{v}^1 + \cdots + N_T \vec{v}^T \text{ such that } A \cdot \vec{v}_{\text{null}} = 0. \tag{2.5}$$

for the 2-form potential and

$$S_{\text{null}} = N_1 S^1 + \cdots + N_T S^T \tag{2.6}$$

for the corresponding linear combination of scalars. Since they specify dimensionful parameters, we get an associated mass scale, which we refer to as M_{string}:

$$S_{\text{null}} = M_{\text{string}}^2. \tag{2.7}$$

Returning to our example from 6D gauge theory, the tension of the solitonic string in Eq. (2.4) is just $1/g_{\text{YM}}^2 = M_{\text{string}}^2$. At energies above M_{string}, our effective field theory is no longer valid, and we must provide a UV completion.

On general grounds, $A \geq 0$ could have many null directions. However, in the case where we have a single interacting theory, i.e., when A is simple, there are further strong restrictions. As explained in Ref. [25], when $A \geq 0$ is simple, all of its minors are positive definite: $A_{\text{minor}} > 0$. Consequently, there is precisely *one* zero eigenvalue, and the eigenvector is a positive linear combination of basis vectors. Consequently, there is only one dimensionful parameter M_{string}. This also means that if we delete any tensor multiplet, we reach a positive definite intersection pairing, and consequently a 6D SCFT. What we have just learned is that if we work in the subspace orthogonal to the ray swept out by S_{null}, then the remaining scalars can all be collapsed to the origin of moduli space. When we do this, we reach the LST limit.

We shall refer to this property of the matrix A as the "tensor-decoupling criterion" for a LST. As we show in subsequent sections, the fact that decoupling any tensor multiplet takes us to an SCFT imposes sharp restrictions.

Even so, our discussion has up to now focussed on some necessary conditions to reach a UV complete theory different from a 6D SCFT. In Refs. [14,21], the specific case of 6D supersymmetric gauge theories was considered, and closely related consistency conditions for UV completing to a LST were presented. Here, we see the same consistency condition $A \geq 0$ appearing for *any* effective theory with (possibly nondynamical) tensor multiplets.

Indeed, simply specifying the tensor multiplet content provides an incomplete characterization of the tensor branch. In addition to this, we will also have vector multiplets and hypermultiplets. For theories with only eight real supercharges, anomaly cancellation often imposes tight consistency conditions.

There is, however, an important difference in the way anomaly cancellation operates in a 6D SCFT compared with a 6D LST. The crucial point is that because A has a zero eigenvalue, there is a nondynamical tensor multiplet which does not participate in the Green-Schwarz mechanism. In other words, on the tensor branch of a LST with T tensor multiplets, at most only $T - 1$ participate. This is not particularly worrisome since, as explained in Ref. [24] and further explored in Ref. [26], there is in general a difference

between the tensor multiplets which participate in anomaly cancellation and those which appear on the tensor branch of a general 6D theory.

Though we have given a number of necessary conditions that any putative LST must satisfy, to truly demonstrate their existence, we must pass beyond effective field theory, embedding these theories in a UV complete framework such as string theory. We therefore now turn to the F-theory realization of little string theories.

III. LSTS FROM F-THEORY

In this section, we spell out the geometric conditions necessary to realize LSTs in F-theory. Recall that in a little string theory, we are dealing with a 6D theory which contains strings with finite tension. As such, they are an intermediate case between the case of a 6D superconformal field theory (which only contains tensionless strings) and the full string theory (i.e., one in which gravity is dynamical).

Any supersymmetric F-theory compactification to six dimensions is defined by an elliptically fibered Calabi-Yau threefold $X \to B$. Here, X is the total space, and B is the base. The elliptic fibration can be described by a local Weierstrass model,

$$y^2 = x^3 + fx + g, \tag{3.1}$$

where f and g are local functions on B, that globally are sections, respectively, of $\mathcal{O}_B(-4K_B)$ and $\mathcal{O}_B(-6K_B)$, K_B being the canonical class of B. The discriminant of the elliptic fibration is

$$\Delta \equiv 4f^3 + 27g^2, \tag{3.2}$$

which globally is a section of $\mathcal{O}_B(-12K_B)$. The discriminant locus $\Delta = 0$ is a divisor, and its irreducible components tell us the locations of degenerations of elliptic fibers. Such singularities determine monodromies for the complex structure parameter τ of the elliptic fiber, which is interpreted in type IIB string theory as the axiodilaton field. In type IIB language, the discriminant locus signals the location of 7-branes in the F-theory model.

In F-theory, decoupling gravity means we will always be dealing with a noncompact base B. When all curves of B are of finite nonzero size, we get a 6D effective theory with a lattice of string charges:

$$\Lambda_{\text{string}} = H_2^{\text{cpct}}(B, \mathbb{Z}). \tag{3.3}$$

The intersection form defines a canonical pairing,

$$A_{\text{intersect}} : \Lambda_{\text{string}} \times \Lambda_{\text{string}} \to \mathbb{Z}, \tag{3.4}$$

which we identify with the Dirac pairing:

$$A_{\text{Dirac}} = A_{\text{intersect}}. \qquad (3.5)$$

We also introduce the "adjacency matrix"

$$A_{\text{adjacency}} = -A_{\text{Dirac}}. \qquad (3.6)$$

To streamline the notation, we shall simply denote the adjacency matrix as A. The 2-form potentials of the 6D theory arise from reduction of the 4-form potential of type IIB string theory. Additionally, the volumes of the various compact two-cycles translate to the real scalars of tensor multiplets:

$$S^I \propto \text{Vol}(\Sigma_I). \qquad (3.7)$$

In the F-theory model, the appearance of a null vector for $A_{\text{intersect}}$ means that some of these moduli are not dynamical in the 6D effective field theory. Rather, they define dimensionful parameters/mass scales. This follows from the Grauert-Artin contractibility criterion in algebraic geometry [27,28], which states that any given curve in a complex surface is contractible if and only if the intersection matrix of its irreducible components is negative definite. This simple geometrical criterion gives a necessary condition ($A > 0$) for engineering SCFTs and implies that any null eigenvalues of A correspond to noncontractible curves, which thus define intrinsic energy scales.

To define an F-theory model, we need to ensure that there is an elliptic Calabi-Yau X in which B is the base. A necessary condition for realizing the existence of an elliptic model is that the collection of curves entering in a base B are obtained by gluing together the "non-Higgsable clusters" (NHCs) of Ref. [29] via $\mathbb{P}^1$'s of self-intersection -1.

Recall that the non-Higgsable clusters are given by collections of up to three $\mathbb{P}^1$'s in which the minimal singular fiber type is dictated by the self-intersection number of the $\mathbb{P}^1$. The self-intersection number and associated gauge symmetry and matter content are as follows:

Self-intersection	-3	-4	-5	-6	-7	-8
Gauge theory	$\mathfrak{su}_3$	$\mathfrak{so}_8$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7 \oplus \frac{1}{2}56$	$\mathfrak{e}_7$

$$(3.8)$$

Self-intersection	-9	-10	-11	-12
Gauge theory	$\mathfrak{e}_8 \oplus 3\text{inst}$	$\mathfrak{e}_8 \oplus 2\text{inst}$	$\mathfrak{e}_8 \oplus 1\text{inst}$	$\mathfrak{e}_8$

$$(3.9)$$

Self-intersection	$-3, -2$	$-2, -3, -2$	$-3, -2, -2$
Gauge theory	$\mathfrak{g}_2 \times \mathfrak{su}_2 \oplus \frac{1}{2}(7+1,2)$	$\mathfrak{su}_2 \times \mathfrak{so}_7 \times \mathfrak{su}_2 \oplus \frac{1}{2}(2,8,1) \oplus \frac{1}{2}(1,8,2)$	$\mathfrak{g}_2 \times \mathfrak{sp}_1 \oplus \frac{1}{2}(7,2) \oplus \frac{1}{2}(1,2)$

$$(3.10)$$

In addition, we can also consider a single -1 curve and configurations of -2 curves arranged either in an ADE Dynkin diagram or its affine extension (in the case of little string theories). The local rules for building up an F-theory base compatible with these NHCs amount to a local gauging condition on the flavor symmetries of a -1 curve: We scan over product subalgebras of the $\mathfrak{e}_8$ flavor symmetry which are also represented by the minimal fiber types of the NHCs. When they exist, we get to "glue" these NHCs together via a -1 curve.

For a general elliptic Calabi-Yau threefold, the curves appearing in a given gluing configuration can lead to rather intricate intersection patterns. For example, two curves may intersect more than once and may therefore form either a closed loop or an intersection with some tangency. Additionally, we may have three curves all meeting at the same point, as in the case of the type IV Kodaira fiber. Finally, a single -1 curve may in general intersect more then just two curves. The possible ways to locally glue together such NHCs has also been worked out explicitly in Ref. [29] (see also Ref. [30]). The main idea, however, is that, since the -1 curve theory defines a 6D SCFT with E_8 flavor symmetry, we must perform a gluing compatible with gauging some product subalgebra of the Lie algebra $\mathfrak{e}_8$.

What this means in general is that the adjacency matrix provides only a partial characterization of intersecting curves in the base of a geometry. To handle these different possibilities, we therefore introduce the following notation:

$$\text{Normal intersection}: \quad a, b \quad \text{or} \quad ab \qquad (3.11)$$

$$\text{Tangent intersection}: \quad a\|b \qquad (3.12)$$

$$\text{Triple intersection}: \quad a\overset{b}{\triangledown}c \qquad (3.13)$$

$$\text{Looplike configuration}: \quad //a_1 \cdots a_k//. \qquad (3.14)$$

Now, decoupling gravity to reach an SCFT or a LST leads to significant restrictions on the possible ways to glue together NHCs. In the case of a 6D SCFT, contractibility of

all curves in the base means first that all of the compact curves are $\mathbb{P}^1$'s and further that a -1 curve can intersect at most two other curves. Additionally, all off-diagonal entries of the intersection pairing are either zero or 1. In the case of LSTs, however, the curves of the base could include a T^2, and a -1 curve can potentially intersect more than two curves. Additionally, there is also the possibility that the off-diagonal entries of the adjacency matrix may be different than just zero or 1.

Again, we stress that the intersection pairing provides only *partial* information. For example, a curve of self-intersection zero could refer either to a $\mathbb{P}^1$ or to a T^2. In the case of a T^2 of self-intersection zero, the normal bundle need not be trivial but could be a torsion line bundle instead. Additionally, an off-diagonal entry in the adjacency matrix which is 2 may either refer to a pair of curves which intersect twice or to a single intersection of higher tangency. The case of tangent intersections violates the condition of normal crossing (which is known to hold for SCFTs [15] but fails for LSTs). An additional type of normal crossing violation appears when we blow down a -1 curve meeting more than two curves. In Appendix B, we determine the types of matter localized when there are violations of normal crossing.

A. Geometry of the gravity-decoupling limit

We now discuss how to obtain limits of F-theory compactifications in which gravity is decoupled, following a program initiated in Ref. [31], worked out in detail in Ref. [32] (see also Ref. [33]), and extended to the case of 6D SCFTs in Ref. [34]. For this purpose, we consider F-theory from the perspective of the type IIB string, with the volume of the base B of the F-theory compactification providing a Planck scale for the compactified theory. We will see that the quest for decoupled gravity leads to the same condition on semidefiniteness of the intersection matrix of the compact curves, and moreover we will see how to ensure that the F-theory base B in such cases has a metric of the appropriate kind.

1. Case of compact base

We begin with the case in which the F-theory base B is a compact surface and suppose we have a sequence of metrics (specified by their Kähler forms ω_i) which decouple gravity in the limit $i \to \infty$. In particular, the volume must go to infinity: $\lim_{i\to\infty} \mathrm{vol}(\omega_i) = \infty$.

To investigate the geometry of this family of metrics, we temporarily rescale them and consider the Kähler forms

$$\tilde{\omega}_i := \frac{\omega_i}{\sqrt{\mathrm{vol}(\omega_i)}}. \tag{3.15}$$

The rescaled metrics all have volume 1, and since the closure of the set of volume 1 Kähler classes on B is compact, there must be a convergenct subsequence of Kähler classes $[\tilde{\omega}_{i_j}]$ of which the limit

$$[\tilde{\omega}_\infty] = \lim_{j\to\infty}[\tilde{\omega}_{i_j}] \tag{3.16}$$

lies in the closure of the Kähler cone. If the original sequence was chosen generically, the limit of the rescaled sequence will be an interior point of the Kähler cone, and in this case, all areas and volumes grow uniformly as we take the limit of the original sequence ω_i. Gravity decouples, but all other physical quantities measured by areas and volumes approach either zero or infinity, leaving us with a trivial theory.

However, if the rescaled limit (3.16) lies on the boundary of the Kähler cone, more interesting things can happen. In favorable circumstances, such as those present in Mori's cone theorem [35] and its generalizations [36], we can form another complex space $\bar{B}$ out of B by identifying pairs of points p and q whenever they are both contained in a curve C of which the area vanishes in the limit. There is a holomorphic map $\pi: B \to \bar{B}$ for which all such curves of zero limiting area are contained in fibers $\pi^{-1}(t)$, $t \in \bar{B}$.

As already pointed out in Ref. [32], there are two qualitatively different cases: $\bar{B}$ might be a surface or it might be a curve. (It is not possible for $\bar{B}$ to be a point since there are some curves $C \subset B$ of which the area does not vanish in the limit.) If $\bar{B}$ is a surface, then the map $\pi: B \to \bar{B}$ contracts some curves to points and may create singularities in $\bar{B}$. It is widely believed, and has been mathematically proven under certain hypotheses [37,38], that the limiting metric $\tilde{\omega}_\infty$ can be interpreted as a metric $\omega_{\bar{B}}$ on the smooth part of $\bar{B}$.

On the other hand, if $\bar{B}$ is a curve, so that π has curves Σ_t, $t \in \bar{B}$ as fibers, then we again expect the limiting metric $\tilde{\omega}_\infty$ to be induced by a metric on $\bar{B}$, although there are fewer mathematical theorems covering this case. (See Ref. [39] for one known theorem of this kind.)

In general, we do not expect the curves contracted by π to necessarily have zero area in the gravity-decoupling limit. This can be achieved by starting with a reference Kähler form ω_0 on B as well as a (possibly degenerate) Kähler form $\omega_{\bar{B}}$ on $\bar{B}$ and constructing a limit of the form

$$\lim_{t\to\infty}(\omega_0 + t\pi^*(\omega_{\bar{B}})). \tag{3.17}$$

In the case of an SCFT, we wish all curves contracted by π to be at zero area in the limit, so in that case we should omit ω_0 and simply scale up $\pi^*(\omega_{\bar{B}})$.

2. Case of noncompact base

Our discussion of the compact bases makes it clear that the decoupling limit only depends on the metric in a (finite volume) neighborhood of a collection of curves on the original F-theory base B, together with a rescaling which takes that neighborhood to infinite volume and smooths out its features in the process. This analysis can be applied to an arbitrary base, compact or noncompact.

If the collection of curves is disconnected, the corresponding points on $\bar{B}$ to which the collection is mapped will be moved infinitely far apart during the rescaling process, thus leading to several decoupled quantum theories. So to study a single theory, it suffices to consider a connected configuration. To reiterate the two cases we have found:

(1) We may have a connected collection of curves Σ_j which can be simultaneously contracted to a singular point on a space $\bar{B}$. (The contractibility implies that the intersection matrix is negative definite.) When the metric on $\bar{B}$ is rescaled, gravity is decoupled, giving a 6D SCFT (in which every curve in the collection remains at zero area).

Alternatively, we can combine this rescaled metric with another reference metric which provides finite area to each Σ_j. This produces a quantum field theory in the Coulomb branch of the SCFT.

(2) Or we may have a connected collection of curves Σ_j which are all contained in a single fiber of a map $\pi : B \to \bar{B}$ and include all components of that fiber. (This implies that the intersection matrix is negative semidefinite, with a one-dimensional zero eigenspace.) We combine a reference metric on B that sets the areas of the individual Σ_j's with a metric on $\bar{B}$ which is rescaled to decouple gravity, yielding a little string theory (with the string provided by a D3-brane wrapping the entire fiber). The overall area of the fibers of π sets the string scale, and the possible areas of the Σ_j's map out the moduli space of the theory. Gravity is decoupled, and we find a LST.

Note that in the second case, there are two distinct possibilities for the fibers of the map π: The general fiber can be a curve of genus 0 or a curve of genus 1. In the case of genus 1, it is possible for the central fiber to have a nontrivial multiplicity; that is, the fiber can take the form $m\Sigma$ for some $m > 1$.

IV. EXAMPLES OF LSTS

In the previous section, we gave the general rules for constructing LSTs. Our plan in this section will be to show how the F-theory realization allows us to recover well-known examples of LSTs previously encountered in the literature.

To this end, we begin by first showing how LSTs with 16 supercharges arise in F-theory constructions. After this, we turn to known constructions of LSTs with eight supercharges (i.e., minimal supersymmetry). This will also serve to illustrate how F-theory provides a single coherent framework for realizing LSTs.

A. Theories with 16 Supercharges

To set the stage, we begin with little string theories with 16 supercharges. In this case, we have two possibilities given by $\mathcal{N} = (2,0)$ supersymmetry or $\mathcal{N} = (1,1)$

supersymmetry. Note that only the former is possible in the context of 6D SCFTs.

One way to generate examples of $\mathcal{N} = (2,0)$ LSTs is to take k M5-branes filling $\mathbb{R}^{5,1}$ and probing the geometry $S^1_\perp \times \mathbb{C}^2$ with $S^1_\perp$ a transverse circle of radius R. To reach the gravity-decoupling limit for a LST, we simultaneously send the radius $R \to 0$ and $M_{pl} \to \infty$ while holding fixed the effective string scale. In this case, it is the effective tension of an M2-brane wrapped over the circle which we need to keep fixed. Performing a reduction along this circle, we indeed reach type IIA string theory with k NS5-branes. By a similar token, we can also consider IIB string theory with k NS5-branes. This realizes LSTs with $\mathcal{N} = (1,1)$ supersymmetry.

T-dualizing the k NS5-branes of type IIA, we obtain type IIB string theory on the local geometry given by a configuration of -2 curves arranged in the affine $\hat{A}_{k-1}$ Dynkin diagram. Similarly, we also get a LST by taking type IIA string theory on the same geometry.

Consider next the F-theory realization of these little string theories. First of all, we reach the aforementioned theories by working with F-theory models of which the associated Calabi-Yau threefold takes the form $T^2 \times S$, in which $S \to C$ is an elliptically fibered (noncompact) Calabi-Yau surface. If we treat the T^2 factor as the elliptic fiber of F-theory, we get IIB on S, and if we treat the elliptic fiber of S as the elliptic fiber of F-theory, we get (after shrinking the T^2 factor to small size) F-theory with base $T^2 \times C$, which is dual to IIA on S. To refer to both cases, it will be helpful to label the auxiliary elliptic curve T^2 as T^2_F (for fiber) and the other elliptic curve as T^2_S (since it lies in the surface S).

Now, by allowing S to develop a singular elliptic fiber, we can realize the same local geometries obtained perturbatively. For example, the $\mathbb{C}^2/\mathbb{Z}_k$ lifts to a Kodaira fiber of type I_k. Resolving this local singularity, we find k compact cycles $\Sigma_I \simeq \mathbb{P}^1$'s which intersect according to the affine $\hat{A}_{k-1}$ Dynkin diagram. In this case, the null divisor class is

$$[\Sigma_{\text{null}}] = [\Sigma_1] + \cdots + [\Sigma_k]; \tag{4.1}$$

that is, it is the ordinary minimal imaginary root of $\hat{A}_{k-1}$. By shrinking T^2_F to small size, this engineers in F-theory the $\mathcal{N} = (2,0)$ LST of k M5-branes or of k NS5-branes in type IIA. [See Fig. 2 for a depiction of the A-type $\mathcal{N} = (2,0)$ LSTs.] In the other case, one obtains F-theory on $T^2 \times C$ of which the fibers have an I_k singularity along $T^2 \times \{0\}$. Then, $[\Sigma_{\text{null}}]$ is precisely the class of the F-theory fiber T^2_S, and supersymmetry enhances to $\mathcal{N} = (1,1)$.

More generally, we can consider any of the degenerations of the elliptic fibration classified by Kodaira, i.e., the type $I_n, II, III, IV, I^*_n, II^*, III^*, IV^*$ fibers, and produce a $\mathcal{N} = (2,0)$ model with that degeneration occurring as a curve configuration on the F-theory base, as well as a
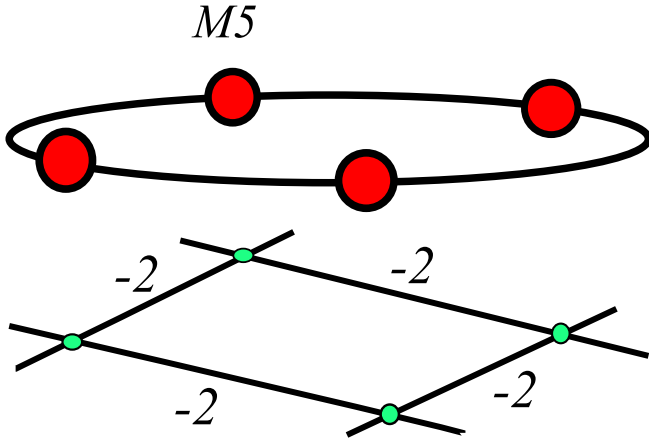
FIG. 2. Depiction of the tensor branch of the $\mathcal{N} = (2,0)$ $\hat{A}_3$ LST. In the top figure, we engineer this example using spacetime filling M5-branes probing the geometry $S^1_\perp \times \mathbb{C}^2$. In the dual F-theory realization, we have four -2 curves in the base, which are arranged as the affine $\hat{A}_3$ Dynkin diagram. The Kähler class of each -2 curve in the F-theory realization corresponds in the M-theory realization to the relative separation between the M5-branes.

$\mathcal{N} = (1,1)$ model with that same degeneration occurring as the F-theory fiber over some T^2 in the base.

As a brief aside, a convenient way to realize examples of both the $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ theories is to consider F-theory on the Schoen Calabi-Yau threefold $dP_9 \times_{\mathbb{P}^1} dP_9$ [40]. Then, we can keep the elliptic fiber on one dP_9 factor generic and allow the other to degenerate. Switching the roles of the two fibers then moves us from the IIA to IIB case. Note that, although this strictly speaking only yields eight real supercharges (as we are on a Calabi-Yau threefold), in the rigid limit used to reach the little string theory, we expect a further enhancement to either $\mathcal{N} = (2,0)$ or $\mathcal{N} = (1,1)$ supersymmetry. The specific chirality of the supersymmetries depends on which elliptic curve we take to be in the base and which to be in the fiber of the corresponding F-theory compactification.

Let us also address whether each of the different Kodaira fiber types leads us to a different little string theory. Indeed, some pairs of Kodaira fiber types lead to identical gauge symmetries in the effective field theory. To illustrate, consider the type IV Kodaira fiber, and compare it with the type I_3 fiber. There is a complex structure deformation which moves the triple intersection appearing in the type IV case out to the more generic-type I_3 case. This modulus, however, is decoupled from the 6D little string theory. The reason is that if we consider a further compactification on a circle, we reach a five-dimensional (5D) gauge theory which is the same for both fiber types. The additional complex structure modulus from deforming IV to I_3 does not couple to any of the modes of the 5D theory. So, there does not appear to be any difference between these theories. In other words, we should classify all of the $\mathcal{N} = (2,0)$

little string theories in terms of affine ADE Dynkin diagrams rather than in terms of Kodaira fiber types.

For the $\mathcal{N} = (1,1)$ little string theories, the absence of a chiral structure actually leads to more possibilities. For example, if we consider M-theory on an ADE singularity compactified on a further circle, we have the option of twisting by an outer automorphism of the simply laced ADE Lie algebra [41]. In other words, for the $\mathcal{N} = (1,1)$ theories, we have an ABCDEFG classification according to all of the simple Lie algebras.

To realize these LSTs in F-theory, we make an orbifold of the previous construction. Suppose that $S \rightarrow C$ is an elliptically fibered (noncompact) Calabi-Yau surface which has compatible actions of $\mathbb{Z}_m$ on the base and on the total space, such that the action on the total space preserves the holomorphic 2-form. Then, $\mathbb{Z}_m$ acts on $T^2_F \times S$ with the action on T^2 being translation by a point of order m. The quotient $X := (T^2_F \times S)/\mathbb{Z}_m$ is then an elliptically fibered Calabi-Yau threefold (with two genus 1 fibrations as before).

The elliptic fibration $X \rightarrow (T^2_F \times C)/\mathbb{Z}_m$ leads to an F-theory model with $\mathcal{N} = (1,1)$ supersymmetry. Note that the base $(T^2_F \times C)/\mathbb{Z}_m$ of the F-theory fibration contains a curve Σ of genus 1 and self-intersection 0 such that $m\Sigma$ can be deformed into a one-parameter family although no smaller multiple can be deformed. Note also that if the action of $\mathbb{Z}_m$ on S preserves the section of the fibration $S \rightarrow C$, then $X \rightarrow (T^2_F \times C)/\mathbb{Z}_m$ also has a section, and as we will explain in Sec. VII A 1, $m \in \{2, 3, 4, 6\}$ since every elliptic fibration with a section has a Weierstrass model [42].

There is a second fibration $X \rightarrow (S/\mathbb{Z}_m)$ which is a genus 1 fibration without a section and leads to theories with $\mathcal{N} = (2,0)$ supersymmetry. We discuss additional details about this second fibration, as well as T-duality for these theories, in Sec. IX.

It is instructive to study the structure of the moduli space of the LSTs with maximal supersymmetry. Recall that the tensor branch for a $\mathcal{N} = (2,0)$ SCFT of ADE type $\mathfrak{g}$ is given by

$$\mathcal{M}_{(2,0)}[\mathfrak{g}] = \mathbb{R}^T_{\geq 0}/\mathcal{W}_\mathfrak{g}, \qquad (4.2)$$

where, in the above, T is the number of tensor multiplets and $\mathcal{W}_\mathfrak{g}$ is the Weyl group of the ADE Lie algebra $\mathfrak{g}$. That is, the moduli space is given by a Weyl chamber of the ADE Lie algebra and is therefore noncompact. In the present case of LSTs, we see that the condition that we have a string scale leads to one further constraint on this moduli space, effectively "compactifying" it to the compact Coxeter box for an affine root lattice [5].

Finally, one of the prominent features of these examples is the manifest appearance of two elliptic fibrations in the geometry. Indeed, in passing from the $\mathcal{N} = (2,0)$ theories to the $\mathcal{N} = (1,1)$ theories, we observe that we have simply

switched the role of the two fibrations. In Sec. IX, we return to this general phenomenon for how T-duality of LSTs is realized in F-theory.

B. Theories with eight supercharges

Several examples of LSTs with minimal, i.e., (1,0) supersymmetry are realized by mild generalizations of the examples reviewed above.

To begin, let us consider again the case of k coincident M5-branes filling $\mathbb{R}^{5,1}$ and probing the geometry $S_{\perp}^1 \times \mathbb{C}^2$. We arrive at a (1,0) LST by instead taking a quotient of the $\mathbb{C}^2$ factor by a nontrivial discrete subgroup $\Gamma_G \subset SU(2)$ so that the geometry probed by the M5-brane is $\mathbb{C}^2/\Gamma_G$. The discrete subgroups admit an ADE classification, and the corresponding simple Lie group G_{ADE} specifies the gauge group factors on the tensor branch. We reach a 6D SCFT by decompactifying the $S_{\perp}^1$. In this limit, we have an emergent $G_L \times G_R$ flavor symmetry. From this perspective, the little string theory arises from gauging a diagonal G subgroup of the flavor symmetry. In the IIB realization of NS5-branes probing the affine geometry, applying S-duality takes us to a stack of D5-branes probing an ADE singularity. On its tensor branch, this leads to an affine quiver gauge theory [43].

The F-theory realization of these LSTs is simply an affine A-type Dynkin diagram of k curves of self-intersection -2 decorated with I_n, I_n^*, IV^*, III^*, II^* fibers, respectively, for $G = A_{n-1}$, D_{n+4}, $E_{6,7,8}$. We reach a 6D SCFT by decompactifying any of the -2 curves in the loop, and we recover a 6D SCFT with an emergent $G_L \times G_R$ flavor symmetry. From this perspective, the little string theory arises from gauging a diagonal G subgroup of the flavor symmetry. Note that for $G \neq A_n$, all these systems involve conformal matter in the sense of Ref. [17].

Another class of LSTs is given by taking k M5-branes in heterotic M-theory, i.e., M-theory on $S^1/\mathbb{Z}_2 \times \mathbb{C}^2$. In this case, we have two E_8 flavor symmetry factors. one for each endpoint of the interval $S^1/\mathbb{Z}_2$. In this case, the gravity-decoupling limit requires us to collapse the size of the interval to zero size (i.e., to reach perturbative heterotic strings), while still holding the effective string scale finite. (The ratios of the lengths of subintervals between the endpoints and the various M5-branes to the length of the total interval will remain finite in the gravity-decoupling limit and provide parameters for the tensor branch.) In perturbative heterotic string theory, we have k NS5-branes probing $\mathbb{C}^2$. A related example is provided by instead working with the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string in the presence of k NS5-branes. Indeed, once suitable Wilson line data have been specified, these two examples are T-dual to one another.

The F-theory realization of the theory of k M5-branes is given by a noncompact base with a configuration of curves,
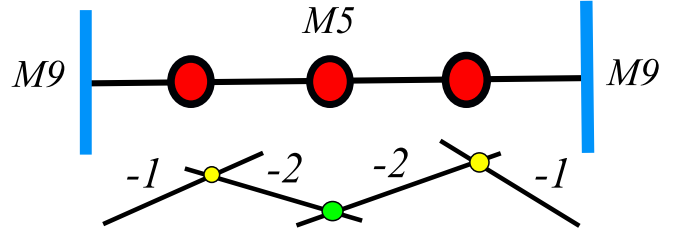

FIG. 3. TOP: Depiction of the LST realized by k M5-branes in between the two Horava-Witten nine-brane walls of heterotic M-theory ($k = 3$ above). This leads to a LST with an $E_8 \times E_8$ flavor symmetry. BOTTOM: The corresponding F-theory base given by the configuration of curves $[E_8], 1, 2, \ldots, 2, 1, [E_8]$ for k total compact curves. In this realization, the E_8 flavor symmetry is localized on two noncompact 7-branes, one intersecting each -1 curve.

$$[E_8]\underbrace{1, 2, \ldots, 2, 1}_{k}[E_8], \qquad (4.3)$$

where we have indicated the flavor symmetry factors in square brackets. In this configuration, we reach the LST limit by holding fixed the volume of the null divisor (given by a sum over each divisor with multiplicity 1) and collapse all other Kähler moduli to zero size. The construction of the T-dual characterization is somewhat more involved, and so we defer a full discussion to Sec. IX and Appendix F. See Fig. 3 for a depiction of the M-theory and F-theory realizations of this LST.

We can also combine the effects of different orbifold group actions. For example, we can consider k M5-branes filling $\mathbb{R}^{5,1}$ and probing the geometry $S^1/\mathbb{Z}_2 \times \mathbb{C}^2/\Gamma_G$. In F-theory terms, this is given by the geometry

$$[E_8]\underbrace{\overset{\mathfrak{g}}{1}, \overset{\mathfrak{g}}{2}, \ldots, \overset{\mathfrak{g}}{2}, \overset{\mathfrak{g}}{1}}_{k}[E_8]; \qquad (4.4)$$

i.e., we decorate by a $\mathfrak{g}$-type ADE gauge symmetry over each curve of self-intersection -1 or -2. This geometry was studied in detail in Ref. [3]. Further blowups in the base are needed for all fibers to remain in Kodaira-Tate form. This leads to conformal matter between each simply laced gauge group factor [17].

Summarizing, we have seen in the above that the various LSTs which have been constructed via perturbative string theory and M-theory all have a natural embedding in the context of specific F-theory constructions. With this in mind, we now turn to a systematic construction of all 6D LSTs in F-theory.

V. CONSTRAINTS FROM TENSOR DECOUPLING

As a first step toward the classification of LSTs, we now show how to classify possible bases using the tensor-decoupling criterion, that is, the requirement that

decoupling any tensor multiplet from a LST must take us to an SCFT. In geometric terms, deleting any curve of the base (with possible fiber enhancements along this curve) must take us back to an SCFT base (with possibly disconnected components). Since all SCFTs have the structure of a treelike graph of intersecting curves [20], our task reduces to scanning over the list of *connected* SCFTs and asking whether adding an additional curve (with possible fiber enhancements on this curve) will produce a LST. This inductive approach to classification will allow us to effectively constrain the overall structure of bases for LSTs.

In this section, we show how the tensor-decoupling criterion constrains many candidate bases for LSTs. We first use this criterion to limit the possible graph topologies of curves in the base. Next, we give a general inductive rule for how to take an SCFT and verify whether it enhances to a LST. We shall refer to this as an inductive classification, since it implicitly accounts for all possible structures for LSTs. In Sec. VI, we use these constraints to present a more explicit construction of possible bases for LSTs.

A. Graph topologies for LSTs

For any compact curve Σ in the base which remains in the gravity-decoupling limit, the self-intersection Σ^2 must be $-n$ for $0 \leq n \leq 12$. Moreover, since having an F-theory model requires that $-4K$, $-6K$, and $-12K$ be effective divisors, if $K \cdot \Sigma + \Sigma^2 > 0$ (so that $K \cdot \Sigma > 0$), then $-4K$ would have multiplicity at least 4 along Σ, $-6K$ would have multiplicity at least 6 along Σ, and $-12K$ would have multiplicity at least 12 along Σ. Since this is not allowed in the Kodaira classification, we conclude that $2g - 2 = K \cdot \Sigma + \Sigma^2 \leq 0$, in other words, that Σ is either $\mathbb{P}^1$ or T^2. We now use the tensor-decoupling criterion to argue that the possible topologies of LST bases are limited to treelike structures and appropriate degenerations of an elliptic curve.

Let us first show that a curve Σ of self-intersection zero (of topology $\mathbb{P}^1$ or T^2) can only appear in isolation; i.e., it cannot intersect any other curve. If it met another curve and we decoupled everything that this curve touches, we would be left with an SCFT base containing a curve of self-intersection zero, a contradiction. If Σ has genus 0, then the base takes the form $\mathbb{C} \times \Sigma$, while if Σ has genus 1, then the base takes the form $(\mathbb{C} \times \Sigma)/\mathbb{Z}_m$, with $\mathbb{Z}_m$ acting on Σ by a translation and on $\mathbb{C}$ by multiplication by a root of unity. Note that if either $g = 0$ or $m = 1$, the base is just a product. Hence, to get a six-dimensional theory, we must wrap 7-branes over Σ, i.e. we *must* include a nontrivial fiber enhancement over this curve, unless $g = 1$ and $m1$. (We will see examples of this latter case in Sec. IX.)

Consider next adjacency matrices in which the off-diagonal entries are different from 0 or 1. For example, this can occur when a -4 and -1 curve form a closed loop (i.e., intersect twice) or when the same curves intersect along a higher order tangency. Again, this possibility is

severely limited because if this were to a occur in a configuration with three or more curves, we would contradict the tensor-decoupling criterion. By the same token, the value of all off-diagonal entries are bounded below by 2,

$$-2 \leq A_{IJ} \leq 0 \quad \text{for } I \neq J, \tag{5.1}$$

and in the case where -2 appears, we are limited to just two curves. The only possibilities for a rank 1 LST base (i.e., with two curves) are therefore

$$1,1 \quad \text{or} \quad //2,2// \quad \text{or} \quad //4,1// \quad \text{or} \quad 2\|2 \quad \text{or} \quad 4\|1. \tag{5.2}$$

In Appendix B, we analyze the possible fiber enhancements which can occur when the two curves meet along a tangency (i.e., do not respect normal crossing), as is the case in the last two configurations.

For all other LST bases, we see that all curves must be constructed from $\mathbb{P}^1$'s of self-intersection $-x$ for $1 \leq x \leq 12$, which all intersect with normal crossings; i.e., all off-diagonal entries of the adjacency matrix are either 0 or 1.

To further constrain the structure, we next observe that the base of any 6D SCFT is always treelike [15]. This means that the graph associated to a LST adjacency matrix can admit at most one loop, and when it contains a loop, there can be no additional curves branching off. This is because the tensor-decoupling criterion would be violated by joining a loop of curves to anything else. We are therefore left with two general types of configurations:

  (i) Treelike LSTs
  (ii) Looplike LSTs.

Note that some of the treelike structures we shall encounter can also be viewed as loops, that is, as degenerations of an elliptic curve.

B. Inductive classification

To proceed further, we now present an inductive strategy for constructing the base of any LST with three or more curves. The main idea is that we simply need to sweep over the list of SCFT bases and ask whether we can append an additional curve of self-intersection $-y$ to such a base. By the remarks on decoupling already noted, we see that this additional curve can intersect either one curve or two curves of an SCFT base. In the latter case, we obtain a looplike configuration of curves in the base. The latter possibility can only occur for an SCFT base which consists of a single line of curves (i.e., no branches emanating off of the primary spine of the base). The main condition we need to check is that after adding this curve, we obtain a positive semidefinite adjacency matrix. In particular, the determinant must vanish. Implicit in this construction is that we only append an additional curve compatible with the gluing rules for bases.

Consider first the case of a LST with adjacency matrix A_{LST} which describes a treelike base given by adding a single curve of self-intersection $-y$ to some SCFT with adjacency matrix A_{SCFT}:

$$A_{\text{LST}}^{\text{tree}} = \begin{pmatrix} y & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & & & & & & & \\ 0 & & & & A_{\text{SCFT}} & & & \\ \vdots & & & & & & & \vdots \\ 0 & & & & & & & \end{pmatrix}. \quad (5.3)$$

Let $A_{\text{SCFT}'}$ be the matrix obtained from A_{SCFT} by removing the first column and the first row. Evaluating the determinant of $A_{\text{LST}}^{\text{tree}}$, we obtain the condition

$$0 = \det(A_{\text{LST}}^{\text{tree}}) = y \det(A_{\text{SCFT}}) - \det(A_{\text{SCFT}'}) \quad (5.4)$$

or

$$y = \det(A_{\text{SCFT}'}) / \det(A_{\text{SCFT}}). \quad (5.5)$$

Consider now the case of a looplike LST. In this case, the only SCFTs we need consider are those constructed from a single line of curves (i.e., no trivalent vertices at all), and we can only add the additional curve to the leftmost and rightmost ends of a candidate SCFT. The adjacency matrix is then of the form

$$A_{\text{LST}}^{\text{loop}} = \begin{pmatrix} y & -1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ -1 & & & & & & & \\ 0 & & & & & & & \\ \vdots & & & & A_{\text{SCFT}} & & & \vdots \\ 0 & & & & & & & \\ -1 & & & & & & & \end{pmatrix}. \quad (5.6)$$

To have a LST, we must have

$$0 = y \det(A_{\text{SCFT}}) - (A_{\text{SCFT}})_{(1,1)} - (A_{\text{SCFT}})_{(N-1,N-1)}$$
$$+ 2(-1)^{N+1}(A_{\text{SCFT}})_{(1,N-1)}, \quad (5.7)$$

where we have denoted the (i, j)th minor of A_{SCFT} by an appropriate subscript. Solving for y, we obtain

$$y = \frac{(A_{\text{SCFT}})_{(1,1)} + (A_{\text{SCFT}})_{(N-1,N-1)} - 2(-1)^{N+1}(A_{\text{SCFT}})_{(1,N-1)}}{\det A_{\text{SCFT}})}. \quad (5.8)$$

The above algorithm allows us to systematically classify LSTs; from this structure, we see that the locations of where we can add an additional curve to an existing SCFT are quite constrained. Indeed, in order to *not* produce another SCFT, but instead a LST, we will typically only be able to add our extra curve at the end of a configuration of curves or at the second to last curve. Otherwise, we could not reach an SCFT upon decoupling other curves in the base.

C. Low rank examples

To illustrate how the algorithm works in practice, we now give some low rank examples. In Table I, we list all of the rank 2 SCFT bases which we attempt to enhance to looplike LSTs. Of the cases where y is an integer, some are further eliminated since the resulting base requires further blowups.[7] The full list of rank 2 LST bases is then

Three curve LST bases:

$$2\overset{2}{\nabla}2 \quad \text{and} \quad 121 \quad \text{and} \quad 212 \quad \text{and} \quad //222//, \quad (5.9)$$

---

[7] For example, configurations such as $//512//$ and $//313//$ require a further blowup. Doing this, we instead reach a four curve LST base, respectively given by $//6131//$ and $//4141//$.

where the first entry denotes a triple intersection of -2 curves (that is, a type *IV* Kodaira degeneration) and $//x_1 x_2 \cdots x_{n+1}//$ denotes a loop in which the two sides are identified.

VI. ATOMIC CLASSIFICATION OF BASES

In principle, the remarks of the previous section provide an implicit way to characterize all LSTs. Indeed, we simply need to sweep over the list of bases for SCFTs obtained in Ref. [20] and then determine whether there is any place to add one additional curve to reach a LST. The self-intersection of this new curve is constrained by the condition that the determinant of the adjacency matrix vanishes, and the location of where we add this curve is likewise constrained by the tensor-decoupling criterion.

In this section, we use the atomic classification of 6D SCFTs presented in Ref. [20] to perform a corresponding atomic classification of bases for LSTs. We now use the explicit structure of 6D SCFTs found in Ref. [20] to further cut down the possibilities. It is helpful to view the bases as built out of smaller "atoms" and "radicals." In particular, we introduce the convention of a node referring to a single curve in which the minimal fiber type leads to a D- or E-type gauge algebra. We refer to a "link" as any collection of curves which does not contain any D- or E-type gauge

TABLE I. Candidate looplike rank 2 LSTs from adding an additional curve to a rank 2 SCFT.

SCFT	12	13	14	15	16	17	18	19	1(10)	1(11)	1(12)	22	23
det(A)	1	2	3	4	5	6	7	8	9	10	11	3	5
y	5	3	7/3	2	9/5	5/3	11/7	3/2	13/9	7/5	15/11	2	7/5

algebras for the minimal fiber type. The results of Ref. [20] amount to a classification of all possible links as well as all possible ways of attaching links to the nodes. Quite remarkably, the general structure of the resulting bases is quite constrained. For all 6D SCFTs, we can filter the theories according to the number of nodes in the graph. These nodes are always arranged along a single line joined by links,

$$S_{0,1} \overset{S_1}{g_1} L_{12} \overset{\mathbf{I}^{\oplus s}}{g_2} L_{2,3} g_3 \dots g_{k-2} L_{k-2,k-1} \overset{\mathbf{I}^{\oplus t}}{g_{k-1}} L_{k-1,k} \overset{\mathbf{I}^{\oplus u}}{g_k} S_{k,k+1};$$

(6.1)

here, the g_i's denote the nodes, the $L_{i,i+1}$'s denote interior links (since they join to two nodes), and the S's are side links as they can only join to one node. The notation $\mathbf{I}^{\oplus m}$ refers to decorating by m small instantons; these are further classified according to partitions of m (i.e., how many of the small instantons are coincident with one another). One of the key points is that for $k \geq 6$, there is no decoration on any of the interior nodes, i.e., for $3 \leq i \leq k-2$. This holds both for the types of links which can attach to these nodes (which are always the minimal ones forced by the resolution algorithm of Ref. [15]), as well as the possible fiber enhancements (there are none). When $k = 5$, it is possible to decorate the middle node g_3 by a single -1 curve. In Ref. [20], the explicit form of all such sequences of g's, as well as the possible side links and minimal links, was classified. An additional important property is that all of the interior links blow down to a trivial endpoint, the blow-down of a single -1 curve.

Turning now to LSTs, we can ask whether we can add one more curve to the base quiver, resulting in yet another treelike graph or in a looplike graph. By inspection, we can either add this additional curve to a side link, an interior link, or a base node.

A. Restrictions on looplike graphs

In fact, a general looplike graph which is a LST is tightly constrained by the tensor-decoupling criterion. The reason is that if we consider the resulting sequence of nodes, we must have a pattern of the form

$$//g_1 L_{12} g_2 L_{2,3} g_3 \dots g_{k-2} L_{k-2,k-1} g_{k-1} L_{k-1,k} g_k L_{k,1}//, \quad (6.2)$$

where the notation "$//$" indicates that the left and the right of the base quiver are joined together to form a loop. Now, another important constraint from Ref. [20] is that the minimal fiber type on the nodes obeys a nested sequence of containment relations. But in a loop, no such ordering is

possible. We therefore conclude that all of the nodes for a looplike LST must be identical and, moreover, that the interior links must all be minimal. We therefore can specify all such loops simply by the type of node (i.e., a -4 curve, a -6 curve, a -8 curve, or a -12 curve) and the number of such nodes.

For this reason, we now confine our attention to treelike graphs, i.e., where we add an additional curve which intersects only one other curve in the base. The main restriction we now derive is that the resulting configuration of curves is basically the same as that of line (6.1). Indeed, we will simply need to impose further restrictions on the possible side links and sequences of nodes which can appear in a LST base.

B. Restrictions on adding to interior links

Our first claim is that we can only possibly add an extra curve to an interior link in a base with two or fewer nodes. Indeed, suppose to the contrary. Then, we will encounter a configuration such as

$$\dots g_i \overset{y}{L_{i,i+1}} g_{i+1} \dots, \quad (6.3)$$

where y denotes our additional curve attached in some way to the link. The notation "$\dots$" denotes the fact that there is at least one more curve in the base. Now, since the interior link blows down to a single -1 curve, we will get a violation of normal crossing. This is problematic if we have one additional curve (as denoted by the "$\dots$"), since deleting that curve would produce a putative SCFT with a violation of normal crossing, a contradiction. By the same token, in a two node base, if any side links are attached to this node, then we cannot add anything to the interior link. This leaves us with the case of just

$$g_1 \overset{y}{L_{1,2}} g_2. \quad (6.4)$$

In this case, it is helpful to simply enumerate once again all of the possible interior links and ask whether we can attach an additional curve. This we do in Appendix D, finding that the options are severely limited. Summarizing, then, we find that we can attach an extra curve to an interior link only in the case where there are two nodes, and then only if these two nodes do not attach to any side links.

C. Restrictions on adding to nodes

Let us next turn to restrictions on adding an extra curve to a node in a base. If we add a curve to a node, we observe

that this extra curve must have self-intersection -1. Note that the endpoint of the SCFT must therefore be trivial in these cases. We now ask which of the nodes of the base can support an additional -1 curve. Since we must be able to delete a single curve and reach a collection of SCFTs, we cannot place this -1 curve too far into the interior of the configuration. More precisely, we see that for $k \geq 7$ nodes, we are limited to adding a -1 curve to the first three or last three nodes. In the specific case where we attach a -1 curve to the third interior node, we see that there cannot be any side links whatsoever. Otherwise, we would find a sub-configuration of curves which is not a 6D SCFT.

D. Restrictions on adding to side links

Consider next restrictions on adding an extra curve to a side link. In the case of a small instanton link such as $1,2\ldots,2$, we can append an additional -1 curve to the rightmost -2 curve, but then it can no longer function as a side link (via the tensor-decoupling criterion). In Appendixes D and E, we determine the full list of LSTs comprised of just adding one more curve to a side link. If we instead attempt to take an existing SCFT and add an additional curve to a side link to reach a LST, then we either produce a new side link (i.e., if the curve has self-intersection -1, -2, -3 or -5) or we produce a base quiver with one additional node (i.e., if the curve has self-intersection -4, -6, -7, -8, -9, -10, -11, -12). Phrased in this way, we see that the rules for which side links can join to an SCFT are slightly different but cannot alter the overall topology of a base quiver from the case of an SCFT.

Summarizing, we see that, unless we have precisely two nodes, and no side links, we cannot decorate any interior link. Moreover, we can only decorate the three leftmost and rightmost nodes in special circumstances. So in other words, the general structure of a treelike LST base is essentially the same as that of a certain class of SCFTs. All that remains is for us to determine the possible sequences of nodes (with no decorations) which can generate an LST and to also determine which of our side links can be attached to an SCFT such that the resulting configuration is a LST.

E. Overview of Appendixes

This final point is addressed in a set of Appendixes. In the Appendixes, we collect a full list of the building blocks for constructing LSTs. The tensor-decoupling criterion prevents a direct gluing of smaller LSTs to reach another LST. Rather, we are always supplementing an SCFT to reach a LST. Along these lines, in Appendix C, we collect the list of bases which are comprised of a single spine of nodes with no further decoration from side links. In Appendix D, we collect the full list of bases in which no nodes appear. Borrowing from the terminology used for 6D SCFTs, these links are "noble" in the sense that they cannot attach to anything else in the base. Finally, in Appendix E, we give a list of LSTs given by attaching a single side link to a single

node. Much as in the classification of 6D SCFTs, the further task of sweeping over all possible ways to decorate a base quiver by side links is left implicit (as dictated by the number of blowdowns induced by a given side link). All of these rules follow directly from Ref. [20].

This completes the classification of bases for LSTs. We now turn to the classification of elliptic fibrations over a given base.

VII. CLASSIFYING FIBERS

Holding fixed the choice of base, we now ask whether we can enhance the singularities over curves of the base while keeping all fibers in Kodaira-Tate form. As this is a purely local question (i.e., compatible with the matter enhancements over the neighboring curves), most of the rules for adding extra gauge groups/matter are fully specified by the rules spelled out in Ref. [20]. Rather than repeat this discussion, we refer the interested reader to these cases for further discussion of the "standard" fiber enhancement rules for curves which intersect with normal crossings.

There are, however, a few cases which cannot be understood using just the SCFT considerations of Ref. [20]. Indeed, we have already seen that a curve of self-intersection zero, an elliptic curve, tangent intersections, and triple intersections of curves can all occur in the base of a LST. We have also seen, however, that all of these cases are comparatively "rare" in the sense that they do not attach to larger structures. Our plan in this section will therefore be to deal with all of these low rank examples. In Appendix A, we give general constraints from anomaly cancellation in F-theory models, and in Appendix B, we present some additional technical material on the localization of matter in the case of tangent intersections such as the 4∥1 and 2∥2 configurations. Finally, compared with the case of 6D SCFTs, the available fiber enhancements over a given base are also comparatively rare. To illustrate this point, we give some examples in which the base is an affine Dynkin diagram of -2 curves. In these cases, the presence of the additional imaginary root (and the constraints from anomaly cancellation) typically dictate a small class of possible fiber enhancements.

A. Low rank LSTs

In this subsection, we give a complete characterization of fiber enhancements for low rank LSTs. To begin, we consider the case of the rank zero LSTs, i.e., those where the F-theory base consists of a single compact curve. In these cases, we *only* get a 6D theory once we wrap some 7-branes over the curve unless the normal bundle of the curve is a torsion line bundle. An interesting feature of this and related examples is that, because the corresponding tensor multiplet is nondynamical, it cannot participate in the Green-Schwarz mechanism, and we must cancel the anomaly using just the content of the gauge theory sector. We then turn to the other low rank examples where other

violations of normal crossing appear. In all of these cases, the F-theory geometry provides a systematic tool for determining which of these structures can embed in a UV complete LST.

1. Rank zero LSTs

In a rank zero LST, we have a single compact curve, which must necessarily have self-intersection zero. There are only a few inequivalent configurations consisting of a single curve with self-intersection zero:

$$\Sigma_{\text{null}} = P, I_0, I_1, II, {}_m I_0, {}_m I_1. \tag{7.1}$$

Here, I_0 (respectively, P) is shorthand for a base B consisting of a smooth torus (respectively, 2-sphere) with trivial normal bundle $T^2 \times \mathbb{C}$ (respectively, $\mathbb{P}^1 \times \mathbb{C}$), while I_1 (respectively, II) is a curve with a node (respectively, a cusp) singularity and trivial normal bundle. These configurations can give rise to LSTs *only if* 7-branes wrap Σ_{null}. Otherwise, we do not have a genuine 6D model. The variants ${}_m I_0$ and ${}_m I_1$ describe curves of which the normal bundle is torsion of order $m > 1$; these can also support 6D theories for $m \in \{2, 3, 4, 6\}$ as discussed below. Observe also that if we apply the tensor-decoupling criterion in these cases, we find that the resulting 6D SCFT is empty, i.e., trivial.

As curves of self-intersection 0 do not show up in 6D SCFTs, it is important to explicitly list the possible singular fiber types which can arise on each curve of line (7.1).[8]

We begin with the "multiple fiber phenomenon"—a genus 1 curve Σ of which the normal bundle is torsion of order $m1$. The F-theory base B is a (rescaled) small neighborhood of Σ, and its canonical bundle $\mathcal{O}_B(K_B)$ must also be torsion of the same order by the adjunction formula. Now, to construct a Weierstrass model, we need sections f and g of $\mathcal{O}_B(-4K_B)$ and $\mathcal{O}_B(-6K_B)$, respectively, but nontrivial torsion, bundles do not have nonzero sections. Thus, in order to have a nonzero f, the order m of the torsion must divide 4, while to have a nonzero g, the order m must divide 6. There are thus three cases:

  (1) If $m = 2$, then both f and g may be nonzero.
  (2) If $m = 3$ or 6, then f must be zero, but g may be nonzero.
  (3) If $m = 4$, then g must be zero, but f may be nonzero.
For any other value of $m > 1$, Weierstrass models do not exist (since f and g are not both allowed to vanish identically).

---

[8]In what follows, we focus on those cases where the fiber enhancement leads to a non-Abelian gauge symmetry, i.e., a gauge theory description. In the cases where we have a type I_1 or type II fiber enhancement, the resulting 6D theory will consist of some number of weakly coupled free hypermultiplets, where the precise number depends on whether the base curve has nontrivial arithmetic and/or geometric genus. Much as in the case of 6D SCFTs, these cases can be covered through a mild extension of the analysis presented in Ref. [20]. See also Ref. [44] for additional information about these theories.

Note that the fact that some quantities obtained from coefficients in a Weierstrass model are sections of torsion bundles also provides the possibility that those sections do not exist (if they are known to be nonzero). As described in Table 4 of Ref. [45], the criterion for deciding whether a given Kodaira type leads to a gauge algebra of which the Dynkin diagram is simply laced or not simply laced reduces in almost every case to a question of whether a certain quantity has a square root.[9] If the desired square root is in fact a section of a 2-torsion bundle, then it cannot exist.

We can give explicit examples of this phenomenon which do not involve enhanced gauge symmetry, using the framework outlined in Sec. IV A. We start with $S = T_S^2 \times \mathbb{C}$, where T_S^2 admits an automorphism of order m which acts faithfully on the holomorphic 1-form. If we extend the action to include multiplication by an appropriate root of unity on $\mathbb{C}$, then the holomorphic 2-form on S is preserved. As is well known, such automorphisms exist exactly for $m \in \{2, 3, 4, 6\}$. As explained in Sec. IV A, the quotient $(T_F^2 \times S)/\mathbb{Z}_m$ has an elliptic fibration $(T_F^2 \times S)/\mathbb{Z}_m \to (T_F^2 \times \mathbb{C})/\mathbb{Z}_m$ of which the fibers over $T_F^2 \times \{0\}$ are all nonsingular elliptic curves, but with $\mathbb{Z}_m$ acting upon them as loops are traversed on T_F^2. This same geometry has a second fibration $(T_F^2 \times S)/\mathbb{Z}_m \to S/\mathbb{Z}_m$ with no section and some multiple fibers in codimension 2, which will be further discussed in Sec. IX.

Turning now to enhancements of fibers, from the relation between anomaly cancellation and enhanced singular fibers [45–48], we find all possible gauge theories compatible with a given choice of base curve. This is summarized in Table II. We find that in general, such theories can support hypermultiplets in the adjoint representation (denoted Adj), the two-index symmetric representation (denoted sym), and the n-index antisymmetric representation (denoted Λ^n).

The greatest novelty here relative to the case of 6D SCFTs is the theories with $n_{Adj} = 1$ or $\mathfrak{su}(N)$ gauge algebra, $n_{\text{sym}} = 1$, $n_{\Lambda^2} = 1$, or, in the special case of $\mathfrak{su}(6)$, $n_f = 1$, $n_{\text{sym}} = 1$, $n_{\Lambda^3} = 1/2$. The first of these cases, with $n_{Adj} = 1$, corresponds simply to a smooth curve of genus 1 in the base. The cases with symmetric representations of $\mathfrak{su}(N)$, on the other hand, arise when the *base* curve is of Kodaira type I_1 (i.e., has a nodal singularity). As reviewed in Appendix B, the notion of genus is ambiguous for singular curves. A type I_1 curve has topological genus 0 but arithmetic genus 1, and as a result, it must support a hypermultiplet in the two-index symmetric representation, rather than one in the adjoint representation.

For LSTs, these are the only examples in which a curve of (arithmetic) genus 1 shows up, and a curve of genus

---

[9]In the remaining case, one must consider a more complicated cubic equation, but in the situation being described here, the question in that case boils down to the existence or non-existence of a cube root. If the bundle of which the desired cube root is a section is a 3-torsion bundle, the cube root cannot exist.

TABLE II. Rank zero LSTs. In the above, Adj refers to the adjoint representation, sym refers to a two-index symmetric representation, and Λ^n refers to an n-index antisymmetric representation.

Base curve	Matter content
I_0	Any simple Lie algebra, $n_{Adj} = 1$
I_1, II	$\mathfrak{su}(N)$, $n_{\mathrm{sym}} = 1$, $n_{\Lambda^2} = 1$
	$\mathfrak{su}(6)$, $n_f = 1$, $n_{\mathrm{sym}} = 1$, $n_{\Lambda^3} = \frac{1}{2}$
P	$\mathfrak{su}(N)$, $N \geq 2$, $n_f = 16$, $n_{\Lambda^2} = 2$
	$\mathfrak{su}(6)$, $n_f = 17$, $n_{\Lambda^2} = 1$, $n_{\Lambda^3} = \frac{1}{2}$
	$\mathfrak{su}(6)$, $n_f = 18$, $n_{\Lambda^3} = 1$
	$\mathfrak{sp}(N)$, $N \geq 1$, $n_f = 16$, $n_{\Lambda^2} = 1$
	$\mathfrak{sp}(3)$, $n_f = 17\frac{1}{2}$, $n_{\Lambda^3} = \frac{1}{2}$
	$\mathfrak{so}(N)$, $N = 6, \ldots, 14$, $n_f = N - 4$, $n_s = \frac{64}{d_s}$
	$\mathfrak{g}_2$, $n_f = 10$
	$\mathfrak{f}_4$, $n_f = 5$
	$\mathfrak{e}_6$, $n_f = 6$
	$\mathfrak{e}_7$, $n_f = 4$
	$\mathfrak{e}_8$, $n_{\mathrm{inst}} = 12$

$g \geq 2$ is never allowed. Note also that a curve of genus 0 and self-intersection 0 cannot itself support an $\mathfrak{e}_8$ gauge algebra; it must be blown up at 12 points, resulting in an $\mathfrak{e}_8$ theory with 12 small instantons:

$$(12), 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2. \tag{7.2}$$

2. Rank 1 LSTs

'Consider next the case of rank 1 LSTs, i.e., those in which there are two compact curves in the base. As we have already remarked, in this and all higher rank LSTs, all the curves of the base will be $\mathbb{P}^1$'s, and moreover, they will have self-intersection $-x$ for $1 \leq x \leq 12$. Now, in the case of two curves, we can have various violations of normal crossing. For example, we can have curves which intersect along a tangency. This occurs in both the $4\|1$ and $2\|2$ configurations, the latter describing a type III Kodaira fiber. Observe that in both cases, there is a smoothing deformation which takes us from an order 2 tangency to a loop; i.e., we can deform to $//4, 1//$ and $//2, 2//$. In addition to these rank 1 LSTs, there is just one more configuration given by $1,1$, which in some sense is the most "conventional" possibility (as all intersections respect normal crossing).

To this end, let us first discuss fiber enhancements for the $1,1$ configuration. We shall then turn to the cases where there is either a violation of normal crossing or a loop configuration. Whenever curves with gauge algebras intersect, matter charged under each gauge algebra will pair up into a mixed representation of the gauge algebras. The mixed anomaly condition places strong constraints on which representations are allowed to pair up. The allowed set of mixed representations for two curves intersecting at a single point is given in Sec. VI. 2 of Ref. [20]. Consider, for example, the $1,1$ base. We have

$$\overset{\mathfrak{g}_L \; \mathfrak{g}_R}{1 \; 1} \tag{7.3}$$

with the following list of allowed gauge algebras:
  (i) $\mathfrak{g}_L = \mathfrak{so}(M)$, $\mathfrak{g}_R = \mathfrak{sp}(N)$, $M = 7, \ldots, 12$, $M - 5 \geq N$, $4N + 16 \geq M$.
  (ii) $\mathfrak{g}_L = \mathfrak{so}(M)$, $\mathfrak{g}_R = \mathfrak{sp}(N)$, $M = 7$, $N \leq 6$.
  (iii) $\mathfrak{g}_L = \mathfrak{g}_2$, $\mathfrak{g}_R = \mathfrak{sp}(N)$, $N \leq 7$.
  (iv) $\mathfrak{g}_L = \mathfrak{sp}(M)$, $\mathfrak{g}_R = \mathfrak{sp}(N)$, $2M + 8 \geq 2N$, $2N + 8 \geq 2M$.
  (v) $\mathfrak{g}_L = \mathfrak{sp}(M)$, $\mathfrak{g}_R = \mathfrak{su}(N)$, $2M + 8 \geq N$, $N + 8 + \delta_{N,3} + \delta_{N,6} \geq 2M$.
  (vi) $\mathfrak{g}_L = \mathfrak{su}(M)$, $\mathfrak{g}_R = \mathfrak{su}(N)$, $M + 8 + \delta_{M,3} + \delta_{M,6} \geq N$, $N + 8 + \delta_{N,3} + \delta_{N,6} \geq M$.
  (vii) $\mathfrak{g}_L = \mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ or $\mathfrak{e}_8$, $\mathfrak{g}_R = \emptyset$.
Here, it is understood that $\mathfrak{sp}(0)$ is the same as an empty -1 curve, and $\mathfrak{e}_8$ on a -1 curve implies that 11 points on the -1 curve have been blown up.

Consider next the configuration $//2, 2//$, i.e., a loop of two -2 curves. In this case, the only gauge algebra enhancement is given by

$$\overset{\mathfrak{su}(N) \; \mathfrak{su}(N)}{// \; 2 \quad 2 \; //} \tag{7.4}$$

with a bifundamental localized at each intersection point.

When the -2 curves intersect tangentially, i.e., in the $2\|2$ configuration, more options are available, as suggested by the effective field theory on the tensor branch. We find the following general possibilities for enhancements of the gauge algebra and matter:
  (i) $\mathfrak{g}_a = \mathfrak{su}(N_a)$, $\mathfrak{g}_b = \mathfrak{so}(N_b)$, $R_a = N_a$, $R_b = N_b$. The only allowed possibilities are $N_a = 6$, $N_b = 12$, and $N_a = 7$, $N_b = 13$.
  (ii) $\mathfrak{g}_a = \mathfrak{su}(4)$, $\mathfrak{g}_b = \mathfrak{so}(7)$, $\mathfrak{so}(8)$, $R_a = \mathbf{4}$, $R_b = \mathbf{8}$.
  (iii) $\mathfrak{g}_a = \mathfrak{su}(4)$, $\mathfrak{g}_b = \mathfrak{g}_2$, $R_a = \mathbf{4}$, $R_b = \mathbf{7}$.
Just as it was necessary to deform the singular Kodaira I_1 curve to get hypermultiplets in the symmetric representation of $\mathfrak{su}(N)$, so it is necessary to deform the I_2 base to the tangentially intersecting Kodaira type III curve configuration to get these matter pairings. In conclusion, for the Kodaira type III configuration $2\|2$, we may enhance the gauge algebras as

$$\overset{\mathfrak{so}(7) \quad \mathfrak{su}(4)}{\underset{[Sp(1)]}{2} \; \| \; 2}$$

$$\overset{\mathfrak{so}(8) \quad \mathfrak{su}(4)}{\underset{[Sp(2)]}{2} \; \| \; 2}$$

$$\overset{\mathfrak{g}_2 \quad \mathfrak{su}(4)}{2 \; \| \; 2}$$

$$\overset{\mathfrak{so}(12) \quad \mathfrak{su}(6)}{\underset{[N_s = 1]}{2} \; \| \; 2}$$

$$\overset{\mathfrak{so}(13) \quad \mathfrak{su}(7)}{\underset{[N_s = 1/2]}{2} \; \| \; 2} .$$

Here, there is a mixed representation in the bifundamental of the two gauge algebras in the last three cases. These representations can only show up when the curves are tangent to each other.

Finally, we turn to the case of the bases $//4, 1//$ and $4\|1$. In both cases, the only enhancement possible is an SO-type algebra over the -4 curve and an Sp-type or SU-type algebra over the -1 curve. The matter content of the theory, however, depends on the type of intersection. Consider first the case of

$$// \overset{\mathfrak{so}(2N+8)}{4} \overset{\mathfrak{sp}(N)}{1} //$$ (7.5)

with a half hypermultiplet localized at each intersection point. Indeed, we can reach this theory by starting from the 6D SCFT,

$$[\mathfrak{so}(2N+8)] \overset{\mathfrak{sp}(N)}{1} [\mathfrak{so}(2N+8)],$$ (7.6)

and gauging the diagonal subalgebra of the flavor symmetry.

In the case of the tangential intersection $4\|1$, we again find novel configurations of matter which are missing from the case of normal crossing. First, we can consider a configuration in which the gauge algebras are the same as those of (7.5):

$$\overset{\mathfrak{so}(2N+8)}{4} \| \overset{\mathfrak{sp}(N)}{1}.$$ (7.7)

However, there is now a single full hypermultiplet in the bifundamental of the two gauge algebras rather than two half hypermultiplets. This configuration cannot be realized in F-theory, since the tangential intersection removes the monodromy from the I_{2N} locus leading to gauge algebra $\mathfrak{su}(2N)$ instead. It would be interesting to find a field-theoretic reason for excluding this case.

Second, there is a similar configuration with a unitary rather than symplectic gauge algebra,

$$\overset{\mathfrak{so}(2N+8)}{4} \| \overset{\mathfrak{su}(2N)}{1},$$ (7.8)

with a hypermultiplet in the bifundamental of the two gauge algebras, as well as a hypermultiplet in the two-index antisymmetric representation of the $\mathfrak{su}(n)$ factor, all of which are located at the collision point between the two branes. This configuration can be realized in F-theory.

3. Rank 2 LSTs

We now turn to the case of rank 2 LSTs, i.e., those with three curves. Here, we can have no tangential intersections. In this case, the adjacency matrix again provides only partial information about the geometry of intersecting curves. In the case where we have normal crossings for all pairwise intersections, the rules of enhancing fiber enhancements follow from Ref. [20] and are also reviewed

in Appendix A. There is, however, also the possibility of a Kodaira type IV configuration of -2 curves, i.e., $2\overset{2}{\triangledown}2$. We shall therefore confine our attention to fiber decorations with this base.

To illustrate, suppose the gauge algebra localized on each of the three -2 curves is $\mathfrak{su}(2)_i$, $i = 1, 2, 3$. Anomaly cancellation dictates that in such a case, there must be a single half-trifundamental $\frac{1}{2}(\mathbf{2}, \mathbf{2}, \mathbf{2})$ plus two fundamentals charged under each $\mathfrak{su}(2)$. We note in passing that the looplike configuration $//222//$ also admits a similar enhancement in the gauge algebras, i.e., with $\mathfrak{su}(2)_i$, $i = 1, 2, 3$ gauge groups, but that the corresponding matter content is given by three bifundamentals $(\mathbf{2}, \mathbf{2}, \mathbf{1})$, $(\mathbf{2}, \mathbf{1}, \mathbf{2})$, $(\mathbf{1}, \mathbf{2}, \mathbf{2})$. These two configurations have the same anomaly polynomials.

But in contrast to the $//222//$ configuration, for $2\overset{2}{\triangledown}2$, no other gauge algebra enhancements are possible. To see this, suppose we have $\mathfrak{su}(N)$ factors on each -2 curve. We would then need N^2 fundamentals of each $\mathfrak{su}(N)$ to get a (N, N, N) representation. However, anomaly cancellation considerations constrain us to $2N$ such fundamentals. In other words, we are limited to $N \leq 2$.

B. Higher rank LSTs

Turning next to the case of LSTs with at least four curves in the base, all of these local violations of normal crossing do not appear. Nevertheless, we encounter such violations when we attempt to blow down -1 curves which touch more than two curves. Even so, the local rules for fiber enhancements follow the same algorithm already spelled out in detail in Ref. [20]. In some cases, however, there can be additional restrictions compared with the fiber enhancements which are possible for 6D SCFTs. To illustrate, we primarily focus on some simple examples, i.e., the affine ADE bases; fiber decorations for the base 1,2,…,2,1; and fiber decorations for the looplike bases.

1. Affine ADE bases

Let us next consider the case of fiber enhancements in which the base is given by an affine Dynkin diagram of -2 curves. If we assume that no further blowups are introduced in the base, we will be limited to just $\mathfrak{su}(N)$ gauge algebras over each curve. In the case of 6D SCFTs, it is typically possible to obtain a rich class of possible sequences of gauge group factors, because 6D anomaly cancellation can be satisfied by introducing an appropriate number of additional flavor symmetry factors. This in turn leads to a notion of a "ramp" in the increase in the ranks [17] (see also Ref. [6]). For an affine quiver, this is much more delicate, since all of these cases can be viewed as a degeneration of an elliptic curve. For example, in the case of an affine $\hat{A}_k$ base (i.e., the I_k Kodaira type), anomaly cancellation tells us that all of the gauge algebra factors are the same $\mathfrak{su}(N)$. By a similar token, 6D anomaly

cancellation tells us that the gauge algebra of any of these cases is $\mathfrak{su}(Nd_i)$, where d_i is the Dynkin label of the node in the affine graph and $N \geq 1$ is an overall integer.[10] In F-theory language, we have fiber enhancements I_{Nd_i} over each node. Indeed, at a formal level we can think of anomaly cancellation being satisfied by introducing $\mathfrak{su}(1)$ gauge algebras.

2. 1,2,...,2,1 base

Compared with the case of affine ADE bases, there are comparatively more options available for fiber enhancements of the base 1,2,…,2,1. In some sense, this is because these bases do not directly arise from the degeneration of a compact elliptic curve but are better viewed as the degeneration of a cylinder.

With this in mind, we now explain how fiber enhancements work for this choice of base. For a large number of -2 curves, the allowed enhancements take a rather simple form, whereas there are outlier LSTs for smaller numbers of -2 curves. In particular, when there are more than five -2 curves, the -2 curves necessarily hold $\mathfrak{su}(N_i)$ gauge algebras:

$$
\begin{array}{ccccccc}
\mathfrak{g}_L & \mathfrak{su}(N_1) & \mathfrak{su}(N_2) & & \mathfrak{su}(N_{k-1}) & \mathfrak{su}(N_k) & \mathfrak{g}_R \\
1 & 2 & 2 & \ldots & 2 & 2 & 1
\end{array}.
$$

The N_i are subject to the convexity conditions $2N_i \geq N_{i-1} + N_{i+1}$, with the understanding that $N_i = 1$ for a curve without a gauge algebra.

The -1 curves in this configuration may hold either $\mathfrak{sp}(M)$ or $\mathfrak{su}(M)$ gauge algebra. If the leftmost -1 curve holds $\mathfrak{sp}(M)$, anomaly cancellation imposes the additional conditions $2N_1 \geq 2M + N_2$, $2M + 8 \geq N_1$. If the leftmost -1 curve holds $\mathfrak{su}(M)$, anomaly cancellation imposes $2N_1 \geq M + N_2$, $M + 8 + \delta_{M,3} + \delta_{M,6} \geq N_1$. Finally, the -1 curve may be empty provided $N_1 \leq 9$. The story is mirrored for the rightmost -1 curve at the other end of the chain.

When there are exactly five -2 curves, we have two additional configurations:

$$
\begin{array}{ccccccc}
 & \mathfrak{su}(2) & & \mathfrak{so}(7) & & \mathfrak{su}(2) & \\
1 & 2 & 2 & 2 & 2 & 2 & 1
\end{array}
$$

and

$$
\begin{array}{ccccccc}
 & \mathfrak{su}(2) & & \mathfrak{g}_2 & & \mathfrak{su}(2) & \\
1 & 2 & 2 & 2 & 2 & 2 & 1
\end{array}.
$$

When there are four -2 curves, we similarly have

$$
\begin{array}{cccccc}
 & \mathfrak{su}(2) & \mathfrak{so}(7) & & \mathfrak{su}(2) & \\
1 & 2 & 2 & 2 & 2 & 1
\end{array}
$$

and

$$
\begin{array}{cccccc}
 & \mathfrak{su}(2) & \mathfrak{g}_2 & & \mathfrak{su}(2) & \\
1 & 2 & 2 & 2 & 2 & 1
\end{array}.
$$

When there are three -2 curves, we have several new configurations:

$$
\begin{array}{ccccc}
 & \mathfrak{su}(2) & \mathfrak{so}(7) & \mathfrak{su}(2) & \\
1 & 2 & 2 & 2 & 1
\end{array}
$$

$$
\begin{array}{ccccc}
 & \mathfrak{su}(2) & \mathfrak{g}_2 & \mathfrak{su}(2) & \\
1 & 2 & 2 & 2 & 1
\end{array}
$$

$$
\begin{array}{ccccc}
\mathfrak{g}_L & \mathfrak{so}(7) & \mathfrak{su}(2) & & \\
1 & 2 & 2 & 2 & 1
\end{array}
$$

and

$$
\begin{array}{ccccc}
\mathfrak{g}_L & \mathfrak{g}_2 & \mathfrak{su}(2) & & \\
1 & 2 & 2 & 2 & 1
\end{array},
$$

with $\mathfrak{g}_L = \mathfrak{sp}(M)$, $M \leq 3$, in each of the last two cases.

When there are two -2 curves, we have

$$
\begin{array}{cccc}
\mathfrak{g}_L & \mathfrak{so}(7) & \mathfrak{su}(2) & \\
1 & 2 & 2 & 1
\end{array}
$$

and

$$
\begin{array}{cccc}
\mathfrak{g}_L & \mathfrak{g}_2 & \mathfrak{su}(2) & \\
1 & 2 & 2 & 1
\end{array},
$$

with $\mathfrak{g}_L = \mathfrak{sp}(M)$, $M \leq 3$, in each of the last two cases.

When there is only a single -2 curve, there are even more possibilities:

$$
\begin{array}{ccc}
\mathfrak{g}_L & \mathfrak{so}(8) & \mathfrak{g}_R \\
1 & 2 & 1
\end{array}
$$

with $\mathfrak{g}_L = \mathfrak{sp}(M_L)$, $M_L \leq 2$, $\mathfrak{g}_R = \mathfrak{sp}(M_R)$, $M_R \leq 2$;

$$
\begin{array}{ccc}
\mathfrak{g}_L & \mathfrak{so}(7) & \mathfrak{g}_R \\
1 & 2 & 1
\end{array}
$$

with $\mathfrak{g}_L = \mathfrak{sp}(M_L)$, $\mathfrak{g}_R = \mathfrak{sp}(M_R)$, $M_L + M_R \leq 4$, or $M_L = 4$, $M_R = 1$;

$$
\begin{array}{ccc}
\mathfrak{g}_L & \mathfrak{g}_2 & \mathfrak{g}_R \\
1 & 2 & 1
\end{array}
$$

with $\mathfrak{g}_L = \mathfrak{sp}(M_L)$, $\mathfrak{g}_R = \mathfrak{sp}(M_R)$, $M_L + M_R \leq 4$; and

$$
\begin{array}{ccc}
\mathfrak{g}_L & \mathfrak{su}(2) & \\
1 & 2 & 1
\end{array}
$$

with $\mathfrak{g}_L = \mathfrak{g}_2$ or $\mathfrak{so}(7)$.

C. Looplike bases

Finally, let us turn to the case of fiber enhancements for the looplike bases. We have already discussed the case of

---

[10]This same observation has already been made in the context of four-dimensional superconformal $\mathcal{N} = 2$ quiver gauge theories [49]. Indeed, in this special case, the condition of vanishing beta functions is identical to the condition that 6D anomalies cancel.

an affine $\hat{A}_k$ base of -2 curves in which we the allowed fiber enhancements are just a uniform I_N fiber. Otherwise, we induce some blowups. Now, if we allow for blowups, we can reach more general looplike configurations. However, as we have already discussed near line (6.2), all of these cases consist of a single type of base node suspended between minimal links. This is a consequence of the fact that in a general 6D SCFT, there are nested containment relations on the minimal fiber types [20]:

$$\mathfrak{g}_1^{\min} \subseteq \cdots \subseteq \mathfrak{g}_m^{\min} \supseteq \cdots \supseteq \mathfrak{g}_k^{\min}. \tag{7.9}$$

However, in a 6D LST, we must *also* demand periodicity of the full configuration. This forces a uniform fiber enhancement on each such node.

As a consequence, we can summarize all of these cases by keeping implicit the blowups associated with conformal matter. We have

$$//\overset{\mathfrak{g}}{2}, \ldots, \overset{\mathfrak{g}}{2}//, \tag{7.10}$$

where we allow for a general fiber enhancement to an ADE-type simple Lie algebra $\mathfrak{g}$ over each of the -2 curves. For all cases other than the $\mathfrak{su}(N)$ gauge algebras, this in turn requires further blowups in the base; i.e., we have a configuration with conformal matter in the sense of Refs. [17,18]. So in other words, all of these looplike configurations are summarized by stating the number of -2 curves and the choice of fiber type over any of the -2 curves.

VIII. EMBEDDINGS AND ENDPOINTS

In the previous sections, we presented a general classification of LSTs in F-theory. One of the crucial ingredients we have used is that decompactifying any curve must return us to a collection of (possibly disconnected) SCFTs. Turning the question around, it is natural to ask whether *all* SCFTs embed in LSTs.

In this section, we show that this is indeed the case. Moreover, there can often be more than one way to complete an SCFT to a LST. To demonstrate such an embedding, we will need to show that there exists a deformation of a given LST F-theory background which takes us to the requisite SCFT. This can involve both Kähler deformations, i.e., motion onto a partial tensor branch, and may also include complex structure deformations, i.e., a Higgsing operation.

With this in mind, we first demonstrate that all of the bases for 6D SCFTs embed in a LST base. A suitable tensor branch flow then takes us from the LST base back to the 6D SCFT base. Then, we proceed to show that the available fiber decorations for LSTs can be Higgsed down to the fiber decorations for an SCFT. The latter issue is somewhat nontrivial since the fiber decorations of an ADE-type base

are comparatively less constrained when compared with their affine counterparts.

A. Embedding the bases

We now show that all bases for 6D SCFTs embed in LST bases. To demonstrate that such an embedding is possible, it is convenient to use the terminology of "endpoints" for SCFTs introduced in Ref. [15], which we can also extend to the case of LSTs. Given a collection of curves for an SCFT base, we can consider blowdowns of all of the -1 curves of the configuration. Doing so, we shift the self-intersection of all curves touching this -1 curve according to the rule $x \rightarrow (x-1)$ for a curve of self-intersection $-x$. In the case where a -1 curve is interposed in between two curves, we have $x, 1, y \rightarrow (x-1), (y-1)$. After this first stage of blowdowns, we can then sometimes generate *new* -1 curves. Iteratively blowing down all such -1 curves, we eventually reach a configuration of curves which we shall refer to as an endpoint. The set of all endpoints has been classified in Ref. [15], and they split up according to four general types:

$$\text{Trivial endpoint: } 1 \rightarrow \mathbb{C}^2 \tag{8.1}$$

$$\text{A-type endpoint: } x_1 \ldots x_k \tag{8.2}$$

$$\text{D-type endpoint: } 2\overset{2}{x}_1 \ldots x_k \tag{8.3}$$

$$\text{E-type endpoint: } 2\overset{2}{2}222, \quad 2\overset{2}{2}2222, \quad 2\overset{2}{2}22222. \tag{8.4}$$

In fact, starting from such an endpoint, we can generate all possible bases of 6D SCFTs by further blowups. Sometimes such blowups are required to define an elliptic Calabi-Yau, while some can be added even when an elliptic fibration already exists. By a similar token, we can also take a fixed base and then decorate by appropriate fibers.

Now, a central feature of this procedure is that the resulting adjacency matrix retains the important property that it is positive definite. Similarly, if we instead have a positive semidefinite adjacency matrix, the resulting matrix will retain this property under further blowups (or blowdowns) of the base.

To demonstrate that we can always embed an SCFT in a LST, it will therefore suffice to show that there is *some* way to add additional curves to an SCFT endpoint such that the resulting LST defines a base. To illustrate the idea, suppose we have a 6D SCFT with a trivial endpoint. Then, before the very last stage of blowdowns, we have a single -1 curve, which we shall call Σ. If we return to the original SCFT, this curve will also be present, but its self-intersection will be different. Hence, to get a LST, we can simply attach one more -1 curve to Σ. For example, the configuration 1,1 defines the base of a LST. Supplementing

the fibers can always be done to realize the case of fiber decorations.

Consider next the case of A-type endpoints. Here, we can always attach a suitable number of "tails" of the form $1,2\ldots,2$ to each curve such that blowing down these instanton links leads us to k curves of self-intersection -2, i.e., $2,\ldots,2$. Attaching an additional -1 curve to the left and to the right, we get a base of the form $1,2,\ldots,2,1$. We therefore conclude that adding such tails again allows an embedding in a LST.

By a similar token, we can append such tails to a D-type endpoint until we reach the Dynkin diagram with just -2 curves. Adding one more -1 curve to this configuration,

$$\text{D-type endpoint} = 2\overset{2}{2}\ldots2,1, \qquad (8.5)$$

leads to a blowdown eventually to the configuration $2,1,2$, which in turn blows down to $1,1$. So again, we conclude that the adjacency matrix is positive semidefinite, and we have arrived at a LST.

This leaves us with the E-type endpoints. In these cases, we just have a configuration of -2 curves, and the possible blowups are severely limited [15]. For example, there are no blowups of the E_8 Dynkin diagram. It therefore suffices to add one additional -2 curve to this configuration to reach its affine extension. Similar considerations also apply for the E_6 and E_7 configurations when no additional blowups are present; i.e., we simply proceed from the Dynkin diagram to its affine extension.

To round out the analysis, we need to demonstrate that if we perform any blowups of an E_6 or E_7 endpoint, we can again attach a -2 curve at the same location, without inducing any further blowups.[11] That this is indeed the case is conveniently summarized by simply writing down the possible blowups. For E_6, there are two other consistent bases, and for E_7, there is one. In both cases, we can indeed still add our -2 curve without inducing extraneous blowups:

$$\overset{\overset{\overset{\overset{2}{3}}{2}}{1}}{232181232} \to \overset{\overset{\overset{2}{3}}{1}}{2315132} \to \overset{\overset{2}{2}}{22222} \qquad (8.6)$$

$$\overset{\overset{5}{1}}{223131322} \to \overset{2}{2222222}. \qquad (8.7)$$

Summarizing, we have just demonstrated that all bases for 6D SCFTs embed in a LST base.

1. LST endpoints

As a brief aside, one of the interesting features of this argument is that we have implicitly relied on the notion of a

LST endpoint. Given that we have already classified all such bases, we can also ask about the possible endpoints for LSTs. Compared with the case of 6D SCFTs, the number of distinct endpoints is comparatively small. Roughly speaking, this is because of the positive semidefinite condition for our adjacency matrix, which in turn means that many configurations will blow down to a single curve of self-intersection zero (as in the $1,2,\ldots,2,1$ configurations).

Combining the tensor-decoupling criterion with the demand that we have a positive semidefinite adjacency matrix means that the total number of endpoints is given by the Kodaira-type intersections of -2 curves, an elliptic curve, as well as a single $\mathbb{P}^1$ of self-intersection zero, which we denote by P. In the latter two cases, we note that we only obtain a 6D LST by having a nontrivial elliptic fibration. Thus, we find the following list of LST endpoints,[12]

$$\text{LST endpoints}: \ P, I_n, II, III, IV, I_n^*, II^*, III^*, IV^*, \quad (8.8)$$

for $n \geq 0$.

B. Embedding the fibers

Suppose next that we have supplemented a base by an additional curve. When we do this, additional nontrivial fibers are sometimes inevitable and can in turn force additional structure on the elliptic fibers. To give a concrete example, consider the case of a 6D SCFT with base given by the E_6 Dynkin diagram of -2 curves. Fiber decorations for this model were studied in Ref. [20] where it was found that, typically, additional flavors can be added so that an appropriate convexity condition on the ranks is obeyed. To extend this to a LST base, we cannot add a -1 curve (as the blowdown is inconsistent). Rather, we must add an additional -2 curve to reach the affine $\hat{E}_6$ Dynkin diagram of -2 curves. When we do this, we must remember that the elliptic fibration of the resulting F-theory model also becomes rigid. So in other words, the available elliptic fibrations are further constrained. It is at this stage that we must include the effects of Higgsing as well as tensor branch deformations to reach the original 6D SCFT. That this is always possible follows from the fact that all representations of the Lie algebra $\mathfrak{e}_6$ embed in representations of its affine extension $\hat{\mathfrak{e}}_6$. Similar considerations apply for the fiber decorations of all of the E-type bases.

In the case of the A- and D-type bases, the analysis is comparatively simpler. The reason is that we can just add additional tails of the form $1,2,\ldots,2$ with trivial fibers and leave the fibers above the curves in the SCFT base as they were. One might worry here about the fact that curves of self-intersection -1 and -2 are not always allowed to have

---

[11]Note that attaching a -1 curve will not work since we would then blowdown to a configuration where the adjacency matrix is not positive semidefinite—a contradiction.

[12]Here we neglect the possibility of torsion in the normal direction to the compact curves of the base.

trivial fibers. For instance, the -1 curve in the sequence 2,2,3,1,5 necessarily has a type *II* fiber. However, such subtleties do not arise in this case; one can always add small instanton links of the form 1,2,…,2 with trivial fibers to an A- or D-type base to get a LST.

Putting together our analysis of tensor branch flows and Higgs branch flows, we conclude that all 6D SCFTs embed in some LST. Indeed, it is also clear that there can sometimes be more than one such embedding.

IX. T-DUALITY

In the previous sections, we used the geometry of F-theory compactifications to tightly constrain the structure of LSTs. In this section, we turn the analysis around and show how the physics of little string theories suggests nontrivial geometric structures for elliptic Calabi-Yau threefolds in which the noncompact base has a negative semidefinite intersection form.

In physical terms, one of the key features of LSTs is that the description as a local quantum field theory must break down near the string scale. A sharp way to probe this structure is by compactifying on a circle. Recall that in T-duality, the theory compactified on a small S^1 of radius R is dual to another string theory compactified on an S^1 of radius $\tilde{R} \sim \alpha'_{\text{eff}}/R$. Based on this, it is natural to expect that all LSTs have a similar T-duality.

This expectation suggests a nontrivial constraint on the geometry of an F-theory realization of a LST. Recall that F-theory compactified on an elliptically fibered Calabi-Yau threefold $X \to B$ leads, upon further compactification on an S^1, to M-theory compactified on the same Calabi-Yau threefold. In the M-theory description, the Kähler class becomes a dynamical modulus (which is taken to zero size to reach the F-theory limit). On the other hand, T-duality tells us that if we take this circle to be very small, we should expect to obtain another LST, this time compactified on a large radius circle. For this to be so, we must have available to us more than one way to reach an F-theory background from a given compactification of M-theory on X. In other words, physical considerations suggest the existence of *another* elliptic fibration for our Calabi-Yau threefold $X \to \tilde{B}$, and the lift from M-theory to F-theory involves collapsing the Kähler class of this other elliptic fiber to zero size.

Our plan in this section will be to give further evidence that T-duality is realized in F-theory constructions of LSTs through the presence of a double elliptic fibration. We shall, however, mainly focus on particular examples of how T-duality is realized geometrically. After this, we give a sketch for how we expect this correspondence to work in general, leaving a complete proof to future work.

A. Examples

As a first example, consider the T-duality between the LST of $k \geq 2$ NS5-branes in IIA [the $\mathcal{N} = (2,0)$ A-type LSTs] and that of k NS5-branes in IIB string theory [the $\mathcal{N} = (1,1)$ A-type LSTs]. The IIA realization just follows from a base with -2 curves arranged in a loop, i.e., as the type I_{k-1} degeneration of an elliptic curve. The F-theory elliptic curve is then a smooth T^2, i.e., an I_0 fiber. Switching the roles of these two curves, we get the type IIB $\mathcal{N} = (1,1)$ LST; i.e., we have $(k-1)$ D7-branes wrapped over a T^2. There is a clear extension of this case to all of the ADE $\mathcal{N} = (2,0)$ LSTs in terms of the corresponding ADE 7-branes wrapped over a T^2.

Another class of examples we have already encountered several times involves the LSTs realized by M5-branes filling $\mathbb{R}^{5,1}$ and probing the geometry $S^1_\perp \times \mathbb{C}^2/\Gamma$ for Γ an ADE subgroup of $SU(2)$. Compactifying on a further circle, we can shrink the $S^1_\perp$ factor to reach IIA string theory. T-duality is then inherited from that of the physical superstring theory.

In the F-theory realization of these systems, we have a base quiver with conformal matter suspended in between the nodes:

$$// \underbrace{g \oplus g \oplus \cdots \oplus g \oplus g}_{k} //. \tag{9.1}$$

As we have already remarked in the classification of such structures, all fiber types and conformal matter are necessarily minimal; no deviations from this rigid structure are possible. This actually means that we can readily identify the other elliptic fiber of this model; it is given by a suitable multiple of an I_{kd_i} fiber, where the d_i denote the Dynkin labels of the affine extension of the g-type Lie algebra. Another consequence of this analysis is that sometimes the absence of other fiber decorations for say the E-type affine bases means we cannot arbitrarily combine these two structures. Rather, since the only fiber enhancements over the E-type bases are I_k-type fibers (i.e., without inducing further blowups), there is again a clear exchange between the roles of the two (singular) elliptic curves.

Quite strikingly, this also entails the existence of several infinite classes of models which are self-T-dual upon toroidal reduction. These are models which have a double elliptic structure consisting of two copies of the same Kodaira fiber, for example, models which have an I_N Kodaira base, with gauge groups $\mathfrak{su}(N)$ on each -2 curve and bifundamentals at the intersections. As a more exotic example, consider the blown up type *IV* degeneration: $33\overset{3}{3}$. On each -3 curve in this configuration, there is an $\mathfrak{su}(3)$ gauge sector which arises precisely from a type *IV* fiber. Clearly, contracting the -1 curve, switching the fiber and base, and then blowing up, we obtain back the same model.[13]

---

[13]See Ref. [44] for a further analysis of this case.

As a final class of examples, consider the F-theory models with a base:

$$[E_8]\underbrace{1, 2, \ldots, 2, 1}_{k}[E_8]. \qquad (9.2)$$

This is realized in heterotic M-theory by a collection of k M5-branes in between the two heterotic walls. Compactifying on a further circle and activating appropriate Wilson lines for the background $E_8 \times E_8$ flavor symmetry, we reach—via T-duality of the physical super-string—the case of k NS5-branes of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string theory. This system can be analyzed in perturbative string theory, and the T-dual LST is therefore realized by an $\mathfrak{sp}(k)$ gauge theory with 32 half-hypermultiplets in the fundamental representation and a single hypermultiplet in the two-index antisymmetric representation.

Demonstrating the presence of the extra elliptic fibration for the F-theory model is somewhat more subtle in this case, but we can see it as descending from a $\mathbb{Z}_2$ quotient of the I_{2k}-type Kodaira configuration of -2 curves:

$$//\underbrace{2, 2, \ldots, 2, 2}_{2k}//\overset{\mathbb{Z}_2}{\to}\underbrace{1, 2, \ldots, 2, 1}_{k}. \qquad (9.3)$$

In Appendix F, we present an explicit analysis of this case of the base 1,1 and also explain its extension to the configuration $1, 2, \ldots, 2, 1$. The corresponding F-theory model is realized by a base given by P, a single $\mathbb{P}^1$ with self-intersection zero, with a fiber enhancement I_{2k}^{ns}; i.e., we get an $\mathfrak{sp}(k)$ 7-brane wrapped over P. We have already classified the matter enhancements for this case in Sec. VII, and indeed we find agreement with the purely heterotic analysis. Note that the $\mathfrak{sp}$-type algebra originates from the $\mathbb{Z}_2$ quotient of an A_{2k} algebra via its outer automorphism.

In fact, a similar observation allows us to extend this to some of the models in which we have a $\mathbb{P}^1$ of self-intersection zero. For example, under a further $\mathbb{Z}_2$ quotient, we can reach some of the gauge theories already encountered via more direct methods,

$$[\mathfrak{su}_8]\overset{\mathfrak{su}_N}{1}, \overset{\mathfrak{su}_N}{1}[\mathfrak{su}_8]\overset{\mathbb{Z}_2}{\to}[\mathfrak{su}_{16}]\overset{\mathfrak{su}_N}{0}[\mathfrak{su}_2], \qquad (9.4)$$

where the $\mathfrak{su}_{16}$ flavor symmetry acts on the 16 hyper-multiplets in the fundamental representation and the $\mathfrak{su}_2$ flavor symmetry acts on the hypermultiplets in the two-index antisymmetric representation. From this perspective, we can still recognize the quotient of an additional elliptic fiber. Schematically, we have

$$//2, 2//\overset{\mathbb{Z}_2}{\to}1, 1\overset{\mathbb{Z}_2}{\to}0. \qquad (9.5)$$

B. Examples involving curves with torsion normal bundle

Recall that in Sec. IV, we found that the $\mathcal{N} = (1, 1)$ LSTs admitted an ABCDEFG classification according to their corresponding affine Lie algebras. In the F-theory realization of these models, we also saw that the base contained a genus 1 curve with a torsion normal bundle. We encountered torsion normal bundles again in Sec. VII A 1 in our discussion of rank zero LSTs. We will now explain these examples in more detail.

By construction, examples in this class admit two genus 1 fibrations: $(T_F^2 \times S)/\mathbb{Z}_m \to (T_F^2 \times C)/\mathbb{Z}_m$ and $(T_F^2 \times S)/\mathbb{Z}_m \to S/\mathbb{Z}_m$. The first fibration has a section with monodromy over the central fiber of $\pi: (T_F^2 \times C)/\mathbb{Z}_m \to C$. The second fibration does not have a section (i.e., it is not an elliptic fibration, in the terminology of Ref. [50]), and we discuss its structure here.

Recall that for a genus 1 fibration $X \to B$ without a section, there is an associated "Jacobian fibration" $J(X/B) \to B$ which has a section and which has precisely the same τ function describing the fibers.[14] As explained in Refs. [12,50,51], the set of X's which share a common Jacobian fibration (and are equipped with an action by the Jacobian fibration, as stressed in Ref. [52]) forms a group[15] which should be identified with the group of connected components of the gauge group in F-theory. That is, compactifying the F-theory model on a circle, there is a discrete choice of one of these X's to serve as the compactification space for M-theory, which is the hallmark of a discrete gauge choice.

In our examples, the action of $\mathbb{Z}_m$ on S has fixed points, leading to A_{m-1} singularities on the quotient $S/\mathbb{Z}_m$. (More generally, there can be $A_{\ell-1}$ singularities for any ℓ dividing m, due to fixed points of subgroups.) Over an $A_{\ell-1}$ point, we have taken the quotient of the fiber by a translation of order ℓ, so that fiber has multiplicity ℓ. Notice that, even though the fiber is multiple, the total space is smooth.

This is precisely the situation analyzed by Mark Gross in Ref. [54], who showed that the Jacobian fibration is fibered over the same base, still with $A_{\ell-1}$ singularities. But once the Jacobian fibration has been taken, it is possible to resolve those singularities of the base (which corresponds

---

[14]A more common notation is $J(X)$, but since we are studying Calabi-Yau threefolds with more than one genus 1 fibration, we indicate the fibration for emphasis.

[15]This group has been incorrectly called the Tate-Shafarevich group in the literature [50,51], for which the fourth author apologizes. The group contains the Tate-Shafarevich group [53], but it can also contain fibrations with isolated multiple fibers [54], a fact which arose in the proof of the finiteness theorem for elliptic Calabi-Yau threefolds [55,56]. The examples presented here have such isolated multiple fibers and so do not belong to the Tate-Shafarevich group but rather to the larger group of Calabi-Yau threefolds sharing a common Jacobian fibration.

physically to giving an expectation value to scalars in the corresponding tensor multiplets). We thus find that these theories are part of a larger family of LSTs, but at special values of the tensor moduli in the larger family, a finite gauge group appears, leading to additional 5D vacua (i.e., the fibrations without a section) corresponding to distinct sectors of Wilson line expectation values.

Let us illustrate this with two concrete examples drawn from Appendix G, where we work out the $\mathcal{N} = (1, 1)$ theories of BCFG type in detail. As a first example, consider a partially blown down graph of type $\hat{E}_8$, illustrated here (and in Fig. 4),

where we blow down each solid circle to an A_1 singularity. As shown in Appendix G, there is a genus 1 fibration X over this base B with a fiber of multiplicity 2 at each A_1 singularity. The fiber never degenerates over this locus.

The Jacobian fibration $J(X/B)$ is simply a Weierstrass fibration over B of which the fibers do not degenerate. There is no obstruction to resolving the singularities of B, moving out into the rest of the moduli space. In fact, this is part of the moduli space of the $\mathcal{N} = (2, 0)$ theory of type E_8, and we claim that when the tensors are tuned to blow down precisely the curves corresponding to solid nodes, a $\mathbb{Z}_2$ gauge symmetry appears in the theory. That gauge symmetry is necessary to explain the additional 5D models which appear when the A_1's are blown down and a twist of the Jacobian fibration is possible. Presumably, moving away from this locus amounts to Higgsing the $\mathbb{Z}_2$ gauge symmetry. Incidentally, this example is the T-dual of the $\mathcal{N} = (1, 1)$ model with group F_4.

As a second example [which is the T-dual of the $\mathcal{N} = (1, 1)$ model with group G_2], consider another partially blown down graph of type $\hat{E}_8$, illustrated here (and in Fig. 5),
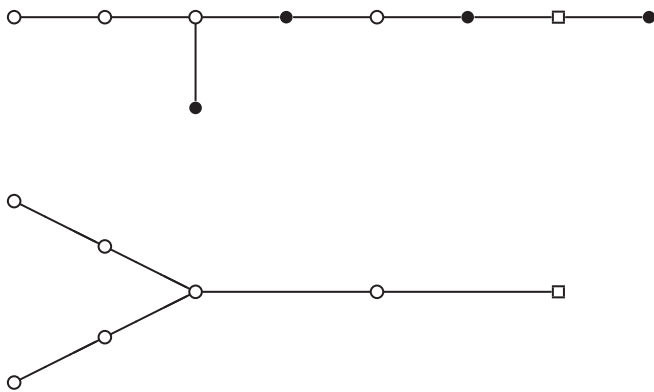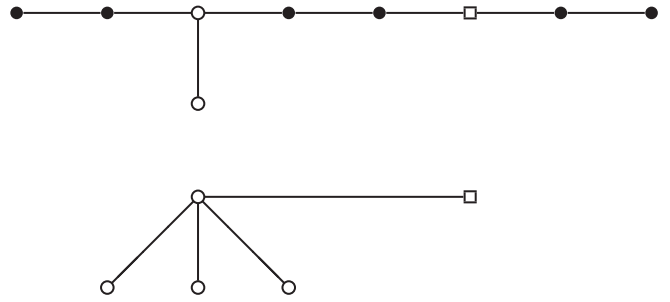

FIG. 4. F_4.

FIG. 5. G_2.

where this time, we blow down each lined pair of solid circles to an A_2 singularity. There is a fibration with multiple fibers of multiplicity 3 over each of those points. The Jacobian fibration is again part of the $\mathcal{N} = (2, 0)$ theory, with a different tuning of the moduli. With this tuning, we find a $\mathbb{Z}_3$ gauge symmetry.

A related example was given in Sec. VII A 1 using the action of $\mathbb{Z}_6$ on a nonsingular elliptic curve with $j = 0$. The quotient $S/\mathbb{Z}_6$ has central fiber which is a rational curve passing through surface singularities of types A_1, A_2, and A_5 with fibers of multiplicity 2, 3, and 6 over those points. Resolving the singularities leads to another affine $\hat{E}_8$ diagram,
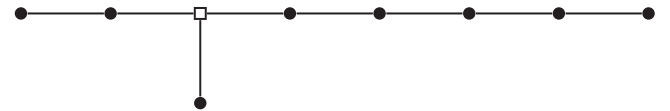
where again the solid circles represent the curves being blown down. Tuning the moduli of the $\mathcal{N} = (2, 0)$ E_8 theory to blow down those curves leads to a locus in moduli with a $\mathbb{Z}_6$ gauge symmetry. Notice that this is the intersection of the previous two loci with $\mathbb{Z}_2$ and $\mathbb{Z}_3$ gauge symmetry.

C. Toward T-duality in the more general case

In the examples from the previous two subsections, we saw that the expected appearance of T-duality for a LST motivates the search for a double elliptic fibration structure in such F-theory models. When the base has a fibration by curves of genus 1, the origin of this second fibration is clear. When the fibration on the base is by curves of genus 0, however, the T-duality is not as readily manifest.

Example (9.2) does have manifest T-duality, as further analyzed in Appendix F. In this case, the Weierstrass model $y^2 = F(x, \psi, [s, t])$ can be regarded as a double cover of a $\mathbb{P}^1$-bundle over the base B, where x denotes the coordinate on the $\mathbb{P}^1$. Since the double cover is branched at four points

along each fiber $\mathbb{P}^1_F$ of this fibration, the total space gets a fibration by curves of genus 1.

Now, the base B of that F-theory model has a fibration $\pi : B \to C$ of its own of which the general fiber is another $\mathbb{P}^1$ which we shall call $\mathbb{P}^1_B$. In Appendix F, after making a birational modification of the base (i.e., blowing it up and down), we find that the double cover is branched along four points of each fiber $\mathbb{P}^1_B$ as well, and this implies that there is a *second* genus 1 fibration on the total space. Note that this echoes the discovery made in Ref. [3] that heterotic T-duality, when viewed from the perspective of F-theory, exchanges the roles of a base and fiber in the heterotic weak coupling limit.

It is therefore natural to seek out a more general geometric exchange symmetry in LSTs with T-duality. We leave a more complete investigation of this possibility to future work.

X. OUTLIERS AND NONGEOMETRIC PHASES

Much as in the case of the classification of SCFTs achieved in Ref. [20], we view the F-theory realization of LSTs as providing a systematic approach to the construction of such models. In some cases, however, we indeed find a few small gaps between what is expected based on field theory considerations and what can be obtained in geometric phases of F-theory.

Our plan in this short section will be to proceed mainly by effective field theory considerations to give a list of such outlying behavior, both for 6D SCFTs and LSTs. We hasten to add that some of these putative theories may end up being inconsistent due to the lack of an embedding in an F-theory (or other string) construction.[16] In some cases, however, this also points to a few additional novel possibly nongeometric structures. Though we shall comment on possible ways to realize these models in F-theory, we leave a more complete analysis to future work.

A. Candidate LSTs and SCFTs

From a bottom up perspective, the primary constraints on the construction of consistent LSTs are the existence of a lattice of string charges with a negative semidefinite Dirac pairing and possible gauge groups "decorating" the associated tensor multiplets.

In some cases, there is a clear indication from F-theory that certain bottom up considerations are too weak; for example, the phenomenon of a $-n$ curve for $n \geq 3$ always implies the existence of a nontrivial gauge group factor, a condition which is not obvious from any anomaly cancellation condition.

There are, however, two intermediate cases suggested by field theory which also have a potential realization in string

theory. The first case deals with the gauge-gravitational anomaly cancellation condition [see (A7)] imposed in all geometrically realized 6D SCFTs and LSTs. This condition has no field-theoretic analog in flat space as there are no *a priori* restrictions on one-loop contribution to the mixed gauge-gravitational anomaly. The nonvanishing of the total mixed gauge-gravitational anomaly implies that the theory is inconsistent when put on a fixed curved spacetime background. It would clearly be troublesome if 6D SCFTs were anomalous in this way. Fortunately, it is always possible to cancel the one-loop contribution to this anomaly against a Green-Schwarz contribution, as was demonstrated in Ref. [58]. This argument does not apply to 6D LSTs, and indeed it can be checked explicitly in many examples of 6D LSTs that there is no way to cancel the one-loop mixed gauge-gravitational anomaly using the Green-Schwarz mechanism. This means that 6D LSTs cannot always be put on a fixed curved spacetime.

One class of such models would arise on

$$12 \ldots 21, \tag{10.1}$$

where we decorate all the tensors with $\mathfrak{su}$-type gauge groups along with antisymmetric matter for the last $\mathfrak{su}$ gauge group and symmetric matter for the first $\mathfrak{su}$ gauge group. With the presence of this symmetric representation, Eq. (A7) is violated. Nonetheless, this model can be constructed by putting type IIA on an interval $S^1/\mathbb{Z}_2$ with an $O8^-$-orientifold plane on each fixed point, D6s stretched along the interval, NS5s embedded in the D6s at various points along the interval, and two of the NS5s stuck, respectively, at the two fixed points.

More generally, to construct new examples of 6D SCFTs and LSTs which violate the one-loop mixed gauge-gravitational anomaly condition (A7), we can do the following: take any F-theory model having a -1 curve or P base curve associated with $\mathfrak{su}$-type gauge group and at least one antisymmetric hyper and 16 fundamental hypers not transforming under any other gauge group. Then, we can replace this set of hypermultiplets with a single hypermultiplet in the symmetric representation. The resulting theory will satisfy gauge anomaly cancellation, but it will have a nonvanishing one-loop gauge-gravitational anomaly.

The second case has to do with the condition of "normal crossing" which is present in all geometrically realized 6D SCFTs and is only mildly violated in LSTs. For example, we have seen that an intersection with an order 2 tangency $4\|1$ leads to a consistent LST base. In the bottom up perspective, we have a negative semidefinite Dirac pairing which has 2 on the off-diagonal entries and -4 and -1 on the diagonals. Generalizing, we can consider constructions such as

---

[16]For a recent example of a seemingly consistent 6D SCFT which is actually inconsistent, see, e.g., Appendix A of Ref. [57].

$$4\|2...2\ 1 \qquad (10.2)$$

$$4\|2...2\|4, \qquad (10.3)$$

all of which have a negative definite Dirac pairing. Though we have not encountered these possibilities in our discussion of geometric phases of F-theory, the first two have been realized in IIA string theory via appropriate suspended brane configurations (see, e.g., Ref. [6]), at least when there are nontrivial gauge group factors over the associated tensor multiplets. For example, we can have $\mathfrak{su}$ type on the -2 charge tensors and $\mathfrak{so}$-type gauge groups on the -4 tensors, with $\mathfrak{sp}$-type gauge groups on the -1 tensors. The crucial ingredient appearing in these suspended brane models is an $O8^+$ orientifold, rather than an $O8^-$ orientifold. The T-dual description in IIB string theory involves $O7^+$-orientifold planes, a case which leads to some nongeometric behavior, a point we shall return to later. The last example $4\|2...2\|4$ resists an embedding in IIA string theory, since it would appear to involve two $O8^+$-orientifold planes. Indeed, if we attempt to decorate these tensor multiplets by $\mathfrak{so}$-type algebras on the -4 tensors and $\mathfrak{su}$-type algebras on the -2 tensors, we cannot cancel gauge theoretic anomalies.

Assuming that configurations such as $4\|2$ can indeed occur in the construction of LSTs, it is natural to ask how many additional models can be obtained, at least from a bottom up perspective. Pairing each tensor multiplet with a simple gauge algebra, these are as follows:

$$\overset{\mathfrak{so}(M)}{4} \overset{\mathfrak{su}(N_1)}{\|} \overset{\mathfrak{su}(N_2)}{2} \;...\; \overset{\mathfrak{su}(N_k)}{2} \overset{\mathfrak{sp}(N_R)}{1} \qquad (10.4)$$

$$\overset{\mathfrak{sp}(N_1)}{1} \overset{\overset{\overset{\mathfrak{sp}(N_T)}{1}}{\mathfrak{so}(M_1)}}{4} \;...\; \overset{\mathfrak{sp}(N_k)}{1} \overset{\mathfrak{so}(M_k)}{4} \overset{\mathfrak{su}(N_R)}{\|} \;2 \qquad (10.5)$$

$$\overset{\mathfrak{su}(N')}{2} \overset{\mathfrak{sp}(N_1)}{1} \overset{\mathfrak{so}(M_1)}{4} \;...\; \overset{\mathfrak{so}(M_k)}{4} \overset{\mathfrak{su}(N_R)}{\|} \;2 \;. \qquad (10.6)$$

It should be noted that gauge and mixed anomalies strongly constrain the allowed gauge algebras in the above list. In particular, the configuration $\overset{\mathfrak{so}(M)}{4} \| \overset{\mathfrak{su}(N)}{2}$ is constrained by mixed anomalies to have a bifundamental $(\mathbf{M}, \mathbf{N})$, requiring $N \leq M - 8$, $2N \geq M$. These conditions lead to strong constraints on the rest of the gauge algebras in the aforementioned theories, as discussed in Ref. [21].

If one also relaxes the condition that there is a gauge group paired with each tensor, even more constructions are possible. In most of these cases, no known embedding in a string construction is available, so we suspect that at least some of these theories are actually inconsistent. Nevertheless, for the sake of completeness, we list them here:

$$2\|4\|2 \qquad (10.7)$$

$$4\ 1\ 4\|2\ 2 \qquad (10.8)$$

$$4\|2\ 2...2\|4 \qquad (10.9)$$

$$4\|2\ 2...2\ 1 \qquad (10.10)$$

$$4\|2\ 2...\overset{2}{2}\ 2 \qquad (10.11)$$

$$\overset{1}{1\ 4}\ 1...4\ 1\ 4\|2 \qquad (10.12)$$

$$1\ 3\ 1...4\ 1\ 4\|2 \qquad (10.13)$$

$$1\ 2\ 3\ 1\ 4...4\ 1\ 4\|2 \qquad (10.14)$$

$$1\ 2\ 2\ 3\ 1\ 4...4\ 1\ 4\|2 \qquad (10.15)$$

$$2\ 1\ 4...4\ 1\ 4\|2. \qquad (10.16)$$

It is worth mentioning that a violation of normal crossing and a violation of gauge-gravitational anomaly cancellation do not appear simultaneously in any of these examples. The lists of LSTs and putative LSTs arising from these violations are small and tightly constrained.

This perspective on LSTs also points to the existence of some additional novel structures for 6D SCFTs. Indeed, starting from a LST, we can consider a combination of tensor decoupling and Higgsing to reach some additional candidate SCFTs.

The theories of this type, which are consistent with anomaly cancellation and with no unpaired tensors, take the form

$$\overset{\mathfrak{so}(M)}{4} \overset{\mathfrak{su}(N_1)}{\|} \overset{\mathfrak{su}(N_2)}{2} \;...\; \overset{\mathfrak{su}(N_k)}{2} \qquad (10.17)$$

$$\overset{\mathfrak{sp}(N')}{1} \overset{\mathfrak{so}(M)}{4} \overset{\mathfrak{su}(N_1)}{\|} \overset{\mathfrak{su}(N_2)}{2} \qquad (10.18)$$

$$\overset{\mathfrak{so}(M_1)}{4} \overset{\mathfrak{sp}(N_1)}{1} \;...\; \overset{\mathfrak{so}(M_k)}{4} \overset{\mathfrak{su}(N_R)}{\|} \;2 \qquad (10.19)$$

$$\overset{\mathfrak{sp}(N_1)}{1} \overset{\mathfrak{so}(M_1)}{4} \;...\; \overset{\mathfrak{so}(M_k)}{4} \overset{\mathfrak{su}(N_R)}{\|} \;2 \;, \qquad (10.20)$$

Notice that (10.18) does not admit a known type IIA construction, whereas the other three do. Another curious thing to notice is that this model does not admit an embedding in a putative LST. The obvious examples of attaching an $\mathfrak{so}$ group to the left or attaching an $\mathfrak{su}$ group to the right are pathological because they necessarily have nonvanishing quartic part of gauge anomaly. Although we have only proved that every 6D SCFT can be embedded in a 6D LST for models arising geometrically within F-theory,

we expect this statement to be true in general. Hence, we suspect that the putative SCFT (10.18) is inconsistent. Finally, concentrating only on positive definiteness, we have the 4‖2 configurations:

$$1\ 4\|2\ 2 \tag{10.21}$$

$$4\|2\ 2...2 \tag{10.22}$$

$$1\ 4\ 1...4\ 1\ 4\|2 \tag{10.23}$$

$$4\ 1\ 4...4\ 1\ 4\|2 \tag{10.24}$$

$$3\ 1...4\ 1\ 4\|2 \tag{10.25}$$

$$2\ 3\ 1\ 4...4\ 1\ 4\|2 \tag{10.26}$$

$$2\ 2\ 3\ 1\ 4...4\ 1\ 4\|2. \tag{10.27}$$

B. Toward an embedding in F-theory

In the above, we encountered some additional candidate SCFTs and LSTs which appear to be consistent with effective field theory considerations. Additionally, some of these models admit an embedding in type IIA suspended brane constructions.

It is therefore important to ask to what extent we should expect F-theory to cover this and related examples. Though we leave a complete characterization to future work, there are some general ingredients we can already identify which point the way to incorporating these additional nongeometric structures.

As we have already mentioned, one crucial ingredient in the IIA realization of the 4‖2 configuration is the appearance of an $O8^+$-plane, which T-dualizes to a pair of $O7^+$-planes in type IIB string theory. Such orientifolds lead to the phenomena of "frozen singularities" in F-theory [11–13]. These are models in which the monodromy of the axiodilaton around the brane is consistent with that of an appropriate I_n^* singularity, but in which the corresponding gauge algebra is not $\mathfrak{so}(2n+8)$.

Another not entirely unrelated phenomenon we have encountered in the construction of the $\mathcal{N}=(1,1)$ LSTs, as well as in some of the low rank LSTs, are models in which the normal bundle of a curve on the base is torsion of finite order. To produce a Weierstrass model, we have found it necessary to impose specific restrictions on the order of these torsion bundles, though the M-theory realization of these A-type $\mathcal{N}=(1,1)$ theories suggests a whole family of models parametrized by rational theta angles [41].

In fact, it is relatively straightforward to engineer all the A-type $\mathcal{N}=(1,1)$ LSTs in type IIB and to lift this back to F-theory. For example, in type IIB language, we have a stack of N D7-branes wrapped on a T^2. Switching on a background value for a flat Ramond

Ramond (RR); Neveu-Schwarz (NS) 2-form potential, we get additional theories parametrized by the ratio of these two periods. In F-theory language, we see this by a choice of how we resolve the affine node of the $\hat{A}_{N+1}$ Dynkin diagram. In physical terms, this resolution comes from compactifying an eight-dimensional model on an additional S^1. Going down on a T^2, we have Wilson lines for this affine node along the A- and B-cycles of the T^2.

The presence of such background B-fields also suggests that similar effects from discrete group actions may also make an appearance in the construction of the 4‖2-type configurations. For example, at the level of the effective field theory, we can consider a base,

$$2\overset{2}{2}2...2, \tag{10.28}$$

with I_N-type fiber decorations on each -2 curve. Now, this field theory admits a $\mathbb{Z}_2$ automorphism in which we combine the outer automorphism of the D-type base with an outer automorphism on the $\mathfrak{su}_N$ factors on the leftmost -2 curve and the top -2 curve. At the level of gauge theory, the outer automorphisms of $\mathfrak{su}_N$ can take us to either an $\mathfrak{so}$- or $\mathfrak{sp}$-type algebra. Combining these two operations, we see that we get an effective field theory where there is one less tensor multiplet, in which the Bogomol'nyi–Prasad–Sommerfield (BPS) charge has doubled, and in which the Dirac pairing between this $\mathbb{Z}_2$ invariant combination and its neighbor is 2, leaving us with a configuration of tensor multiplets 4‖2...2. Taking into account the algebra assignment (i.e., the would-be fibers), we can in principle have either $\mathfrak{so}$- or $\mathfrak{sp}$-type algebras, of which only the former is compatible with anomaly cancellation.[17] Similar considerations also apply for the LST tensor multiplet configurations such as 4‖2...21.

Let us stress that effective field theory considerations do not directly inform us of the actual nongeometric realization of these models in F-theory. Indeed, what is particularly remarkable is that, *even if* we allow these additional structures, the total number of additional candidate SCFTs and LSTs is quite small, with the vast majority being covered by geometric phases of F-theory. This suggests that, whatever the mild deformation of known F-theory backgrounds are that produce these models, the structures encountered in this paper and in earlier work remain quite robust.

---

[17]Let us recall that in F-theory, conjugation by an outer automorphism apparently leads to a geometrically ambiguous assignment for the quotient algebra. This ambiguity has been resolved by appealing to anomaly cancellation considerations, in which the opposite conclusion is reached, i.e. a non-split I_N fiber realizes an $\mathfrak{sp}$-type algebra [59]. However, effective field theory considerations suggest that when combined with a quotient on the tensor multiplets, there may be a generalization of this construction available in which we instead reach an $\mathfrak{so}$-type algebra. Incorporation of an $O7^+$ plane or of discrete B-fields might provide a route to such a generalization.

XI. CONCLUSIONS

In this paper, we have given a systematic approach to realizing supersymmetric little string theories via compactifications of F-theory. Much as in the case of 6D SCFTs, these theories arise from working with F-theory on a noncompact base, in which some collection of curves are simultaneously collapsed to small size. The key difference with a 6D SCFT is that the intersection pairing for these curves defines a negative semidefinite quadratic form on the lattice of string charges. So, in contrast to the case of SCFTs, the associated theories contain a dimensionful parameter which is naturally promoted to a non-ynamical tensor multiplet. After spelling out all necessary conditions to geometrically realize LSTs in F-theory, we have given a classification of all such theories. On the one hand, these theories can all be viewed as arising from extending 6D SCFTs by one or more additional curves. As such, they also admit an atomic classification, much as in Ref. [20]. We have also seen that the general expectation that all 6D SCFTs embed in a LST is indeed realized via the explicit embedding in an F-theory compactification. Finally, we have seen that T-duality of a LST is realized via a double elliptic fibration in the corresponding F-theory model. In the remainder of this section, we discuss some avenues for future investigation.

Perhaps the most important issue left open by our analysis is the small gap between theories realized by geometric phases of F-theory and the list of effective field theories which can potentially complete to a LST (or SCFT). It would be interesting to establish to what extent nongeometric deformations can enter in such F-theory models and, conversely, how many of these putatively consistent LSTs (and SCFTs) are actually excluded by further nontrivial consistency conditions.

One of the key simplifications in our analysis of LSTs is that decoupling any curve in the base takes us back to a collection of (possibly decoupled) 6D SCFTs and scale invariant theories (i.e., when we have free vector multiplets). This strongly suggests that the common notions of renormalization group flows for local quantum field theories extend to *nonlocal* LSTs. Developing the details of such a structure would provide a rather striking vantage point on what it means to "UV complete" a quantum field theory in the first place.

As a preliminary step in this direction, it is also natural to ask whether there is a notion of monotonic loss in the degrees of freedom in such conjectural flows from LSTs to SCFTs. For example, in many of the cases studied in this paper, we can weakly gauge both diffeomorphisms as well as an $SU(2)$ field strength, which in the context of a 6D SCFT would be identified with the R-symmetry of the theory. It is tempting to conjecture that there is a formal extension of conformal anomalies to these cases as well. It would be interesting to study whether an extension of the methods presented in Refs. [60,61] (see also Ref. [62]) would provide insight into such generalizations of renormalization group flows.

Finally, one of the hallmarks of the systems we have encountered is the appearance of an effective T-duality upon compactification on a further circle. Given that there are now concrete methods for extracting the partition functions for some 6D SCFTs (see, e.g., Ref. [63]), it would be quite natural to study this and related structures for LSTs.

APPENDIX A: BRIEF REVIEW OF ANOMALY CANCELLATION IN F-THEORY

The allowed gauge algebras and matter content for a given tensor branch structure is heavily constrained by anomaly cancellation. In six dimensions, anomalies are related to four-point amplitudes of external currents J_a associated with continuous symmetry group G_a. In such a four-point amplitude, insertions of external currents J_a for a given gauge group G_a must come in pairs, so we need only consider the anomalies related to the four-point functions $\langle J_a J_a J_a J_a \rangle$ (gauge anomaly cancellation) and $\langle J_a J_a J_b J_b \rangle$ (mixed gauge anomaly cancellation).

For a representation R of some gauge group G, we introduce constants Ind_R, x_R, and y_R relating the quadratic and quartic Casimirs of G as follows:

$$\text{Tr}_R F^2 = Ind_R \text{tr} F^2,$$
$$\text{Tr}_R F^4 = x_R \text{tr} F^4 + y_R (\text{tr} F^2)^2. \tag{A1}$$

Here, tr indicates the trace in a defining representation of the group.[18]

The constraints on gauge and mixed anomalies take the form [45–47,64]

$$Ind_{Adj_a} - \sum_R Ind_{R_a} n_{R_a} = 6(10-n)\frac{\Omega_{IJ}a^I b_a^J}{\Omega_{IJ}a^I a^J} \quad (A2)$$

$$y_{Adj_a} - \sum_R y_{R_a} n_{R_a} = -3(10-n)\frac{\Omega_{IJ}b_a^I b_a^J}{\Omega_{IJ}a^I a^J} \quad (A3)$$

$$x_{Adj_a} - \sum_R x_{R_a} n_{R_a} = 0 \quad (A4)$$

$$\sum_{R,R'} Ind_{R_a} Ind_{R'_b} n_{R_a R'_b} = (10-n)\frac{\Omega_{IJ}b_a^I b_b^J}{\Omega_{IJ}a^I a^J}. \quad (A5)$$

Here, Ω_{IJ} is the natural metric on the space of antisymmetric tensors, and a^I, b_a^J are related to the anomaly 8-form I_8 via

$$I_8 = \frac{1}{2}\Omega_{IJ}X^I X^J, \qquad X^I = \frac{1}{2}a^I \text{tr}R^2 + 2b_a^I \text{tr}F_a^2. \quad (A6)$$

These field-theoretic conditions can be translated into F-theory language as restrictions on the allowed gauge algebras and matter for a given base. Any gauge algebra summand $\mathfrak{g}_a$ in the theory is paired with a tensor multiplet, which is in turn associated with a curve Σ_a in the base. In these terms, the anomaly cancellation conditions become

$$Ind_{Adj_a} - \sum_R Ind_{R_a} n_{R_a} = 6(K \cdot \Sigma_a)$$
$$= 6(2g_a - 2 - \Sigma_a \cdot \Sigma_a) \quad (A7)$$

$$y_{Adj_a} - \sum_R y_{R_a} n_{R_a} = -3(\Sigma_a \cdot \Sigma_a) \quad (A8)$$

$$x_{Adj_a} - \sum_R x_{R_a} n_{R_a} = 0 \quad (A9)$$

$$\sum_{R,R'} Ind_{R_a} Ind_{R'_b} n_{R_a R'_b} = \Sigma_a \cdot \Sigma_b. \quad (A10)$$

Here, K is the canonical divisor of the base, and g_a is the genus of Σ_a. The adjunction formula $K \cdot \Sigma_a = 2(g_a - 1) - \Sigma_a \cdot \Sigma_a$ has been used in the second equality of (A7).

Gauge anomaly cancellation may be used to constrain the gauge groups paired with a given tensor node, and mixed anomaly cancellation constrains the gauge groups allowed on neighboring tensor nodes. However, tensors need not be paired with gauge groups. In particular, curves of self-intersection 0, −1, or −2 can be devoid of a gauge group entirely. Curves of self-intersection zero cannot touch any other curves, so this case is relatively uninteresting. Curves of self-intersection −1 and −2, on the other hand, can touch other curves.

APPENDIX B: MATTER FOR SINGULAR BASES AND TANGENTIAL INTERSECTIONS

In this Appendix, we review the relationship between matter, singularities, and tangential intersections. Further information can be found in Ref. [48].

Consider some curve Σ. When this curve becomes singular, there are two distinct notions of genus. Geometric genus, denoted p_g, is the topological genus of the curve after all singularities have been resolved. Arithmetic genus, denoted g, is the quantity related to the intersection theory of the curve by

$$2g - 2 = K \cdot \Sigma + \Sigma^2. \quad (B1)$$

The arithmetic genus is the one that shows up in the adjunction formula and hence enters the anomaly cancellation equation (A7). These two notions of the genus are related via

$$g = p_g + \sum_P \frac{m_P(m_P - 1)}{2}. \quad (B2)$$

Here, the sum runs over all singular points P of the curve, and m_P is the multiplicity of the singularity at P. For a curve with a nodal singularity (of Kodaira type I_1) or a cusp singularity (of Kodaira type II), there is a single singular point of multiplicity $m_P = 2$. Hence, in each of these cases, we have $g = p_g + 1$. The only values of g and p_g that can actually arise in our classification are $g = 1$ and $p_g = 0$.

Although p_g does not show up directly in the anomaly cancellation conditions, it still determines the F-theory matter content. Namely, p_g is precisely the number of adjoint hypermultiplets charged under the gauge algebra paired with this curve. For *smooth* LST bases, we see that there is one adjoint whenever the genus $g = p_g$ of the curve is 1. For singular curves, on the other hand, the arithmetic genus g can be 1 even without any adjoint hypermultiplets. For curves with gauge algebra $\mathfrak{su}(N)$, one instead finds symmetric and antisymmetric representations, as noted in Ref. [46].

We now consider the Kodaira type III base configuration, which consists of two −2 curves intersecting tangentially. First, we show that two curves meeting tangentially

can indeed carry fiber types I_0^* and I_4, which correspond to gauge algebras $\mathfrak{g}_2/\mathfrak{so}(7)/\mathfrak{so}(8)$ and $\mathfrak{su}(4)$, respectively. For this, we consider the Weierstrass model:

$$y^2 = x^3 + (zu + \sigma^2)x^2 + (zu + \sigma^2)^2 z^2 v x$$
$$+ (zu + \sigma^2)^3 z^4 w. \tag{B3}$$

Here, z and σ are local coordinates on the base, and u, v, and w are functions that do not vanish at $(z, \sigma) = (0, 0)$. From this, we compute f, g, and Δ to be

$$f = (zu + \sigma^2)^2(-1/3 + z^2 v)$$
$$g = (zu + \sigma^2)^3(2/27 - (1/3)z^2 v + z^4 w)$$
$$\Delta = z^4(zu + \sigma^2)^6(-v^2 - 18z^2 wv + 4z^2 v^3$$
$$+ 27z^4 w^2 + 4w). \tag{B4}$$

From this, we see that the curve $z = 0$ has Kodaira type I_4, whereas the curve $zu + \sigma^2$ has Kodaira type I_0^*. These curves meet at $(z, \sigma) = (0, 0)$, where they are tangent to each other. The multiplicities of f, g, and Δ at $(0,0)$ are 2, 3, and 10, respectively. On the other hand, had we attempted to intersect transversely curves with fibers of singularity types I_0^* and I_4, we would have found that the multiplicities of f, g, and Δ would have been 4, 6, and 12, respectively. Hence, curves with these fiber types cannot meet transversely, only tangentially.

We can enhance the gauge algebras on these tangent curves to $\mathfrak{so}(12)$ and $\mathfrak{su}(6)$, corresponding to Kodaira fiber types I_2^* and I_6, respectively. Here, the Weierstrass model is

$$y^2 = x^3 + (zu + \sigma^2)x^2 + (zu + \sigma^2)^3 z^3 v x. \tag{B5}$$

This yields

$$f = -\frac{1}{3}(zu + \sigma^2)^2 + vz^3(zu + \sigma^2)^3$$
$$g = \frac{2}{27}(zu + \sigma^2)^3 - \frac{1}{3}vz^3(zu + \sigma^2)^4$$
$$\Delta = -v^2 z^6(zu + \sigma^2)^8 + 4v^3 z^9(zu + \sigma^2)^9. \tag{B6}$$

Thus, from the degrees of vanishing of f, g, and Δ, we see the curve $z = 0$ has Kodaira type I_6, whereas the curve $zu + \sigma^2$ has Kodaira type I_2^*.

We also get $\mathfrak{so}(13)$ and $\mathfrak{su}(7)$ from Kodaira fiber types I_3^* and I_7, respectively, using the Weierstrass model,

$$y^2 = x^3 + (zu + \sigma^2)x^2 + w(zu + \sigma^2)^6 z^7. \tag{B7}$$

This model has

$$f = -\frac{1}{3}(zu + \sigma^2)^2$$

$$g = \frac{2}{27}(zu + \sigma^2)^3 + wz^7(zu + \sigma^2)^6$$

$$\Delta = 4wz^7(zu + \sigma^2)^9 + 27w^2 z^{12}(zu + \sigma^2)^{16}. \tag{B8}$$

The fact that we got $\mathfrak{so}(13)$ in this case and $\mathfrak{so}(12)$ in the previous case follows from Table 7 of Ref. [45].

Using these Weierstrass models, we are able to realize the gauge algebra enhancements on two tangent -2 curves discussed in Sec. VII.

APPENDIX C: NOVEL DE-TYPE BASES

Appendixes B and C [20] provided a complete list of DE-type bases that can be used to construct 6D SCFTs. All of these bases can show up in LSTs as well when certain non-DE-type links are suitably attached. However, there are also some novel DE-type bases, which blow down to 0.

We use an abbreviated notation to describe these bases. Namely, we specify curves of self-intersection -4, -6, -8, and -12 according to

$$D \simeq 4 \tag{C1}$$

$$E_6 \simeq 6 \tag{C2}$$

$$E_7 \simeq 8 \tag{C3}$$

$$E_8 \simeq 12. \tag{C4}$$

The allowed types of conformal matter between DE-type nodes are represented as follows:

$$D1D \simeq D \overset{1,1}{\oplus} D \tag{C5}$$

$$E_6131E_6 \simeq E_6 \overset{2,2}{\oplus} E_6 \tag{C6}$$

$$E_712321E_6 \simeq E_7 \overset{3,3}{\oplus} E_6 \tag{C7}$$

$$E_712321E_7 \simeq E_7 \overset{3,3}{\oplus} E_7 \tag{C8}$$

$$E_71231D \simeq E_7 \overset{3,2}{\oplus} D \tag{C9}$$

$$E_812231D \simeq E_8 \overset{4,2}{\oplus} D \tag{C10}$$

$$E_61315131E_6 \simeq E_6 \overset{3,3}{\circ} E_6 \tag{C11}$$

$$E_613151321E_7 \simeq E_6 \overset{3,4}{\oplus} E_7 \tag{C12}$$

$$E_6131513221E_8 \simeq E_6 \overset{3,5}{\oplus} E_8 \tag{C13}$$

$$E_7 123151321 E_7 \simeq E_7 \overset{4,4}{\oplus} E_7 \tag{C14}$$

$$E_7 1231513221 E_8 \simeq E_7 \overset{4,5}{\oplus} E_8 \tag{C15}$$

$$E_8 12231513221 E_8 \simeq E_8 \overset{5,5}{\oplus} E_8. \tag{C16}$$

We omit the subscripts above the $\oplus$ when dealing with the minimal type of conformal matter:

$$D \oplus D \simeq D \overset{1,1}{\oplus} D \tag{C17}$$

$$D \oplus E_6 \simeq D \overset{2,2}{\oplus} E_6 \tag{C18}$$

$$D \oplus E_7 \simeq D \overset{2,3}{\oplus} E_7 \tag{C19}$$

$$D \oplus E_8 \simeq D \overset{2,4}{\oplus} E_8 \tag{C20}$$

$$E_6 \oplus E_6 \simeq E_6 \overset{2,2}{\oplus} E_6 \tag{C21}$$

$$E_6 \oplus E_7 \simeq E_6 \overset{3,3}{\oplus} E_7 \tag{C22}$$

$$E_6 \oplus E_8 \simeq E_6 \overset{3,5}{\oplus} E_8 \tag{C23}$$

$$E_7 \oplus E_7 \simeq E_7 \overset{3,3}{\oplus} E_7 \tag{C24}$$

$$E_7 \oplus E_8 \simeq E_7 \overset{4,5}{\oplus} E_8 \tag{C25}$$

$$E_8 \oplus E_8 \simeq E_8 \overset{5,5}{\oplus} E_8. \tag{C26}$$

Using this notation, we now list the novel DE-type bases for LSTs. All of these bases are positive semidefinite with a single zero eigenvalue. We begin with configurations with only D nodes:

$$D \overset{3,3}{\oplus} D \tag{C27}$$

$$D \overset{3,3}{\bigcirc} D. \tag{C28}$$

The configurations with only E_6 nodes are

$$E_6 \overset{5,5}{\oplus} E_6. \tag{C29}$$

The configurations with D and E_6 are

$$D \overset{3,5}{\oplus} E_6 \tag{C30}$$

$$D \oplus E_6^{\oplus 2} \overset{3,3}{\oplus} E_6 \tag{C31}$$

$$D \oplus E_6^{\oplus 2} \overset{3,3}{\bigcirc} E_6 \tag{C32}$$

$$D \oplus E_6 \overset{3,4}{\oplus} E_6 \tag{C33}$$

$$D \oplus E_6 \overset{3,2}{\oplus} D \tag{C34}$$

$$D^{\oplus 2} \oplus E_6 \oplus D. \tag{C35}$$

The configurations with D and E_7 are

$$D \overset{3,3}{\oplus} E_7^{\oplus n} \overset{3,3}{\oplus} D, \qquad n = 1, 2, \ldots \tag{C36}$$

$$D \oplus E_7^{\oplus 3} \overset{4,4}{\oplus} E_7 \tag{C37}$$

$$D \oplus E_7^{\oplus 2} \overset{4,5}{\oplus} E_7 \tag{C38}$$

$$D^{\oplus 2} \oplus E_7 \oplus D \tag{C39}$$

$$D \overset{2,4}{\oplus} E_7 \oplus D \tag{C40}$$

$$D \oplus 1\overset{2}{3}21 \oplus E_7. \tag{C41}$$

The configurations with D and E_8 are

$$D \overset{3,5}{\oplus} E_8^{\oplus n} \overset{5,3}{\oplus} D, \qquad n = 1, 2, \ldots \tag{C42}$$

$$D^{\oplus 3} \oplus E_8 \oplus D^{\oplus 2} \tag{C43}$$

$$D^{\oplus 4} \oplus E_8 \oplus D. \tag{C44}$$

The configurations with E_6 and E_8 are

$$E_6 \overset{5,5}{\oplus} E_8^{\oplus n} \overset{5,5}{\oplus} E_6, \qquad n = 1, 2, \ldots \tag{C45}$$

$$E_6 \overset{4,5}{\oplus} E_8^{\oplus 2} \oplus E_6^{\oplus 2}. \tag{C46}$$

The configurations with D, E_6, and E_7 are

$$D \oplus E_7 \overset{4,4}{\oplus} E_6 \tag{C47}$$

$$D \oplus E_7^{\oplus 2} \overset{4,3}{\oplus} E_6 \tag{C48}$$

$$D \oplus E_6^{\oplus 2} \overset{3,5}{\oplus} E_7 \tag{C49}$$

$$D \oplus E_6^{\oplus 3} \overset{3,4}{\oplus} E_7 \tag{C50}$$

$$D \oplus E_6^{\oplus 4} \oplus E_7 \tag{C51}$$

$$D \oplus E_7^{\oplus 2} \oplus E_6^{\oplus 2} \tag{C52}$$

$$D \oplus E_6 \oplus E_7 \oplus E_6^{\oplus 2}. \tag{C53}$$

The configurations with D, E_6, and E_8 are

$$D \overset{4,5}{\oplus} E_8^{\oplus n} \overset{5,5}{\oplus} E_6, \qquad n = 1, 2, \ldots \tag{C54}$$

$$D \oplus E_6^{\oplus 6} \oplus E_8 \tag{C55}$$

$$D^{\oplus 2} \oplus E_8^{\oplus 2} \oplus E_6^{\oplus 2} \tag{C56}$$

$$D^{\oplus 3} \oplus E_8 \overset{5,4}{\oplus} E_6. \tag{C57}$$

The configurations with D, E_7, and E_8 are

$$D \oplus E_7^{\oplus 6} \overset{4,5}{\oplus} E_8 \tag{C58}$$

$$D^{\oplus 2} \oplus E_8^{\oplus 3} \oplus E_7^{\oplus 2}. \tag{C59}$$

The configurations with E_6, E_7, and E_8 are

$$E_6 \overset{4,5}{\oplus} E_8 \oplus E_7 \oplus E_6 \tag{C60}$$

$$E_6 \overset{4,5}{\oplus} E_8^{\oplus 3} \oplus E_7^{\oplus 2}. \tag{C61}$$

The configurations with D, E_6, E_7, and E_8 are

$$D \oplus E_8 \oplus E_7^{\oplus 2} \oplus E_6 \tag{C62}$$

$$D^{\oplus 2} \oplus E_8 \oplus E_7 \oplus E_6. \tag{C63}$$

APPENDIX D: NOVEL NON-DE-TYPE BASES

The following is the list of novel bases constructed solely from curves of self-intersection -1, -2, -3, and -5:

$$12\ldots.21 \tag{D1}$$

$$5 \oplus 1^{\oplus 5} \tag{D2}$$

$$\overset{1}{\underset{1}{1}}512 \tag{D3}$$

$$2\overset{1}{1}512 \tag{D4}$$

$$1312 \tag{D5}$$

$$\overset{1}{1}31 \tag{D6}$$

$$131\overset{1}{5}12 \tag{D7}$$

$$131\overset{1}{\underset{1}{5}}12 \tag{D8}$$

$$123\overset{1}{5}12 \tag{D9}$$

$$123\overset{1}{\underset{1}{5}}12 \tag{D10}$$

$$1223\overset{1}{5}12 \tag{D11}$$

$$1223\overset{1}{\underset{1}{5}}12 \tag{D12}$$

$$2\overset{1}{3}1512 \tag{D13}$$

$$2\overset{1}{3}1\overset{1}{5}1 \tag{D14}$$

$$3131512 \tag{D15}$$

$$313\overset{1}{1}51 \tag{D16}$$

$$215131512 \tag{D17}$$

$$2151\overset{1}{3}151 \tag{D18}$$

$$\overset{1}{1}513\overset{1}{1}51 \tag{D19}$$

$$\overset{1}{1}5131512 \tag{D20}$$

$$\overset{1}{1}513\overset{1}{1}51 \tag{D21}$$

$$1315131512 \tag{D22}$$

$$13151\overset{1}{3}151 \tag{D23}$$

$$12315131512 \tag{D24}$$

$$123151\overset{1}{3}151 \tag{D25}$$

$$122315131512 \tag{D26}$$

$$1223151\overset{1}{3}151 \tag{D27}$$

$$151231512 \tag{D28}$$

$$15123\overset{1}{1}51 \tag{D29}$$

$$512231512 \tag{D30}$$

$$5122315\overset{1}{1} \tag{D31}$$

$$12321512 \tag{D32}$$

$$12321\overset{1}{5}1 \tag{D33}$$

$$1321512 \tag{D34}$$

$$13215\overset{1}{1} \tag{D35}$$

$$13\overset{1}{2}2 \tag{D36}$$

$$\overset{2}{2}2\ldots21 \tag{D37}$$

$$151\overset{1}{3}22 \tag{D38}$$

$$1512\overset{1}{3}2 \tag{D39}$$

$$512222 \tag{D40}$$

$$151222 \tag{D41}$$

$$3122 \tag{D42}$$

$$1315122 \tag{D43}$$

$$12315122 \tag{D44}$$

$$122315122 \tag{D45}$$

$$215122 \tag{D46}$$

$$\overset{1}{1}5122 \tag{D47}$$

$$13151\overset{1}{3}2 \tag{D48}$$

$$123151\overset{1}{3}2 \tag{D49}$$

$$1223151\overset{1}{3}2 \tag{D50}$$

$$2151\overset{1}{3}2 \tag{D51}$$

$$\overset{1}{1}51\overset{1}{3}2 \tag{D52}$$

$$32132 \tag{D53}$$

$$12231321 \tag{D54}$$

$$23123 \tag{D55}$$

$$12313221 \tag{D56}$$

$$1313221 \tag{D57}$$

$$1512231 \tag{D58}$$

$$13213 \tag{D59}$$

$$123131 \tag{D60}$$

$$13131 \tag{D61}$$

$$312321 \tag{D62}$$

$$3132151 \tag{D63}$$

$$2313151 \tag{D64}$$

$$151232151 \tag{D65}$$

$$1232151321 \tag{D66}$$

$$5123132 \tag{D67}$$

$$3215123 \tag{D68}$$

$$123151231 \tag{D69}$$

$$13151231 \tag{D70}$$

$$1321513221 \tag{D71}$$

$$5131322 \tag{D72}$$

$$13215131 \tag{D73}$$

$$3221513 \tag{D74}$$

$$1512313 \tag{D75}$$

$$5122313 \tag{D76}$$

$$1231512321 \tag{D77}$$

$$12231512321 \tag{D78}$$

$$131512321 \tag{D79}$$

$$12315132151 \tag{D80}$$

$$1315132151 \tag{D81}$$

$$315123151 \tag{D82}$$

$$122315132215 \tag{D83}$$

$$321513151 \tag{D84}$$

$$315123215 \tag{D85}$$

$$513215132 \tag{D86}$$

$$51223151321 \tag{D87}$$

$$512315123 \tag{D88}$$

$$151231513221 \tag{D89}$$

$$3131513221 \tag{D90}$$

$$1512315131 \tag{D91}$$

$$5122315131 \tag{D92}$$

$$512321513 \tag{D93}$$

$$123151313 \tag{D94}$$

$$13151313 \tag{D95}$$

$$513151232 \tag{D96}$$

$$15131513215 \tag{D97}$$

$$15131513151 \tag{D98}$$

$$12231513151321 \tag{D99}$$

$$12315131513221 \tag{D100}$$

$$1315131513221 \tag{D101}$$

$$123151315131 \tag{D102}$$

$$13151315131 \tag{D103}$$

$$51315131513. \tag{D104}$$

APPENDIX E: NOVEL GLUINGS

At times, non-DE-type side links or noble atoms can attach to DE-type nodes in ways they could not for 6D SCFTs. For instance, the side link 2151321 could never attach to a D node in a 6D SCFT, since it induces four blowdowns on the -4 curve. However, this is allowed for LSTs. The full list of these novel gluings of one side link to a single node is as follows:

For gluing to a D-type node (i.e., a -4 curve), we have

$$2221 \oplus D \tag{E1}$$

$$2151321 \oplus D \tag{E2}$$

$$3215131 \oplus D \tag{E3}$$

$$1512321 \oplus D \tag{E4}$$

$$3151231 \oplus D \tag{E5}$$

$$3\overset{1}{1}5131 \oplus D \tag{E6}$$

$$2\overset{1}{2}31 \oplus D \tag{E7}$$

$$1\overset{1}{5}1321 \oplus D \tag{E8}$$

$$151315131 \oplus D \tag{E9}$$

$$512315131 \oplus D. \tag{E10}$$

For gluing to an E_6-type node (i.e., a -6 curve), we have

$$222221 \oplus E_6 \tag{E11}$$

$$22\overset{1}{3}15131 \oplus E_6 \tag{E12}$$

$$231512321 \oplus E_6 \tag{E13}$$

$$232151321 \oplus E_6 \tag{E14}$$

$$321513221 \oplus E_6 \tag{E15}$$

$$31\overset{1}{5}13221 \oplus E_6 \tag{E16}$$

$$23\overset{1}{1}51321 \oplus E_6 \tag{E17}$$

$$15131513221 \oplus E_6 \tag{E18}$$

$$51231513221 \oplus E_6 \tag{E19}$$

$$3151315121 \oplus E_6 \tag{E20}$$

$$23151315131 \oplus E_6. \tag{E21}$$

For gluing to an E_7-type node (i.e., a -8 curve), we have

$$\underbrace{22222221}_{8} \oplus E_7 \tag{E22}$$

$$2231513151321 \oplus E_7 \tag{E23}$$

$$223151^{1}3221 \oplus E_7 \tag{E24}$$

$$223151^{1}3221 \oplus E_7, \tag{E25}$$

while for an E_8-type node (i.e., a -12 curve), we have

$$\underbrace{222222222221}_{12} \oplus E_8. \tag{E26}$$

We remark that in the case of the E_7 and E_8 nodes, we can also delete some of the aforementioned curves, using instead a "primed" node (i.e., by adding a small instanton link elsewhere).

APPENDIX F: T-DUALITY IN THE 1,2,…,2,1 MODEL

Let us consider the LST of which the F-theory base B has a chain of rational curves consisting of two curves of self-intersection -1 at the ends of the chain and $k \geq 0$ curves of self-intersection -2 in between meet each other. The union of those $k + 2$ curves on the base deforms to a nonsingular rational curve of self-intersection 0 and provides a fibration $\pi : B \to C$ on the base of which the general fiber is $\mathbb{P}^1$. The total space of the elliptic fibration looks like a one parameter family of elliptic K3 surfaces (over general fibers of π) degenerating to a pair of dP_9's with k intermediate elliptic ruled surfaces [over $\pi^{-1}(0)$], which (when $k = 0$) is precisely the degeneration which appears in the analysis of heterotic/F-theory duality in Ref. [65].

Remarkably, the total space of this elliptic fibration admits a second elliptic fibration, at least birationally. The computation which shows this was written out in Ref. [66], although it has some earlier antecedents in the math literature [67–70].[19]

Consider a base of the form $\mathbb{P}^1 \times \mathbb{C}$ where the homogeneous coordinates on $\mathbb{P}^1$ are $[\sigma, \tau]$ on the coordinate on $\mathbb{C}$ is ψ. We write a Weierstrass equation of the form

$$Y^2 = X^3 + a\sigma^4\tau^4 X + (\psi^{k+1}\sigma^5\tau^7 + c\sigma^6\tau^6 + \sigma^7\tau^5), \tag{F1}$$

---

[19]There was a subsequent extension of this computation to a more general case [70–72], which will undoubtedly be useful for understanding additional T-dualities of LSTs.

where a and c are constants. (In Ref. [66], only the $\tau = 1$ affine chart appears.) Notice that we get a small instanton with instanton number $k + 1$ when $\psi = \sigma = 0$, so to get the $1, 2, …, 2, 1$ model, we should blow up the point $\psi = \sigma = 0$ $k + 1$ times. We keep this implicit in what follows.

Note that we have Kodaira type II^* at both $\sigma = 0$ and $\tau = 0$, and since those represent noncompact curves, we see $E_8 \times E_8$ global symmetry.

Now, there is a remarkable coordinate change:

$$X = st^{-5}x^2 \tag{F2}$$

$$Y = t^{-8}x^2 y \tag{F3}$$

$$\sigma = t^{-3}x \tag{F4}$$

$$\tau = t. \tag{F5}$$

(This is only a "rational map" because of the division by powers of t.) The result of the substitution is

$$t^{-16}x^4y^2 = s^3t^{-15}x^6 + ast^{-13}x^6 \\ + (\psi^{k+1}t^{-8}x^5 + ct^{-12}x^6 + t^{-16}x^7). \tag{F6}$$

If we multiply the resulting equation by t^{16}/x^4, we obtain

$$y^2 = s^3tx^2 + ast^3x^2 + (\psi^{k+1}t^8x + ct^4x^2 + x^3) \tag{F7}$$

$$= x^3 + (s^3t + ast^3 + ct^4)x^2 + \psi^{k+1}t^8x. \tag{F8}$$

We interpret this as a family elliptic curves over $\mathbb{P}^1_{[s,t]} \times \mathbb{C}$. (Note that this is a very different base, with coordinates $[s, t]$ which mix the former base and fiber coordinates.) Remarkably, this is the equation for the F-theory dual of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string with a small instanton at $t = \psi = 0$ (as derived in Ref. [3,65]). We have a global symmetry algebra $\mathfrak{so}(32)$ along $t = 0$. Note that we should also blow up $t = \psi = 0$.

Thus, in a quite subtle way, there is T-duality for this pair of models, in which the two different elliptic fibrations on the semilocal total space are exchanged, after a birational change. Note that the coordinate change given above is (rationally) invertible. The inverse is

$$x = \sigma\tau^3 \tag{F9}$$

$$y = \sigma^{-2}\tau Y \tag{F10}$$

$$s = \sigma^{-2}\tau^{-1}X \tag{F11}$$

$$t = \tau. \tag{F12}$$

TABLE III.   Cyclic actions on ALE spaces.

	r	Type	φ	Description	Gauge group	θ
(1)	Any	$\frac{1}{r}(1,-1,0;0)$	$xy+z^n$	$A_{n-1} \xrightarrow{r\text{-to-1}} A_{rn-1}$	$SU(n)$	$\in \pi\mathbb{Q}$
(2)	4	$\frac{1}{4}(1,3,2;2)$	$x^2+y^2+z^{2n-1}$	$A_{2n} \xrightarrow{4\text{-to-1}} D_{2n+3}$	$Sp(n)$	π
(3)	2	$\frac{1}{2}(0,1,1;0)$	$x^2+y^2+z^{2n}$	$A_{2n-1} \xrightarrow{2\text{-to-1}} D_{n+2}$	$Sp(n)$	0
(4)	3	$\frac{1}{3}(0,1,2;0)$	$x^2+y^3+z^3$	$D_4 \xrightarrow{3\text{-to-1}} E_6$	G_2	0
(5)	2	$\frac{1}{2}(1,1,0;0)$	$x^2+y^2z+z^n$	$D_{n+1} \xrightarrow{2\text{-to-1}} D_{2n}$	$SO(2n+1)$	0
(6)	2	$\frac{1}{2}(1,0,1;0)$	$x^2+y^3+z^4$	$E_6 \xrightarrow{2\text{-to-1}} E_7$	F_4	0

APPENDIX G: F-THEORY CONSTRUCTION OF $\mathcal{N}=(1,1)$ LSTs

The M-theory construction for little string theories with $\mathcal{N}=(1,1)$ supersymmetry is described in Ref. [41]. Such a theory can be seen as M-theory compactified on a spacetime of the form $(\mathbb{C}^2 \times S^1)/\Gamma$ for $\Gamma \subset SU(2)$ a finite subgroup. The action on S^1 is by rotations, and there is a subgroup Γ' of Γ which acts trivially on S^1, leading to a short exact sequence of groups,

$$0 \to \Gamma' \to \Gamma \to \mathbb{Z}_r \to 0. \tag{G1}$$

Geometrically, there is an action of $\mathbb{Z}_r$ on the asymptotically locally Euclidean (ALE) space $\mathbb{C}^2/\Gamma'$ corresponding to an automorphism of the corresponding Lie group; the gauge group of the little string theory is the subgroup commuting with the outer automorphism.

The mathematical description of these groups has been known for a long time and is nicely summarized in a table on p. 376 of Ref. [73] which we reproduce as Table III. We have made a minor correction to the table (already noted in footnote 15 of Ref. [41]), and we have added the information about the gauge group and theta angle as discussed in Ref. [41]. In the table, the cyclic group action on $\mathbb{C}^3$ with coordinates (x,y,z) is described in terms of exponents $(\frac{a}{r},\frac{b}{r},\frac{c}{r})$ of the generators; the notation also indicates how the equation transforms under the action.

To construct these theories using F-theory, we use a base B which is a neighborhood of an elliptic curve Σ of which the normal bundle is a torsion line bundle of order r. The base has a finite unramified cover of degree r which is a

product $\Sigma \times \mathbb{C}$, and the cyclic group $\mathbb{Z}_r$ will act on the elliptic fibration over $\Sigma \times \mathbb{C}$ (which we take to be in Weierstrass form). Thus, the classification is analogous— we must find cyclic group actions on Weierstrass elliptic fibrations which induce the corresponding actions on the ADE singularities.

Note that, as observed in Sec. VII A 1, this construction requires $r \in \{2,3,4,6\}$, so that most of the instances of case (1) are ruled out. In fact, we have been unable to find a conventional F-theory construction of any instance of case (1) (which would correspond, in the interpretation of Ref. [41], to an $SU(n)$ theory with a rational theta angle). Instead, as mentioned in Sec. X B, we anticipate an F-theory construction for these models involving B-field expectation values.

In Table IV, we present explicit forms of these group actions in cases 3, 4, and 6, using a Weierstrass equation φ with variables (x,y,t). The quotient can be described in terms of a Weierstrass equation Φ of which the variables (X,Y,T) are expressed in terms of (x,y,t) in the table.

We now explain the geometry of the group actions by means of figures illustrating cases 2, 3, 4, and 6. We will explain the cases in the reverse order from the one given in Table III.

We begin with case 6 leading to gauge symmetry F_4, illustrated in Fig. 4. The M-theory construction only involved the Dynkin diagram of the singularities, but here we must consider the entire Kodaira fiber as represented by the affine Dynkin diagram. Thus, in the bottom half of Fig. 4, we see $\hat{E}_6$, with the affine node represented by a square rather than a circle. This diagram has been obtained as a double cover of the affine E_8 diagram shown in the top

TABLE IV.   Group actions on Weierstrass models. Here, u and w represent invariant functions of t which do not vanish at $t=0$.

	Type	φ	(X,Y,T)	Φ
(3)	$\frac{1}{2}(0,1,1)$	$-y^2+x^3+ux^2+wt^{2n}$	(t^2x,t^3y,t^2)	$-Y^2+X^3+uTX^2+wT^{n+3}$
(4)	$\frac{1}{3}(1,0,2)$	$-y^2+x^3+wt^3$	(t^4x,t^6y,t^3)	$-Y^2+X^3+wT^5$
(6)	$\frac{1}{2}(0,1,1)$	$-y^2+x^3+wt^4$	(t^2x,t^3y,t^2)	$-Y^2+X^3+wT^5$

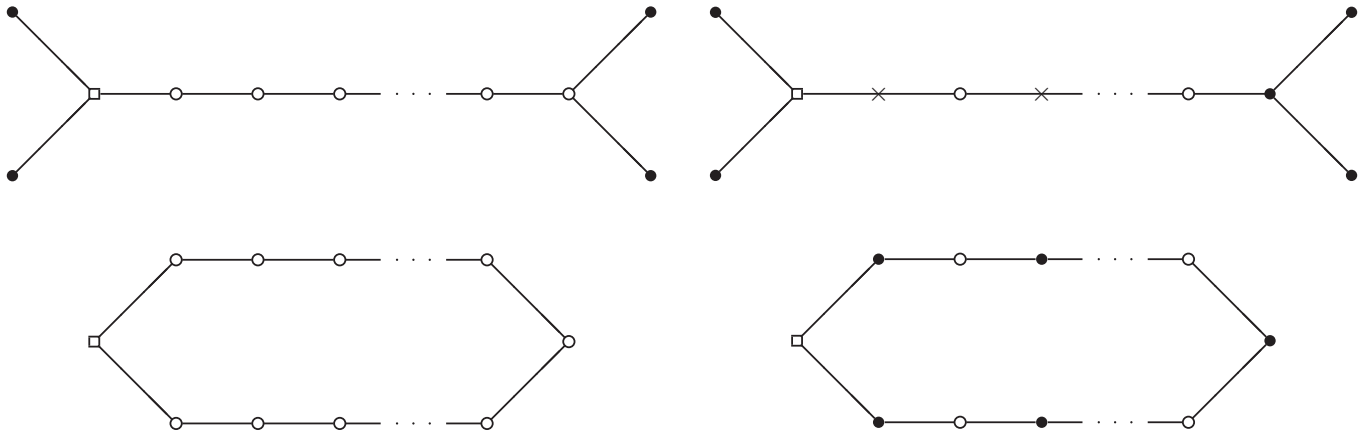FIG. 6. $Sp(n)$, $\theta = 0$.

FIG. 7. $Sp(n)$, $\theta = \pi$.

half of Fig. 4, in which the solid circles represent curves along which we branch as we construct a double cover. The two curves on the left are duplicated in the bottom half because they do not meet the trivalent vertex at a branch point of the double cover. After taking the double cover, the solid curves have become -1 curves and are to be blown down. (Alternatively, the solid curves can be contracted to A_1 singularities prior to taking the double cover).

Note that the quotient involves an affine E_8 diagram rather than an affine E_7 diagram. This is because the $\mathbb{Z}_2$ action on the "extra" curve in the Kodaira fiber (corresponding to the image of the affine node) has two fixed points, one giving an E_7 singularity and the other giving an A_1 singularity. The two together fit into an affine E_8 diagram.

We next treat case 4 leading to gauge symmetry G_2, depicted in Fig. 5. The bottom half of the figure this time is $\hat{D}_4$ with the affine node again represented by a square. This diagram has been obtained as a *triple* cover of the affine E_8 diagram shown in the top half of Fig. 5, in which the solid circles represent curves along which we branch as we construct a triple cover. (For an adjacent pair of solid curves, the branching is via 1/3 on one curve and 2/3 on the other). The lowest curve in the top half of the diagram is triplicated in the bottom half because it does not meet the trivalent vertex at a branch point of the triple cover. After taking the triple cover, the solid curves have become -1, -2 pairs and are to be blown down. (Alternatively, the solid curves can be contracted to A_2 singularities prior to taking the triple cover.) Note that the $\mathbb{Z}_3$ action on the image of the affine node has two fixed points, one giving an E_6
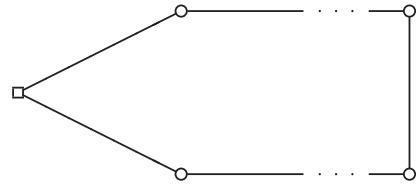
singularity and the other giving an A_2 singularity; the two together fit into an affine E_8 diagram.

Now, we consider case 3, depicted in Fig. 6. The bottom half is $\hat{A}_{2n-1}$, obtained as a branched double cover of the top half, which is $\hat{D}_{n+4}$. The square denotes the affine node (or its image), and the solid circles in the top half denote curves along which the double cover is branched. This time, the $\mathbb{Z}_2$ action on the affine node has two fixed points unrelated to the singularity we are studying, creating the two extra solid curves at the left of the diagram.

Finally, we treat case 2, depicted in Fig. 7, which is the most complicated case. Here, we have a $\mathbb{Z}_4$ action on $\hat{A}_{2n}$ at the bottom of the figure, and we describe the quotient in two stages: the quotient by the $\mathbb{Z}_2$ subgroup (shown in the middle of the figure as an $\hat{A}_{4n+1}$ diagram) and the quotient by the full $\mathbb{Z}_4$ (shown at the top of the figure as a $\hat{D}_{2n+5}$ diagram). The diagram at the top includes a resolved A_3 singularity (three connected solid dots), two resolved A_1 singularities (at the far left) which are to be branched along during the first double cover, and points marked by $\times$ which are A_1 singularities not branched during the first double cover (from top to middle) but branched during the second double cover (from middle to bottom).

[1] E. Witten, Some Comments On String Dynamics, arXiv: hep-th/9507121.

[2] P. S. Aspinwall, Point-like instantons and the $Spin(32)/\mathbb{Z}_2$ heterotic string, Nucl. Phys. **B496**, 149 (1997).

[3] P. S. Aspinwall and D. R. Morrison, Point-like instantons on K3 orbifolds, Nucl. Phys. **B503**, 533 (1997).

[4] N. Seiberg, New theories in six-dimensions and matrix description of M theory on T^5 and $T^5/\mathbb{Z}_2$, Phys. Lett. B **408**, 98 (1997).

[5] K. A. Intriligator, New string theories in six-dimensions via branes at orbifold singularities, Adv. Theor. Math. Phys. **1**, 271 (1997).

[6] A. Hanany and A. Zaffaroni, Branes and six-dimensional supersymmetric theories, Nucl. Phys. **B529**, 180 (1998).

[7] I. Brunner and A. Karch, Branes at orbifolds versus Hanany Witten in six-dimensions, J. High Energy Phys. 03 (1998) 003.

[8] A. Kapustin, On the universality class of little string theories, Phys. Rev. D **63**, 086005 (2001).

[9] K. A. Intriligator, Compactified little string theories and compact moduli spaces of vacua, Phys. Rev. D **61** (2000) 106005.

[10] O. Aharony, M. Berkooz, D. Kutasov, and N. Seiberg, Linear dilatons, NS five-branes and holography, J. High Energy Phys. 10 (1998) 004.

[11] E. Witten, Toroidal compactification without vector structure, J. High Energy Phys. 02 (1998) 006.

[12] J. de Boer, R. Dijkgraaf, K. Hori, A. Keurentjes, J. Morgan, D. R. Morrison, and S. Sethi, Triples, fluxes, and strings, Adv. Theor. Math. Phys. **4**, 995 (2000).

[13] Y. Tachikawa, Frozen, arXiv:1508.06679.

[14] N. Seiberg, Nontrivial fixed points of the renormalization group in six-dimensions, Phys. Lett. B **390**, 169 (1997).

[15] J. J. Heckman, D. R. Morrison, and C. Vafa, On the Classification of 6D SCFTs and Generalized ADE Orbifolds, J. High Energy Phys. 05 (2014) 028; 06 (2015) 17(E).

[16] D. Gaiotto and A. Tomasiello, Holography for (1,0) theories in six dimensions, J. High Energy Phys. 12 (2014) 003.

[17] M. Del Zotto, J. J. Heckman, A. Tomasiello, and C. Vafa, 6d conformal matter, J. High Energy Phys. 02 (2015) 054.

[18] J. J. Heckman, More on the Matter of 6D SCFTs, Phys. Lett. B **747**, 73 (2015).

[19] M. Del Zotto, J. J. Heckman, D. R. Morrison, and D. S. Park, 6D SCFTs and gravity, J. High Energy Phys. 06 (2015) 158.

[20] J. J. Heckman, D. R. Morrison, T. Rudelius, and C. Vafa, Atomic classification of 6D SCFTs, Fortschr. Phys. **63**, 468 (2015).

[21] L. Bhardwaj, Classification of 6d $\mathcal{N} = (1,0)$ gauge theories, J. High Energy Phys. 11 (2015) 002.

[22] D. S. Park, private communication.

[23] S. Hohenegger, A. Iqbal, and S.-J. Rey, Instanton-Monopole Correspondence from M-Branes on $\mathbb{S}^1$ and Little String Theory, arXiv:1511.02787.

[24] N. Seiberg and W. Taylor, Charge lattices and consistency of 6D supergravity, J. High Energy Phys. 06 (2011) 001.

[25] È. B. Vinberg, Discrete linear groups that are generated by reflections, Izv. Akad. Nauk SSSR Ser. Mat. **35**, 1072 (1971).

[26] M. Del Zotto, J. J. Heckman, D. S. Park, and T. Rudelius, On the Defect Group of a 6D SCFT, arXiv:1503.04806.

[27] H. Grauert, Über modifikationen und exzeptionelle analytische mengen, Math. Ann. **146**, 331 (1962).

[28] M. Artin, Some numerical criteria for contractability of curves on algebraic surfaces, Am. J. Math. **84**, 485 (1962).

[29] D. R. Morrison and W. Taylor, Classifying bases for 6D F-theory models, Central Eur. J. Phys. **10**, 1072 (2012).

[30] D. R. Morrison and W. Taylor, Toric bases for 6D F-theory models, Fortschr. Phys. **60**, 1187 (2012).

[31] C. Beasley, J. J. Heckman, and C. Vafa, GUTs and exceptional branes in F-theory—II: Experimental predictions, J. High Energy Phys. 01 (2009) 059.

[32] C. Córdova, Decoupling gravity in F-theory, Adv. Theor. Math. Phys. **15**, 689 (2011).

[33] J. J. Heckman and H. Verlinde, Evidence for F(uzz) theory, J. High Energy Phys. 01 (2011) 044.

[34] M. Bertolini, P. R. Merkx, and D. R. Morrison, On the global symmetries of 6D superconformal field theories, arXiv:1510.08056.

[35] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. Math. **116**, 133 (1982).

[36] Y. Kawamata, The cone of curves of algebraic varieties, Ann. Math. **119**, 603 (1984).

[37] S. Donaldson and S. Sun, Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, Acta Math. **213**, 63 (2014).

[38] S. Donaldson and S. Sun, Gromov-Hausdorff limits of Kahler manifolds and algebraic geometry, II, arXiv:1507.05082.

[39] M. Gross and P. M. H. Wilson, Large complex structure limits of $K3$ surfaces, J. Diff. Geom. **55**, 475 (2000).

[40] C. Schoen, On fiber products of rational elliptic surfaces with section, Math. Z. **197**, 177 (1988).

[41] E. Witten, New, "gauge", theories in six dimensions, J. High Energy Phys. 01 (1998) 001.

[42] N. Nakayama, *On Weierstrass Models*, Algebraic Geometry and Commutative Algebra (Kinokuniya, Tokyo, 1988), Vol. II, p. 405.

[43] M. R. Douglas and G. W. Moore, D-branes, Quivers, and ALE Instantons, arXiv:hep-th/9603167.

[44] D. R. Morrison, D. S. Park, and W. Taylor (unpublished).

[45] A. Grassi and D. R. Morrison, Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds, Commun. Num. Theor. Phys. **6**, 51 (2012).

[46] V. Sadov, Generalized Green-Schwarz mechanism in F theory, Phys. Lett. B **388**, 45 (1996).

[47] A. Grassi and D. R. Morrison, Group representations and the Euler characteristic of elliptically fibered Calabi–Yau threefolds, J. Algebraic Geom. **12**, 321 (2003).

[48] D. R. Morrison and W. Taylor, Matter and singularities, J. High Energy Phys. 01 (2012) 022.

[49] N. Nekrasov and V. Pestun, Seiberg-Witten geometry of four dimensional $N = 2$ quiver gauge theories, arXiv:1211.2240.

[50] V. Braun and D. R. Morrison, F-theory on genus-one fibrations, J. High Energy Phys. 08 (2014) 132.

[51] D. R. Morrison and W. Taylor, Sections, multisections, and $U(1)$ fields in F-theory, J. Singularities (to be published).

[52] M. Cvetič, R. Donagi, D. Klevers, H. Piragua, and M. Poretschkin, F-theory vacua with $\mathbb{Z}_3$ gauge symmetry, Nucl. Phys. **B898**, 736 (2015).

[53] I. Dolgachev and M. Gross, Elliptic threefolds. I. Ogg-Shafarevich theory, J. Algebraic Geom. **3**, 39 (1994).

[54] M. Gross, Elliptic three-folds. II. Multiple fibres, Trans. Am. Math. Soc. **349**, 3409 (1997).

[55] A. Grassi, Log contractions and equidimensional models of elliptic threefolds, J. Algebraic Geom. **4**, 255 (1995).

[56] M. Gross, A finiteness theorem for elliptic Calabi-Yau threefolds, Duke Math. J. **74**, 271 (1994).

[57] K. Ohmori, H. Shimizu, Y. Tachikawa, and K. Yonekura, 6d $\mathcal{N} = (1, 0)$ theories on S^1/T^2 and class S theories: part II, arXiv:1508.00915.

[58] K. Ohmori, H. Shimizu, Y. Tachikawa, and K. Yonekura, Anomaly polynomial of general 6d SCFTs, Prog. Theor. Exp. Phys. **2014**, 103B07 (2014).

[59] P. S. Aspinwall, S. H. Katz, and D. R. Morrison, Lie groups, Calabi-Yau threefolds, and F theory, Adv. Theor. Math. Phys. **4**, 95 (2000).

[60] J. J. Heckman, D. R. Morrison, T. Rudelius, and C. Vafa, Geometry of 6D RG flows, J. High Energy Phys. 09 (2015) 052.

[61] J. J. Heckman and T. Rudelius, Evidence for C-theorems in 6D SCFTs, J. High Energy Phys. 09 (2015) 218.

[62] C. Cordova, T. T. Dumitrescu, and K. Intriligator, Anomalies, Renormalization Group Flows, and the a-Theorem in Six-Dimensional (1,0) Theories, arXiv:1506.03807.

[63] B. Haghighat, G. Lockhart, and C. Vafa, Fusing E-strings to heterotic strings: $E + E \rightarrow H$, Phys. Rev. D **90**, 126012 (2014).

[64] M. Bershadsky and C. Vafa, Global Anomalies and Geometric Engineering of Critical Theories in Six Dimensions, arXiv:hep-th/9703167.

[65] D. R. Morrison and C. Vafa, Compactifications of F theory on Calabi-Yau threefolds. 2., Nucl. Phys. **B476**, 437 (1996).

[66] J. McOrist, D. R. Morrison, and S. Sethi, Geometries, nongeometries, and fluxes, Adv. Theor. Math. Phys. **14**, 1515 (2010).

[67] D. R. Morrison, On K3 surfaces with large Picard number, Inventiones Mathematicae **75**, 105 (1984).

[68] K. Oguiso, On Jacobian fibrations on the Kummer surfaces of the product of nonisogenous elliptic curves, J. Math. Soc. Jpn. **41**, 651 (1989).

[69] A. Clingher and C. F. Doran, Modular invariants for lattice polarized $K3$ surfaces, Michigan Math. J. **55**, 355 (2007).

[70] A. Kumar, $K3$ surfaces associated with curves of genus two, Int. Math. Res. Not. **2008** (2008).

[71] A. Clingher and C. F. Doran, Lattice polarized K3 surfaces and Siegel modular forms, Adv. Math. **231**, 172 (2012).

[72] A. Malmendier and D. R. Morrison, K3 surfaces, modular forms, and non-geometric heterotic compactifications, Lett. Math. Phys. **105**, 1085 (2015).

[73] M. Reid, Young person's guide to canonical singularities, in Algebraic geometry, Bowdoin, 1985, Vol. 46 of Proc. Sympos. Pure Math., pp. 345–414. Amer. Math. Soc., 1987.