

Axion electrodynamics and nonrelativistic photons in nuclear and quark matter

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We argue that the effective theory for electromagnetic fields in spatially varying meson condensations in dense nuclear and quark matter is given by the axion electrodynamics. We show that one of the helicity states of photons there has the nonrelativistic gapless dispersion relation $\omega \sim k^2$ at small momentum, while the other is gapped. This “nonrelativistic photon” may also be realized at the interface between topological and trivial insulators in condensed matter systems.

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I. INTRODUCTION

Recently, topologically nontrivial states of matter, such as the topological insulators and topological superconductors [1,2], have attracted much attention. The nontrivial topology of the fully gapped bulk is reflected in the physics of boundaries. The well-known example is the boundary between a topological insulator and an ordinary (or topologically trivial) insulator [3], where electromagnetic responses are modified and are described by the axion electrodynamics [4].

In this paper, we point out yet another realization of the essentially same physics of this surface state in a different system: spatially varying meson condensations in dense nuclear and quark matter, characterized by a nonzero gradient of some meson field (say φ), $\langle \nabla\varphi \rangle \neq 0$. Such phases appear in nuclear matter in a strong magnetic field (known as the pion domain wall) [5] and in color superconducting quark matter even in the absence of the magnetic field (known as the meson supercurrent phase) [6,7], which may be realized inside neutron stars or quark stars.

We argue, based only on the symmetries and topology of the system, that the low-energy dynamics of electromagnetic fields there is described by the axion electrodynamics. Similarly to the boundary of the topological insulator, the anomalous magnetoelectric effects (e.g., the quantum Hall effect) appear in these phases.

We also show that, due to the modifications of the electromagnetic dynamics, photons behave differently depending on the helicity at small momentum; one of the helicity states has the nonrelativistic gapless dispersion relation $\omega \sim k^2$, while the other is gapped [see Eq. (23)]. Such behavior, including the “nonrelativistic gapless photon,” is a universal feature of the axion electrodynamics not only there, but also at the interface between topological and ordinary insulators. To the extent of our knowledge, this is a new result, even in the latter context, which may be tested in table-top experiments.

In this paper, we take the units $\hbar = c = e = 1$.

II. SPATIALLY VARYING MESON CONDENSATIONS

We first review the possible spatially varying meson condensations in high density matter [5–7]. These examples in an external magnetic field include the π^0 domain wall in nuclear matter, the η meson domain wall in two-flavor color superconductivity (2SC), and the η' meson domain wall in color-flavor locked (CFL) phase [5]. The space-dependent meson condensation may also appear *even without* the external magnetic field at intermediate density between 2SC and CFL phases due the Fermi surface splitting induced by the large strange quark mass ($m_s \gg m_{u,d}$); in this context, it is called the meson supercurrent phase [6,7].

Our argument below is applicable to all of them, but for simplicity and concreteness, we will consider a space-dependent π^0 field (in nuclear matter) as an example. Applications of our argument to other possible spatially varying meson condensations are straightforward.

At low energy, the dynamics of QCD can be systematically described by the effective theory—the chiral perturbation theory for pions. Let us concentrate on the dynamics of neutral pion (which we denote by π^0), as the other degrees of freedom (π^\pm) will be irrelevant in our discussion. The effective Lagrangian is given by

$$\mathcal{L}_{\pi^0} = \frac{1}{2} (\partial_\mu \pi^0)^2 + m_\pi^2 f_\pi^2 \cos\left(\frac{\pi^0}{f_\pi}\right), \quad (1)$$

where f_π the pion decay constant and m_π is the pion mass.

The equation of motion for $\phi \equiv \pi^0/f_\pi$ derived from Eq. (1) is given by the sine-Gordon equation,

$$\partial_t^2 \phi - \partial_z^2 \phi + m_\pi^2 \sin \phi = 0, \quad (2)$$

where, for simplicity, we assume ϕ depends only on z . This has the domain-wall solution,

$$\phi(z) = 4 \tan^{-1} e^{m_\pi z}. \quad (3)$$

From this solution, it is easy to see that ϕ changes 2π across the wall and the thickness of the wall is $\sim 1/m_\pi$. The energy density per unit area is

$$\frac{E}{S} = 8f_\pi^2 m_\pi. \quad (4)$$

At finite baryon chemical potential μ_B in an external magnetic field, we have the additional ‘‘Wess-Zumino-Witten (WZW) term’’ related to quantum anomalies in the effective Lagrangian [5],

$$\mathcal{L}_B = C\mu_B \mathbf{B}_{\text{ex}} \cdot \nabla \pi^0, \quad (5)$$

where

$$C = \frac{1}{4\pi^2 f_\pi}. \quad (6)$$

This term gives the domain wall a nonzero baryon density given by

$$n_B = C\mathbf{B}_{\text{ex}} \cdot \nabla \pi^0, \quad (7)$$

so that baryon number per unit surface area is

$$\frac{N_B}{S} = \frac{B_{\text{ex}}}{2\pi}, \quad (8)$$

where $B_{\text{ex}} \equiv |\mathbf{B}_{\text{ex}}|$.

Combining Eqs. (4) and (8), the energy of the domain wall per baryon number is

$$\frac{E}{N_B} = \frac{16\pi f_\pi^2 m_\pi}{B_{\text{ex}}}. \quad (9)$$

When this energy becomes smaller than the baryon mass m_N , namely,

$$B_{\text{ex}} > \frac{16\pi f_\pi^2 m_\pi}{m_N}, \quad (10)$$

the π^0 domain wall is energetically more favorable than nuclear matter [5]; the ground state becomes the π^0 domain wall. A more elaborate analysis shows that the most energetically favorable ground state is the periodic array of π^0 domain walls that spontaneously break parity and continuous translational symmetries [8]. The structure of this state is mathematically similar to the ‘‘chiral magnetic soliton lattice’’ experimentally observed in chiral magnets CrNb_3S_6 in an external magnetic field [9].

In the chiral limit ($m_\pi = 0$), Eq. (10) shows that nuclear matter would be unstable against the decay into the space-dependent π^0 condensation in an infinitesimally small magnetic field. In this case, the Hamiltonian density is given by [10]

$$\mathcal{H} = \frac{1}{2}(\nabla \pi^0)^2 - C\mu_B \mathbf{B}_{\text{ex}} \cdot \nabla \pi^0. \quad (11)$$

This is minimized when

$$\langle \nabla \pi^0 \rangle = C\mu_B \mathbf{B}_{\text{ex}}, \quad (12)$$

and the minimum is

$$\langle \mathcal{H} \rangle = -\frac{1}{2}(C\mu_B \mathbf{B}_{\text{ex}})^2 < 0. \quad (13)$$

This is indeed lower than the energy density of nuclear matter for $\mu_B \approx m_N$ [5].

III. EFFECTIVE THEORY FOR ELECTROMAGNETIC FIELDS

Let us consider the low-energy effective theory of *dynamical* electromagnetic fields in spatially varying meson condensations. We stress that our argument here is independent of the microscopic origin of these phases. As an example, we here consider the presence of the space-dependent π^0 condensation and investigate its consequences.

Note that the low-energy effective theory does not respect the full Lorentz symmetry due to the presence of the medium. So the effective Lagrangian can be written, using the gauge-invariant electromagnetic fields, as¹

$$\mathcal{L}_{\text{EM}} = \frac{\epsilon}{2} \mathbf{E}^2 - \frac{1}{2\mu} \mathbf{B}^2, \quad (14)$$

with some constants ϵ and μ that depend on microscopic details. In the usual electromagnetism, ϵ and μ are called the permittivity and permeability, respectively. We assume that ϵ and μ do not depend on spatial coordinates and $\epsilon \sim \mu \sim O(1)$.

At low energy, we also need to take into account light meson degrees of freedom which couple to electromagnetic fields. In the presence the pion background field, we have the following WZW term due to the quantum anomalies [11,12]:

$$\mathcal{L}_{\text{WZW}} = C\pi^0 \mathbf{E} \cdot \mathbf{B}. \quad (15)$$

In the QCD vacuum, this term is related to the decay of a neutron pion to two photons, $\pi^0 \rightarrow 2\gamma$ [13,14]. The

¹For the π^0 domain wall in an external magnetic field B_{ex} in the z direction, the rotational symmetry is explicitly broken by the presence of B_{ex} . The effective Lagrangian for dynamical electromagnetic fields in this case is

$$\mathcal{L}_{\text{EM}} = \frac{\epsilon}{2} \mathbf{E}_\perp^2 + \frac{\epsilon'}{2} E_z^2 - \frac{1}{2\mu} \mathbf{B}_\perp^2 - \frac{1}{2\mu'} B_z^2,$$

where \mathbf{E}_\perp and \mathbf{B}_\perp are the electromagnetic fields in the xy plane. As a result, ϵ and μ in the z component in Eqs. (17) and (18) below are replaced by ϵ' and μ' . Note however that this replacement does not affect the discussion in Sec. IV.

coefficient C is uniquely determined by the anomaly matching and is given by Eq. (6) [11,12].

If $\langle\pi^0(\mathbf{x})\rangle$ is a constant in space as usual, this anomalous term is just a total derivative and does not affect the dynamics of electromagnetic fields.² However, when $\langle\pi^0(\mathbf{x})\rangle$ varies spatially, it modifies the equations of motion for electromagnetic fields. From the full Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{WZW}} + A_\mu j^\mu, \quad (16)$$

with the electric current density $j^\mu = (\rho, \mathbf{j})$, the equations of motion for the electromagnetic fields are derived as

$$\epsilon \nabla \cdot \mathbf{E} = \rho - C \langle \nabla \pi^0 \rangle \cdot \mathbf{B}, \quad (17)$$

$$\frac{1}{\mu} \nabla \times \mathbf{B} = \epsilon \partial_t \mathbf{E} + \mathbf{j} + C \langle \nabla \pi^0 \rangle \times \mathbf{E}. \quad (18)$$

The final terms in Eqs. (17) and (18) are the additional contributions to the usual Gauss's law and Ampère's law, respectively, due to the presence of the space-dependent π^0 field. The former correction was already found in Ref. [5] through a different path. On the other hand, the latter correction, which stands for the electric current in the direction perpendicular to the electric field, has not yet been appreciated. As we will see later, this anomalous Hall current modifies the dispersion relation of photons.

Note that, if $\langle\pi^0\rangle$ were time-dependent, $\langle\partial_t\pi^0\rangle$ would play the role of the so-called chiral chemical potential and generates the ‘‘chiral magnetic effect,’’ $\mathbf{j} \sim \langle\partial_t\pi^0\rangle\mathbf{B}$ [15].

The form of Lagrangian in Eq. (16) is equivalent to the axion electrodynamics [4]. The axion electrodynamics is also found to emerge at the interface between a topological insulator and a normal insulator in condensed matter physics, where the ‘‘axion field’’ $\theta(\mathbf{x})$ varies in space from $\theta = 0$ in the normal insulator to $\theta = \pi$ in the topological insulator [3]. In the specific case of the π^0 domain wall above, the field $\phi(z) = \pi^0(z)/f_\pi$ plays the role of the ‘‘axion field,’’ which varies from 0 to 2π across the wall.

IV. NONRELATIVISTIC PHOTONS

We now consider an electromagnetic wave (or a photon) in the spatially varying meson condensation. As an illustration, we consider the simplest case where $\langle\nabla\pi^0\rangle$ is homogeneous. Such a configuration is achieved, e.g., by Eq. (12) in a homogeneous external magnetic field [10]. Our argument may also be extended to periodic meson

²We note that, when $\langle\pi^0(\mathbf{x})\rangle \neq 0$, the WZW term above leads to anomalous magnetoelectric responses similar to topological insulators [3], such as the magnetization induced by the electric field, $\mathbf{M} = C\langle\pi^0(\mathbf{x})\rangle\mathbf{E}$, and the electric polarization induced by the magnetic field, $\mathbf{P} = C\langle\pi^0(\mathbf{x})\rangle\mathbf{B}$. When $\langle\pi^0(\mathbf{x})\rangle$ is a constant, however, these modifications do not affect the equations of motion for electromagnetic fields.

condensations [8]. We assume the condition of local electric charge neutrality, $n = 0$, and $\mathbf{j} = \mathbf{0}$.³ We take the variation of the meson condensation in the z direction, $\langle\partial_z\pi^0\rangle > 0$, and consider the perturbation of electromagnetic fields in the xy plane. In this setup, we have $\langle\nabla\pi^0\rangle \cdot \mathbf{B} = 0$, and so $\nabla \cdot \mathbf{E} = 0$ from Eq. (17); the electromagnetic wave is a transverse wave. Below we will consider the electromagnetic wave propagating in the positive z direction.

Combining Eq. (18) and Faraday's law, $\nabla \times \mathbf{E} = -\partial_t\mathbf{B}$, and eliminating \mathbf{B} , we obtain a closed equation in terms of \mathbf{E} as

$$\partial_t^2 \mathbf{E} = v^2 \nabla^2 \mathbf{E} - \frac{C}{\epsilon} \langle \nabla \pi^0 \rangle \times \partial_t \mathbf{E}, \quad (19)$$

where $v \equiv 1/\sqrt{\epsilon\mu}$ is the velocity of light in medium. Without the meson condensation, this reduces to the conventional wave equation for electromagnetic waves.

We then look for the plane-wave solution of the form, $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t + ikz}$, where $\mathbf{E}_0 = E_x \mathbf{e}_x + E_y \mathbf{e}_y$ and $\omega, k > 0$. Substituting it into Eq. (19), we have

$$\omega^2 E_x = v^2 k^2 E_x + i\omega \frac{C}{\epsilon} \langle \partial_z \pi^0 \rangle E_y, \quad (20)$$

$$\omega^2 E_y = v^2 k^2 E_y - i\omega \frac{C}{\epsilon} \langle \partial_z \pi^0 \rangle E_x. \quad (21)$$

From these equations, we get the dispersion relation,

$$\omega^2 - v^2 k^2 \pm \frac{C}{\epsilon} \langle \partial_z \pi^0 \rangle \omega = 0. \quad (22)$$

Here the signs \pm correspond to the positive and negative helicity states of the photon, $E_y = \pm iE_x$, which we denote as $\lambda = \pm 1$.

For $k \gg C\langle\partial_z\pi^0\rangle$, the dispersion relation (22) reduces to $\omega = vk$ independently of the helicity. This is the same dispersion as the usual electromagnetic wave in medium; the wavelength $1/k$ is so short that the wave is not affected by the structure of the meson condensation.

For $k \ll C\langle\partial_z\pi^0\rangle$, on the other hand, the dispersion relation (22) are different depending on the helicity; the dispersion relations for $\lambda = \pm 1$ are given by⁴

$$\omega_+ = \frac{v^2 k^2}{m^*}, \quad \omega_- = m^*, \quad (23)$$

³When the Ohmic current is added, the dispersion relation that we will derive below has the imaginary part and the electromagnetic wave is damped by dissipation.

⁴For the electromagnetic wave propagating in the negative z direction ($\omega < 0$ and $k > 0$), we have $\omega_+ = -m^*$ and $\omega_- = -v^2 k^2/m^*$.

respectively, where $m^* = C\langle\partial_z\pi^0\rangle/\epsilon$. The photon with $\lambda = +1$ is gapless, $\lim_{k\rightarrow 0}\omega_+(k) = 0$, while the one with $\lambda = -1$ is gapped. Note also that the former has the “nonrelativistic” dispersion. In this way, the spatially varying meson condensation can distinguish between the two helicity states of photons.

This should be contrasted with Maxwell-Chern-Simons theories (topologically massive gauge theories) in two spatial dimensions [16] and superconductors. In these systems, the dispersion relation of photons is $\omega^2 = k^2 + m^2$ with m the screening mass, which reduces to a gapped dispersion $\omega = \pm m$ for $k \ll m$ and is not the same as Eq. (23).

It is known that the usual photons with the dispersion relation $\omega^2 = k^2$ in the vacuum can be understood as the gapless Nambu-Goldstone (NG) modes [17,18]. The nonrelativistic gapless photons in Eq. (23) may be understood as the so-called type-II NG modes [19]⁵ due to the presence of spatially varying meson condensation, in a way similar to Ref. [23]. This will be reported elsewhere.

V. DISCUSSION

In this paper, we argued that the low-energy effective theory for electromagnetic fields in spatially varying meson condensations is given by the axion electrodynamics. This is analogous to the physics at the interface between topological and ordinary insulators. We also found that such meson condensations can distinguish between the helicity states of photons, and, in particular, one of them is a nonrelativistic gapless photon. Our finding suggests that, if such phases are realized in neutrons stars, the

⁵See also Refs. [20–22] for the general counting rule of type-I and type-II NG modes.

electromagnetic properties of neutron stars should be modified by the quantum Hall effect.

In this paper, we consider the time-independent (but space-dependent) meson condensation. If the meson condensation is time-dependent, $\langle\partial_t\pi^0\rangle \neq 0$, the chiral magnetic effect of the form $\mathbf{j} \sim \langle\partial_t\pi^0\rangle\mathbf{B}$ would lead to an unstable mode, which then tends to erase the time dependence of $\langle\pi^0\rangle$ and generates helical magnetic fields. This is an analogue of the chiral instability [24,25] in the meson condensation, which may have relevance to the origin of the strong magnetic fields in magnetars [26,27].

Finally, we emphasize once again that essentially the same physics should also be realized at the surface of topological insulators. It would be interesting to experimentally observe the nonrelativistic gapless photons, which is a novel signature of the “axion field” θ , in condensed matter systems.

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Note added.—After this work was being completed, we learned that a closely related observation was found in Ref. [28], where the axion electrodynamics emerges in the dual chiral density wave (DCDW). While Ref. [28] uses an NJL-type model and considers the DCDW, our argument is based on the low-energy effective theory and is model-independent; our assumption is rather the presence of the gradient of a meson field alone, and it does not necessarily require the CDW itself. Also, the nonrelativistic gapless photons found in this paper were not discussed in Ref. [28].

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